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The Ramsey and the Ordering Property for Classes of Lattices and Semilattices

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Abstract

The class of finite distributive lattices, as many other classes of structures, does not have the Ramsey property. It is quite common, though, that after expanding the structures with appropriately chosen linear orders the resulting class has the Ramsey property. So, one might expect that a similar result holds for the class of all finite distributive lattices. Surprisingly, Kechris and Sokić have proved in 2012 that this is not the case: no expansion of the class of finite distributive lattices by linear orders satisfies the Ramsey property. In this paper we prove that the variety of distributive lattices is not an exception, but an instance of a more general phenomenon. We show that for almost all nontrivial locally finite varieties of lattices no "reasonable" expansion of the finite members of the variety by linear orders gives rise to a Ramsey class. The responsibility for this lies not with the lattices as structures, but with the lack of algebraic morphisms: if we consider lattices as partially ordered sets (and thus switch from algebraic embeddings to embeddings of relational structures) we show that every variety of lattices gives rise to a class of linearly ordered posets having both the Ramsey property and the ordering property. It now comes as no surprise that the same is true for varieties of semilattices.

Keywords Ramsey property · Ordering property · Varieties of lattices · Varieties of semilattices

1 Introduction

Generalizing the classical results of F. P. Ramsey from the late 1920's, the structural Ramsey theory originated at the beginning of 1970's in a series of papers (see [15] for references). We say that a class **K** of finite structures has the *Ramsey property (with respect to subobjects)* if the following holds: for any number $k \geqslant 2$ of colors and all $\mathcal{A}, \mathcal{B} \in \mathbf{K}$ such that $\mathcal{A}$ embeds into $\mathcal{B}$ there is a $\mathcal{C} \in \mathbf{K}$ such that no matter how we color the copies of $\mathcal{A}$ in $\mathcal{C}$ with

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k colors, there is a *monochromatic* copy $\mathcal{B}'$ of $\mathcal{B}$ in $\mathcal{C}$ (that is, all the copies of $\mathcal{A}$ that fall within $\mathcal{B}'$ are colored by the same color).

Another possibility is to color embeddings instead of subobjects. Then a class **K** of finite structures has the *Ramsey property (with respect to embeddings)* if the following holds: for any number $k \geqslant 2$ of colors and all $\mathcal{A}, \mathcal{B} \in$ **K** such that $\mathcal{A}$ embeds into $\mathcal{B}$ there is a $\mathcal{C} \in$ **K** such that no matter how we color the set $\mathrm{Emb}(\mathcal{A}, \mathcal{C})$ of all the embeddings $\mathcal{A} \hookrightarrow \mathcal{C}$ with k colors, there is an embedding $w : \mathcal{B} \hookrightarrow \mathcal{C}$ such that the set $\{w \circ f : f \in \mathrm{Emb}(\mathcal{A}, \mathcal{B})\}$ is monochromatic. In this context the embedding $w : \mathcal{B} \hookrightarrow \mathcal{C}$ can be thought of as "a pointer" to the copy $\mathcal{B}'$ of $\mathcal{B}$ in $\mathcal{C}$, while the embeddings $\{w \circ f : f \in \mathrm{Emb}(\mathcal{A}, \mathcal{B})\}$ can be thought of as "pointers" to the copies of $\mathcal{A}$ that fall within $\mathcal{B}'$.

The two notions are closely related but not equivalent. For example, the Finite Ramsey Theorem claims that the class of all finite sets has the Ramsey property with respect to subobjects, and it is easy to see that this class does not have the Ramsey property with respect to embeddings. The same holds for the class of all finite complete graphs and the the class of all finite boolean lattices. Actually, all the examples of classes having the Ramsey property with respect to subobjects but not with respect to embeddings that we have are about classes of objects that are nothing but finite sets in disguise. On the other hand, for classes of rigid finite structures the two notions coincide (a structure is *rigid* if its automorphism group is trivial). Unless explicitly stated otherwise, whenever we refer to the Ramsey property in this paper we mean *the Ramsey property for embeddings.*

Many natural classes of structures (such as finite graphs, metric spaces and partially ordered sets, just to name a few) do not have the Ramsey property, and lattices as algebras with two binary operations satisfying certain algebraic laws are not an exception: the class of all finite lattices, the class of all finite distributive lattices and the class of all finite modular lattices do not have the Ramsey property (see [21, 23]). This is not surprising as all these classes contain non-rigid structures, and it has been established relatively recently that a necessary condition for a class of finite structures to have the Ramsey property is that all its elements be rigid (that is, have trivial automorphism groups) [14, 16].

It is quite common, though, that after expanding the structures under consideration with appropriately chosen linear orders, the resulting class of expanded structures has the Ramsey property. For example, the class of all finite linearly ordered graphs $(V, E, <)$ where (V, E) is a finite graph and $<$ is a linear order on the set V of vertices of the graph has the Ramsey property [1, 19]. The same is true for metric spaces [17]. In case of finite posets we consider the class of all finite linearly ordered posets $(P, \preccurlyeq, <)$ where $(P, \preccurlyeq)$ is a finite poset and $<$ is a linear order on P which extends $\preccurlyeq$ [4, 22]. Moreover, in [26] several classes of semilattices have been shown to have the Ramsey property if the semilattices in the class are expanded by appropriate linear orders. So, one might expect that a similar result holds for finite lattices. Surprisingly, this is not the case. In [12] the authors prove that no expansion of the class of finite distributive lattices by linear orders satisfies the Ramsey property.

