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Cambrian Acyclic Domains: Counting c-singletons

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Abstract
We study the size of certain acyclic domains that arise from geometric and combinatorial constructions. These acyclic domains consist of all permutations visited by commuting equivalence classes of maximal reduced decompositions if we consider the symmetric group and, more generally, of all c-singletons of a Cambrian lattice associated to the weak order of a finite Coxeter group. For this reason, we call these sets *Cambrian acyclic domains*. Extending a closed formula of Galambos–Reiner for a particular acyclic domain called Fishburn's alternating scheme, we provide explicit formulae for the size of any Cambrian acyclic domain and characterize the Cambrian acyclic domains of minimum or maximum size.

Keywords Acyclic sets · Enumeration · Generalized permutahedra · Pseudoline arrangements · Sortable elements · Coxeter groups

1 Introduction

Examples of c-singletons include certain acyclic domains in social choice theory, natural partial orders of crossings in pseudoline arrangements as well as certain vertices of particular convex polytopes called permutahedra and associahedra in discrete geometry. We first describe these objects and outline the relationship between these incarnations.

Acyclic domains are of great interest in social choice theory because of their importance for the following voting process: voters choose among a given collection of linear orders on m candidates and the result of the ballot obeys the order imposed by the majority for each pair of candidates. As already mentioned by the Marquis de Condorcet in 1785 [6], not every collection of linear orders yields a transitive order on the candidates in every

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election. Collections that do guarantee transitivity are called *acyclic domains* or *Condorcet domains*. According to Fishburn [11, Introduction], the fundamental problem to determine the maximum cardinality of an acyclic domain for a given number of candidates is one of most fascinating and intractable combinatorial problems in social choice theory. Abello as well as Chameni-Nembua describe different constructions of “large” acyclic sets. They use maximal chains of the weak order on the symmetric group Σ_m [1] and study covering distributive sublattices of the weak order on Σ_m [5].

Galambos and Reiner [13] prove that maximal acyclic domains constructed by Abello coincide with those of Chameni-Nembua and describe them in terms of higher Bruhat orders. Moreover, they show that acyclic domains obtained from Fishburn’s alternating scheme [10] are a special case of Chameni-Nembua’s construction and prove that the cardinality of Fishburn’s acyclic domain is given by

$$\mathsf{fb}(m) = 2^{m-3}(m+3) - \begin{cases} \frac{m-1}{2}\binom{m-1}{\frac{m-1}{2}} & \text{for odd } m, \\ \frac{2m-3}{2}\binom{m-2}{\frac{m-2}{2}} & \text{for even } m. \end{cases} \tag{1}$$

Weakening a conjecture of Fishburn [10, Conjecture 2], Galambos and Reiner conjecture that $\mathsf{fb}(m)$ is a tight upper bound on the cardinality of acyclic sets described in terms of higher Bruhat orders [13, Conjecture 1]. We notice that Knuth had a conjecture related to the one of Galambos and Reiner discussing his Equation (9.8) [22, p. 39]. Felsner and Valtr as well as Danilov, Karzanov and Koshevoy mention counterexamples to these conjectures [7, 9]. Galambos and Reiner base the formula for $\mathsf{fb}(m)$ and their conjecture on counting extensions of a certain pseudoline arrangement by adding a new pseudoline and relate these extensions to elementarily equivalent maximal chains in the weak order on Σ_m.

Planar pseudoline arrangements with contact points as well as pseudo- and multitriangulations are systematically studied by Pilaud and Pocchiola using the framework of *networks* [31]. Subsequently, Pilaud and Santos construct polytopes from a given network and relate their combinatorics to the combinatorics of triangulations of point configurations [32]. For well-chosen networks, they construct associahedra (or Stasheff polytopes) which essentially coincide with a family of realizations obtained from the permutahedron by Hohlweg and Lange [17]. This family provides a geometric interpretation of Reading’s Cambrian lattices [36]. Cambrian lattices are remarkable as they generalize the Tamari lattice as lattice quotient of the weak order on Σ_m in two ways. First, distinct lattice quotients are obtained by choosing different Coxeter elements c and yield distinct realizations of the associahedron from the permutahedron. Second, the construction of distinct lattice quotients extends from the symmetric group Σ_m to the weak order of any finite Coxeter group W. Hohlweg, Lange and Thomas then identify *c-singletons* as fundamental objects of Cambrian lattices and use them to derive distinct polytopal realizations of generalized associahedra from W-permutahedra [18]. Generalized associahedra are CW-complexes defined in the context of cluster algebras of finite type [12] that coincide with associahedra in type A. Finally, Pilaud and Stump extend the construction of polytopes from Pilaud and Santos to any finite Coxeter group and, analogous to type A, essentially reobtain realizations of generalized associahedra discovered by Hohlweg, Lange and Thomas [33] which was generalized and further studied by Hohlweg, Pilaud and Stella [19].

Two interpretations of c-singletons described in [18] are fundamental for our work. First, the geometric construction of generalized associahedra from W-permutahedra exhibits c-singletons as the common vertices of both polytopes and, second, c-singletons are combinatorially described as prefixes of a certain reduced expression for the longest element

$w_\circ \in W$ up to commutations. For Coxeter groups of type A, the latter interpretation translates to higher Bruhat orders: the set of c-singletons for a fixed Coxeter element $c \in \Sigma_m$ is precisely the set of all elements $w \in \Sigma_m$ visited by the maximal chains contained in a certain equivalence class of elementarily equivalent maximal chains determined by c. Galambos and Reiner showed in type A that these elements coincide with certain maximal acyclic domains and for this reason we define a Cambrian acyclic domain as the set of c-singletons for a given Coxeter element c of a finite Coxeter group W. The main results of this article are

- Theorem 3 that provides a combinatorial description for the cardinality of a Cambrian acyclic domain for any finite Coxeter system (W, S) and any Coxeter element c.
- Theorem 4 that characterizes the possible choices of c to minimize and maximize the cardinality of a Cambrian acyclic domain for any finite Coxeter system (W, S).

These results solve Problem 3.1 of [16]. Even though we mentioned above that the conjecture of Galambos and Reiner is not true in general, Theorem 4 proves that the conjecture holds if it is restricted to the large subclass of acyclic domains: Fishburn's alternating scheme yields the maximum cardinality for Cambrian acyclic domains of type A.

The article is organized as follows. In Section 2, we summarize and discuss the objects and results in type A. Sections 2.1–2.3 provide a unified description of Cambrian acyclic domains as geometric entities in terms of vertices of convex polytopes, as pseudoline arrangements, and as certain order ideals for type A. Moreover, we derive formulae for the cardinality of Cambrian acyclic domains and give a new proof of Eq. 1 using hypergeometric sums in Section 2.4. Section 3 generalizes the discussion from type A to other finite types. We introduce and discuss relevant notions in Sections 3.1–3.5 before proving Theorem 3 in Section 3.6. More precisely, a poset called *natural partial order* by Galambos and Reiner [13] as well as *heap* by Viennot [43] and Stembridge [41] is introduced in Section 3.1. In Section 3.2 we introduce c-singletons of a finite Coxeter system (W, S) and show that the weak order on c-singletons is isomorphic to the lattice of order ideals of a well-chosen natural partial order. In Section 3.3, Hasse diagrams of natural partial orders are embedded in a cylindrical oriented graph that we call 2-cover. The 2-cover replaces the network used in type A as framework to count c-singletons in arbitrary type. The extension of a pseudoline arrangements considered by Galambos and Reiner in type A is replaced by cut paths introduced in Section 3.3. It turns out that the total number of cut paths in the 2-cover exceeds the size S_c of Cambrian domains and the difference can be expressed in terms of "crossing" cut paths discussed in Section 3.4. In Section 4, we illustrate Theorem 3: we explicitly compute the cardinality of Cambrian acyclic domains for various finite types and different choices of Coxeter elements. In Section 5, we finally derive lower and upper bounds for the cardinality of Cambrian acyclic domains. The examples discussed in Section 4 cover all possibilities to minimize and maximize the size S_c of Cambrian domains.

We assume familiarity with basic notions of convex polytopes and of Coxeter group theory and refer to [45] as well as [20] for details.

2 Associahedra, Pseudoline Arrangements and *c*-singletons in Type *A*

This section presents c-singletons in three different ways as well as a counting strategy for them in type A. It should be thought as a preparation for the general definitions in Sections 3 and 5.

2.1 Associahedra and *c*-singletons

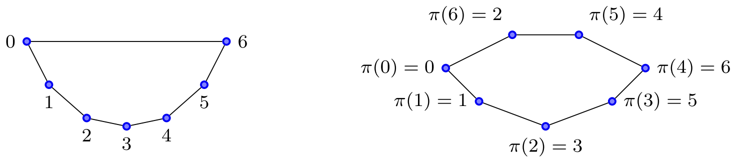
An associahedron is a simple convex polytope of a particular combinatorial type. The underlying combinatorial structure relates to various branches of mathematics as mentioned in [27, 42] or [40]. We follow Lee and consider triangulations of a convex $(n+3)$-gon to define the combinatorics of an *n-dimensional associahedron* [24]. A plethora of distinct polytopal realizations is known for the associahedron, e.g. [3, 8, 14, 15, 17, 32], we focus on a family of realizations described by Hohlweg and Lange [17, 23] that generalizes [25] and relates directly to triangulations of a labeled $(n+3)$-gon P and to the symmetric group Σ_{n+1}. The resulting n-dimensional associahedra and the labelings of P are parametrized by Coxeter elements $c \in \Sigma_{n+1}$. We refer to the labeled polygons as P_c and to the various polytopal realizations of associahedra as Asso_c.

Assume that P is a convex $(n+3)$-gon in the plane with no two vertices on a vertical line. To obtain the labeled polygon P_c, we label the vertices of P from smallest to greatest x-coordinate using the integers 0 to $n+2$. Without loss of generality, we assume that the vertices labeled 0 and $n+2$ lie on a horizontal line. This induces a partition of the $(n+1)$-element set $\{1, 2, \dots, n+1\}$ into a down set $D_c = \{d_1 < d_2 < \cdots < d_k\}$ and an up set $U_c = \{u_1 < u_2 < \cdots < u_\ell\}$ where the vertices in U_c lie in the upper hull of P_c and the vertices in D_c in the lower hull. We have special notation in the following two situations. If $U_c = \varnothing$ then we replace the subscript c by Lod to remind of Loday who gave a combinatorial interpretation of the vertex coordinates of $\mathsf{Asso}_{\mathrm{Lod}}$ [25]. If $U_c = \{d \in \mathbb{N} \mid 0 < d < n+2 \text{ and } d \text{ even}\}$ then we replace the subscript c by “alt”. This reminds of alternating (or bipartite) Coxeter elements and relates to Fishburn’s alternating scheme.

The labeled $(n+3)$-gons P_c are characterized by certain permutations π_c with one peak which describe a relabeling to obtain P_c from P_{Lod}, see Fig. 1 for two examples.

Moreover, the set of labeled $(n+3)$-gons P_c is in bijection with orientations of Coxeter graphs of type A and all Coxeter elements c of Σ_{n+1} [38]. This bijection is crucial to extend the construction of associahedra to generalized associahedra for arbitrary finite Coxeter groups [18].

Any proper diagonal δ of P_c yields a facet-defining inequality $H_{\geq}^{\delta}$ for Asso_c as follows. Let B_δ be the label set of vertices of P_c which lie strictly below the line supporting δ and include the endpoints of δ which are in U_c and set $\left[H_{\geq}^{\delta} := \left\{x \in \mathbb{R}^{n+1} \;\middle|\; \sum_{i \in B_\delta} x_i \geq \binom{|B_\delta|+1}{2}\right\}\right]$.



$U_{\mathrm{Lod}} = \varnothing$ and $D_{\mathrm{Lod}} = \{1, 2, 3, 4, 5\}$ $\quad$ $U_{\mathrm{alt}} = \{2, 4\}$ and $D_{\mathrm{alt}} = \{1, 3, 5\}$

$\pi_{\mathrm{Lod}} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$ $\quad$ $\pi_{\mathrm{alt}} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 1 & 3 & 5 & 6 & 4 & 2 \end{pmatrix}$

Fig. 1 Two examples of labeled heptagons P_c

Theorem 1 ([17, Proposition 1.3],[23, Theorem 7])
For every labeled $(n+3)$-gon P_c, the polytope

$$\mathsf{Asso}_c = \left\{ x \in \mathbb{R}^{n+1} \;\middle|\; \begin{array}{c} \sum_{i \in [n+1]} x_i = \frac{(n+1)(n+2)}{2} \text{ and} \\ x \in H^{\delta}_{\geq} \text{ for any proper diagonal } \delta \text{ of } P_c \end{array} \right\}$$

is a particular realization of an n-dimensional associahedron.

Each associahedron Asso_c is an instance of a generalized permutahedron, introduced by Postnikov, as it is obtained from the classical permutahedron

$$\begin{aligned} \mathsf{Perm}_n &= \mathsf{conv}\left\{ (\pi(1), \ldots, \pi(n+1))^\top \in \mathbb{R}^{n+1} \;\middle|\; \pi \in \Sigma_{n+1} \right\} \\ &= \left\{ x \in \mathbb{R}^{n+1} \;\middle|\; \begin{array}{c} \sum_{i \in [n+1]} x_i = \frac{(n+1)(n+2)}{2} \text{ and} \\ \sum_{i \in I} x_i \geq \binom{|I|+1}{2} \text{ for any nonempty } I \subset [n+1] \end{array} \right\} \end{aligned}$$

by discarding some facet-defining inequalities [34, 35]. Following [18, Section 2.3], a *c-singleton* is a common vertex of Perm_n and of Asso_c.

From Fig. 2, where the 3-dimensional polytopes Perm_4, $\mathsf{Asso}_{\mathrm{Lod}}$ and $\mathsf{Asso}_{\mathrm{alt}}$ are shown, it is immediate that the number of c-singletons as well as the number of paths from $(1,2,3,4)^\top$ to $(4,3,2,1)^\top$ in the 1-skeleton of Asso_c visiting only c-singletons depends on c. For later use, we remark that the realization of Asso_c is completely determined by $U_c \cap \{2, 3, \ldots, n+1\}$ and assume without loss of generality

$$D_c = \{d_1 = 1 < d_2 < \cdots < d_k\} \qquad \text{and} \qquad U_c = \{u_1 < u_2 < \cdots < u_\ell\}.$$

2.2 Pseudoline Arrangements and c-singletons

Using the duality of points and lines in the Euclidean plane, we now describe Asso_c and c-singletons in terms of pseudoline arrangements, see [31] and [32] for details. We visualize pseudoline arrangements on an alternating sorting network $\mathcal{N}_c$ that encodes the combinatorics of the point configuration of P_c: $\mathcal{N}_c$ consists of $n+3$ horizontal lines and $\binom{n+3}{2} = \frac{(n+3)(n+2)}{2}$ *commutators* which are vertical line segments connecting consecutive horizontal lines in an alternating way. A commutator is at *level* i if it connects the horizontal lines i and $i+1$ of $\mathcal{N}_c$ (counted from bottom to top starting with 0). Additionally, we

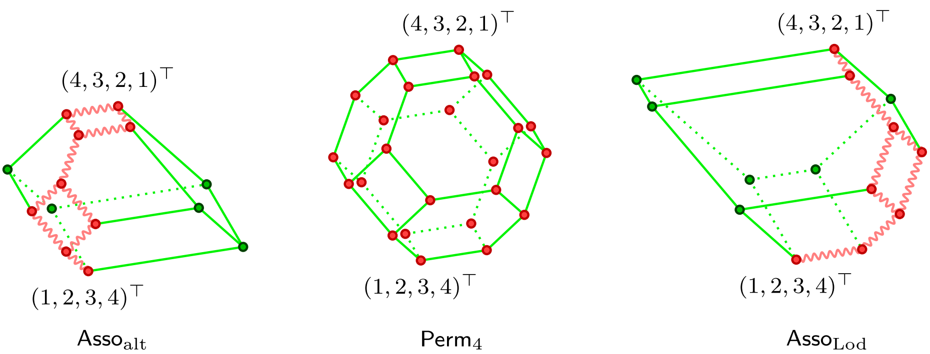


Fig. 2 The permutahedron Perm_4 and the associahedra $\mathsf{Asso}_{\mathrm{alt}}$ and $\mathsf{Asso}_{\mathrm{Lod}}$. Red vertices of the associahedra indicate c-singletons and maximal paths from $(1,2,3,4)^\top$ to $(4,3,2,1)^\top$ along red zig-zag edges of associahedra correspond to elementarily equivalent maximal chains in the weak order

label the ends of the horizontal lines from 0 to $n+2$ bottom to top at the left end of $\mathcal{N}_c$ and from 0 to $n+2$ top to bottom at the right end of $\mathcal{N}_c$. Figure 3 illustrates these notions for $\mathcal{N}_{\text{Lod}}$ and $\mathcal{N}_{\text{alt}}$ which correspond to P_{Lod} and P_{alt} of Fig. 1.

Now, a *pseudoline* (supported by $\mathcal{N}_c$) is an abscissa monotone path on $\mathcal{N}_c$ starting on the left at some label and ending right at the same label, i.e. traversing the path yields monotone x-coordinates. A pseudoline supported by $\mathcal{N}_c$ is labeled i if it starts on the left on the horizontal line i. A *pseudoline arrangement* on $\mathcal{N}_c$ is a collection of pseudolines such that any pair of pseudolines intersects precisely along one commutator. A commutator traversed by two pseudolines of a pseudoline arrangement is called a *crossing*. For example, there is a unique pseudoline arrangement with $n+3$ pseudolines on $\mathcal{N}_c$ and every commutator is a crossing of this arrangement. This arrangement induces a labeling of all commutators of $\mathcal{N}_c$ by the two unique pseudolines that traverse it, see Fig. 3. A pseudoline arrangement on $\mathcal{N}_c$ with fewer than $n+3$ pseudolines has commutators that are traversed by no pseudoline but touched in its end points by at most two pseudolines. A commutator that is touched by two pseudolines and traversed by none is called a *contact*. The reader may prove the following facts:

i) labeled commutators of $\mathcal{N}_c$ at levels 0 and $n+1$ are in bijection to boundary diagonals of P_c.
ii) $\mathcal{N}_c$ is determined by $\mathcal{N}_{\text{Lod}}$ and π_c. The inversions of π_c label commutators in the bottom right of $\mathcal{N}_{\text{Lod}}$ which can be moved to the upper left part if we temporarily consider $\mathcal{N}_{\text{Lod}}$ as Möbius strip by identifying its sides. Relabeling the commutators by π_c yields $\mathcal{N}_c$.

