Order (2022) 39:1–6
https://doi.org/10.1007/s11083-021-09559-2

Invariant Computation in a Poset

Convergence to a Chain

DJamel Talem[1] · Bachir Sadi[1]

Received: 6 October 2018 / Accepted: 18 February 2021 / Published online: 10 March 2021

Abstract
Kong and Ribemboim (1994) define for every poset P a sequence $P = D^0(P)$, $D(P)$, $D^2(P)$, $D^3(P)\dots$ of posets, where $D^i(P) = D(D^{i-1}(P))$ consists of all maximal antichains of $D^{i-1}(P)$. They prove that for a finite poset P, there exists an integer $i \geq 0$ such that $D^i(P)$ is a chain. In this paper, for every finite poset P, we show how to calculate the smallest integer i for which $D^i(P)$ is a chain.

Keywords Partially ordered set · Chain · Antichain

1 Introduction

Let P be a partially ordered set. By $D(P)$, we denote the set of maximal antichains of P. We define an order on $D(P)$ as follows: for $A, B \in D(P)$, $A <_{D(P)} B$ if and only if for each a in A, there exists b in B such that $a <_P b$. In the same way, we define an order on the set of maximal antichains of $D(P)$, and so on. So, we generate a sequence of posets $P, D(P), D^2(P), \dots, D^i(P), \dots$, where $D^i(P) = D(D^{i-1}(P))$ consists of all maximal antichains of $D^{i-1}(P)$.

T. Y. Kong and P. Ribemboim [4], proved that for a finite ordered set P, the above sequence converges to a total order. More precisely, there exists a smallest natural number i, noted $cdev(P)$, such that $D^{cdev(P)}(P)$ is a chain. We have $cdev(P) \leq 2ht(P) + 1$, where $ht(P)$ denotes the height of P. In this paper, for a finite poset P which is neither an antichain nor a linear sum, we show that the value of the parameter $cdev(P)$ is given by the distance in the incomparability graph of P between two additional, artificial vertices 0_P and 1_P, where 0_P is adjacent to all minimal elements of P and 1_P is adjacent to all maximal elements of P. We also show how to handle the latter two particular cases. Finally, we

✉ DJamel Talem
talemdja@yahoo.com

Bachir Sadi
sadibach@yahoo.fr

1 Operation Research Laboratory and Mathematics of Decision, University Mouloud MAMMERI, Tizi Ouzou, Algeria

show that the parameter "$cdev$" is a comparability invariant, i.e orders with the same comparability graph have the same "$cdev$". See [1, 2] for more information on comparability invariants of finite orders.

2 Definitions and Notations

In this paper, $(P, \leq_P)$ denotes a finite poset. We write $x \sim_P y$ and call the pair comparable if either $x <_P y$ or $y <_P x$. Otherwise, we write $x \|_P y$ and call them incomparable. We denote $\oplus_{1 \leq i \leq h} P_i$ the linear sum of posets $(P_1, \leq_1), (P_2, \leq_2), \ldots, (P_h, \leq_h)$ [1].

For every $x \in P$, we define $Succ(x) = \{y \in P, x <_P y\}$, $Pred(x) = \{y \in P, y <_P x\}$, $MaxP = \{x, Succ(x) = \emptyset\}$ and $MinP = \{x \in P, Pred(x) = \emptyset\}$.

Denote by $(D(P), <_{D(P)})$ the poset on the set of maximal antichains of P which is defined as follows: for $A, B \in D(P)$, $A <_{D(P)} B$ if and only if for each a in A, there exists b in B such that $a <_P b$. It has been shown in [4] that this definition is equivalent to this one: $A <_{D(P)} B$ if and only if for each b in B, there exists a in A such that $a <_P b$.

Lemma 1 *If A and B are incomparable in $D(P)$, then $\forall a \in A, \exists b \in B$ such that $a \|_P b$*

Proof If $A \cap B \neq \emptyset$, it is obvious.

If $A \cap B = \emptyset$, by definition of $D(P)$, and since A and B are incomparable, then $\exists a_1, a_2 \in A$ and $\exists b_1, b_2 \in B$ such that $a_1 <_P b_1$ and $b_2 <_P a_2$.

Suppose that $\exists a \in A$ such that $a \sim_P b, \forall b \in B$. Since B is an antichain, it results that $a <_P b, \forall b \in B$ or $b <_P a, \forall b \in B$.