We start Section 3 by showing that the variety of distributive lattices is not an exception, but an instance of a more general phenomenon. We show that for an arbitrary nontrivial locally finite variety **V** of lattices distinct from the variety of all the lattices and the variety of distributive lattices, no "reasonable" expansion of $\mathbf{V}^{fin}$ (= the class of all the finite lattices in **V**) by linear orders has the Ramsey property. So, it seems that lattices are simply not fit for the Ramsey property. Our main goal in Section 3 is to demonstrate that the responsibility for this lies not with the lattices as structures, but with the lack of algebraic morphisms: if we consider lattices as partially ordered sets (and thus switch from algebraic embeddings to embeddings of relational structures) we show that every variety of lattices gives rise to a class of linearly ordered posets having both the Ramsey property and the ordering property

(a property related to the Ramsey property which we define in Section 2 along with other notions we use in the paper). Namely, there are much more embeddings between two lattices understood as relational structures than there are embeddings between the same two lattices understood as algebras, and this abundance of embeddings between relational structures is the key reason we are able to prove that all the varieties of lattices have the Ramsey property in their "relational alter ego". It now comes as no surprise that the same is true for varieties of semilattices.

In many particular cases the ordering property is implied by nontrivial Ramsey properties [16], and one such particular case is demonstrated in Section 3. Using the standard Sierpinski-style coloring obtained by comparing two linear orders we can derive the ordering property for a class of linearly ordered posets from the fact that it has the Ramsey property. In Section 4 we generalize this idea to arbitrary classes of first-order structures which satisfy a model-theoretic requirement we refer to as the *weak triangle condition*. This is a weaker form of the *triangle condition* introduced in [11] in connection to understanding the consequences of the ordering property. As an example we apply the main result of Section 4 to show the ordering property for a class of finite structures consisting of a set together with several partially ordered sets that conform to a given template. This example is instructive since we do not see an easy way to derive the ordering property for this class directly.

We conclude the paper by a discussion of the importance of the Ramsey and the ordering properties in the context of the Kechris-Pestov-Todorčević correspondence [11], an intricate interplay of discrete mathematics, model theory and topological dynamics. As the final example we present a new infinite family of topological groups whose universal minimal flows can be computed using this correspondence.

2 Preliminaries

First-Order Structures Let $\Theta = \Theta_R \cup \Theta_F$ be a first-order language where Θ_R is a set of finitary relational symbols, and Θ_F is a set of finitary functional symbols. Whereas we do not allow relational symbols of arity 0, functional symbols of arity 0 are welcome and correspond to constants. A Θ-*structure* $\mathcal{A} = (A, \Theta^{\mathcal{A}})$ is a set A together with a set $\Theta^{\mathcal{A}}$ of finitary relations on A and finitary functions on A which are the interpretations of the corresponding symbols in Θ. A *relational language* is a first order language Θ where $\Theta_F = \varnothing$. An *algebraic language* is a first order language Θ where $\Theta_R = \varnothing$. If Θ is a relational language, Θ-structures are then usually referred to as Θ-*relational structures*; and if Θ is an algebraic language, Θ-structures are then usually referred to as Θ-*algebras*.

Structures will be denoted by script letters $\mathcal{A}$, $\mathcal{B}_1$, $\mathcal{C}^*$, ..., and the underlying set of a structure $\mathcal{A}$, $\mathcal{B}_1$, $\mathcal{C}^*$, ... will always be denoted by its roman letter A, B_1, C^*, ... respectively. A structure $\mathcal{A}$ is *finite (countably infinite)* if A is a finite (countably infinite) set. For a class **K** of structures, by **K**^{fin} we denote the class of all the finite structures in **K**.

An *embedding* $f : \mathcal{A} \hookrightarrow \mathcal{B}$ between two Θ-structures is every injective map $f : A \to B$ satisfying the following:

- for every $\theta \in \Theta_R$ we have that $(a_1, \ldots, a_r) \in \theta^{\mathcal{A}} \Leftrightarrow (f(a_1), \ldots, f(a_r)) \in \theta^{\mathcal{B}}$, where r is the arity of θ; and
- for every $\varphi \in \Theta_F$ we have that $f(\varphi^{\mathcal{A}}(a_1, \ldots, a_r)) = \varphi^{\mathcal{B}}(f(a_1), \ldots, f(a_r))$, where r is the arity of φ.

By $\mathrm{Emb}(\mathcal{A}, \mathcal{B})$ we denote the set of all the embeddings $\mathcal{A} \hookrightarrow \mathcal{B}$. Surjective embeddings are *isomorphisms*. Structures $\mathcal{A}$ and $\mathcal{B}$ are *isomorphic*, and we write $\mathcal{A} \cong \mathcal{B}$, if there is an isomorphism $\mathcal{A} \to \mathcal{B}$. An *automorphism* is an isomorphism $\mathcal{A} \to \mathcal{A}$. By $\mathrm{Aut}(\mathcal{A})$ we denote the set of all the automorphisms of a structure $\mathcal{A}$. A structure $\mathcal{A}$ is *rigid* if $\mathrm{Aut}(\mathcal{A}) = \{\mathrm{id}_A\}$.

A structure $\mathcal{A}$ is a *substructure* of a structure $\mathcal{B}$, and we write $\mathcal{A} \leqslant \mathcal{B}$, if $A \subseteq B$ and the identity map $a \mapsto a$ is an embedding of $\mathcal{A}$ into $\mathcal{B}$. A *substructure of a structure $\mathcal{A}$ generated by $S \subseteq A$* is the least (with respect to inclusion) substructure $\mathcal{B}$ of $\mathcal{A}$ such that $S \subseteq B$. We denote by $\langle S \rangle_{\mathcal{A}}$ the substructure of $\mathcal{A}$ generated by $S \subseteq A$. A structure $\mathcal{A}$ is *locally finite* if every finitely generated substructure of $\mathcal{A}$ is finite.