The *1-kernel* $\mathcal{K}_c$ of the network $\mathcal{N}_c$ is the network obtained from $\mathcal{N}_c$ by deletion of the horizontal lines 0 and $n+2$ as well as all commutators touching these lines. On $\mathcal{K}_c$, we use notions induced by $\mathcal{N}_c$, for example, the level of a commutator, its label or the label of a pseudoline are inherited from $\mathcal{N}_c$. Triangulations of P_c are now in bijection to pseudoline arrangements with $n+1$ pseudolines supported by $\mathcal{K}_c$: diagonals of a triangulation correspond to the contacts of a unique pseudoline arrangement on $\mathcal{K}_c$ [31, Theorem 23]. The simple fact that a commutator labeled by the endpoints of a proper diagonal δ of P_c is at level $|B_\delta|$ extends [17, Proposition 1.4] and [23, Proposition 20] by statement c) below:

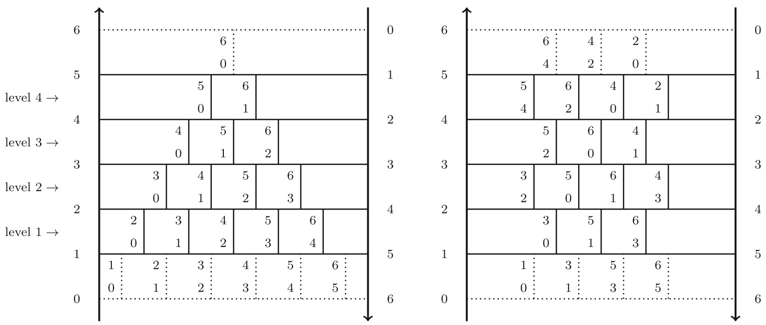


Fig. 3 Two networks for pseudoline arrangements with labeled commutators: $\mathcal{N}_{\text{Lod}}$ (left) and $\mathcal{N}_{\text{alt}}$ (right) correspond to the labeled heptagons of Fig. 1. The 1-kernels $\mathcal{K}_{\text{Lod}}$ and $\mathcal{K}_{\text{alt}}$ are obtained by deletion of the dotted line segments

Proposition 1 *Let v be a vertex of* Asso_c *with corresponding triangulation T_v of P_c and let C_v be the set of commutators of $\mathcal{N}_c$ labeled by proper diagonals of T_v. The following statements are equivalent:*

a) *v is a vertex of* Perm_n.
b) *The proper diagonals δ_i of T_v can be ordered such that*

$$\varnothing \subset B_{\delta_1} \subset \ldots \subset B_{\delta_n} \subset [n+1].$$

c) *C_v contains one commutator from each level of $\mathcal{K}_c$ and commutators from consecutive levels are adjacent.*

Proposition 1 shows that a c-singleton for Asso_c corresponds to a path which traverses the 1-kernel $\mathcal{K}_c$ from bottom to top and ascents whenever possible, that is the y-coordinates are monotone increasing. We call such a path a *greedy ordinate monotone path* on $\mathcal{K}_c$. In the theory of pseudoline arrangements, a greedy ordinate monotone path is an instance of a "cut" or "pseudoline from the south pole to the north pole" which extends the original arrangement by a new pseudoline. We avoid the term "cut" in this context and emphasize tha a greedy ordinate monotone path corresponds to a cut path in Section 3.

2.3 Order Ideals and *c*-singletons

Combining Proposition 1 with [18, Theorem 2.2], c-singletons can be described using neighbouring transpositions $s_i = (i \;\; i+1)$ for $1 \le i \le n$. A permutation $\pi \in \Sigma_{n+1}$ is a c-singleton of Asso_c if and only if there is a reduced word for π in the generators $s_1, \ldots, s_n$ that is a prefix up to commutation of a particular reduced expression $\boldsymbol{w}_\circ^{\mathbf{c}}$ of the reverse permutation $w_\circ = [n+1, n, \ldots, 1]$ (given in one-line notation). This point of view will be used in Definition 2 of Section 3.2 to define c-singletons for arbitrary irreducible finite Coxeter systems (W, S). According to Proposition 3, the poset of c-singletons (ordered by the weak order on Σ_{n+1}) is isomorphic to the lattice of order ideals of $(\mathcal{L}_{\boldsymbol{w}_\circ^{\mathbf{c}}}, \prec_{\boldsymbol{w}_\circ^{\mathbf{c}}})$ associated to a Coxeter triple $(W, S, \mathbf{c})$ in type A defined in Section 3.1. To illustrate this aspect of c-singletons in type A, we describe the poset $(\mathcal{S}, \prec_c)$ that is isomorphic to $(\mathcal{L}_{\boldsymbol{w}_\circ^{\mathbf{c}}}, \prec_{\boldsymbol{w}_\circ^{\mathbf{c}}})$ in type A and can be constructed directly from $\mathcal{K}_c$.

The set $\mathcal{S}$ corresponds to the bounded regions of $\mathcal{K}_c$: we first count the bounded regions at level i left-to-right starting with 1 and label the j^{th} bounded region of level i by (s_i, j). Then $\mathcal{S}$ is defined as the set of all such pairs and the partial order $\prec_c$ on $\mathcal{S}$ is the transitive closure of the covering relation $(s_i, j) \to (s_k, l)$ that satisfies the following three conditions:

i) $|i-k| = 1$;
ii) the bounded regions (s_i, j) and (s_k, l) intersect in a (nonempty) horizontal line segment;
iii) the commutator that bounds region (s_i, j) to the left is on the left of the region associated to (s_k, l).

Although closely related, the undirected Hasse diagram of $(\mathcal{S}, \prec_c)$ differs essentially from the adjacency graph of all bounded regions of $\mathcal{K}_c$. The Hasse diagram of $(\mathcal{S}, \prec_c)$ does not contain edges $(s_i, j) \to (s_i, j+1)$ but all edges of this type are edges of the adjacency graph.

Two Hasse diagrams for $(\mathcal{S}, \prec_c)$ associated to $\mathcal{K}_{\text{alt}}$ are illustrated in Fig. 4. To simplify Fig. 4 and later illustrations, we identify each bounded region of $\mathcal{K}_c$ at level i not with a pair (s_i, j) but simply with the corresponding generator s_i, since j can be easily retrieved

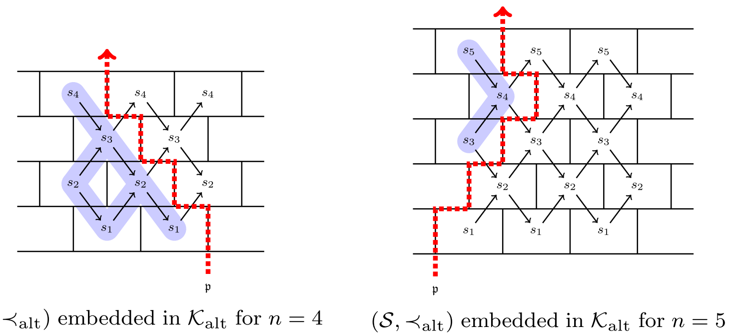


Fig. 4 The Hasse diagram of $(\mathcal{S}, \prec_{\text{alt}})$. An ordinate monotone path $\mathfrak{p}$ (dashed red line) determines an order ideal $\mathcal{S}_\mathfrak{p}$ of $(\mathcal{S}, \prec_{\text{alt}})$ (shaded region)

from the 1-kernel. Each greedy ordinate monotone path $\mathfrak{p}$ in $\mathcal{K}_c$ is a cut of the Hasse diagram $(\mathcal{S}, \prec_c)$ that partitions the vertex set $\mathcal{S}$ of this oriented graph in two sets and the set $\mathcal{S}_\mathfrak{p} \subseteq \mathcal{S}$ below $\mathfrak{p}$ is an order ideal of $(\mathcal{S}, \prec_c)$. Such an order ideal $\mathcal{S}_\mathfrak{p}$ corresponds to a c-singleton according to Proposition 3 if we believe $(\mathcal{S}, \prec_c) \cong (\mathcal{L}_{\boldsymbol{w}_\circ^\mathbf{c}}, \prec_{\boldsymbol{w}_\circ^\mathbf{c}})$.

We briefly indicate the relation to the work of Galambos and Reiner as well as Manin and Schechtmann. For a given Coxeter element c, each c-singleton w determines a unique greedy ordinate monotone path $\mathfrak{p}_w$ in the 1-kernel $\mathcal{K}_c$ as well as a unique pseudoline arrangement $\mathcal{A}_w$ supported by $\mathcal{K}_c$ (where contacts are the commutators traversed by $\mathfrak{p}_w$). The arrangements $\mathcal{A}_w$ differ only by their embedding in $\mathcal{K}_c$. They describe a unique pseudoline arrangement $\mathcal{A}_c$ that depends only on c. Galambos and Reiner consider the "natural partial order" $\mathcal{P}_\mathcal{A}$ on the crossings of $\mathcal{A}_c$ which coincides with $(\mathcal{S}, \prec_c)$ described above. Moreover, they show that the order ideals of $\mathcal{P}_\mathcal{A}$ encode an equivalence class of elementarily equivalent maximal chains in the weak order on Σ_{n+1} as defined by Manin and Schechtmann [26]. Hence the set of c-singletons for a given Coxeter element c corresponds to an equivalence class of elementarily equivalent admissible permutations of $\binom{[n+1]}{2}$ also studied by Ziegler [44].

2.4 Counting *c*-singletons

This section provides the fundamental idea how the enumeration problem for c-singletons is approached and should be thought of as a preparation for the general definitions in Sections 3 and 5. The focus on c-singletons in type A allows us to circumvent some technical details and we are able to provide a complete alternative proof of Formula (1) for the cardinality of Fishburn's acyclic domain. The fundamental idea is to embed the 1-kernel $\mathcal{K}_c$ in a larger alternating sorting network, called trapeze network $\mathcal{T}_c$, for which the total number of greedy ordinate monotone paths is easily determined and then count all greedy ordinate monotone paths that do not remain in $\mathcal{K}_c$. Then the number of c-singletons is simply the difference of these two quantities.

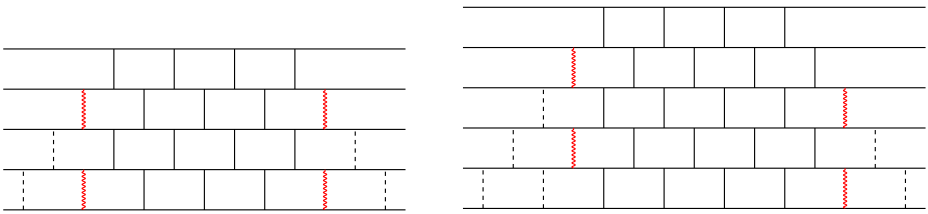
To count c-singletons for general types in Section 3, we shall switch the point of view and consider c-singletons as order ideals of $(\mathcal{L}_{\boldsymbol{w}_\circ^\mathbf{c}}, \prec_{\boldsymbol{w}_\circ^\mathbf{c}})$ instead of greedy ordinate monotone paths in $\mathcal{K}_c$. The poset $(\mathcal{L}_{\boldsymbol{w}_\circ^\mathbf{c}}, \prec_{\boldsymbol{w}_\circ^\mathbf{c}})$ generalizes $(\mathcal{S}, \prec_c)$ that corresponds to $\mathcal{K}_c$ in type A. The trapeze network that extends $\mathcal{K}_c$ will then be replaced by the 2-cover $\mathcal{C}^2_\mathbf{c}$, a directed graph that extends the Hasse diagram of $(\mathcal{L}_{\boldsymbol{w}_\circ^\mathbf{c}}, \prec_{\boldsymbol{w}_\circ^\mathbf{c}})$, and greedy ordinate monotone paths in $\mathcal{K}_c$ and $\mathcal{T}_c$ will be replaced by cut paths of the 2-cover $\mathcal{C}^2_\mathbf{c}$. Similar to the total number of greedy ordinate monotone paths of $\mathcal{T}_c$, the total number of cut paths of $\mathcal{C}_{\mathbf{c}}^2$ are easily counted. Moreover, this number depends not on the choice of the Coxeter element c but it does depend on the type. It will turn out that c-singletons correspond to cut paths that "remain" in $(\mathcal{L}_{\boldsymbol{w}_\circ^{\mathbf{c}}}, \prec_{\boldsymbol{w}_\circ^{\mathbf{c}}})$ and that their number is the difference between the number of all cut paths of $\mathcal{C}_{\mathbf{c}}^2$ and the cut paths of $\mathcal{C}_{\mathbf{c}}^2$ that do not "enter" $(\mathcal{L}_{\boldsymbol{w}_\circ^{\mathbf{c}}}, \prec_{\boldsymbol{w}_\circ^{\mathbf{c}}})$. Taking a symmetry of the 2-cover into account, the final enumeration problem will be solved in Section 3 by counting cut paths of $\mathcal{C}_{\mathbf{c}}^2$ that cross the primary or secondary cut path. The primary and secondary cut paths are special cut paths that characterize $(\mathcal{L}_{\boldsymbol{w}_\circ^{\mathbf{c}}}, \prec_{\boldsymbol{w}_\circ^{\mathbf{c}}})$ within $\mathcal{C}_{\mathbf{c}}^2$.

We now derive a general formula for the number S_c of c-singletons in type A by enumeration of greedy ordinate monotone paths of $\mathcal{K}_c$ with $n+1$ horizontal lines. To this respect, we define the *trapeze network* $\mathcal{T}_c$ associated to $\mathcal{K}_c$ as an extension of $\mathcal{K}_c$ by adding commutators at levels 1 to $n-1$ (called trapeze commutators) such that every commutator at level n is contained in precisely 2^{n-1} ordinate monotone paths starting at a commutator located at level 1. More precisely, we define the trapeze network $\mathcal{T}_c$ as the maximal alternating sorting network with $n+1$ horizontal lines that satisfies:

i) $\mathcal{T}_c$ contains $\mathcal{K}_c$ and has the same commutators at level n as $\mathcal{K}_c$,
ii) $\mathcal{T}_c$ is an alternating network, i.e. commutators at level i and $i+1$ alternate,
iii) every commutator of $\mathcal{T}_c$ is included in a greedy ordinate monotone path in $\mathcal{T}_c$ that contains a commutator of $\mathcal{K}_c$ at level n.

A commutator in $\mathcal{T}_c$ which is not in $\mathcal{K}_c$ is called a *trapeze commutator*. If $\mathcal{K}_c = \mathcal{K}_{\text{Lod}}$ then $\mathcal{T}_{\text{Lod}} = \mathcal{K}_{\text{Lod}}$, so $\mathcal{T}_{\text{Lod}}$ contains no trapeze commutators. When $\mathcal{K}_c = \mathcal{K}_{\text{alt}}$, the trapeze network $\mathcal{T}_{\text{alt}}$ contains 8 trapeze commutators for $n = 4$ and contains 10 trapeze commutators for $n = 5$, see Fig. 5.

Clearly, any greedy ordinate monotone path in $\mathcal{T}_c$ ends at some commutator of $\mathcal{K}_c$ at level n and is a greedy ordinate monotone path in $\mathcal{T}_c$ that either traverses only commutators of $\mathcal{K}_c$ or traverses at least one trapeze commutator. As there are $|U_c| + 2$ commutators of $\mathcal{K}_c$ at level n and since there are precisely 2^{n-1} distinct ordinate monotone paths in $\mathcal{T}_c$ that end at any given commutator at level n, we conclude that the number of greedy ordinate monotone paths in $\mathcal{T}_c$ equals $(|U_c| + 2)2^{n-1}$. To determine the number of greedy ordinate monotone paths that remain in $\mathcal{K}_c$, we aim to subtract the number of ordinate monotone paths that traverse at least one trapeze commutator. Clearly, the paths that traverse at least one trapeze commutator are naturally partitioned by the last (top-most) trapeze commutator traversed. Let Θ_c denote the set of trapeze commutators that appear as last (top-most) trapeze commutator of some greedy ordinate monotone path in $\mathcal{T}_c$. For every $t \in \Theta_c$, we



$\mathcal{T}_{\text{alt}}$ associated to $\mathcal{K}_{\text{alt}}$ for $n = 4$ $\qquad$ $\mathcal{T}_{\text{alt}}$ associated to $\mathcal{K}_{\text{alt}}$ for $n = 5$

Fig. 5 The trapeze network $\mathcal{T}_{\text{alt}}$ with trapeze commutators drawn dashed or zig-zag. The trapeze commutators in Θ_{alt} are drawn zig-zag denote by γ_t the number of greedy ordinate monotone paths in $\mathcal{T}_c$ (traversing $\mathcal{T}_c$ bottom-to-top) that start at t and stay in $\mathcal{K}_c$ after traversing t at level ℓ_t. There are $\gamma_t 2^{\ell_t-1}$ greedy ordinate monotone paths in $\mathcal{T}_c$ with $t \in \Theta_c$ as last trapeze commutator as there are 2^{ℓ_t-1} greedy ordinate monotone paths that start at level 1 and end in t. We conclude

$$\mathsf{S}_c = (|U_c| + 2)2^{n-1} - \sum_{t\in\Theta_c} \gamma_t 2^{\ell_t-1}.$$

A trivial consequence is $\mathsf{S}_{\mathrm{Lod}} = 2^n$ as $\Theta_{\mathrm{Lod}} = U_{\mathrm{Lod}} = \varnothing$ and $D_{\mathrm{Lod}} = [n+1]$. For a less trivial example, we prove $\mathsf{S}_{\mathrm{alt}} = \mathsf{fb}(n+1)$. Assume first that $n = 2k$. Then

$$|U_{\mathrm{alt}}| = \frac{n}{2} \qquad \text{and} \qquad (|U_{\mathrm{alt}}| + 2)2^{n-1} = (n+4)2^{n-2}.$$

For $0 \le r \le k-1$ there are exactly two distinct commutators in Θ_{alt} at odd level $\ell_t = 2r+1$ and no commutator at even level $\ell_t = 2r+2$, see Fig. 5. For $t \in \Theta_{\mathrm{alt}}$ to the left of $\mathcal{K}_{\mathrm{alt}}$ at level $\ell_t = 2r+1$, any greedy ordinate monotone path $\mathfrak{p}$ with last trapeze commutator t contains a greedy ordinate monotone path $\tilde{\mathfrak{p}}$ from t to a commutator of $\mathcal{K}_{\mathrm{alt}}$ at level n that uses only commutators in $\mathcal{K}_{\mathrm{alt}}$. The path $\tilde{\mathfrak{p}}$ traverses $n - \ell_t$ commutators (after t), where the first commutator (at level $\ell_t + 1$) is determined since $\tilde{\mathfrak{p}}$ must take an "east" step after t. Moreover, at any position $\tilde{\mathfrak{p}}$ must have taken strictly more "east" steps than "west" steps. The number of such paths $\tilde{\mathfrak{p}}$ is $\binom{2(k-r-1)}{k-r-1}$, see [29, Corollary 6] for details.