If $a <_P b, \forall b \in B$, then $a <_P b_2$, and so $a <_P a_2$; if $b <_P a, \forall b \in B$, then $b_1 <_P a$, and so $a_1 <_P a$. In both cases, we have a contradiction with the fact that A is an antichain. $\square$

Given a poset P, we generate a sequence $(D^n(P))_{n \geq 0}$ of orders by setting $D^0(P) = P$, and $D^{n+1}(P) = D(D^n(P))$ for every $n \geq 0$. Kong and Ribenboim proved that for a finite poset P, there exists $n \geq 0$ such that $D^n(P)$ is a total order. We denote by "$cdev(P)$" the smallest integer such that $D^{cdev(P)}(P)$ is a total order. Figure 1 shows how the sequence of posets $P, D(P), D^2(P), \ldots, D^i(P)$ is formed. In this example, $D^3(P)$ is a chain. This means that $cdev(P) = 3$.

Remark 1 Note that for any poset P, $MinP \in D(P)$. Moreover, by definition of the poset $D(P)$, $\forall A \in D(P)$, $Pred_{D(P)}(A) = \emptyset$ if and only if $A \cap MinP \neq \emptyset$. Therefore $MinD(P) = \{A \in D(P), A \cap MinP \neq \emptyset\}$. Similarly, $MaxD(P) = \{A \in D(P), A \cap MaxP \neq \emptyset\}$.

Let P be a poset and let $Inc(P)$ be its incomparability graph. We define the graph $G(P)$ in adding to the graph $Inc(P)$ two vertices 0_P and 1_P such that 0_P is adjacent to all elements of $MinP$ and 1_P is adjacent to all elements of $MaxP$ (see Fig. 2). For $x, y \in G(P)$, $d_{G(P)}(x, y)$ denotes the distance between x and y in the graph $G(P)$.

Remark 2 Note that the graph $Inc(P)$ is connected if and only if P is not a linear sum of posets and so the distance between vertices 0_p and 1_P in the graph $G(P)$ is defined if and only if P is not a linear sum of posets. In this case, it is easy to see that if μ is a shortest

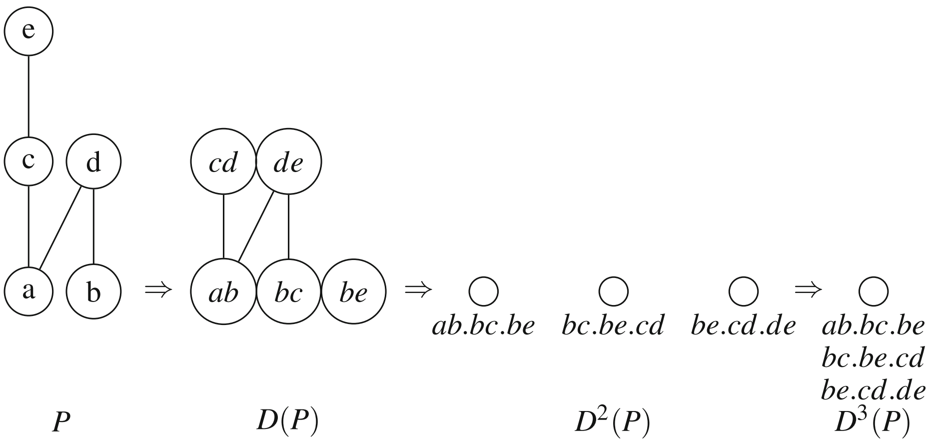

Fig. 1 A sequence of posets obtained from order P

path between 0_P and 1_P in $G(P)$, then it satisfies $|\mu \cap MinP| = |\mu \cap MaxP| = 1$, since the vertices 0_P, 1_P are adjacent to any vertex of $MinP$ and $MaxP$ respectively.