A poset $\mathcal{E} = (E, \sqsubseteq)$ is a *lattice* if every pair of elements of E has the greatest lower bound and the least upper bound. Alternatively, an algebra $\mathcal{L} = (L, \wedge, \vee)$ with two binary operations is a *lattice* if both operations are idempotent, commutative and associative, and the absorptive laws hold. These two points of view are closely related: every poset $(E, \sqsubseteq)$ which is a lattice uniquely determines operations $\wedge, \vee : E^2 \to E$ such that the algebra $(E, \wedge, \vee)$ is a lattice (take $a \wedge b$, resp. $a \vee b$, to be the greatest lower bound, resp. the least upper bound, for a and b). Conversely, every algebra $(L, \wedge, \vee)$ which is a lattice uniquely determines the partial order $\sqsubseteq \subseteq L^2$ such that the poset $(L, \sqsubseteq)$ is a lattice (take $a \sqsubseteq b$ if and only if $a \wedge b = a$). What makes these two approaches to lattices different are the embeddings. Clearly, every embedding $f : (L_1, \wedge_1, \vee_1) \to (L_2, \wedge_2, \vee_2)$ between algebras is also an embedding $(L_1, \sqsubseteq_1) \to (L_2, \sqsubseteq_2)$, where $\sqsubseteq_1$ and $\sqsubseteq_2$ are the corresponding derived lattice-ordering relations. The converse, however, is not true.

A poset $\mathcal{E} = (E, \sqsubseteq)$ is a *(meet) semilattice* if every pair of elements of E has the greatest lower bound. An algebra $\mathcal{S} = (L, \wedge)$ with one binary operation is a *(meet) semilattice* if the operation is idempotent, commutative and associative.

Classes of Structures A *variety* of algebras is a class of algebras over a fixed algebraic language which is closed with respect to taking homomorphic images, subalgebras and products of arbitrary families of algebras from the class. The study of varieties of lattices is a deep and active research field in modern algebra, and we refer the reader to [10] for more insight into the typical problems addressed in this context. Clearly, the class **S** of all the semilattices as algebras (of all cardinalities) is a variety of semilattices and the class **L** of all the lattices as algebras (of all cardinalities) is a variety of lattices. Let **D** denote the variety of all the distributive lattices.

Fraïssé theory is a deep structural theory of classes of relational structures. The *age* of a countably infinite structure $\mathcal{A}$ is the class of all the finite structures that embed into $\mathcal{A}$. The age of $\mathcal{A}$ will be denoted by $\mathrm{Age}(\mathcal{A})$. A class **K** of finite structures is an *age* if there is countably infinite structure $\mathcal{A}$ such that $\mathbf{K} = \mathrm{Age}(\mathcal{A})$. It is easy to see that a class **K** of finite structures is an age if and only if **K** is an abstract class (that is, closed for isomorphisms), there are at most countably many pairwise nonisomorphic structures in **K**, **K** has the *hereditary property*:

(HP)   if $\mathcal{A} \in \mathbf{K}$ and $\mathcal{B} \hookrightarrow \mathcal{A}$ then $\mathcal{B} \in \mathbf{K}$;

and **K** has the *joint embedding property*:

(JEP)   for all $\mathcal{A}, \mathcal{B} \in \mathbf{K}$ there is a $\mathcal{C} \in \mathbf{K}$ such that $\mathcal{A} \hookrightarrow \mathcal{C}$ and $\mathcal{B} \hookrightarrow \mathcal{C}$.

An age **K** is a *Fraïssé age* (= Fraïssé class = amalgamation class) [5, 6] if **K** satisfies the *amalgamation property*:

(AP) for all $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbf{K}$ and embeddings $f : \mathcal{A} \hookrightarrow \mathcal{B}$ and $g : \mathcal{A} \hookrightarrow \mathcal{C}$ there exist $\mathcal{D} \in \mathbf{K}$ and embeddings $f' : \mathcal{B} \hookrightarrow \mathcal{D}$ and $g' : \mathcal{C} \hookrightarrow \mathcal{D}$ such that $f' \circ f = g' \circ g$.

A structure $\mathcal{C}$ is *ultrahomogeneous* if for every finitely generated structure $\mathcal{A}$ and every pair of embeddings $f, g : \mathcal{A} \hookrightarrow \mathcal{C}$ there is an automorphism $h \in \mathrm{Aut}(\mathcal{C})$ such that $f = h \circ g$. The age of every locally finite countably infinite ultrahomogeneous structure is a Fraïssé age [5, 6]. Conversely, for every Fraïssé age $\mathbf{K}$ there is a unique (up to isomorphism) countably infinite locally finite ultrahomogeneous structure $\mathcal{A}$ such that $\mathbf{K} = \mathrm{Age}(\mathcal{A})$ [5, 6]. We say that $\mathcal{A}$ is the *Fraïssé limit* of $\mathbf{K}$. For details on Fraïssé theory and further model theoretic background we refer the reader to [9].

Ramsey Theory The leitmotif of Ramsey theory is to prove the existence of regular patterns that occur when a large structure is considered in a restricted context. It started with the nowadays famous Ramsey theorem whose finite version takes the following form where for a set S and a positive integer k by $\binom{S}{k}$ we denote the set of all the k-element subsets of S:

Theorem 2.1 (Finite Ramsey Theorem [24]) *For positive integers k, m and r there exists an integer n such that for every coloring $\chi : \binom{n}{k} \to \{1, 2, \ldots, r\}$ there exists a set $S \in \binom{n}{m}$ such that $\chi(X) = \chi(Y)$ for all $X, Y \in \binom{S}{k}$.*

The Graham-Rothschild Theorem (Theorem 2.2 below) is one of the most powerful tools in Ramsey theory. It formulation requires some preparation. Let A be a finite alphabet. A word u of length n over A can be thought of as an element of A^n but also as a mapping $u : \{1, 2, \ldots, n\} \to A$. In the latter case $u^{-1}(a)$, $a \in A$, denotes the set of all the positions in u where a appears.