A similar argument applies if $t \in \Theta_{\mathrm{alt}}$ with $\ell_t = 2r+1$ is located to the right of $\mathcal{K}_{\mathrm{alt}}$, so $\gamma_t = \binom{2(k-r-1)}{k-r-1} = \binom{n-\ell_t-1}{(n-\ell_t-1)/2}$ for all $t \in \Theta_{\mathrm{alt}}$ relates to the sequence of central binomial coefficients `A000984` of [28]. Setting $p := k-r-1$ and using Gosper's algorithm to obtain a closed form for the hypergeometric sum [30, Chapter 5], we conclude

$$\begin{aligned}\sum_{t\in\Theta_{\mathrm{alt}}} \gamma_t 2^{\ell_t-1} &= 2\sum_{r=0}^{k-1} \binom{2(k-r-1)}{k-r-1} 2^{2r} \\ &= 2^{n-1}\sum_{p=0}^{k-1}\binom{2p}{p}2^{-2p} = k\binom{2k}{k} = \frac{n}{2}\binom{n}{\frac{n}{2}}.\end{aligned}$$

If we now assume $n = 2k-1$ then

$$|U_{\mathrm{alt}}| = \frac{n-1}{2} \qquad \text{and} \qquad (|U_{\mathrm{alt}}| + 2)2^{n-1} = (n+3)2^{n-2}.$$

For $0 \le r \le k-2$ there is precisely one commutator in Θ_{alt} at even level $\ell_t = 2r+2$. Since $n - \ell_t$ is odd, we get $\gamma_t = \binom{n-\ell_t-1}{(n-\ell_t-1)/2} = \binom{2(k-r-2)}{k-r-2}$. Further there is precisely one commutator in Θ_{alt} at odd level $\ell_t = 2r+1$, a similar argument yields $\gamma_t = \binom{2(k-r-2)+1}{(k-r-2)+1}$. Thus $\gamma_t = \binom{n-\ell_t-1}{\lfloor\frac{n-\ell_t-1}{2}\rfloor}$ relates to sequence `A001405` of [28]. Setting $p := k-r-2$ and again using Gosper's algorithm we conclude

$$\sum_{t\in\Theta_{\mathrm{alt}}} \gamma_t 2^{\ell_t-1} = 2^{n-3}\sum_{p=0}^{k-2}\frac{(4p+3)}{p+1}\binom{2p}{p}2^{-2p} = -2^{n-2} + \frac{2n-1}{2}\binom{n-1}{\frac{n-1}{2}}.$$

Finally, we set $n = m-1$ to prove the claim $\mathsf{S}_{\mathrm{alt}} = \mathsf{fb}(n+1)$. This provides an alternative proof for the cardinality of Fishburn's acyclic domain (1).

Of course, it is possible to avoid Gosper's algorithm and prove the closed form for the hypergeometric sums by induction if candidates for the closed form of the hypergeometric sums are known. The strength of Gosper's algorithm is to provide a desired closed formula and the algorithm can be used for different choices of the Coxeter element c. As we again need (non-obvious) closed formulae for certain hypergeometric sums in Section 4.1, we decided to mention the general tool, i.e. Gosper's algorithm, here explicitly.

3 Enumeration of *c*-singletons – General Case

In order to provide formulae to enumerate singletons in the general case of arbitrary finite Coxeter systems, we first generalize the poset $(\mathcal{S}, \prec_c)$ of Section 2.3 for type A to general type, and present a planar drawing of its Hasse diagram in Section 3.2. In Section 3.2, we describe two equivalence relations on words and their class representatives indexed by Coxeter elements. In Section 3.3, we present a graph called the *2-cover* that we embed on a cylinder. In Section 3.4, we count *cut paths* in this embedding and provide a correspondence to Coxeter elements. In Section 3.5, we define when two cut paths are *crossing*. Finally, we obtain a formula for the cardinality S_c of a Cambrian acyclic domain by counting certain cut path that do not cross the cut path corresponding to c in Section 3.6

Consider an *irreducible finite Coxeter system* (W, S) of rank n with generators $s_1, \ldots, s_n$ and length function ℓ. A *Coxeter element* $c \in W$ is the product of n distinct generators of S in some order and $\mathsf{Cox}(W, S)$ is the set of all Coxeter elements of (W, S). The *Coxeter number* h is the smallest positive integer such that c^h is the identity of W and is independent of the choice of c. As proposed by Shi [38], we identify Coxeter elements $c \in W$ and *orientations* Γ_c *of the Coxeter graph* Γ associated to W: an edge $\{s, t\}$ of Γ is directed from s to t if and only if $s, t \in S$ do not commute and s comes before t in (any reduced expression of) c. A *word* $\mathbf{w}$ in S is a concatenation $\sigma_1 \ldots \sigma_k$ for some nonnegative integer k and $\sigma_i \in S$, a *subword* of $\mathbf{w} = \sigma_1 \ldots \sigma_k$ is a word $\sigma_{i_1} \ldots \sigma_{i_r}$ with $1 \le i_1 < \ldots < i_r \le k$ and the *support* $\operatorname{supp}(\mathbf{w})$ of $\mathbf{w}$ is the set of generators that appear in $\mathbf{w}$. Of particular interest is the unique element $w_\circ \in W$ of maximum length $\ell(w_\circ) = N := \frac{nh}{2}$ which is called *longest element*. A reduced expression $\mathbf{w}_\circ = \sigma_1 \ldots \sigma_N$ is called *longest word*.

3.1 Natural Partial Order

A *Coxeter triple* $(W, S, \mathbf{w})$ is an irreducible finite Coxeter system (W, S) together with a word $\mathbf{w}$ in S such that $\operatorname{supp}(\mathbf{w}) = S$. Any Coxeter triple $(W, S, \mathbf{w})$ induces a unique reduced expression $\mathbf{c_w}$ of a Coxeter element $c_\mathbf{w}$ where the elements of S appear according to their first appearance in $\mathbf{w}$. In particular, $\mathbf{w}$ induces a canonical orientation on Γ. We define now the natural partial order $\prec_\mathbf{w}$ on the set $\mathcal{L}_\mathbf{w} := [k] = \{1, 2, \ldots, k\}$ of positions of letters of the word $\mathbf{w} = \sigma_1 \ldots \sigma_k$. The map $\sigma : \mathcal{L}_\mathbf{w} \to S$ assigns to each position $i \in \mathcal{L}_\mathbf{w}$ the letter $\sigma_i \in S$ of $\mathbf{w}$ at position i. Often we identify $\mathcal{L}_\mathbf{w}$ with the set $\{(i, \sigma_i) \mid 1 \le i \le k\}$, replace (i, σ_i) by the generator $s_j \in S$ equal to σ_i and keep its original position i in mind. The natural partial order appeared in [43, Section 2], [41, Section 2.2], [13, Definition 6].

Definition 1 (Natural partial order)

The *natural partial order* $\prec_\mathbf{w}$ on $\mathcal{L}_\mathbf{w}$ is defined for any Coxeter triple $(W, S, \mathbf{w})$ as follows: $r \prec_\mathbf{w} s$ if and only if there is a subword $\sigma_{i_1} \ldots \sigma_{i_k}$ of $\mathbf{w}$ such that $i_1 = r$, $i_k = s$ and $\sigma_{i_j}\sigma_{i_{j+1}} \ne \sigma_{i_{j+1}}\sigma_{i_j}$ for all $1 \le j \le k-1$. The Hasse diagram of $(\mathcal{L}_\mathbf{w}, \prec_\mathbf{w})$ is an oriented graph denoted by $\mathcal{G}_\mathbf{w}$.

Example 1 Let $(\Sigma_5, S, \mathbf{w})$ be the Coxeter triple with generators $s_i = (i \;\; i+1)$ and

$$\mathbf{w} := s_3 s_2 s_1 s_2 s_3 s_4 s_2 s_3 s_2 s_1 s_2 s_3 s_4 s_3 s_2 s_1 s_3 s_2 s_3 s_4.$$

The induced reduced word for the Coxeter element $c_{\mathbf{w}}$ is $\mathbf{c_w} = s_3 s_2 s_1 s_4$ and $\mathcal{L}_{\mathbf{w}}$ has 20 elements. Moreover, $(\mathcal{L}_{\mathbf{w}}, \prec_{\mathbf{w}})$ consists of 21 covering relations that determine the graph $\mathcal{G}_{\mathbf{w}}$ illustrated in Fig. 6.

Remark 1

a) Let $(W, S, \mathbf{w}_1)$ and $(W, S, \mathbf{w}_2)$ be Coxeter triples for (W, S) with $\mathbf{w}_1 \neq \mathbf{w}_2$, and assume that $\mathbf{w}_1$ is obtained from $\mathbf{w}_2$ by a sequence of commutation relations (and no deletions). Then $\mathbf{c}_{\mathbf{w}_1} \neq \mathbf{c}_{\mathbf{w}_2}$ are reduced expressions for the same Coxeter element $c \in W$ and the posets $(\mathcal{L}_{\mathbf{w}_1}, \prec_{\mathbf{w}_1})$ and $(\mathcal{L}_{\mathbf{w}_2}, \prec_{\mathbf{w}_2})$ are isomorphic.
b) To simplify the drawing of Hasse diagrams of natural partial orders, we prefer to indicate vertices by generators $s_j \in S$, rather than indicating positions explicitly. See Fig. 6 where positions *and* generators are used.

The next result gives a crossing-free straight-line planar drawing of the Hasse diagram $\mathcal{G}_{\mathbf{w}}$ for any Coxeter triple $(W, S, \mathbf{w})$ where $\mathbf{w} = \mathbf{c}^k$ is a concatenation of k copies of $\mathbf{c}$. A crossing-free straight-line planar drawing of the Hasse diagram $\mathcal{G}_{\mathbf{w}}$ for an arbitrary Coxeter triple $(W, S, \mathbf{w})$ will be obtained in Section 3.2.

Proposition 2 *Let k be a positive integer and $(W, S, \boldsymbol{c})$ be a Coxeter triple where $\mathbf{c}$ is a reduced expression for $c \in \mathsf{Cox}(W, S)$. The graph $\mathcal{G}_{\mathbf{c}^k}$ is connected and has a crossing-free straight-line planar drawing using integer vertex coordinates such that the x-coordinate is strictly increasing in direction of every oriented edge.*

Proof If $k = 1$ then $\mathcal{G}_{\mathbf{c}}$ is isomorphic (as oriented graph) to the Coxeter graph $\Gamma_{\mathbf{c}}$ and the isomorphism is induced by the bijection $\mathsf{g} : \mathcal{L}_{\mathbf{w}} = [n] \to [n]$ where $\mathsf{g}(i)$ is defined by $\sigma_i = s_{\mathsf{g}(i)}$. The classification of finite Coxeter groups implies that Γ is a planar tree. Without loss of generality, we assume that $s_1, \ldots, s_p \in S$ of Γ are successive vertices of Γ along a path of maximum length. We have $p = n - 1$ if (W, S) is of type D or E and $p = n$ otherwise. If $p = n - 1$ we may additionally assume that the path is labeled such that the remaining vertex s_n is adjacent to s_r where $r = n - 2$ (type D) or $r = n - 3$ (type E). We now construct a particular crossing-free straight-line planar drawing of $\mathcal{G}_{\mathbf{c}}$. First locate s_1 at $(0, 0)$ and determine coordinates (x_j, y_j) for s_j with $j \leq p$ inductively from (x_{j-1}, y_{j-1}) for s_{j-1} via

$$x_j := \begin{cases} x_{j-1} + 1 & \text{if } \mathsf{g}^{-1}(j-1) \prec_{\mathbf{c}} \mathsf{g}^{-1}(j), \\ x_{j-1} - 1 & \text{if } \mathsf{g}^{-1}(j) \prec_{\mathbf{c}} \mathsf{g}^{-1}(j-1), \end{cases} \qquad \text{and} \qquad y_j := y_{j-1} + 1.$$

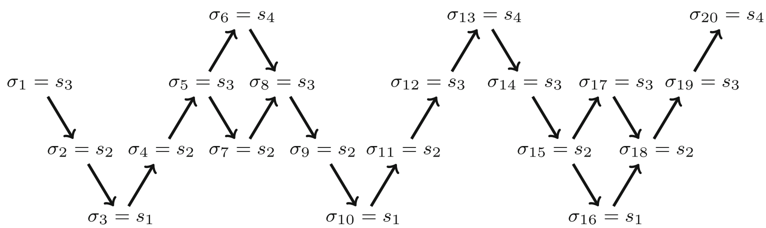


Fig. 6 Hasse diagram of the natural partial order for the word $\mathbf{w}$ of Example 1

If $p = n - 1$, then coordinates for the remaining point s_n are

$$x_n := \begin{cases} x_r + 1 & \text{if } \mathsf{g}^{-1}(r) \prec_{\mathbf{c}} \mathsf{g}^{-1}(n), \\ x_r - 1 & \text{if } \mathsf{g}^{-1}(n) \prec_{\mathbf{c}} \mathsf{g}^{-1}(r), \end{cases} \qquad \text{and} \qquad y_n := y_r.$$

The construction is illustrated for (W, S) of type D_5 and $k = 1$ in Example 2.

We now inductively construct a planar drawing of $\mathcal{G}_{\mathbf{c}^{k+1}}$ from a drawing of $\mathcal{G}_{\mathbf{c}^k}$. From $\mathbf{c}^{k+1} = \mathbf{c}^k \sigma_{nk+1} \cdots \sigma_{n(k+1)}$, we have $S = \{\sigma_{nk+1}, \ldots, \sigma_{n(k+1)}\}$ and

$$\mathcal{L}_{\mathbf{c}^{k+1}} \setminus \mathcal{L}_{\mathbf{c}^k} = \{nk+1, nk+2, \ldots, n(k+1)\}.$$

Any covering relation $j \prec\!\!\cdot_{\mathbf{c}^{k+1}} j'$ of $(\mathcal{L}_{\mathbf{c}^{k+1}}, \prec_{\mathbf{c}^{k+1}})$ is now characterized by one of the following three statements:

i) $j, j' \in \mathcal{L}_{\mathbf{c}^k}$ and $j \prec\!\!\cdot_{\mathbf{c}^k} j'$;
ii) $j, j' \in \mathcal{L}_{\mathbf{c}^{k+1}} \setminus \mathcal{L}_{\mathbf{c}^k}$ and $(\sigma_j, \sigma_{j'})$ is an oriented edge of $\Gamma_{\mathbf{c}}$;
iii) $j' \in \mathcal{L}_{\mathbf{c}^{k+1}} \setminus \mathcal{L}_{\mathbf{c}^k}$, $j \in \mathcal{L}_{\mathbf{c}^k} \setminus \mathcal{L}_{\mathbf{c}^{k-1}}$ and $(\sigma_{j'}, \sigma_j)$ is an oriented edge of $\Gamma_{\mathbf{c}}$.

To obtain a drawing of $\mathcal{G}_{\mathbf{c}^{k+1}}$, we choose (x_j, y_j) as in $\mathcal{G}_{\mathbf{c}^k}$ for $j \in [nk]$, set

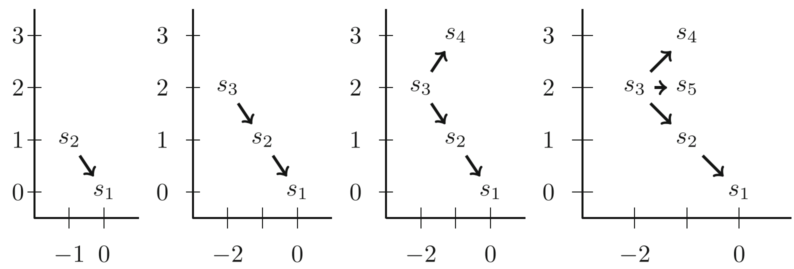
$$(x_{nk+j}, y_{nk+j}) := (x_{n(k-1)+j} + 2, y_{n(k-1)+j}) \quad \text{for all } nk + j \in \mathcal{L}_{\mathbf{c}^{k+1}} \setminus \mathcal{L}_{\mathbf{c}^k}$$

and add oriented edges according to the covering relation of $\prec_{\mathbf{c}^{k+1}}$. □

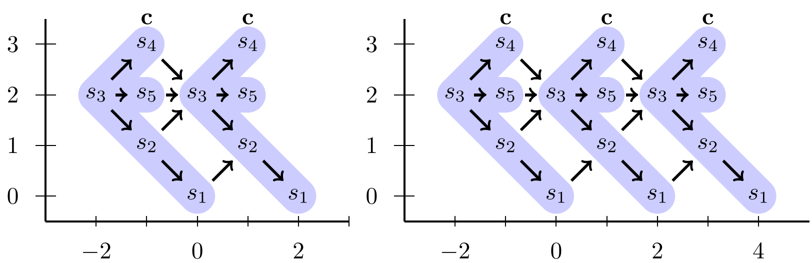
Example 2 If $W = D_5$, $\mathbf{c} = s_3 s_2 s_4 s_5 s_1$ then

$$\sigma_1 = s_3, \quad \sigma_2 = s_2, \quad \sigma_3 = s_4, \quad \sigma_4 = s_5 \quad \text{and} \quad \sigma_5 = s_1$$

and the inductive construction sequence for $\mathcal{G}_{\mathbf{c}}$ is



and the construction sequence from $\mathcal{G}_{\mathbf{c}^k}$ to $\mathcal{G}_{\mathbf{c}^{k+1}}$ is illustrated for $k = 2$



This example extends easily to all possible situations in type D and E.