3 Characterization of the Parameter *cdev*(*P*)

The interest of the graph $G(P)$ is to show that $cdev(P)$ equals the distance between vertices 0_P and 1_P in $G(P)$ if P is neither an antichain nor a linear sum of posets (Theorem 2). Now,

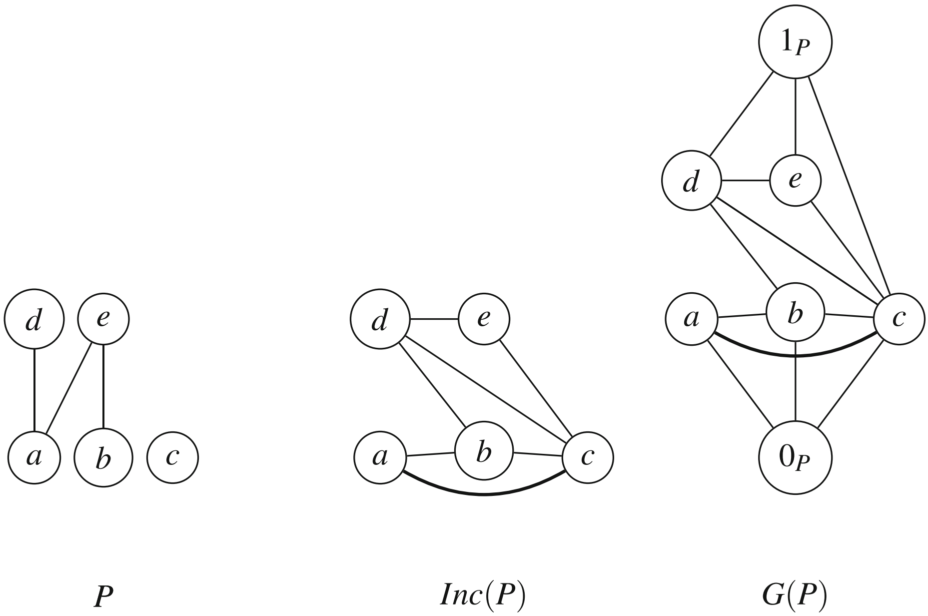

Fig. 2 A poset P and its associated graphs

it is clear that $cdev(P) = 0$ if P is a chain and $cdev(P) = 1$ if P is an antichain having at least two elements. However, if P is a linear sum of posets $P_1, P_2 \dots P_h$, Theorem 1 shows that there exists $i \in \{1, \dots h\}$ such that $cdev(P) = cdev(P_i)$.

Theorem 1 *Given a poset P. If P is a linear sum of posets $(P_1, \leq_1), (P_2, \leq_2), \dots, (P_h, \leq_h)$, then:*

$$cdev(P) = \max\{cdev(P_i), 1 \leq i \leq h\}.$$

Proof Let $P = \oplus_{1 \leq i \leq h} P_i$ and $\alpha = \max\{cdev(P_i), 1 \leq i \leq h\}$. Note that $A \in D(P)$ if and only if $\exists i \in \{1, 2, \dots h\}$ such that $A \in D(P_i)$, so $D(P) = \oplus_{1 \leq i \leq h} D(P_i)$. It resultes that $D^{\alpha}(P) = \oplus_{1 \leq i \leq h} D^{\alpha}(P_i)$ is a chain. Therefore, $cdev(P) = \alpha = \max\{cdev(P_i), 1 \leq i \leq h\}$. □

Remark 3 By definition of the graph $G(P)$, it is easy to see that $d_{G(P)}(0_P, 1_P) = 2$ if and only if P contains at least one isolated element.

Lemma 2 *Given a poset P which is not an antichain. Then $d_{G(P)}(0_P, 1_P) = 2$ if and only if $D(P)$ is an antichain having at least two elements.*

Proof By definition of the graph $G(P)$ and since P is not an antichain, $d_{G(P)}(0_P, 1_P) = 2$ implies, firstly $MaxP \neq MinP$ and so $D(P)$ contains at least two elements which are $MinP$ and $MaxP$, secondly there exists at least one isolated element x of P and so x belongs to all maximal antichains of P, i.e. the maximal antichains of P are pairwise incomparable in $D(P)$. Hence, $D(P)$ is an antichain having at least two elements.