Let $X = \{x_1, x_2, \ldots\}$ be a countably infinite set of variables and let A be a finite alphabet disjoint from X. An *m-parameter word over A of length n* is a word $w \in (A \cup \{x_1, x_2, \ldots, x_m\})^n$ satisfying the following:

- each of the letters $x_1, \ldots, x_m$ appears at least once in w, and
- $\min(w^{-1}(x_i)) < \min(w^{-1}(x_j))$ whenever $1 \leqslant i < j \leqslant m$.

Let $W_m^n(A)$ denote the set of all the m-parameter words over A of length n. For $u \in W_m^n(A)$ and $v = v_1 v_2 \ldots v_m \in W_k^m(A)$ let

$$u \cdot v = u[v_1/x_1, v_2/x_2, \ldots, v_m/x_m] \in W_k^n(A)$$

denote the word obtained by replacing each occurrence of x_i in u with v_i, simultaneously for all $i \in \{1, \ldots, m\}$.

Theorem 2.2 (Graham, Rothschild [7]) *Let A be a finite alphabet and let $m, \ell \geqslant 1$ and $k \geqslant 2$. Then there exists an n such that for every partition $W_\ell^n(A) = \mathcal{X}_1 \cup \ldots \cup \mathcal{X}_k$ there exist a $u \in W_m^n(A)$ and j such that $\{u \cdot v : v \in W_\ell^m(A)\} \subseteq \mathcal{X}_j$.*

Structural Ramsey Theory Generalizing the Finite Ramsey Theorem, the structural Ramsey theory originated at the beginning of 1970's in a series of papers (see [15] for references). Let Θ be a first-order language, let $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ be finite Θ-structures and let $k \geqslant 2$ be an integer. Let $\binom{\mathcal{B}}{\mathcal{A}} = \{\tilde{\mathcal{A}} : \tilde{\mathcal{A}} \leqslant \mathcal{B} \text{ and } \tilde{\mathcal{A}} \cong \mathcal{A}\}$. We write $\mathcal{C} \xrightarrow{sub} (\mathcal{B})_k^{\mathcal{A}}$ to denote the following: for every partition $\binom{\mathcal{C}}{\mathcal{A}} = \mathcal{X}_1 \cup \ldots \cup \mathcal{X}_k$ there exist $\tilde{\mathcal{B}} \in \binom{\mathcal{C}}{\mathcal{B}}$ and $j \in \{1, \ldots, k\}$

such that $\binom{\tilde{\mathcal{B}}}{\mathcal{A}} \subseteq \mathcal{X}_j$. A class **K** of finite Θ-structures has the *Ramsey property (with respect to subobjects)* if the following holds:

(RP) for all $\mathcal{A}, \mathcal{B} \in \mathbf{K}$ such that $\mathcal{A} \hookrightarrow \mathcal{B}$ and any integer $k \geqslant 2$ there is a $\mathcal{C} \in \mathbf{K}$ such that $\mathcal{C} \xrightarrow{sub} (\mathcal{B})_k^{\mathcal{A}}$.

We write $\mathcal{C} \xrightarrow{emb} (\mathcal{B})_k^{\mathcal{A}}$ to denote the following: for every partition $\mathrm{Emb}(\mathcal{A}, \mathcal{C}) = \mathcal{X}_1 \cup \ldots \cup \mathcal{X}_k$ there exist $w \in \mathrm{Emb}(\mathcal{B}, \mathcal{C})$ and $j \in \{1, \ldots, k\}$ such that $w \circ \mathrm{Emb}(\mathcal{A}, \mathcal{B}) \subseteq \mathcal{X}_j$. A class **K** of finite Θ-structures has the *Ramsey property (with respect to embeddings)* if the following holds:

(RPE) for all $\mathcal{A}, \mathcal{B} \in \mathbf{K}$ such that $\mathcal{A} \hookrightarrow \mathcal{B}$ and any integer $k \geqslant 2$ there is a $\mathcal{C} \in \mathbf{K}$ such that $\mathcal{C} \xrightarrow{emb} (\mathcal{B})_k^{\mathcal{A}}$.

If a class **K** of finite structures has (RPE) then all the structures in **K** have to be rigid (see [14] for proof). Then it easily follows that for any class of finite rigid structures the properties (RP) and (RPE) are equivalent. So, when working with classes of finite rigid structures we shall simply write $\mathcal{C} \longrightarrow (\mathcal{B})_k^{\mathcal{A}}$ instead of $\mathcal{C} \xrightarrow{emb} (\mathcal{B})_k^{\mathcal{A}}$.

The Ordering Property The ordering property is a property related to the Ramsey property and in many particular cases is implied by nontrivial Ramsey properties [16]. The ordering property was introduced in [18, 20] and has since played an important role in Structural Ramsey theory.

Let Θ be a first-order language and let $< \notin \Theta$ be a new binary relational symbol. Let $\Theta^* = \Theta \cup \{<\}$. Given a Θ^*-structure $\mathcal{A}$, we shall always interpret $<$ in $\mathcal{A}$ as a linear order on A. A class $\mathbf{K}^*$ of Θ^*-structures is an *order expansion* of the class **K** of Θ-structures if for every $(\mathcal{A}, <) \in \mathbf{K}^*$ we have $\mathcal{A} \in \mathbf{K}$, and for every $\mathcal{A} \in \mathbf{K}$ there is at least one linear order $<$ on A such that $(\mathcal{A}, <) \in \mathbf{K}^*$.

For a class $\mathbf{K}^*$ of Θ^*-structures let $\mathbf{K}^*{\restriction}_\Theta = \{\mathcal{A} : (\mathcal{A}, <) \in \mathbf{K}^*\}$. Clearly, $\mathbf{K}^*{\restriction}_\Theta$ is a class of Θ-structures.