3.2 Equivalence Classes and c-sorting Words

As in [41], we now define the equivalence relations $\approx$ and $\sim$ on words in S as well as representatives for the equivalence classes $[\mathbf{w}]_\approx$ and $[\mathbf{w}]_\sim$ that are determined by a reduced expression $\mathbf{c}$ for $c \in \mathsf{Cox}(W, S)$.

First, we write $\mathbf{u} \approx \mathbf{v}$ if and only if $\mathbf{u}, \mathbf{v}$ are reduced words that represent the same element $w \in W$. The equivalence class $[\mathbf{u}]_\approx$ depends only on w, so we often write $[w]_\approx$ instead of $[\mathbf{w}]_\approx$. Following Reading [37], we define the $\mathbf{c}$-sorting word $\boldsymbol{w}^{\mathbf{c}}$ of w as the lexicographically first subword of the infinite word $\mathbf{c}^\infty = \mathbf{ccc}\ldots$ (as a sequence of positions) which belongs to $[w]_\approx$.

Second, $\mathbf{u} \sim \mathbf{v}$ if and only if $\mathbf{u}, \mathbf{v}$ are words that *coincide up to commutations*, that is, one is obtained from the other by a sequence of commutation relations (and no deletions). The ***c**-sorting word* $\mathbf{w}^{\mathbf{c}}$ of $\mathbf{w}$ is defined as the element of $[\mathbf{w}]_\sim$ that appears first lexicographically as a subword of the infinite word $\mathbf{c}^\infty = \mathbf{ccc}\ldots$ (as a sequence of positions).

Due to the similar definition, $\boldsymbol{w}^{\mathbf{c}}$ and $\mathbf{w}^{\mathbf{c}}$ are both called $\mathbf{c}$-sorting word. We emphasize that $\boldsymbol{w}^{\mathbf{c}}$ represents $[w]_\approx$ while $\mathbf{w}^{\mathbf{c}}$ represents $[\mathbf{w}]_\sim$ and, by definition, $\ell(\mathbf{u}) = \ell(\mathbf{v})$ if $\mathbf{u} \sim \mathbf{v}$ but $\mathbf{u}$ and $\mathbf{v}$ are not necessarily reduced. Although the definition of $\boldsymbol{w}^{\mathbf{c}}$ depends on a reduced expression $\mathbf{c}$ for c, we have $\boldsymbol{w}^{\mathbf{c}_1} \sim \boldsymbol{w}^{\mathbf{c}_2}$ if $\mathbf{c}_1 \sim \mathbf{c}_2$.

Example 3 The words $\mathbf{u} = s_1 s_2 s_1$ and $\mathbf{v} = s_2 s_1 s_2$ are reduced words for the longest element $w_\circ \in \Sigma_3$ of the Coxeter system (Σ_3, S) with generators $s_i = (i \;\; i+1)$ for $i \in \{1, 2\}$. Thus $\mathbf{u} \approx \mathbf{v}$. As both words do not coincide up to commutations, we have $\mathbf{u} \not\sim \mathbf{v}$. More precisely, if $\mathbf{c} = s_1 s_2$ we have

$$\boldsymbol{w}_\circ^{\mathbf{c}} = s_1 s_2 | s_1, \qquad \mathbf{u}^{\mathbf{c}} = s_1 s_2 | s_1 \qquad \text{and} \qquad \mathbf{v}^{\mathbf{c}} = s_2 | s_1 s_2,$$

and if $\mathbf{c} = s_2 s_1$ then

$$\boldsymbol{w}_\circ^{\mathbf{c}} = s_2 s_1 | s_2, \qquad \mathbf{u}^{\mathbf{c}} = s_1 | s_2 s_1, \qquad \text{and} \qquad \mathbf{v}^{\mathbf{c}} = s_2 s_1 | s_2.$$

We write $|$ to distinguish between copies of $\mathbf{c}$ in $\mathbf{c}^\infty$.

Example 4 (Example 1 continued)

Recall that $\mathbf{c_w} = s_3 s_2 s_1 s_4$. The $\mathbf{c_w}$-sorting word of $\mathbf{w}$ is $[\mathbf{w}^{\mathbf{c_w}} = s_3 s_2 s_1 | s_2 | s_3 s_2 s_4 | s_3 s_2 s_1 | s_2 | s_3 s_4 | s_3 s_2 s_1 | s_3 s_2 | s_3 s_4]$.

Lemma 1 *Let $(W, S, \mathbf{w})$ be a Coxeter triple. The oriented graph $\mathcal{G}_\mathbf{w}$ is the Hasse diagram of a subposet of $\mathcal{L}_{\mathbf{c}_\mathbf{w}^m}$ for some positive integer m.*

Proof Let $\widetilde{\mathbf{w}}$ be the subword of $\mathbf{c_w c_w c_w}\ldots$ that is lexicographically first (as a sequence of positions) among all subwords of $\mathbf{c_w c_w c_w}\ldots$ that coincide with $\mathbf{w}$ up to commutations and let m be the minimum integer such that $\widetilde{\mathbf{w}}$ is a subword of $\mathbf{c}_\mathbf{w}^m = (\mathbf{c_w})^m$. Then $(\mathcal{L}_{\widetilde{\mathbf{w}}}, \prec_{\widetilde{\mathbf{w}}})$ and $(\mathcal{L}_\mathbf{w}, \prec_\mathbf{w})$ are isomorphic, so their Hasse diagrams coincide and $\mathcal{G}_\mathbf{w}$ is the Hasse diagram of a subposet of $\mathcal{L}_{\mathbf{c}_\mathbf{w}^m}$. □

We often write $\mathcal{G}_\mathbf{w}$ for the graph $\mathcal{G}_\mathbf{w}$ embedded according to Lemma 1.

Example 5 (Example 1 continued)

As $\mathbf{w}$ is a subword of $\mathbf{c}_\mathbf{w}^9$, we obtain a planar drawing of $\mathcal{G}_\mathbf{w}$ induced from the planar drawing of $\mathcal{G}_{\mathbf{c}_\mathbf{w}^9}$ as shown in Fig. 7.

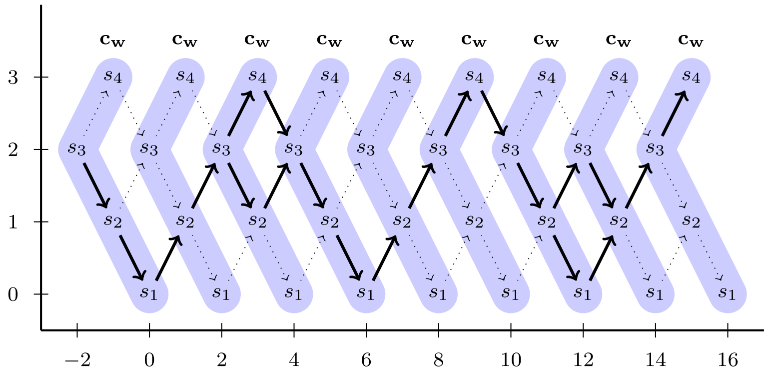


Fig. 7 The crossing-free straight-line drawing of $\mathcal{G}_{\mathbf{c}_{\mathbf{w}}^{9}}$ described in Proposition 2 together with $\mathcal{G}_{\mathbf{w}}$ as a subposet according to Lemma 1 which is also a subgraph in this case

Remark 2

a) If $\mathbf{u} \sim \mathbf{v}$ then $(\mathcal{L}_{\mathbf{u}}, \prec_{\mathbf{u}})$ and $(\mathcal{L}_{\mathbf{v}}, \prec_{\mathbf{v}})$ are isomorphic posets and $\mathcal{G}_{\mathbf{u}}$ and $\mathcal{G}_{\mathbf{v}}$ are isomorphic directed graphs.
b) Let $(W, S, \mathbf{c})$ be a Coxeter triple of type A and $\mathbf{w} \in [\boldsymbol{w}_{\circ}^{\mathbf{c}}]_{\sim}$. Then $(\mathcal{L}_{\mathbf{w}}, \prec_{\mathbf{w}})$ is isomorphic to $(\mathcal{S}, \prec_c)$ described in Section 2.3.
c) The Auslander–Reiten quiver associated to $c \in \mathsf{Cox}(W, S)$ is isomorphic to the graph $\mathcal{G}_{\boldsymbol{w}_{\circ}^{\mathbf{c}}}$ and $\mathcal{G}_{\mathbf{c}^k}$ is a finite truncation of the repetition quiver described by Keller [21, Section 2.2] for all positive integers k.

A word $\sigma_1 \ldots \sigma_r$ is a *prefix up to commutations* of a word $\mathbf{w}$ if and only if there is a word $\mathbf{w}' \sim \mathbf{w}$ such that the first r letters of $\mathbf{w}'$ are $\sigma_1 \ldots \sigma_r$. The following characterization of *c-singletons* serves as definition and does not depend on $\mathbf{c}$ but on $c \in \mathsf{Cox}(W, S)$.

Definition 2 (**c**-singletons [18, Theorem 2.2])
Let $(W, S, \mathbf{c})$ be a Coxeter triple. An element $w \in W$ is a **c**-singleton if and only if some reduced expression of w is a prefix of $\boldsymbol{w}_{\circ}^{\mathbf{c}}$ up to commutations. The number of **c**-singletons is denoted by $\mathsf{S}_{\mathbf{c}}$.

Definition 3 (Cambrian acyclic domains)
Let $(W, S, \mathbf{c})$ be a Coxeter triple. The set $\mathsf{Acyc}_{\mathbf{c}}$ of c-singletons is called *Cambrian acyclic domain* and its cardinality is $\mathsf{S}_{\mathbf{c}}$.

The set of **c**-singletons, endowed with the weak order inherited from (W, S), forms a distributive lattice $L_{\mathbf{c}}$, [18, Proposition 2.5]. Any distributive lattice L is isomorphic to the lattice of order ideals of a poset $(P, \leq)$ which is unique up to isomorphism, [39, Theorem 3.4.1]. Before we show that $(\mathcal{L}_{\boldsymbol{w}_{\circ}^{\mathbf{c}}}, \prec_{\boldsymbol{w}_{\circ}^{\mathbf{c}}})$ is such a poset $(P, \leq)$ for $L_{\mathbf{c}}$, we recall that an *order ideal* (or down-set or semi-ideal) of $(P, \leq)$ is a subset $I \subseteq P$ such that $t \in I$ and $s \leq t$ implies $s \in I$ and that antichains of a finite poset P are in bijection with order ideals of P [39, Section 3.1]. A generator $s \in S$ is called *initial* (resp. *final*) in $w \in W$ if and only if $\ell(sw) < \ell(w)$ (resp. $\ell(ws) < \ell(w)$).

Proposition 3 *Let $(W, S, \boldsymbol{c})$ be a Coxeter triple. The lattice of order ideals of $(\mathcal{L}_{\boldsymbol{w}_\circ^{\mathbf{c}}}, \prec_{\boldsymbol{w}_\circ^{\mathbf{c}}})$ is isomorphic to the poset of c-singletons ordered by the weak order.*

Proof If $w \in W$ is a $\mathbf{c}$-singleton then w is represented by a prefix $\sigma_{i_1} \dots \sigma_{i_k}$ of the c-sorting word $\boldsymbol{w}_\circ^{\mathbf{c}} = \sigma_1 \dots \sigma_N$ up to commutations and the set $F_w \subseteq \{\sigma_{i_1}, \dots, \sigma_{i_k}\}$ of final letters for w is an antichain of $(\mathcal{L}_{\boldsymbol{w}_\circ^{\mathbf{c}}}, \prec_{\boldsymbol{w}_\circ^{\mathbf{c}}})$. Conversely, if $\{a_1, \dots, a_k\}$ is an antichain of $(\mathcal{L}_{\boldsymbol{w}_\circ^{\mathbf{c}}}, \prec_{\boldsymbol{w}_\circ^{\mathbf{c}}})$ then let I_j be the order ideal of $(\mathcal{L}_{\boldsymbol{w}_\circ^{\mathbf{c}}}, \prec_{\boldsymbol{w}_\circ^{\mathbf{c}}})$ generated by a_j for $1 \leq j \leq k$ and consider the order ideal $I := \bigcup_{j=1}^{k} I_j$ of $(\mathcal{L}_{\boldsymbol{w}_\circ^{\mathbf{c}}}, \prec_{\boldsymbol{w}_\circ^{\mathbf{c}}})$ together with some linear extension of $\prec_{\boldsymbol{w}_\circ^{\mathbf{c}}}$ on I. The product of elements of I with respect to this linear order yields a prefix up to commutations of $\boldsymbol{w}_\circ^{\mathbf{c}} = \sigma_1 \dots \sigma_N$ with final letters $\{a_1, \dots, a_k\}$. □

3.3 2-covers

The longest element $w_\circ$ of (W, S) defines an automorphism $\psi : W \to W$ defined via $w \mapsto w_\circ^{-1} w w_\circ$ that preserves length and adapts to words $\mathbf{w} = \sigma_1 \dots \sigma_r$ via $[\psi(\mathbf{w}) := \psi(\sigma_1) \dots \psi(\sigma_r)]$. Let $\operatorname{rev}(\mathbf{w}) := \sigma_r \dots \sigma_1$ denote the *reverse word* of $\mathbf{w}$ and let $\boldsymbol{w}_\circ^{\mathbf{c}} \psi(\boldsymbol{w}_\circ^{\mathbf{c}})$ denote the concatenation of $\boldsymbol{w}_\circ^{\mathbf{c}}$ and $\psi(\boldsymbol{w}_\circ^{\mathbf{c}})$. By Remark 7.6 of [4],

$$\boldsymbol{w}_\circ^{\psi(\mathbf{c})} \sim \operatorname{rev}\left(\boldsymbol{w}_\circ^{\operatorname{rev}(\mathbf{c})}\right) \qquad \text{and} \qquad \mathbf{c}^h \sim \boldsymbol{w}_\circ^{\mathbf{c}} \boldsymbol{w}_\circ^{\psi(\mathbf{c})} = \boldsymbol{w}_\circ^{\mathbf{c}} \psi(\boldsymbol{w}_\circ^{\mathbf{c}}).$$

In combination with Remark 2, we obtain the next lemma.

Lemma 2 *Let $(W, S, \boldsymbol{c})$ be a Coxeter triple. The graphs $\mathcal{G}_{\mathbf{c}^h}$ and $\mathcal{G}_{\boldsymbol{w}_\circ^{\mathbf{c}} \psi(\boldsymbol{w}_\circ^{\mathbf{c}})}$ are isomorphic as oriented graphs. In particular, $\mathcal{G}_{\boldsymbol{w}_\circ^{\mathbf{c}} \psi(\boldsymbol{w}_\circ^{\mathbf{c}})}$ depends on $c \in \mathsf{Cox}(W, S)$ but not on the reduced expression $\boldsymbol{c}$.*

In the rest of the article, we make use of the above isomorphism between $(\mathcal{L}_{\mathbf{c}^h}, \prec_{\mathbf{c}^h})$ and $(\mathcal{L}_{\boldsymbol{w}_\circ^{\mathbf{c}} \psi(\boldsymbol{w}_\circ^{\mathbf{c}})}, \prec_{\boldsymbol{w}_\circ^{\mathbf{c}} \psi(\boldsymbol{w}_\circ^{\mathbf{c}})})$ without mentioning it explicitly.

Example 6 (Example 1 continued)
Consider the longest word $\widetilde{\mathbf{w}_\circ} := s_3 s_2 s_1 s_2 s_3 s_4 s_2 s_3 s_2 s_1$ of (Σ_5, S). Then $\mathbf{w}$ is the concatenation $\widetilde{\mathbf{w}_\circ} \psi(\widetilde{\mathbf{w}_\circ})$ and a direct computation yields

$$\widetilde{\mathbf{w}}_\circ^{\mathbf{c}_\mathbf{w}} = s_3 s_2 s_1 | s_2 | s_3 s_2 s_4 | s_3 s_2 s_1 \qquad \text{and} \qquad \psi(\widetilde{\mathbf{w}}^{\mathbf{c}_\mathbf{w}}) = s_2 s_3 s_4 s_3 s_2 s_3 s_1 s_2 s_3 s_4.$$

Hence, $\mathcal{G}_{\mathbf{c}^h}$ and $\mathcal{G}_{\widetilde{\mathbf{w}}_\circ^{\mathbf{c}_\mathbf{w}} \psi(\widetilde{\mathbf{w}}_\circ^{\mathbf{c}_\mathbf{w}})}$ are not isomorphic as oriented graphs. On the other hand,

$$\boldsymbol{w}_\circ^{\mathbf{c}_\mathbf{w}} = s_3 s_2 s_1 s_4 | s_3 s_2 s_1 s_4 | s_3 s_4 \qquad \text{and} \qquad \psi(\boldsymbol{w}_\circ^{\mathbf{c}_\mathbf{w}}) = s_2 s_3 s_4 s_1 s_2 s_3 s_4 s_1 s_2 s_1$$

show that $\mathbf{c}^5 \sim \boldsymbol{w}_\circ^{\mathbf{c}_\mathbf{w}} \psi(\boldsymbol{w}_\circ^{\mathbf{c}_\mathbf{w}})$. Thus $\mathcal{G}_{\mathbf{c}^5}$ and $\mathcal{G}_{\boldsymbol{w}_\circ^{\mathbf{c}_\mathbf{w}} \psi(\boldsymbol{w}_\circ^{\mathbf{c}_\mathbf{w}})}$ are isomorphic as oriented graphs.