Conversely, if $D(P)$ is an antichain having at least two elements then, firstly P is not an antichain, otherwise $D(P)$ would be a chain, secondly P contains at least one isolated element, otherwise $MinP <_{D(P)} MaxP$ and so $D(P)$ would not be an antichain. Hence, $d_{G(P)}(0_P, 1_P) = 2$. □

For $n \geq 2$, let $\mu = (0_{D(P)}, A_1, A_2 \dots A_n, 1_{D(P)})$ be a shortest path between $0_{D(P)}$ and $1_{D(P)}$ in $G(D(P))$. By definition, and according to Remark 2, we have $A_1 \in MinD(P)$ and $A_2 \notin MinD(P)$. This shows that $MinP <_{D(P)} A_2$ (Remark 1), and so the maximal antichain $MinP$ can not belong to the shortest path between $0_{D(P)}$ and $1_{D(P)}$. For the same reasons, the antichain $MaxP$ does not belong to this shortest path.

Lemma 3 *If P is a poset such that $d_{G(P)}(0_P, 1_P) > 2$, then:*

$$d_{G(P)}(0_P, 1_P) = d_{G(D(P))}(0_{D(P)}, 1_{D(P)}) + 1.$$

Proof Let $\mu = (0_P, a_1 \dots a_n, 1_P)$ be a shortest path between 0_P and 1_P in $G(P)$. Then $n > 1$ because $d_{G(P)}(0_P, 1_P) > 2$, and according to Lemma 2, $D(P)$ is not an antichain. Furthermore, $D(P)$ is not a chain because $a_2 \notin MinP$ (Remark 2) and so there exists an antichain A in P such that $\{a_1, a_2\} \subseteq A$ and the maximal antichains A, $MinP$ are incomparable in $D(P)$. It results that $\mu' = (0_{D(P)}, A_1, \dots, A_{n-1}, 1_{D(P)})$, where $\{a_i, a_{i+1}\} \subseteq A_i$ for $i = 1 \dots n-1$ is a path in $G(D(P))$. Therefore, $d_{G(D(P))}(0_{D(P)}, 1_{D(P)}) \leq d_{G(P)}(0_P, 1_P) - 1$.

Conversely, let $\mu' = (0_{D(P)}, A_1, \dots, A_{n-1}, 1_{D(P)})$ be a shortest path between $0_{D(P)}$ and $1_{D(P)}$ in $G(D(P))$, and let $a_1 \in A_1 \cap MinP$. Then, by Lemma 1, for $i = 2, \dots n-1$, $\exists a_i \in A_i$ such that $a_{i-1} \|_P a_i$. Note that $a_{n-1} \notin MaxP$, otherwise, $\mu" = (0_{D(P)}, B_1, \dots, B_{n-2}, 1_{D(P)})$, where $\{a_i, a_{i+1}\} \subseteq B_i$ for $i = 1 \dots n-2$

would be smaller than μ'. So, by Lemma 1, $\exists a_n \in MaxP$ such that $a_{n-1} \|_P a_n$, and so $(0_P, a_1, \ldots, a_{n-1}, a_n, 1_P)$ is a path in $G(P)$. This implies that $d_{G(P)}(0_P, 1_P) \leq d_{G(D(P))}(0_{D(P)}, 1_{D(P)}) + 1$. Hence, $d_{G(P)}(0_P, 1_P) = d_{G(D(P))}(0_{D(P)}, 1_{D(P)}) + 1$. □

Theorem 2 *Given a poset P. If P is neither an antichain nor a linear sum of posets, then:*

$$cdev(P) = d_{G(P)}(0_P, 1_P)$$

Proof Note that for every poset P, $cdev(P) = 1 + cdev(D(P))$.

We use induction on $d_{G(P)}(0_P, 1_P)$. If $d_{G(P)}(0_P, 1_P) = 2$, and since P is not an antichain then $D(P)$ is an antichain having at least two elements (Lemma 2), and so $cdev(D(P)) = 1$. Therefore $cdev(P) = 1 + 1 = d_{G(P)}(0_P, 1_P)$.

Assume that for every order P such that $d_{G(P)}(0_P, 1_P) = m \geq 2$, $cdev(P) = d_{G(P)}(0_P, 1_P)$. Let Q be a poset such that $d_{G(Q)}(0_Q, 1_Q) = m + 1$. By Lemma 3, we have $d_{G(Q)}(0_Q, 1_Q) = 1 + d_{G(D(Q))}(0_{D(Q)}, 1_{D(Q)})$, so $d_{G(D(Q))}(0_{D(Q)}, 1_{D(Q)}) = m = cdev(D(Q)$ (by hypothesis). It results that $cdev(Q) = 1 + cdev(D(Q)) = 1 + d_{G(D(Q))}(0_{D(Q)}, 1_{D(Q)}) = d_{G(Q)}(0_Q, 1_Q)$. □