An order expansion $\mathbf{K}^*$ of **K** is *reasonable* [11] if for all $\mathcal{A}, \mathcal{B} \in \mathbf{K}$, every embedding $f : \mathcal{A} \hookrightarrow \mathcal{B}$ and every linear order $<$ on A such that $(\mathcal{A}, <) \in \mathbf{K}^*$ there is a linear order $\sqsubset$ on B such that $(\mathcal{B}, \sqsubset) \in \mathbf{K}^*$ and f is an embedding of $(\mathcal{A}, <)$ into $(\mathcal{B}, \sqsubset)$. An order expansion $\mathbf{K}^*$ of **K** is a *reasonable (JEP)-expansion* of **K** if $\mathbf{K}^*$ is a reasonable expansion of **K** and $\mathbf{K}^*$ has (JEP).

The class $\mathbf{K}^*$ of finite Θ^*-structures has the *ordering property* if the following holds, where $\mathbf{K} = \mathbf{K}^*{\restriction}_\Theta$:

(OP) for every $\mathcal{A} \in \mathbf{K}$ there is a $\mathcal{B} \in \mathbf{K}$ such that $(\mathcal{A}, <) \hookrightarrow (\mathcal{B}, \sqsubset)$ for every linear order $<$ on A such that $(\mathcal{A}, <) \in \mathbf{K}^*$, and every linear order $\sqsubset$ on B such that $(\mathcal{B}, \sqsubset) \in \mathbf{K}^*$. We say that $\mathcal{B}$ is a *witness for the ordering property for* $\mathcal{A}$.

3 Varieties of Lattices and Semilattices as Classes of Relational Structures

It was shown in [12] that no order expansion of $\mathbf{D}^{fin}$ has the Ramsey property. We shall now show that this is an ubiquitous phenomenon when it comes to locally finite varieties of lattices.

Theorem 3.1 *Let* **V** *be a nontrivial locally finite variety of lattices distinct from* **L** *and* **D**. *Then no reasonable (JEP)-expansion of* **V**^{fin} *has the Ramsey property (either with respect to subobjects or with respect to embeddings).*

Proof Let **V** be a nontrivial locally finite variety of lattices distinct from **L** and **D**, and let **W** be a reasonable (JEP)-expansion of **V**^{fin}. Assume that **W** has the Ramsey property with respect to subobjects or with respect to embeddings. Because **W** has (JEP) we know from [16] that **W** has (AP) in either case.

Let us show that **V**^{fin} has (AP). Take any $\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2 \in$ **V**^{fin} and embeddings $f_1 : \mathcal{A} \hookrightarrow \mathcal{B}_1$ and $f_2 : \mathcal{A} \hookrightarrow \mathcal{B}_2$. Because **W** is a reasonable expansion of **V**^{fin}, there exists a linear order $\sqsubset$ such that $(\mathcal{A}, \sqsubset) \in$ **W**, and then there exist linear orders $\sqsubset_1$ and $\sqsubset_2$ such that $(\mathcal{B}_1, \sqsubset_1), (\mathcal{B}_2, \sqsubset_2) \in$ **W** and $f_1 : (\mathcal{A}, \sqsubset) \hookrightarrow (\mathcal{B}_1, \sqsubset_1)$, $f_2 : (\mathcal{A}, \sqsubset) \hookrightarrow (\mathcal{B}_2, \sqsubset_2)$ are embeddings. As we have just seen, **W** has (AP), so there is a $(\mathcal{C}, \prec) \in$ **W** and embeddings $g_1 : (\mathcal{B}_1, \sqsubset_1) \hookrightarrow (\mathcal{C}, \prec)$ and $g_2 : (\mathcal{B}_2, \sqsubset_2) \hookrightarrow (\mathcal{C}, \prec)$ such that $g_1 \circ f_1 = g_2 \circ f_2$. Then, clearly, $g_1 : \mathcal{B}_1 \hookrightarrow \mathcal{C}$ and $g_2 : \mathcal{B}_2 \hookrightarrow \mathcal{C}$ are embeddings satisfying $g_1 \circ f_1 = g_2 \circ f_2$. This completes the proof that **V**^{fin} has (AP).

It is a well known fact in lattice theory (see [8, Corollary 509]) that if **X** is a locally finite variety of lattices then **X** has (AP) if and only if **X**^{fin} has (AP). Therefore, **V** has (AP). But by the famous result of Day and Ježek [2], the only nontrivial varieties of lattices with (AP) are **L** and **D**. Contradiction. □

The main goal of this section is to show that every variety of lattices and every variety of semilattices gives rise to a class of finite linearly ordered posets having both the Ramsey and the ordering property. The idea is to replace the algebraic structure by the corresponding relational one. So, the structures we will be working with are lattices understood as partially ordered sets expanded by linear orders that extend the partial order. Note that in this case (RP) and (RPE) coincide since the structures are rigid.

Let **P** denote the class of all the posets (of all cardinalities). A *linearly ordered poset* is a structure $\mathcal{A} = (A, \sqsubseteq, <)$ where $(A, \sqsubseteq)$ is a poset and $<$ is a linear order on A which extends $\sqsubseteq$ (that is, if $a \sqsubseteq b$ and $a \neq b$ then $a < b$). Let $\overrightarrow{\textbf{P}}$ denote the class of linearly ordered posets (of all cardinalities). The Ramsey property for the class $\overrightarrow{\textbf{P}}^{fin}$ was established in two steps: first the ordering property for $\overrightarrow{\textbf{P}}^{fin}$ was established in [22], and then in [4] the ordering property was used to prove the Ramsey property.

An alternative proof that the class $\overrightarrow{\textbf{P}}$ has the Ramsey property was presented in [13]. Whereas the original proof in [22] of the ordering property for $\overrightarrow{\textbf{P}}^{fin}$ relies on the Dual Ramsey Theorem, the alternative proof in [13] derives the the Ramsey property for $\overrightarrow{\textbf{P}}^{fin}$ as a direct consequence of the Graham-Rothschild Theorem (Theorem 2.2). Interestingly, the class $\overrightarrow{\textbf{P}}$ was the only known Ramsey class of structures where the proof of the Ramsey property relied on proving first that the class has the ordering property. The proof presented in [13] is new not only because new proof strategies were used, but also because it does not not rely on the ordering property.