Definition 4 (2-cover)
Let $(W, S, \mathbf{c})$ be a Coxeter triple.

a) The *2-cover* $\mathcal{C}_{\mathbf{c}}^2$ is a directed graph with vertex set $\mathcal{L}_{\mathbf{c}^h}$ and two types of directed edges. The first type of edges are the edges of $\mathcal{G}_{\mathbf{c}^h}$ and the second type consists of edges given by (j', j) where $(\sigma_{j'}, \sigma_j)$ is an oriented edge of $\Gamma_{\mathbf{c}}$, $j \in [n]$, and $j' \in [nh] \setminus [n(h-1)]$.
b) Let $V_{\boldsymbol{w}_\circ^{\mathbf{c}}}$ be the vertices of $\mathcal{C}_{\mathbf{c}}^2$ that correspond to the first $\frac{nh}{2}$ letters of $\boldsymbol{w}_\circ^{\mathbf{c}} \psi(\boldsymbol{w}_\circ^{\mathbf{c}})$ and $V_{\psi(\boldsymbol{w}_\circ^{\mathbf{c}})}$ be the vertices of $\mathcal{C}_{\mathbf{c}}^2$ that correspond to the last $\frac{nh}{2}$ letters of $\boldsymbol{w}_\circ^{\mathbf{c}} \psi(\boldsymbol{w}_\circ^{\mathbf{c}})$. Then $\mathcal{C}_{\boldsymbol{w}_\circ^{\mathbf{c}}}$ is the subgraph of $\mathcal{C}_{\mathbf{c}}^2$ induced by $V_{\boldsymbol{w}_\circ^{\mathbf{c}}}$ and $\mathcal{C}_{\psi(\boldsymbol{w}_\circ^{\mathbf{c}})}$ is the subgraph of $\mathcal{C}_{\mathbf{c}}^2$ induced by $V_{\psi(\boldsymbol{w}_\circ^{\mathbf{c}})}$.

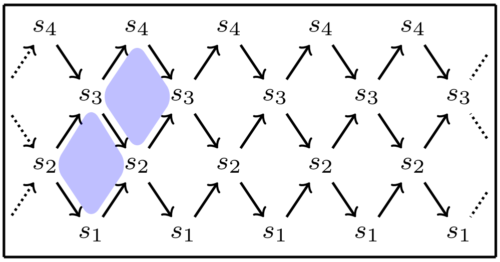


Fig. 8 The 2-cover for $(\Sigma_5, S, \mathbf{c} = s_2s_4s_1s_3)$ with the second type of edges used to define the 2-cover are shown with dotted edges and two shaded tiles as defined in Definition 5

The second type of edges of the 2-cover are illustrated in Fig. 8.

The planar drawing for $\mathcal{G}_{\mathbf{c}^h}$ described in Proposition 2 and the simple observation that $\mathcal{G}_{c^h}$ is isomorphic to some induced subgraph of $\mathcal{G}_{\tilde{\mathbf{c}}^\infty}$ for all $\mathbf{c}, \tilde{\mathbf{c}} \in \mathsf{Cox}(W, S)$ imply the following lemma.

Lemma 3 *Let $(W, S, \boldsymbol{c})$ and $(W, S, \tilde{\boldsymbol{c}})$ be Coxeter triples.*

a) *The 2-cover $\mathcal{C}^2_{\boldsymbol{c}}$ has a crossing-free drawing on the open cylinder $S^1 \times \mathbb{R}$.*
b) *The 2-covers $\mathcal{C}^2_{\boldsymbol{c}}$ and $\mathcal{C}^2_{\tilde{\boldsymbol{c}}}$ are isomorphic as directed graphs.*

We refer to a particular embedding of $\mathcal{C}^2_{\mathbf{c}} \subset S^1 \times \mathbb{R} \subset \mathbb{R}^2 \times \mathbb{R}$ which can be visualized as 'wrapping the plane drawing of $\mathcal{G}_{\mathbf{c}^k}$ around a cylinder'. Without loss of generality, the y-direction for the plane drawing of $\mathcal{G}_{\mathbf{c}^k}$ is parallel to the z-axis of $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$ which coincides with the axis of the cylinder $S^1 \times \mathbb{R} \subset \mathbb{R}^3$. For this reason, we identify y-coordinates in the plane drawing of $\mathcal{G}_{\mathbf{c}^k}$ with third coordinates in $\mathbb{R}^3$. This cylindrical embedding $\mathcal{C}^2_{\mathbf{c}} \subset \mathbb{R}^3$ has the following obvious properties:

i) copies of s_i have strictly smaller third coordinate than copies of s_j for $1 \leq i < j \leq p$ (where p is defined in the proof of Proposition 2);
ii) for fixed $i \in [n]$, the third coordinate of all copies of s_i coincide;
iii) if $p < n$ then the third coordinate of s_r and s_n coincide.

We say that copies of the generator s_1 are located at the bottom of $\mathcal{C}^2_{\mathbf{c}}$ while the copies of the generator s_p are located at the top of $\mathcal{C}^2_{\mathbf{c}}$. Lemma 3 makes the following definition possible.

Definition 5 (Tiles and their boundary)
Let $(W, S, \mathbf{c})$ be a Coxeter triple. A *tile* of $\mathcal{C}^2_{\mathbf{c}}$ is the closure of a bounded connected component of $S^1 \times \mathbb{R} \setminus \mathcal{C}^2_{\mathbf{c}}$, see Fig. 8. The *boundary* of T is denoted by ∂T.

The following lemma is a direct consequence of Definition 5 and Lemma 3.

Lemma 4 *Let T be a tile of $\mathcal{C}^2_{\boldsymbol{c}}$ for a Coxeter triple $(W, S, \boldsymbol{c})$. The boundary ∂T defines an induced subgraph of $\mathcal{C}^2_{\boldsymbol{c}}$ with four vertices; one vertex is a source of out-degree 2 and one vertex is a sink of in-degree 2. In particular, the source and sink of this subgraph are letters of $\boldsymbol{c}^h$ of consecutive copies of $\mathbf{c}$ that represent the same generator of S.*

3.4 Cut Paths

Definition 6 (Cut paths, primary and secondary cut paths)
Let $(W, S, \mathbf{c})$ be a Coxeter triple and $\mathcal{C}^2_{\mathbf{c}}$ the associated 2-cover.

a) A *cut path* κ of $\mathcal{C}^2_{\mathbf{c}}$ is a set of edges of $\mathcal{C}^2_{\mathbf{c}}$ such that every directed cycle of $\mathcal{C}^2_{\mathbf{c}}$ contains precisely one edge of κ. The set of all cut paths of $\mathcal{C}^2_{\mathbf{c}}$ is denoted by $\mathsf{CP}(\mathcal{C}^2_{\mathbf{c}})$.

b) The *primary cut path* $\kappa_{\mathbf{c}}$ of $\mathcal{C}^2_{\mathbf{c}}$ is the cut path that consists of all edges of the 2-cover pointing from $V_{\psi(\boldsymbol{w}^{\mathbf{c}}_\circ)}$ to $V_{\boldsymbol{w}^{\mathbf{c}}_\circ}$. The *secondary cut path* $\kappa^*_{\mathbf{c}}$ is the cut path that consists of all edges of $\mathcal{C}^2_{\mathbf{c}}$ that point from $V_{\boldsymbol{w}^{\mathbf{c}}_\circ}$ to $V_{\psi(\boldsymbol{w}^{\mathbf{c}}_\circ)}$.

The primary and secondary cut paths are disjoint because we have $\mathrm{supp}(\mathbf{w}^{\mathbf{c}}_\circ) = \mathrm{supp}(\psi(\mathbf{w}^{\mathbf{c}}_\circ)) = S$. Primary and secondary cut paths relate to cuts of a graph as $\kappa_{\mathbf{c}} \cup \kappa^*_{\mathbf{c}}$ partitions the 2-cover $\mathcal{C}^2_{\mathbf{c}}$ into two connected components induced by $V_{\boldsymbol{w}^{\mathbf{c}}_\circ}$ and $V_{\psi(\boldsymbol{w}^{\mathbf{c}}_\circ)}$.

Example 7 The notions of Definition 6 are illustrated in Fig. 9 for $\mathbf{c}_1 = \mathbf{c}_{\mathrm{Lod}}$ and $\mathbf{c}_2 = \mathbf{c}_{\mathrm{alt}}$ in (Σ_5, S). The 2-covers $\mathcal{C}^2_{\mathbf{c}}$ coincide in both situations and are shown in a planar drawing where vertex $i \in \mathcal{L}_{\boldsymbol{w}^{\mathbf{c}}_\circ \psi(\boldsymbol{w}^{\mathbf{c}}_\circ)}$ of $\mathcal{C}^2_{\mathbf{c}}$ is labeled by the corresponding generator $\sigma_i \in S$ and oriented edges of $\mathcal{C}^2_{\mathbf{c}}$ contained in a cut path are indicated by $\rightsquigarrow$. The primary and secondary cut paths $\kappa_{\mathbf{c}_i}$ and $\kappa^*_{\mathbf{c}_i}$ for $i \in \{1, 2\}$ are edges $\rightsquigarrow$ intersected by a dashed red line.

Every cut path of $\mathcal{C}^2_{\mathbf{c}}$ that avoids edges of $\mathcal{C}_{\psi(\boldsymbol{w}^{\mathbf{c}}_\circ)}$, equivalently every cut path of $\mathcal{C}^2_{\mathbf{c}}$ that avoids edges of the Hasse diagram of $(\mathcal{L}_{\psi(\boldsymbol{w}^{\mathbf{c}}_\circ)}, \prec_{\psi(\boldsymbol{w}^{\mathbf{c}}_\circ)})$, defines an antichain of $(\mathcal{L}_{\boldsymbol{w}^{\mathbf{c}}_\circ}, \prec_{\boldsymbol{w}^{\mathbf{c}}_\circ})$. Thus, we obtain the following characterization of order ideals of $(\mathcal{L}_{\boldsymbol{w}^{\mathbf{c}}_\circ}, \prec_{\boldsymbol{w}^{\mathbf{c}}_\circ})$.

Lemma 5 *Let $(W, S, \boldsymbol{c})$ be a Coxeter triple. The set of c-singletons is in bijection with the set of cut paths of $\mathcal{C}^2_{\mathbf{c}}$ that avoid edges of $\mathcal{C}_{\psi(\boldsymbol{w}^{\mathbf{c}}_\circ)}$.*

In particular, if the number of all cut paths and the number of cut paths that contain edges of $\mathcal{C}_{\psi(\boldsymbol{w}^{\mathbf{c}}_\circ)}$ are known, Lemma 5 implies a formula for the cardinality S_c. A formula for $|\mathsf{CP}(\mathcal{C}^2_{\mathbf{c}})|$ is obtained from the next theorem in Corollary 3 and a formula for S_c will be derived in Section 3.5.

Moreover, the next theorem shows that every cut path κ induces a sequence ('path') of tiles that cuts the 2-cover $\mathcal{C}^2_{\mathbf{c}}$ from bottom to top: consider the set of tiles such that consecutive tiles have at least one common edge in κ.

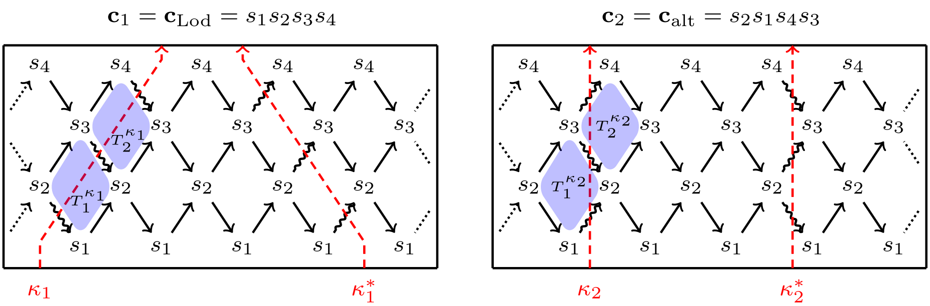


Fig. 9 The primary and secondary cut paths $\kappa_{\mathbf{c}_i}, \kappa^*_{\mathbf{c}_i}$ depicted in a planar drawing of the 2-cover $\mathcal{C}^2_{\mathbf{c}}$ for $\mathbf{c}_1 = s_1s_2s_3s_4$ and $\mathbf{c}_2 = s_2s_4s_1s_3$. For better readability, we denote κ_i for $\kappa_{\mathbf{c}_i}$ in the figures

Theorem 2 *Let $(W, S, \boldsymbol{c})$ be a Coxeter triple, with Coxeter graph Γ, and $\kappa \in \mathsf{CP}(\mathcal{C}^2_{\boldsymbol{c}})$.*

a) *If $e = \{s_i, s_j\}$ is an edge of Γ, then there exists a unique directed edge $e_\kappa = (a, b) \in \kappa$ with $\{\sigma_a, \sigma_b\} = \{s_i, s_j\}$.*

b) *Let s_j be a vertex of degree 2 of Γ with incident edges*

$$e = \{s_i, s_j\} \text{ and } \widetilde{e} = \{s_j, s_k\}.$$

There is a unique tile T that contains the corresponding directed edges

$$e_\kappa = (a, b) \text{ and } \widetilde{e}_\kappa = \left(\widetilde{a}, \widetilde{b}\right),$$

of κ with $\{\sigma_a, \sigma_b\} = \{s_i, s_j\}$ and $\sigma_{\tilde{a}}, \sigma_{\tilde{b}} = \{s_j, s_k\}$ from a).

c) *Let s_r be a vertex of degree 3 of Γ with incident edges*

$$e = \{s_{r-1}, s_r\}, \qquad \widetilde{e} = \{s_r, s_{r+1}\} \text{ and } \qquad \overline{e} = \{s_r, s_n\}$$

and corresponding directed edges of κ from i)

$$e_\kappa = (a, b), \qquad \widetilde{e}_\kappa = \left(\widetilde{a}, \widetilde{b}\right) \text{ and } \qquad \overline{e}_\kappa = \left(\overline{a}, \overline{b}\right).$$

There is a unique pair of tiles T_1, T_2 such that $e_\kappa, \overline{e}_\kappa \in \partial T_1$, $\widetilde{e}_\kappa, \overline{e}_\kappa \in \partial T_2$ and $\partial T_1 \cap \partial T_2$ consists of two edges of $\mathcal{C}^2_{\boldsymbol{c}}$.

Proof a) For every edge $\{s_i, s_j\}$ of the Coxeter graph Γ there exists a directed cycle in $\mathcal{C}^2_{\mathbf{c}}$ that visits all and only vertices corresponding to s_i and s_j. This implies for every edge $\{s_i, s_j\}$ of Γ that a cut path κ must contain precisely one oriented edge (a, b) of $\mathcal{C}^2_{\mathbf{c}}$ such that $\{\sigma_a, \sigma_b\} = \{s_i, s_j\}$.

b) By i), there are unique oriented edges $e_\kappa = (a, b)$, $\widetilde{e}_\kappa = \left(\widetilde{a}, \widetilde{b}\right) \in \kappa$ with $[\{\sigma_a, \sigma_b\} = \{s_i, s_j\}$ and $\{\sigma_{\tilde{a}}, \sigma_{\tilde{b}}\} = \{s_j, s_k\}]$. Suppose there is no tile T with $e_\kappa, \widetilde{e}_\kappa \in \partial T$. Without loss of generality, let T be the unique tile such that $e_\kappa \in \partial T$ and the two other vertices of T correspond to the generators s_j and s_k. Then consider the directed cycle that only uses edges (a, b) of $\mathcal{C}^2_{\mathbf{c}}$ with $\{\sigma_a, \sigma_b\} = \{s_i, s_j\}$ where the two edges of ∂T are replaced by the other two edges of ∂T. Clearly, no edge of this cycle is an edge of κ. This contradicts the assumption that κ is a cut path.

c) The argument to prove ii) can be used to show that there are unique tiles T_1 and T_2 such that $e_\kappa, \overline{e}_\kappa \in \partial T_1$ and $\widetilde{e}_\kappa, \overline{e}_\kappa \in \partial T_2$. But ∂T_1 and ∂T_2 clearly share a directed edge (c, d) of $\mathcal{C}^2_{\mathbf{c}}$ with $\{\sigma_c, \sigma_d\} = \{s_r, s_n\}$ that is distinct from $\overline{e}_\kappa$. □

Corollary 1 *Each cut path $\kappa \in \mathsf{CP}(\mathcal{C}^2_{\boldsymbol{c}})$ determines a unique set of $n-2$ tiles:*

$$\mathsf{tile}(\kappa) = \left\{ T_1, \ldots, T_k \;\middle|\; \begin{smallmatrix} \partial T_i \text{ contains} \\ 2 \text{ edges of } \kappa \end{smallmatrix} \right\}.$$

We tacitly order the tiles $T_i \in \mathsf{tile}(\kappa)$ from bottom to top in $\mathcal{C}^2_{\mathbf{c}} \subset S^1 \times \mathbb{R}$: if z_i denotes the smallest third coordinate of all points in T_i then $1 \leq i < j \leq n-2$ implies $z_i \leq z_j$.

Lemma 3 states that $\mathcal{C}^2_{\mathbf{c}}$ is isomorphic to $\mathcal{C}^2_{\tilde{\mathbf{c}}}$ for all $\mathbf{c}, \tilde{\mathbf{c}}$, so it is impossible to recover $c \in \mathsf{Cox}(W, S)$ from $\mathcal{C}^2_{\mathbf{c}}$. As any cut path κ provides one oriented edge for every edge of Γ, we have an induced Coxeter element c_κ, its reduced expressions are uniquely determined up to commutations.

Corollary 2 *Let $(W, S, \boldsymbol{c})$ be a Coxeter triple, κ_c be the associated primary cut path and $c_{\kappa_{\mathbf{c}}}$ be the Coxeter element obtained from $\kappa_{\mathbf{c}}$. For any reduced expression $\mathbf{w}$ of c_κ, we have* $\mathbf{w} \sim \mathrm{rev}(\boldsymbol{c})$.