Remark 4 According to Theorems 1 and 2, if P is a linear sum of posets $P_1, P_2 \ldots P_h$ such that there exists i for which P_i is not an antichain, then :

$$cdev(P) = max\{d_{G(P_i)}(0_{P_i}, 1_{P_i}), 0 \leq i \leq h\},$$

where $G(P_i)$ is the graph of P_i as defined already. If P is a linear sum of antichains $A_1, \ldots A_h$ (in this case P is called a weak order [6]), Theorem 1 implies that $cdev(P) = cdev(A_1) = \ldots = cdev(A_h) = 1$ (the same result was found in [5]).

It remains to show that "$cdev$" is a comparability invariant, i.e. if P and Q are two posets with the same comparability graph, then $cdev(P) = cdev(Q)$. For that, it suffices to show that P and Q differ only by the reversal of some order autonomous subset S (see [3]).

Remark 5 Let S be an order autonomous subset of the poset P, and let Q be the poset resulting from the reversal of S in P.

1. By definition of S, if $S \cap MinP \neq \emptyset$ (resp. $S \cap MaxP \neq \emptyset$) then $S^d \cap MinQ \neq \emptyset$ (resp. $S^d \cap MaxQ \neq \emptyset$);
2. If $S \neq P$ and $\mu = (0_P, a_1, \ldots, a_n, 1_P)$ is a shortest path from 0_P to 1_P in $G(P)$ then $|\mu \cap S| \leq 1$. In fact, for example, if a_i, a_j in S, with $i < j$ then a_{i-1} and a_j (or a_i and a_{j+1}) are incomparable and so $\mu' = \mu \setminus \{a_i\}$ (or $\mu" = \mu \setminus \{a_j\}$ respectively) would be a path from 0_P to 1_P with length smaller.

Corollary 1 *The parameter "cdev" is a comparability invariant.*

Proof Let S be an order autonomous subset of the poset P, and let Q be the poset resulting from the reversal of S in P. To show that $cdev(P) = cdev(Q)$, it suffices to show that for any shortest path μ in $G(P)$ from 0_P to 1_P there is some path in $G(Q)$ from 0_Q to 1_Q with length at most the length of μ.

Let $\mu = (0_P, a_1 \ldots a_n, 1_P)$ be a shortest path between 0_P and 1_P in $G(P)$. If $S = P$, then $Q = P^d$ and so $\mu' = (0_Q, a_n, a_{n-1} \ldots a_2, a_1, 1_Q)$ is a path from 0_Q to 1_Q in $G(Q)$. If $S \neq P$, There are two cases: if $\mu \cap S = \emptyset$ or $\mu \cap S = \{a_i\}$, with a_i is neither a minimal nor a maximal element in P (in other words, $a_i \neq a_1$ and $a_i \neq a_n$), then

$\mu' = (0_Q, a_1, \ldots a_n, 1_Q)$ forms a path in $G(Q)$; if $\mu \cap S = \{a_1\}$ (resp. $\mu \cap S = \{a_n\}$), then according to Remark 5, there is some element s in $MinQ \cap S^d$ (resp. in $MaxQ \cap S^d$) such that $\mu' = (0_Q, s, a_2, \ldots a_n, 1_Q)$ (resp. $\mu" = (0_Q, a_1, \ldots, s, 1_Q)$) forms a path in $G(Q)$. $\square$

Data Availability Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

References

1. Caspard, N., Leclerc, B., Monjardet, B.: Finite Ordered Sets: Concepts, Results and Uses. Springer, New York (2007)
2. Habib, M.: Comparability invariants. Ann. Discrete Math. **23**, 371–386 (1984)
3. Kelly, D.: Invariants of finite comparability graphs. Order **3**, 155–158 (1986)
4. Kong, T.Y., Ribemboim, P.: Chaining of Partially Ordered Sets. C. R. Acad. Sci. Paris, t. 319, Série I 533–537 (1994)
5. Sadi, B.: Suite d'Ensembles Partiellement Ordonnés. Arima **4**, 66–71 (2006)
6. Schröder, B. Ordered Sets, 2nd edn. Springer, New York (2016)