We shall now present an extract of this proof restructured so as to enable us to reason about the ordering property not only of the class $\overrightarrow{\textbf{P}}$, but also some of its subclasses. In this presentation we refrain from the explicit use of the machinery of category theory which was the main language used in [13] and instead rephrase the proof in terms of first-order structures and embeddings.

Let $<_{alex}$ and $<_{\overline{lex}}$ denote the *anti-lexicographic* and *complemented lexicographic* ordering on $\mathcal{P}(\{1, \ldots, n\})$, respectively, defined as follows:

$$A <_{alex} B \text{ iff } A \subseteq B \text{ or}$$

$$\max(A \setminus B) < \max(B \setminus A) \text{ in case } A \text{ and } B \text{ are incomparable,}$$

$$A <_{\overline{lex}} B \text{ iff } \{1, \ldots, n\} \setminus A <_{lex} \{1, \ldots, n\} \setminus B$$

$$\text{iff } A \supseteq B \text{ or}$$

$$\min(A \setminus B) < \min(B \setminus A) \text{ in case } A \text{ and } B \text{ are incomparable,}$$

where $<$ denotes the usual linear order on the integers. It is easy to see that all three are linear orders on $\mathcal{P}(\{1, \ldots, n\})$.

Theorem 3.2 (*cf.* [13, Theorem 4.1]) *For $n \in \mathbb{N}$ let Π_n denote the following linearly ordered poset:* $\Pi_n = \left(\mathcal{P}(\{1, \ldots, n\}), \supseteq, <_{\overline{lex}}\right)$.

(a) *For every $k \geqslant 2$ and all finite linearly ordered posets $\mathcal{A}, \mathcal{B} \in \overrightarrow{\mathbf{P}}$ such that $\mathcal{A} \hookrightarrow \mathcal{B}$ there is an $N \in \mathbb{N}$ such that $\Pi_N \longrightarrow (\mathcal{B})_k^{\mathcal{A}}$.*

(b) *Let $\mathbf{K}^*$ be a subclass of $\overrightarrow{\mathbf{P}}$ such that $\Pi_n \in \mathbf{K}^*$ for all $n \in \mathbb{N}$. Then $\mathbf{K}^*$ has the Ramsey property. In particular, $\overrightarrow{\mathbf{P}}$ has the Ramsey property (see [4, 22] for the original proof).*

Proof (Sketch) (*a*) Let $k \geqslant 2$ and let $\mathcal{A} = (A, \sqsubseteq, <)$ and $\mathcal{B} = (B, \sqsubseteq, <)$ be finite linearly ordered posets such that $\mathcal{A} \hookrightarrow \mathcal{B}$.

A *downset* in a poset $\mathcal{A}$ is a subset $D \subseteq A$ such that $x \in D$ and $y \sqsubseteq x$ implies $y \in D$. For $a \in A$ let $\downarrow_\mathcal{A} a = \{x \in A : x \sqsubseteq a\}$. Clearly, $\downarrow_\mathcal{A} a$ is always a downset in $\mathcal{A}$, but not all the downsets are of the form $\downarrow_\mathcal{A} a$. To see this, take two $a, b \in A$ incomparable with respect to $\sqsubseteq$. Then $\downarrow_\mathcal{A} a \cup \downarrow_\mathcal{A} b$ is a downset in $\mathcal{A}$ which is not of the form $\downarrow_\mathcal{A} x$ for some $x \in A$.

Let $\mathcal{A}$ have $m_\mathcal{A}$ nonempty downsets and let $\mathcal{B}$ have $m_\mathcal{B}$ nonempty downsets. According to the Graham-Rothschild Theorem (Theorem 2.2) there exists an N such that for every partition $W_{m_\mathcal{A}}^N(\{0\}) = \mathcal{X}_1 \cup \ldots \cup \mathcal{X}_k$ there is a $u \in W_{m_\mathcal{B}}^N(\{0\})$ and a j satisfying $\{u \cdot h : h \in W_{m_\mathcal{A}}^{m_\mathcal{B}}(\{0\})\} \subseteq \mathcal{X}_j$. Let us show that $\Pi_N \longrightarrow (\mathcal{B})_k^\mathcal{A}$.

Let $D_1, \ldots, D_{m_\mathcal{A}}$ be all the nonempty downsets in $\mathcal{A}$ and let $D_1 <_{alex} D_2 <_{alex} \ldots <_{alex} D_{m_\mathcal{A}}$. For $u \in W_{m_\mathcal{A}}^N(\{0\})$, let $X_i = u^{-1}(x_i)$, $1 \leqslant i \leqslant m_\mathcal{A}$, and let $a_i = \bigcup\{X_\alpha : i \in D_\alpha\}$, $i \in A$. It was shown in the proof of [13, Theorem 4.1] that for every $u \in W_{m_\mathcal{A}}^N(\{0\})$ the mapping $\Phi_{\mathcal{A}, N}(u) : \mathcal{A} \to \Pi_N : i \mapsto a_i$ is an embedding.

Take any partition $\binom{\Pi_N}{\mathcal{A}} = \mathcal{X}_1 \cup \ldots \cup \mathcal{X}_k$. Let $W_{m_\mathcal{A}}^N(\{0\}) = \mathcal{X}_1' \cup \ldots \cup \mathcal{X}_k'$ be a partition constructed as follows: for $w \in W_{m_\mathcal{A}}^N(\{0\})$ let

$$w \in \mathcal{X}_j' \text{ if and only if } \text{im}(\Phi_{\mathcal{A}, N}(w)) \in \mathcal{X}_j \tag{3.1}$$

(here, $\text{im}(f)$ denotes the image of f as a substructure of its codomain). By the construction of N, there exist a $u \in W_{m_\mathcal{B}}^N(\{0\})$ and a j such that

$$\{u \cdot h : h \in W_{m_\mathcal{A}}^{m_\mathcal{B}}(\{0\})\} \subseteq \mathcal{X}_j'. \tag{3.2}$$

Then $\tilde{\mathcal{B}} = \text{im}(\Phi_{\mathcal{B}, N}(u))$ is a copy of $\mathcal{B}$ in Π_N because $\Phi_{\mathcal{B}, N}(u)$ is an embedding. Let us show that $\binom{\tilde{\mathcal{B}}}{\mathcal{A}} \subseteq \mathcal{X}_j$.