Proof The edges of $\mathcal{C}^2_{\mathbf{c}}$ pointing from $V_{\psi(\boldsymbol{w}^{\mathbf{c}}_\circ)}$ to $V_{\boldsymbol{w}^{\mathbf{c}}_\circ}$ define a Coxeter element that correspond to the equivalence class $[\mathrm{rev}(\mathbf{c})]_\sim$. □

Example 8 (Example 7 continued)

Figure 9 also illustrates Corollaries 1 and 2. First, the set of tiles $\mathsf{tile}(\kappa_1)$ and $\mathsf{tile}(\kappa_2)$ associated to κ_1 and κ_2 according to Corollary 1 are illustrated. This example shows that the set of tiles can coincide even if $\kappa_1 \neq \kappa_2$. Moreover, the Coxeter element c_{κ_1} is represented by $s_4s_3s_2s_1$ and $s_4s_3s_2s_1 \in [\mathrm{rev}(\mathbf{c}_{\mathrm{Lod}})]_\sim$. Similarly, c_{κ_2} is represented by $s_1s_3s_2s_4$ and $s_1s_3s_2s_4 \in [\mathrm{rev}(\mathbf{c}_{\mathrm{alt}})]_\sim$.

Corollary 3 *The map $\Phi : \mathsf{CP}(\mathcal{C}^2_{\mathbf{c}}) \to \mathsf{Cox}(W, S)$ sending a cut path κ to its corresponding Coxeter element c_κ is surjective and satisfies $|\Phi^{-1}(c)| = h$ for each $c \in \mathsf{Cox}(W, S)$. In particular, $|\mathsf{CP}(\mathcal{C}^2_{\mathbf{c}})| = 2^{n-1}h$.*

Proof We only prove $|\Phi^{-1}(c)| = h$. There are h choices in $\mathcal{C}^2_{\mathbf{c}}$ to pick a vertex a with $\sigma_a = s_1$. Now a determines a unique tile T_1 with $a \in \partial T_1$ and there is a unique directed edge e_1^κ of ∂T_1 that reflects the order of s_1 and s_2 in c. Now consider the unique tile T_2 whose vertices map to s_2, s_3 and s_4 under σ such that the orientation of $T_1 \cap T_2$ reflects the order of s_2 and s_3 in c and proceed similarly with the following generators until all generators have been considered. This process determines a unique cut path κ with $\Phi(\kappa) = c$ after choosing one of the h possible initial vertices σ at the bottom of $\mathcal{C}^2_{\mathbf{c}}$. □

3.5 Crossings of Cut Paths

Definition 7 (crossing of cut paths, initial and final side, crossing tile)

Let $(W, S, \mathbf{c})$ be a Coxeter triple and $\kappa_{\mathbf{c}}, \kappa^*_{\mathbf{c}} \in \mathsf{CP}(\mathcal{C}^2_{\mathbf{c}})$ be the associated primary and secondary cut paths.

a) A cut path κ *crosses* $\kappa_{\mathbf{c}}$ if $\mathsf{tile}(\kappa) \cap \mathsf{tile}(\kappa_{\mathbf{c}}) \neq \varnothing$ and there are edges $e_1, e_2 \in \kappa$ with $e_1 \in \mathcal{C}_{w_\circ^{\mathbf{c}}}$ and $e_2 \in \mathcal{C}_{\psi(w_\circ^{\mathbf{c}})}$.
b) Let κ be a cut path that crosses $\kappa_{\mathbf{c}}$. The *initial side* of κ is the connected component of $\mathcal{C}^2_{\mathbf{c}} \setminus (\kappa_{\mathbf{c}} \cup \kappa^*_{\mathbf{c}})$ that contains the edge of $\kappa \setminus (\kappa_{\mathbf{c}} \cup \kappa^*_{\mathbf{c}})$ whose midpoint has minimal third coordinate. The *final side* of κ is the connected component of $\mathcal{C}^2_{\mathbf{c}} \setminus (\kappa_{\mathbf{c}} \cup \kappa^*_{\mathbf{c}})$ that is not the initial side of κ.
c) Let κ be a cut path that crosses $\kappa_{\mathbf{c}}$. The *crossing tile* $T^{\kappa,\mathbf{c}}$ of κ in $\mathcal{C}^2_{\mathbf{c}}$ is the first tile of $\mathsf{tile}(\kappa)$ (with respect to the bottom-to-top order) that contains an edge of κ in the final side of κ.

Example 9 (Example 7 continued)

We illustrate Definition 7 in Fig. 10. The cut path $\widetilde{\kappa}_1$ crosses κ_1 while $\widetilde{\kappa}_2$ does not cross κ_2. In both cases, the shared tiles are shaded. The two edges in $\widetilde{\kappa}_1$ and $\widetilde{\kappa}_2$ on the boundary of the shared tiles are indicated by $\rightsquigarrow$. In the first case, the two edges lie in different connected components of $\mathcal{C}^2_{\mathbf{c}_1} \setminus (\kappa_1 \cup \kappa^*_1)$, hence the cut paths cross. The initial side of $\widetilde{\kappa}_1$ is located over the brace, and coincidentially the crossing tile is exactly the shaded region. In the second case, $(s_2, s_1), (s_2, s_3) \in \widetilde{\kappa}_2$ do not lie on different connected components, hence the cut paths $\widetilde{\kappa}_2$ does not cross κ_2.

Definition 8 (Initial and final segments)

Let $(W, S, \mathbf{c})$ be a Coxeter triple, $\kappa_{\mathbf{c}} \in \mathsf{CP}(\mathcal{C}^2_{\mathbf{c}})$ be the associated primary cut path, $T^{\mathbf{c}} \in \mathsf{tile}(\kappa_{\mathbf{c}})$ and $\kappa \in \mathsf{CP}(\mathcal{C}^2_{\mathbf{c}})$ with $\mathsf{tile}(\kappa) = \{T_1, \ldots, T_{n-2}\}$.

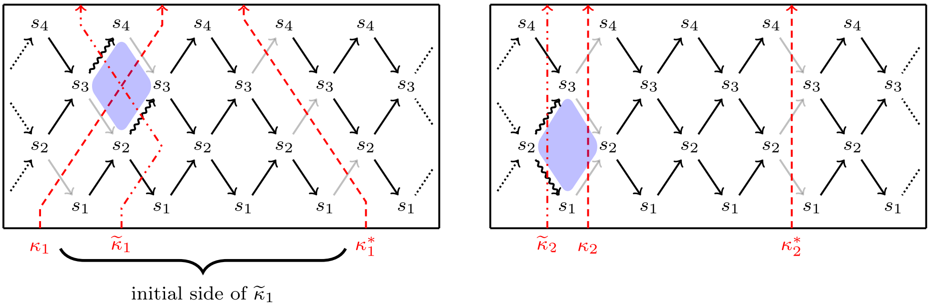


Fig. 10 The cut path $\widetilde{\kappa}_1$ crosses the primary cut path κ_1 (left) and the cut path $\widetilde{\kappa}_2$ does not cross the primary cut path κ_2 (right)

a) Let $i \in [n-1]$. The *initial segment of κ up to T_i* is defined as

$$\{e \in \kappa \;|\; e \in \partial T_j \text{ for } j \in [i-1]\} \cup \left\{(a,b) \in \kappa \;\middle|\; \sigma_a = s_1 \text{ or } \sigma_b = s_1^A\right\}$$

and the *final segment of κ starting at T_i* is defined as

$$\{e \in \kappa \;|\; e \in \partial T_j \text{ with } j > i\} \cup \left\{(a,b) \in \kappa \;\middle|\; \sigma_a = s_p \text{ or } \sigma_b = s_p^A\right\}$$

where $s_1, \ldots, s_p$ are successive vertices of Γ along a path of maximum length.

b) Let $e_1 = (a_1, b_1)$ and $e_2 = (a_2, b_2)$ be the distinct edges of $\partial T^{\mathbf{c}} \setminus \kappa_{\mathbf{c}}$ such that the midpoint of e_1 has smaller third coordinate than the midpoint of e_2. The connected component of $\mathcal{C}_{\mathbf{c}}^2 \setminus (\kappa_{\mathbf{c}} \cup \kappa_{\mathbf{c}}^*)$ that contains e_2 is denoted by $\mathsf{out}(T^{\mathbf{c}})$.

c) Let $I(T^{\mathbf{c}})$ be the number of distinct initial segments of cut paths κ up to $T^{\mathbf{c}}$ with edges contained in $\mathcal{C}_{\mathbf{c}}^2 \setminus \mathsf{out}(T^{\mathbf{c}})$.

d) Let $F(T^{\mathbf{c}})$ be the number of distinct final segments of cut paths κ starting at $T^{\mathbf{c}}$ that contain e_2.

In Definitions 7 and 8, any cut path $\kappa \in \mathsf{CP}(\mathcal{C}_{\mathbf{c}}^2)$ can replace the primary cut path κ_c. Moreover, concatenation of an initial segment counting towards $I(T^{\mathbf{c}})$ that differs from the initial segment of $\kappa_{\mathbf{c}}$ and a final segment counting towards $F(T^{\mathbf{c}})$ yields a cut path that crosses $\kappa_{\mathbf{c}}$.

Example 10 (Example 7 continued)

We illustrate Definition 8 in Fig. 11. The edge in the initial segment of $\widetilde{\kappa}_1$ up to $T_1^{\widetilde{\kappa}_1}$ is indicated by $\rightsquigarrow$. The two edges in the final segment of $\widetilde{\kappa}_1$ starting at $T_1^{\widetilde{\kappa}_1}$ are represented by dashed arrows. A straightforward counting of crossing cut paths then verifies

$$I(T_1^{\kappa_1}) = 2 \text{ and } I(T_2^{\kappa_1}) = 4 \qquad \text{as well as} \qquad I(T_1^{\kappa_2}) = 1 \text{ and } I(T_2^{\kappa_2}) = 2.$$

The reasoning of Section 2.4 yields formulae for all positive integers i:

$$I(T_i^{c_{\text{Lod}}}) = I(T_i^{\text{rev}(c_{\text{Lod}})}) = 2^i \qquad \text{and} \qquad I(T_i^{c_{\text{alt}}}) = \begin{cases} \binom{2j}{j}, & \text{if } i = 2j, \\ \frac{1}{2}\binom{2j}{j}, & \text{if } i = 2j-1. \end{cases}$$

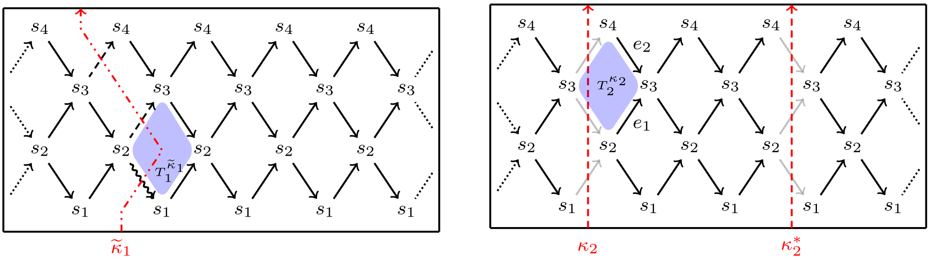


Fig. 11 Two planar drawings of the 2-cover for Σ_5. The cut path $\widetilde{\kappa}_1$ crosses the primary cut path κ_1 (left) and the cut path $\widetilde{\kappa}_2$ does not cross the primary cut path κ_2 (right)

3.6 Enumerating *c*-singletons

Theorem 3 *Let* $(W, S, \mathbf{c})$ *be a Coxeter triple with associated primary cut path* $\kappa_{\mathbf{c}}$ *and set of tiles* $\mathsf{tile}(\kappa_{\mathbf{c}}) = \{T_1^c, \ldots, T_{n-2}^c\}$. *The cardinality of the Cambrian acyclic domain* $\mathsf{Acyc}_{\mathbf{c}}$ *is*

$$\mathsf{S}_c = 2^{n-2}(h+1) - \sum_{i\in[n-2]} 2^{n-2-i} I(T_i^c).$$

Proof We count the cut paths of $\mathsf{CP}(\mathcal{C}_{\mathbf{c}}^2)$ twice. First, the cardinality of $\mathsf{CP}(\mathcal{C}_{\mathbf{c}}^2)$ equals $2^{n-1}h$, by Corollary 3. On the other hand, $\kappa \in \mathsf{CP}(\mathcal{C}_{\mathbf{c}}^2)$ satisfies precisely one of the following statements:

i) κ crosses κ_c or κ_c^*, but not both;
ii) $\kappa \subseteq \mathcal{C}_{w_\circ^{\mathbf{c}}} \cup \kappa_c \cup \kappa_c^*$ or $\kappa \subseteq \mathcal{C}_{\psi(w_\circ^{\mathbf{c}})} \cup \kappa_c \cup \kappa_c^*$ but $\kappa \notin \{\kappa_c, \kappa_c^*\}$;
iii) $\kappa \in \{\kappa_c, \kappa_c^*\}$.

We first claim that the number Q_c of cut paths that cross κ_c equals the number of cut paths that cross κ_c^*. Indeed, the automorphism ψ maps c to $\psi(c)$ and induces an involution φ between $\mathsf{tile}(\kappa_c)$ and $\mathsf{tile}(\kappa_c^*)$ that extends to an involution between cut paths that cross κ_c and cut paths that cross κ_c^* (where $\varphi(T^{\kappa,c})$ is the crossing tile of $\varphi(\kappa)$). Thus, the number of cut paths satisfying i) equals $2\mathsf{Q}_c$. Moreover, this involution extends to an involution on $\mathsf{CP}(\mathcal{C}_{\mathbf{c}}^2)$ that maps cut paths contained in $\mathcal{C}_{w_\circ^{\mathbf{c}}} \cup \kappa_c \cup \kappa_c^*$ to cut paths contained in $\mathcal{C}_{\psi(w_\circ^{\mathbf{c}})} \cup \kappa_c \cup \kappa_c^*$. As κ_c and κ_c^* are the only cut paths that correspond simultaneously to a c-singleton and a $\psi(c)$-singleton, the number of cut paths satisfying ii) equals $2\mathsf{S}_c - 4$ by Lemma 5. Thus we established

$$2\mathsf{Q}_c + (2\mathsf{S}_c - 4) + 2 = 2^{n-1}h \quad \text{or equivalently} \quad \mathsf{S}_c = 2^{n-2}h - \mathsf{Q}_c + 1. \tag{2}$$

To analyze Q_c, we partition the cut paths that cross κ_c according to their crossing tile T_i^c and observe that $I(T_i^c)$ exceeds the number of cut paths with crossing tile T_i^c by one (the initial segment of κ_c is counted by $I(T_i^c)$ but it is not the initial segment of a cut path with crossing tile T_i^c). The number of final segments $F(T_i)$ of cut paths starting at T_i^c satisfies $F(T_i) = 2^{n-2-i}$ because each final segment consists of $n-2-i$ tiles (not counting T_i^c) and there are 2 valid choices to exit each tile. This gives

$$\mathsf{Q}_c = \sum_{i\in[n-2]} \left(I(T_i^c) - 1\right) F(T_i^c) = \sum_{i\in[n-2]} 2^{n-2-i} I(T_i^c) - \left(2^{n-2} - 1\right).$$

Substitution in Eq. (2) yields the claim. □

In Section 4, we first provide explicit formulae for $\mathsf{S}_{\mathbf{c}}$ for two families of Cambrian acyclic domains, and then characterize in Section 5 the Coxeter elements of (W, S) that minimize and maximize $\mathsf{S}_{\mathbf{c}}$. These results are based on the following corollary obtained in the previous proof.

Corollary 4 *Let (W, S) be an irreducible finite Coxeter system of rank n and $c \in \mathsf{Cox}(W, S)$. Then the number of cut paths κ that cross κ_c is*

$$\mathsf{Q}_c = \sum_{i\in[n-2]} 2^{n-2-i} I(T_i^c) - \left(2^{n-2} - 1\right).$$

Remark 3 For reducible finite Coxeter groups, the cardinality of a Cambrian acyclic domain is the product of the cardinalities of the acyclic domains for each irreducible component with respect to the corresponding parabolic Coxeter elements.

4 Examples

In this section, we determine explicit formulae for the cardinality S_c of a Cambrian acyclic domain $\mathsf{Acyc}_{\mathbf{c}}$ when c is a Coxeter element that minimizes or maximizes the total number of sources and sinks of Γ_c.

Definition 9 ($\mathsf{Cox}_{\mathsf{max}}$, $\mathsf{Cox}_{\mathsf{max}}$, path-like Coxeter system)
Let $\mathsf{Cox}_{\mathsf{min}}$ denote the subset of Coxeter elements $c \in \mathsf{Cox}(W, S)$ whose oriented Coxeter graph Γ_c minimizes the total number of sources and sinks. Similarly, let $\mathsf{Cox}_{\mathsf{max}}$ denote the subset of Coxeter elements $c \in \mathsf{Cox}(W, S)$ whose oriented Coxeter graph Γ_c maximizes the total number of sources and sinks. We call a Coxeter system *path-like* if Γ is a path.

4.1 Maximum Total Number of Sources and Sinks

If c provides a bipartition of Γ then every node of Γ_c is a source or a sink and there are n sources and sinks in total. Theorem 2.3 of [2] implies that S_c does not depend on c as the associated associahedra Asso_c are isometric. In particular, S_c depends only on the type and rank of (W, S).

Proposition 4 *Let (W, S) be an irreducible Coxeter system of rank $n > 1$ and $c \in \mathsf{Cox}_{\mathsf{max}}$.*

a) *If (W, S) is path-like then*

$$\mathsf{S}_c = \begin{cases} 2^{n-2}(h+3) - n \cdot \binom{n-1}{\frac{n}{2}}, & n \text{ even,} \\ 2^{n-2}(h+3) - \frac{2n-1}{2} \cdot \binom{n-1}{\frac{n-1}{2}}, & n \text{ odd.} \end{cases}$$

b) *If (W, S) is of type D_n then*

$$\mathsf{S}_c = \begin{cases} 2^{n-2}(h+3) - n \cdot \binom{n-1}{\frac{n}{2}} + \frac{1}{2} \cdot \binom{n-2}{\frac{n-2}{2}}, & n \text{ even,} \\ 2^{n-2}(h+3) - (n-1)\binom{n-1}{\frac{n-1}{2}} - \binom{n-3}{\frac{n-3}{2}}, & n \text{ odd.} \end{cases}$$

c) *If* (W,S) *is of type* E_6, E_7 *or* E_8 *then*

$$\mathsf{S}_c=\begin{cases}2^{n-2}(h+3)-2(n-2)\binom{n-2}{\frac{n-2}{2}}-2\cdot\binom{n-4}{\frac{n-4}{2}}-(n-3)(n-4), & n \text{ even},\\ 2^{n-2}(h+3)-(n-1)\binom{n-1}{\frac{n-1}{2}}+\binom{n-3}{\frac{n-3}{2}}-(n-3)(n-4), & n \text{ odd}.\end{cases}$$

Proof We aim for explicit formulae for $I(T_i^c)$, $1\le i\le n-2$, apply Theorem 3 and simplify the result using the closed form of a hypergeometric sum used in Section 2.4.

Suppose that (W,S) is path-like and recall from Example 10 that

$$I(T_i^c)=\binom{2j}{j}\quad \text{if } i=2j, \qquad \text{and} \qquad I(T_i^c)=\frac{1}{2}\binom{2j}{j}\quad \text{if } i=2j-1.$$

We prove the claim for $n=2k-1$, the proof is along the same lines if $n=2k$. Theorem 3 and $2^{2n+1}\sum_{i=0}^{n}2^{-2i}\binom{2i}{i}=(n+1)\binom{2n+2}{n+1}$ imply

$$\begin{aligned}\mathsf{S}_c&=2^{n-2}(h+1)-\sum_{i\in[n-2]}2^{n-2-i}I(T_i^c)\\ &=2^{n-2}(h+1)-2^{n-1}\left(\sum_{j\in[k-2]}\left(2^{-2j}I(T_{2j-1}^c)+2^{-(2j+1)}I(T_{2j}^c)\right)+2^{-(n-1)}I(T_{n-2}^c)\right)\\ &=2^{n-2}(h+1)-2^{n-1}\left(\sum_{j\in[k-2]}2^{-2j}\binom{2j}{j}+2^{-(2k-1)}\binom{2(k-1)}{k-1}+1-1\right)\\ &=2^{n-2}(h+3)-2^{n-1}\left(2^{-(2k-3)}(k-1)\binom{2(k-1)}{k-1}+2^{-(2k-1)}\binom{2(k-1)}{k-1}\right)\\ &=2^{n-2}(h+3)-\frac{2n-1}{2}\binom{n-1}{\frac{n-1}{2}}.\end{aligned}$$

Suppose that (W,S) is of type D_n. Then $I(T_{n-2}^c)=I(T_{n-3}^c)$ as well as

$$I(T_i^c)=\binom{2j}{j}\quad \text{if } i=2j, \qquad \text{and} \qquad I(T_i^c)=\frac{1}{2}\binom{2j}{j}\quad \text{if } i=2j-1,$$

for $1\le i\le n-3$. Substitution of $I(T_i^c)$ in the formula for S_c of Theorem 3 and a computation similar to the previous case yields the claim.

Finally, we assume that (W,S) is of type E_n for $n\in\{6,7,8\}$. If $i\in[n-4]$ then

$$I(T_i^c)=\binom{2j}{j}\quad \text{if } i=2j, \qquad \text{and} \qquad I(T_i^c)=\frac{1}{2}\binom{2j}{j}\quad \text{if } i=2j-1,$$

as well as $I(T_{n-3}^c)=I(T_{n-4}^c)$ and $I(T_{n-2}^c)=(n-3)(n-4)$. Theorem 3 implies the claim. □

Remark 4

a) Notice that Proposition 4 yields Eq. (1) of Galambos and Reiner if (W,S) is of type A as $W\cong\Sigma_{n+1}$, $c_{\text{alt}}\in\mathsf{Cox}_{\max}$ and $h=n+1=m$.
b) For the Coxeter groups of type $I_2(m)$ we obtain $\mathsf{S}_c=m+1$ for $c\in\mathsf{Cox}_{\max}$.
c) Substitution of the relevant Coxeter numbers yields S_c for exceptional finite Coxeter groups and $c\in\mathsf{Cox}_{\max}$:

(W, S)	H_3	H_4	F_4	E_6	E_7	E_8
S_c	21	120	48	182	546	1840

4.2 Minimum Total Number of Sources and Sinks

Since Γ is a tree with at most one branching point, the minimum number of sources and sinks of Γ_c is two or three: if Γ is a path then $|\mathsf{Cox}_{\mathsf{min}}| = 2$ while $|\mathsf{Cox}_{\mathsf{min}}| = 6$ if Γ has a branching point. In the latter case, we partition $\mathsf{Cox}_{\mathsf{min}}$ into $\mathsf{Cox}_{\mathsf{a}}$, $\mathsf{Cox}_{\mathsf{b}}$ and $\mathsf{Cox}_{\mathsf{c}}$ where each set consists of the Coxeter element shown in Fig. 12 together with $\mathsf{rev}(c)$.

The characterization of isometry classes of associahedra Asso_c in [2] implies that if (W, S) is of type D or E then there are three distinct isometry classes of associahedra Asso_c that correspond to the sets $\mathsf{Cox}_{\mathsf{a}}$, $\mathsf{Cox}_{\mathsf{b}}$ and $\mathsf{Cox}_{\mathsf{c}}$ unless

– (W, S) is of type D_4, where $\mathsf{Cox}_{\mathsf{min}}$ provides one isometry class;
– (W, S) is of type D_n with $n \geq 5$, where $\mathsf{Cox}_{\mathsf{a}}$ and $\mathsf{Cox}_{\mathsf{b}} \cup \mathsf{Cox}_{\mathsf{c}}$ provide two isometry classes;
– (W, S) is of type E_6, where the $\mathsf{Cox}_{\mathsf{a}} \cup \mathsf{Cox}_{\mathsf{b}}$ and $\mathsf{Cox}_{\mathsf{c}}$ provide two isometry classes.

The next proposition provides explicit formulae for $\mathsf{S}_{\mathbf{c}}$ and $\mathbf{c} \in \mathsf{Cox}_{\mathsf{min}}$.

Proposition 5 *Let (W, S) be an irreducible Coxeter system of rank $n > 1$ and $c \in \mathsf{Cox}_{\mathsf{min}}$.*

a) *Suppose that (W, S) is path-like then*

$$\mathsf{S}_c = 2^{n-2}(h - n + 3).$$

b) *Suppose (W, S) is of type D_n and $n \geq 4$ then*

$$\mathsf{S}_c = \begin{cases} 2^{n-2}\big(h - n + \frac{7}{2}\big), & c \in \mathsf{Cox}_{\mathsf{a}}, \\ 2^{n-2}\big(h - n + 4\big) - 2, & c \in \mathsf{Cox}_{\mathsf{b}} \cup \mathsf{Cox}_{\mathsf{c}}. \end{cases}$$

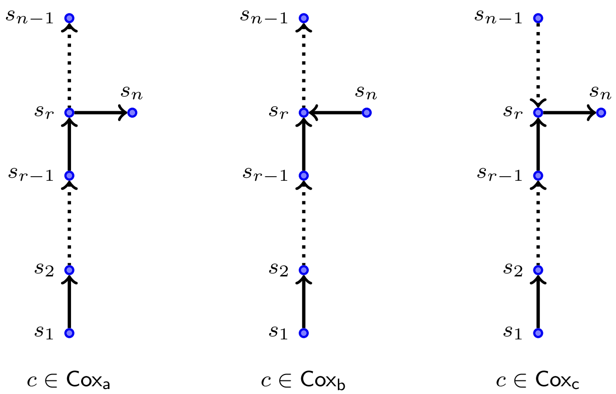


Fig. 12 Oriented Coxeter graphs Γ_c with branching point and minimum total number of sources and sinks ($r = n - 2$ in type D and $r = n - 3$ in type E)

c) *Suppose* (W, S) *is of type* E_6, E_7 *or* E_8. *Then*

$$\mathsf{S}_c = \begin{cases} 2^{n-2}(h-n+4) - 2^{n-4}, & c \in \mathsf{Cox_a}, \\ 2^{n-2}(h-n+4) - 4, & c \in \mathsf{Cox_b}, \\ 2^{n-2}(h-n+4) + 2^{n-2} - 2n, & c \in \mathsf{Cox_c}. \end{cases}$$

Proof Again, all claims follow from Theorem 3 together with the following formulae for $I(T_i^c)$.

First, if (W, S) is path-like then $I(T_i^{\mathbf{c}}) = 2^i$ for all $i \in [n-2]$ and all $c \in \mathsf{C_{min}}$.

Second, if (W, S) is of type D_n and $n \geq 4$. Then $I(T_i^{\mathbf{c}}) = 2^i$ for all $i \leq [n-4]$ and $\mathbf{c} \in \mathsf{Cox_{min}}$ as well as

$$I(T_i^{\mathbf{c}_\mathsf{a}}) = \begin{cases} 2^{n-3}, & i = n-3, \\ 2^{n-3}, & i = n-2, \end{cases} \quad I(T_i^{\mathbf{c}_\mathsf{b}}) = \begin{cases} 1, & i = n-3, \\ 2^{n-2}, & i = n-2, \end{cases} \quad \text{and} \quad I(T_i^{\mathbf{c}_\mathsf{c}}) = \begin{cases} 2^{n-3}, & i = n-3, \\ 2, & i = n-2 \end{cases}$$

where $c_\mathsf{a} \in \mathsf{Cox_a}$, $c_\mathsf{b} \in \mathsf{Cox_b}$ and $c_\mathsf{c} \in \mathsf{Cox_c}$.

Third, if (W, S) is of type $E_n \in \{E_6, E_7, E_8\}$. Then $I(T_i^{\mathbf{c}}) = 2^i$ for $i \in [n-5]$ and all $\mathbf{c} \in \mathsf{Cox_{min}}$ as well as

$$I(T_i^{c_\mathsf{a}}) = \begin{cases} 2^{n-4}, & i = n-4, \\ 2^{n-4}, & i = n-3, \\ 3 \cdot 2^{n-4}, & i = n-2 \end{cases} \quad I(T_i^{c_\mathsf{b}}) = \begin{cases} 1, & i = n-4, \\ 2^{n-3}, & i = n-3, \\ 2^{n-2}, & i = n-2 \end{cases} \quad \text{and} \quad I(T_i^{c_\mathsf{c}}) = \begin{cases} 2^{n-4}, & i = n-4, \\ 2, & i = n-3, \\ 2n-4, & i = n-2, \end{cases}$$

where $c_\mathsf{a} \in \mathsf{Cox_a}$, $c_\mathsf{b} \in \mathsf{Cox_b}$ and $c_\mathsf{c} \in \mathsf{Cox_c}$. □

Remark 5

a) If (W, S) is of type $I_2(m)$ then $\mathsf{Cox_{min}} = \mathsf{Cox_{max}}$. Notice that Proposition 4 and 5 yield $\mathsf{S}_c = m + 1$ in both cases for all m.

b) Consider (W, S) for D_4. Then the formula for $\mathsf{Cox_b} \cup \mathsf{Cox_c}$ satisfies

$$2^{n-2}\big(h - n + 4\big) - 2 = 2^{n-2}\big(h - n + 4 - \tfrac{1}{2}\big),$$

so both formulae of Proposition 5 yield $\mathsf{S}_{\mathbf{c}} = 22$. Moreover, if $n \geq 5$ and $\mathbf{c} \in \mathsf{Cox_{min}}$ then $\mathsf{S}_{\mathbf{c}}$ is minimal if and only if $\mathbf{c} \in \mathsf{Cox_a}$.

c) Let (W, S) be of type E_n. The number of singletons S_c are:

n	$\mathsf{Cox_a}$	$\mathsf{Cox_b}$	$\mathsf{Cox_c}$
6	156	156	164
7	472	476	498
8	1648	1660	1904

d) The minimum numbers of S_c for all exceptional finite Coxeter groups and $c \in \mathsf{Cox_{min}}$ are:

(W, S)	H_3	H_4	F_4	E_6	E_7	E_8
S_c	20	116	44	156	472	1648

5 Lower and Upper Bounds

In this section, we use Theorem 3 and Corollary 4 to prove upper and lower bounds for the cardinality S_c of a Cambrian acyclic domain $\mathsf{Acyc}_{\mathbf{c}}$ by identification of minimizers and maximizers for $\sum_{i=1}^{n-2} 2^{n-2-i} I(T_i^c)$. As done in the proof of Proposition 2, we fix a Coxeter triple $(W, S, \mathbf{c})$ and label the generators of S along a longest path of Γ such that $s_1, \ldots, s_p$ with $p \in \{n-1, n\}$ are successive vertices and in types D and E we have $p = n-1$ and the vertex s_n is connected to s_r where $r = n-2$ (type D_n) or $r = n-3$ (type E).

5.1 Cut Functions

Definition 10 (Cut function)
A function $f : S \to \mathbb{Z}$ is a *cut function* of (W, S) if $|f(s) - f(t)| = 1$ for all non-commuting pairs $s, t \in S$ and $f(s_1)$ is odd. We write $(f(s_1), \ldots, f(s_n))$ for the cut function f and denote the set of all cut functions of (W, S) by $\mathsf{CF}(W, S)$. A generator $s \in S$ is an *extremum* of the cut function f if $f(s) - f(t)$ has the same sign for all $t \in S$ that do not commute with s.

Since $|f(s) - f(t)| = 1$ for all non-commuting pairs $s, t \in S$, any cut function f determines a unique Coxeter element $c_f \in \mathsf{Cox}(W, S)$ such that $f(s) < f(t)$ if and only if (s, t) is a directed edge of Γ_{c_f} and every Coxeter element determines a cut function up to an even constant. Moreover, sources and sinks of Γ_{c_f} correspond to extrema of f and, among the h cut paths of $\Phi^{-1}\big(\mathrm{rev}(c_f)\big)$ from Corollary 3, the cut function f determines a unique cut path κ_f as follows. Let b be the vertex of $\mathcal{C}_{\mathbf{c}}^2$ with $\sigma_b = s_2$ such that the x-coordinate x_b of b as vertex of $\mathcal{G}_{\mathbf{c}^h}$ satisfies $x_b \equiv f(s_1) \mod 2h$ and let (a, b) and $(b, \widetilde{a})$ be the two edges of $\mathcal{C}_{\mathbf{c}}^2$ with $\sigma_a = \sigma_{\widetilde{a}} = s_1$. Now κ_f contains (a, b) if $f(\sigma_a) > f(\sigma_b)$ and $(b, \widetilde{a})$ if $f(\sigma_b) < f(\sigma_{\widetilde{a}})$. We say that the cut path κ_f represents the cut function f.

Example 11 Consider (W, S) of type A_4, the 2-cover $\mathcal{C}_{\mathbf{c}}^2$ with $\mathbf{c} = s_2 s_1 s_4 s_3$. The cut functions f and g with $f(s_1, s_2, s_3, s_4) = (-1, 0, 1, 2)$ and $g(s_1, s_2, s_3, s_4) = (1, 0, 1, 0)$ determine the Coxeter elements $c_f = s_1 s_2 s_3 s_4$ and $c_g = s_2 s_1 s_4 s_3$ indicated as shaded subgraphs in the planar drawing of $\mathcal{C}_{\mathbf{c}}^2$ in Fig. 13. As $f(s_1) = -1 \equiv 9 \mod 10$ and $f(s_1) < f(s_2)$ as well as $g(s_1) = 1 \equiv 1 \mod 10$ and $g(s_1) > g(s_2)$, we obtain the cut

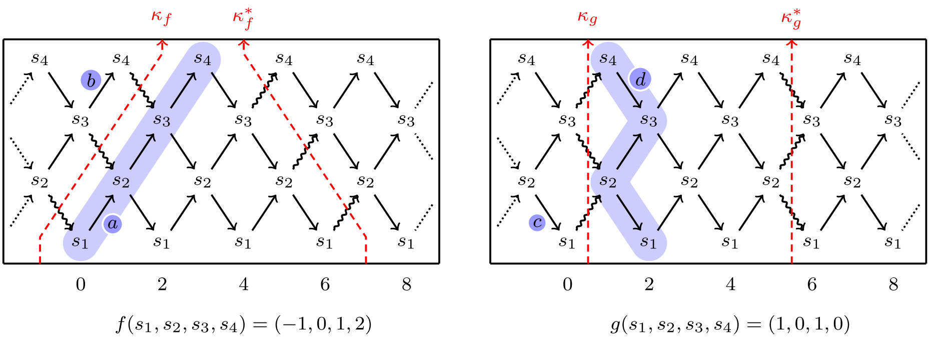


Fig. 13 The cut functions f and g and their associated cut paths κ_f and κ_g and Coxeter elements c_f and c_g in type A_4

paths κ_f and κ_g that represent the cut functions f and g as indicated. The extrema of f are s_1 and s_4, while all generators in S are extrema of g.

Definition 11 (equivalence and crossing of cut functions)
Let $f, g : S \to \mathbb{Z}$ be cut functions of the finite and irreducible Coxeter system (W, S).

a) f and g are *equivalent*, $f \simeq g$, if and only if $f(s) \equiv g(s) \mod 2h$ for all $s \in S$.
b) f and g *cross* if and only if there exist $s, t \in S$ and $\widetilde{f} \simeq f$ such that $\widetilde{f}(s) < g(s)$ and $\widetilde{f}(t) > g(t)$.

Before showing that the notion of crossing cut functions f and g coincides with the crossing of cut paths κ_f and κ_g that represent f and g in the next lemma, we illustrate the definition with an example.

Example 12 (Example 11 continued)
Consider $f'(s_1, s_2, s_3, s_4) = (19, 20, 21, 22)$ and $f''(s_1, s_2, s_3, s_4) = (0, 1, 2, 3)$. Then $f \simeq f'$ and $f \not\simeq f''$. Moreover, f' crosses g as $f' \simeq f$ and $f(s_1) < g(s_1)$ and $f(s_4) > g(s_4)$. Further, notice that the edges a and b required by Definition 7 show that κ_g crosses κ_f while the edges c and d show that κ_f crosses κ_g.