Take any $\tilde{\mathcal{A}} \in \binom{\check{\mathcal{B}}}{\mathcal{A}}$. Then there is an embedding $f : \mathcal{A} \hookrightarrow \mathcal{B}$ such that $\mathrm{im}(\Phi_{\mathcal{B},N}(u) \circ f) = \tilde{\mathcal{A}}$. It was shown in the proof of [13, Theorem 4.1] that one can then find a word $h = h_1 h_2 \ldots h_{m_\mathcal{B}} \in W_{m_\mathcal{A}}^{m_\mathcal{B}}(\{0\})$ such that $\Phi_{\mathcal{B},N}(u) \circ f = \Phi_{\mathcal{A},N}(u \cdot h)$. From Eq. 3.2 we know that $u \cdot h \in \mathcal{X}'_j$, whence $\mathrm{im}(\Phi_{\mathcal{A},N}(u \cdot h)) \in \mathcal{X}_j$ by Eq. 3.1. Therefore, $\mathrm{im}(\Phi_{\mathcal{B},N}(u) \circ f) = \tilde{\mathcal{A}} \in \mathcal{X}_j$.

(*b*) Directly from (*a*). □

Using the standard Sierpinski-style coloring obtained by comparing two linear orders we can deduce the ordering property for a class from the fact that it has the Ramsey property.

Theorem 3.3 *Let* $\mathbf{K}^*$ *be a subclass of* $\overrightarrow{\mathbf{P}}$ *such that* $\Pi_n \in \mathbf{K}^*$ *for all* $n \in \mathbb{N}$. *Then* $\mathbf{K}^*$ *has the ordering property. In particular, the class* $\overrightarrow{\mathbf{P}}$ *has the ordering property (see [22, 25] for the original proof).*

Proof Let $\mathbf{K}^*$ be a subclass of $\overrightarrow{\mathbf{P}}$ such that $\Pi_n \in \mathbf{K}^*$ for all $n \in \mathbb{N}$. It is easy to see that (OP) is equivalent to the following whenever $\mathbf{K}^*$ has (JEP):

(OP')   for every $(\mathcal{A}, <) \in \mathbf{K}^*$ there is a $\mathcal{B} \in \mathbf{K}$ such that $(\mathcal{A}, <) \hookrightarrow (\mathcal{B}, \sqsubset)$ for every linear order $\sqsubset$ on B with $(\mathcal{B}, \sqsubset) \in \mathbf{K}^*$. We say that $\mathcal{B}$ is a *witness for the ordering property for* $(\mathcal{A}, <)$.

Let us show that $\mathbf{K}^*$ has (JEP) so that we can use (OP'). One of the byproducts of Theorem 3.2 (*a*) is the following: for every $\mathcal{A} \in \overrightarrow{\mathbf{P}}$ there is an $n \in \mathbb{N}$ such that $\mathcal{A} \hookrightarrow \Pi_n$. So, take any $\mathcal{A}, \mathcal{B} \in \mathbf{K}^*$. As we have just seen, $\mathcal{A} \hookrightarrow \Pi_n$ and $\mathcal{B} \hookrightarrow \Pi_m$ for some $n, m \in \mathbb{N}$. Without loss of generality we can take that $n \leqslant m$. Therefore, $\mathcal{A} \hookrightarrow \Pi_n \hookrightarrow \Pi_m \hookleftarrow \mathcal{B}$.

So, let us show (OP') for $\mathbf{K}^*$. Let $\mathcal{B} = (B, \sqsubseteq, <)$ be a finite linearly ordered poset. If $(B, \sqsubseteq)$ is an antichain then $(B, \sqsubseteq)$ is a witness for the ordering property for $\mathcal{B}$. Assume, therefore, that $\mathcal{B}$ is not an antichain and take any $x, y \in B$ such that $x \sqsubset y$. Add a new element $z \notin B$ to $\mathcal{B}$ to obtain a finite linearly ordered poset $\mathcal{B}_1 = (B_1, \sqsubseteq_1, <_1)$ as follows: $B_1 = B \cup \{z\}$, $\sqsubseteq_1 = \sqsubseteq \cup \{(z, z)\}$ (in other words, z is incomparable with every $b \in B$), and $<_1$ is an extension of $<$ such that $x <_1 z <_1 y$. Let $\mathcal{A} = (\{0, 1\}, =, <)$ be a two-element linearly ordered antichain ($0 < 1$). By Theorem 3.2 (*a*) there is an $N \in \mathbb{N}$ such that $\Pi_N \longrightarrow (\mathcal{B})_2^{\mathcal{A}}$. For notational convenience, let $\Pi_N = (\pi_N, \sqsubseteq_N, <_N)$. Let us show that $(\pi_N, \sqsubseteq_N) \in \mathbf{K}$ is a witness for the ordering property for $\mathcal{B}_1$ and hence a witness for the ordering property for $\mathcal{B}$ since $\mathcal{B} \hookrightarrow \mathcal{B}_1$. Take any linear order $\prec$ on π_N which extends $\sqsubseteq_N$ and consider the coloring $\binom{\Pi_N}{\mathcal{A}} = X_0 \cup X_1$ as follows. Let $\tilde{\mathcal{A}} = (\{\tilde{0}, \tilde{1}\}, =, <) \in \binom{\Pi_N}{\mathcal{A}}$ where $\tilde{0} < \tilde{1}$. Put

$$\tilde{\mathcal{A}} \in X_0 \quad \text{if} \quad \tilde{0} <_N \tilde{1} \text{ and } \tilde{0} \prec \tilde{1}, \text{ or } \tilde{1} <_N \tilde{0} \text{ and } \tilde{1} \prec \tilde{0};$$
$$\tilde{\mathcal{A}} \in X_1 \quad \text{if} \quad \tilde{0} <_N \tilde{1} \text{ and } \tilde{1} \prec \tilde{0}, \text{ or } \tilde{1} <_N \tilde{0} \text{ and } \tilde{0} \prec \tilde{1}.$$