Lemma 6 *Let $(W, S, \boldsymbol{c})$ be a Coxeter triple with associated 2-cover $\mathcal{C}^2_{\mathbf{c}}$ as well as cut paths κ_f and κ_g representing cut functions f and g. The cut paths κ_f and κ_g cross if and only if the cut functions f and g cross.*

Proof Assume that the cut path κ_f crosses κ_g on $\mathcal{C}^2_{\mathbf{c}}$. Since κ_f and κ_g are crossing, they have a common tile T and there are edges $e_1 \in \kappa_g$ in the initial side of κ_f and $e_2 \in \kappa_g$ in the final side of κ_f. These edges are also in $\mathcal{C}^2_{\mathbf{c}} \setminus \kappa_f$. The common tile T allows us to get specific representatives $\widetilde{f}$ and $\widetilde{g}$ for f and g by going down to the first tiles of κ_f and κ_g. Indeed, consider the (unoriented) edge $\{a, b\} \in \kappa_f$ such that $\sigma_b = s_2$ and $\sigma_a = s_1$ and use the x-coordinate x_b of b as the value $\widetilde{f}(s_1) = \widetilde{f}(\sigma_a) := x_b \equiv f(s_1) \mod 2h$. This determines $\widetilde{f} \simeq f$. Proceed similarly to obtain $\widetilde{g}$. Further, since we have edges e_1 and e_2 on distinct connected components of $\mathcal{C}^2_{\mathbf{c}} \setminus \left(\kappa_f \cup \kappa_f^*\right)$, there exist $s, t \in S$ such that

$$\widetilde{f}(s) < \widetilde{g}(s) \quad \text{and} \quad \widetilde{f}(t) > \widetilde{g}(t) \qquad \text{or} \qquad \widetilde{f}(s) > \widetilde{g}(s) \quad \text{and} \quad \widetilde{f}(t) < \widetilde{g}(t)$$

depending if the initial side of κ_f is "on the right" or "on the left" of κ_f.

Now suppose that f and g are crossing cut functions with equivalent cut functions $\widetilde{f} \simeq f$ and $\widetilde{g} \simeq g$ such that $\widetilde{f}(s) < \widetilde{g}(s)$ and $\widetilde{f}(t) > \widetilde{g}(t)$ for some $s, t \in S$. Since the Coxeter graph Γ is a tree and since a cut function h satisfies $|h(a) - h(b))| = 1$ for non-commuting $a, b \in S$, every integral value between $\widetilde{f}(s)$ and $\widetilde{f}(t)$ is in the image of $\widetilde{f}$. Because of the latter inequalities for $\widetilde{f}(s)$ and $\widetilde{g}(s)$, this implies that there exists a generator $u \in S$ such that $\widetilde{f}(u) = \widetilde{g}(u)$. Following the procedure to obtain the cut path κ_h from a cut function h, this implies that the representing cut paths κ_f and κ_g will have a common tile once we get to a tile with vertex label u. Further, the inequalities guarantee that there will be one edge in the initial side of κ_f and one edge in the final side of κ_f taken by κ_g. □

5.2 Upper and Lower Bounds for the Cardinality of Cambrian Acyclic Domains

To obtain lower and upper bounds for $\mathsf{S}_c = |\mathsf{Acyc}_{\mathbf{c}}|$, the next lemma about the minimum and maximum number of extrema of a cut function f is essential. Recall that the maximum number of extrema is equal to n while the minimum number of extrema is equal to 2 if (W, S) is path-like and equal to 3 if (W, S) is of type D or E.

Lemma 7 *Let f be a cut function of the finite irreducible Coxeter system (W, S).*

a) *The number of cut functions that cross f is minimum if and only if the number of extrema of f is maximum.*
b) *If (W, S) is path-like, then the number of cut functions that cross f is maximum if and only if the number of extrema of f is minimum.*
If (W, S) is of type D or E, then the number of cut functions that cross f is maximum implies that the number of extrema of f is minimum.

Proof a) For $m \in \mathbb{Z}$, the *m-reflection* $\mathcal{R}_m : \mathsf{CF}(W, S) \to \mathsf{CF}(W, S)$ is the bijection

$$\mathcal{R}_m(f)(s) := m - (f(s) - m) = 2m - f(s) \qquad \text{for all } s \in S.$$

If $f \in \mathsf{CF}(W, S)$ has less than n extrema then set

$$f' : S \longrightarrow \mathbb{Z} \qquad \text{via} \qquad s \longmapsto \begin{cases} f(s), & \text{if } f(s) < \max_{t\in S} f(t), \\ f(s) - 2, & \text{if } f(s) = \max_{t\in S} f(t). \end{cases}$$

Clearly, every extremum of f is an extremum of f' and there is at least one $s \in S$ that is extremal for f' but not for f, see Fig. 14. Now define

$$\mathcal{F} := \{g \in \mathsf{CF}(W, S) \mid g \text{ crosses } f\} \text{ and } \mathcal{F}' := \left\{g \in \mathsf{CF}(W, S) \;\middle|\; g \text{ crosses } f'\right\}.$$

We first prove $|\mathcal{F}'| < |\mathcal{F}|$ by showing that $|\mathcal{F}' \setminus \mathcal{F}| < |\mathcal{F} \setminus \mathcal{F}'|$. To see this, let $\mu := \max_{t\in S} f(t)$ and notice that

$$\mathcal{R}_{\mu-1}(g) \in \mathcal{F} \setminus \mathcal{F}' \qquad \text{for all } g \in \mathcal{F}' \setminus \mathcal{F},$$

so the reflection $\mathcal{R}_{\mu-1}$ is an injection $\mathcal{F}' \setminus \mathcal{F} \hookrightarrow \mathcal{F} \setminus \mathcal{F}'$ which is not surjective because $f \notin \mathcal{F}' \setminus \mathcal{F}$ and $\mathcal{R}_{\mu-1}(f) \in \mathcal{F} \setminus \mathcal{F}'$. Thus $|\mathcal{F}'| < |\mathcal{F}|$ and iterating this process yields a cut function where every generator of S is extremal. To complete the proof, we show that $|\mathcal{F}| = |\mathcal{G}|$ if f and g are cut functions where every $s \in S$ is extremal. This follows

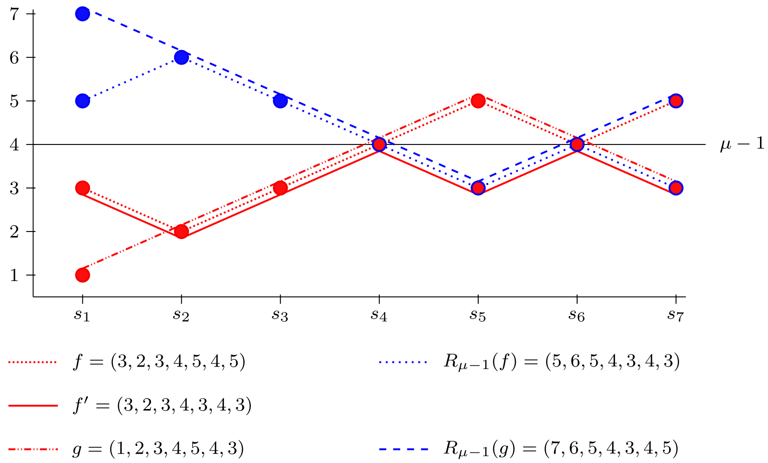


Fig. 14 $\mathcal{F} \supset \mathcal{F}'$ for cut functions f and f' as in the proof of Lemma 7. We have $R_{\mu-1}(f) \in \mathcal{F} \setminus \mathcal{F}'$ as well as $g \in \mathcal{F}' \setminus \mathcal{F}$ implies $R_{\mu-1}(g) \in \mathcal{F} \setminus \mathcal{F}'$

from the observation that two cut functions where every $s \in S$ is extremal differ only by translation and reflection.

b) For $n = 1$ and $n = 2$ there is nothing to prove, as all generators are extremal for each cut function. We therefore assume $n \geq 3$ and prove the claim first if (W, S) is path-like. A cut function f determines the Coxeter element $\mathbf{c}_f$ and the number of cut functions that cross f is equal to the number $\mathsf{Q}_{\mathrm{rev}(\mathbf{c}_f)}$ of cut paths that cross $\kappa_{\mathrm{rev}(\mathbf{c}_f)}$ in $\mathcal{C}^2_{\mathbf{c}}$ by Lemma 6. By Corollary 4, we have

$$\mathsf{Q}_{\mathrm{rev}(\mathbf{c}_f)} = \sum_{i \in [n-2]} 2^{n-2-i} I\big(T_i^{\mathrm{rev}(\mathbf{c}_f)}\big) - \big(2^{n-2} - 1\big),$$

where $T_i^{\mathrm{rev}(\mathbf{c}_f)}$ is tile i of $\kappa_{\mathrm{rev}(\mathbf{c}_f)}$ and $I\big(T_i^{\mathrm{rev}(\mathbf{c}_f)}\big)$ is the number of distinct initial segments of cut paths κ up to $T_i^{\mathrm{rev}(\mathbf{c}_f)}$ with edges contained in $\mathcal{C}^2_{\mathbf{c}} \setminus \mathsf{out}\big(T_i^{\mathrm{rev}(\mathbf{c}_f)}\big)$. The reasoning of Example 10 shows

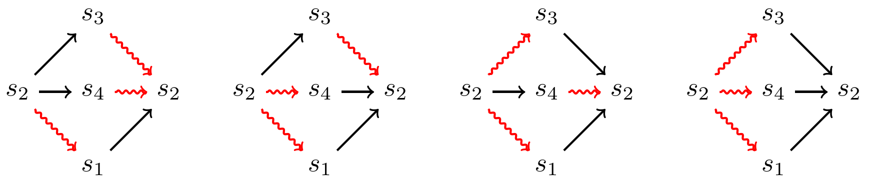
$$\mathrm{rev}(\mathbf{c}_f) \in \{s_1 s_2 \cdot \cdots \cdot s_n, s_n s_{n-1} \cdot \cdots \cdot s_1\}$$

implies

$$I\left(T_i^{\mathrm{rev}(\mathbf{c}_f)}\right) = 2^i \quad \text{for all } \ i \in [n-2].$$

These are clearly the only Coxeter elements with $I\big(T_i^{\mathrm{rev}(\mathbf{c}_f)}\big) = 2^i$ for all $i \in [n-2]$ and these values are maximum. Thus $I\big(T_i^{\mathrm{rev}(\mathbf{c}_f)}\big)$ attains its maximum value for each $i \leq n-2$ if and only if f is strictly monotone on $[i+2]$. In particular, $\mathsf{Q}_{\mathrm{rev}(\mathbf{c}_f)}$ is maximum if and only if the cut function f is strictly monotone. Thus, we conclude for any path-like Coxeter system (W, S) that f has precisely two extrema if and only if the number of cut functions that cross f is maximum.

To analyze type D, we first consider D_4. Without loss of generality, we analyze the following four cases for c_f and $\kappa_{\mathrm{rev}(c_f)}$ as they (together with their reverse words) represent all cut functions f in type D_4:



$c_f = c_a \in \mathsf{Cox}_\mathsf{a}$ $\quad$ $c_f = c_b \in \mathsf{Cox}_\mathsf{b}$ $\quad$ $c_f = c_c \in \mathsf{Cox}_\mathsf{c}$ $\quad$ $c_f = c_{max} \in \mathsf{Cox}_\mathsf{max}$

Corollary 4 implies

$$\mathsf{Q}_{c_a} = 6 - 3 \qquad \mathsf{Q}_{c_b} = 6 - 3 \qquad \mathsf{Q}_{c_c} = 6 - 3 \qquad \mathsf{Q}_{c_{max}} = 3 - 3.$$

This shows that if the number of crossing cut functions of $f \in \mathsf{CF}(D_4, S)$ is maximum then f has three extrema which is the minimum number of extrema in this situation.

We now consider an extension from (D_n, S) to $(D_{n+1}, \widetilde{S})$ with $n \geq 4$ by adding a new vertex adjacent to the leaf s_1 of Γ_{D_n} and appropriate relabeling of generators. Thus Γ_{D_n} corresponds to the subgraph of $\Gamma_{D_{n+1}}$ induced by $\widetilde{S} \setminus \{s_1\}$ and every Coxeter element $c_n \in \mathsf{Cox}(D_n, S)$ can be extended in precisely two ways to a Coxeter element $c_{n+1} \in \mathsf{Cox}(D_{n+1}, \widetilde{S})$. Clearly, we have

$$I(T_1^{c_{n+1}}) \in \{1, 2\} \quad \text{as well as} \quad I(T_i^{c_n}) \leq I(T_{i+1}^{c_{n+1}}) \leq 2I(T_i^{c_n}) \text{ for } i \in [n-2].$$

Thus

$$\begin{aligned}
\mathsf{Q}_{c_{n+1}} &= \sum_{i\in[n-1]} 2^{n-1-i} I\big(T_i^{c_{n+1}}\big) - \big(2^{n-1}-1\big) \\
&\le 2^{n-1-1}\cdot 2 + \sum_{i\in[n-2]} 2^{n-2-i}\cdot \Big(2I\big(T_i^{c_n}\big)\Big) - 2\big(2^{n-2}-1\big) - 1 \\
&= 2^{n-1} + 2\mathsf{Q}_{c_n} - 1
\end{aligned}$$

with equality if $I(T_{i+1}^{c_{n+1}}) = 2I(T_i^{c_n})$ for all $i \in [n-2]$ and $I(T_1^{c_{n+1}}) = 2$ which happens if and only if $\mathsf{out}(T_k^{c_{n+1}})$ and $\mathsf{out}(T_\ell^{c_{n+1}})$ coincide for all $1 \le k, \ell \le n-1$. Thus, if $Q_{c_{n+1}}$ is maximum then $c_{n+1} \in \mathsf{Cox_a} \subseteq \mathsf{Cox_{min}}$. In other words, if the number of cut functions that cross f is maximum then f has three extrema. This proves the claim if (W, S) is of type D.

Finally, we prove the claim in type E. We first analyze E_6. Clearly, removing the vertex $s_p = s_5$ from Γ_{E_6} yields a Coxeter graph of type D_5. Let c be a Coxeter element for type E_6 and $\widetilde{c}$ be the corresponding Coxeter element for $(\widetilde{W}, S \setminus \{s_5\})$ of type D_5. Since $I(T_k^c) = I(T_k^{\widetilde{c}})$ for $1 \le k \le 3$, we obtain $\mathsf{Q}_c = 2\mathsf{Q}_{\widetilde{c}} + I(T_4^c) - 1$.

A case analysis reveals that Q_c is maximum in type E_6 if and only if $c \in \mathsf{Cox_a} \cup \mathsf{Cox_b}$. Thus, if f is a cut function with the maximum number of crossing cut functions then $\mathrm{c}_f \in \mathsf{Cox_a} \cup \mathsf{Cox_b}$, that is, f has three extrema. To solve the remaining cases E_7 and E_8 we extend from type E_6 to E_7 and from type E_7 to E_8 similarly to the induction step from D_n to D_{n+1}. Again, we obtain $\mathsf{Q}_{c_{n+1}} \le 2^{n-1} + 2\mathsf{Q}_{c_n} - 1$ with equality if and only if $\mathsf{out}(T_k^{c_{n+1}})$ and $\mathsf{out}(T_\ell^{c_{n+1}})$ coincide for all $1 \le k, \ell \le n-1$. Therefore, if $Q_{c_{n+1}}$ is maximum then $c_{n+1} \in \mathsf{Cox_a} \subseteq \mathsf{Cox_{min}}$. This proves the claim if (W, S) is of type E_7 and E_8. □

We now characterize the Coxeter elements c that maximize and minimize the cardinality S_c of a Cambrian acyclic domain Acyc_c.

Theorem 4

Let (W, S) be a finite irreducible Coxeter system, $c \in \mathsf{Cox}(W, S)$ and $\mathsf{S}_c = |\mathsf{Acyc}_{\boldsymbol{c}}|$.

a) *The cardinality S_c of Acyc_c is maximum if and only if $c \in \mathsf{Cox_{max}}$.*
b) *The cardinality S_c of Acyc_c is minimum if and only if*

   i) *$c \in \mathsf{Cox_{min}}$ and (W, S) is path-like or of type D_4;*
   ii) *$c \in \mathsf{Cox_a} \cup \mathsf{Cox_b}$ and (W, S) is of type E_6;*
   iii) *$c \in \mathsf{Cox_a}$ and (W, S) is of type E_7, E_8 or D_n for $n \ge 5$.*

Proof a) This is a consequence of Theorem 3 and Corollary 4 combined with Lemma 7.
b) When (W, S) is path-like or of type D_4, it follows immediately from Lemma 7. Otherwise, to decide the remaining cases for type D and E, we use the relevant values for $I\big(T_i^{\mathbf{c}}\big)$ from the proof of Proposition 5. We have to analyze $\mathbf{c}_\mathsf{a} \in \mathsf{Cox_a}$, $\mathbf{c}_\mathsf{b} \in \mathsf{Cox_b}$ and $\mathbf{c}_\mathsf{c} \in \mathsf{Cox_c}$. If (W, S) is of type D, we obtain

$$\begin{aligned}
\mathsf{Q}_{\mathbf{c}_\mathsf{a}} &= (n-4)2^{n-1} + 2^{n-3} + 1, \\
\mathsf{Q}_{\mathbf{c}_\mathsf{b}} &= (n-4)2^{n-1} + 3, \text{ and} \\
\mathsf{Q}_{\mathbf{c}_\mathsf{c}} &= (n-4)2^{n-1} + 3.
\end{aligned}$$

The maximum is achieved by $\mathbf{c}_\mathsf{a}$, $\mathbf{c}_\mathsf{b}$ and $\mathbf{c}_\mathsf{c}$ if $n = 4$ and only by $\mathbf{c}_\mathsf{a}$ if $n \geq 5$. If (W, S) is of type E, we similarly obtain

$$\begin{aligned} \mathsf{Q}_{\mathbf{c}_\mathsf{a}} &= (n-5)2^{n-2} + 5 \cdot 2^{n-4} + 1, \\ \mathsf{Q}_{\mathbf{c}_\mathsf{b}} &= (n-5)2^{n-2} + 2^{n-2} + 5, \text{ and} \\ \mathsf{Q}_{\mathbf{c}_\mathsf{c}} &= (n-5)2^{n-2} + 2n + 1. \end{aligned}$$

The maximum is achieved by $\mathbf{c}_\mathsf{a}$ and $\mathbf{c}_\mathsf{b}$ if $n = 6$ and by $\mathbf{c}_\mathsf{a}$ if $n \in \{7, 8\}$. In particular, this shows that the number of cut functions that cross f is not always maximized if the number of extrema of f is minimized. □

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