Then there is a $\tilde{\mathcal{B}}_1 \in \binom{\mathcal{C}}{\mathcal{B}_1}$ such that $\binom{\tilde{\mathcal{B}}_1}{\mathcal{A}}$ is monochromatic. Let us show that $\binom{\tilde{\mathcal{B}}_1}{\mathcal{A}} \subseteq X_0$. Suppose, to the contrary, that $\binom{\tilde{\mathcal{B}}_1}{\mathcal{A}} \subseteq X_1$. Let $\tilde{x}, \tilde{y}, \tilde{z}$ be the elements of $\tilde{\mathcal{B}}_1$ which correspond to x, y, z in $\mathcal{B}_1$. Then $\{\tilde{x}, \tilde{z}\}$ and $\{\tilde{y}, \tilde{z}\}$ are antichains in $\mathcal{C}$ such that $\tilde{x} <_N \tilde{z} <_N \tilde{y}$ (because this is their order in $\mathcal{B}_1$) and $\tilde{x} \succ \tilde{z} \succ \tilde{y}$ (because $\binom{\tilde{\mathcal{B}}_1}{\mathcal{A}} \subseteq X_1$). On the other hand, $\tilde{x} \sqsubseteq_N \tilde{y}$ – contradiction with the fact that $\prec$ extends $\sqsubseteq_N$.

Therefore, $\binom{\tilde{\mathcal{B}}_1}{\mathcal{A}} \subseteq X_0$, which means that $<_N$ and $\prec$ coincide on all 2-element antichains of $\tilde{\mathcal{B}}_1$. We already know that $<_N$ and $\prec$ coincide on all comparable pairs of elements because $\prec$ extends $\sqsubseteq_N$. So, $\tilde{\mathcal{B}}_1 \leqslant (\pi_N, \sqsubseteq_N, \prec)$, whence $\mathcal{B}_1 \hookrightarrow (\pi_N, \sqsubseteq_N, \prec)$. $\square$

With all the technical results in place, we conclude the section by showing that the "relational alter ego" of every variety of lattices and every variety of semilattices has both the Ramsey property and the ordering property. Let us start by introducing a bit of notation. For a finite lattice $\mathcal{L} = (L, \wedge, \vee)$ let $\mathrm{rel}(\mathcal{L}) = (L, \sqsubseteq)$ be the corresponding finite poset (where we set $a \sqsubseteq b$ if and only if $a \wedge b = a$). With the slight abuse of set notation, for a variety $\mathbf{V}$ of lattices let $\mathrm{rel}(\mathbf{V}^{fin}) = \{\mathrm{rel}(\mathcal{L}) : \mathcal{L} \in \mathbf{V}^{fin}\}$. On the other hand, let $\overrightarrow{\mathrm{rel}}(\mathbf{V}^{fin})$ denote the class of all the finite linearly ordered posets $(L, \sqsubseteq, \prec)$ where $(L, \sqsubseteq) = \mathrm{rel}(\mathcal{L})$ for some $\mathcal{L} \in \mathbf{V}^{fin}$. So, the elements of $\overrightarrow{\mathrm{rel}}(\mathbf{V}^{fin})$ are all the posets from $\mathrm{rel}(\mathbf{V}^{fin})$ where each poset is expanded by all possible linear extensions of the partial order.

Analogously, for a finite semilattice $\mathcal{S} = (S, \wedge)$ let $\mathrm{rel}(\mathcal{S}) = (S, \sqsubseteq)$ be the corresponding finite poset (where we set $a \sqsubseteq b$ if and only if $a \wedge b = a$), let $\mathrm{rel}(\mathbf{V}^{fin}) = \{\mathrm{rel}(\mathcal{S}) : \mathcal{S} \in \mathbf{V}^{fin}\}$ for a variety $\mathbf{V}$, and let $\overrightarrow{\mathrm{rel}}(\mathbf{V}^{fin})$ denote the class of all the finite linearly ordered posets $(S, \sqsubseteq, \prec)$ where $(S, \sqsubseteq) = \mathrm{rel}(\mathcal{S})$ for some $\mathcal{S} \in \mathbf{V}^{fin}$.

Theorem 3.4 (a)   *Let* $\mathbf{V}$ *be a nontrivial variety of lattices. Then* $\overrightarrow{\mathrm{rel}}(\mathbf{V}^{fin})$ *has both the Ramsey property and the ordering property.*

(b)   *Let* $\mathbf{V}$ *be a nontrivial variety of semilattices. Then* $\overrightarrow{\mathrm{rel}}(\mathbf{V}^{fin})$ *has both the Ramsey property and the ordering property.*

Proof (a)   Every nontrivial variety of lattices contains the two-element lattice $\mathcal{L}_2 = (\{0, 1\}, \wedge, \vee)$ where $0 < 1$, and hence all the finite powers of $\mathcal{L}_2$. Since $(\mathcal{P}(A), \cup, \cap) \cong \mathcal{L}_2^{|A|}$ for every finite set A, it follows that every nontrivial variety of lattices contains all the lattices of the form $(\mathcal{P}(A), \cup, \cap)$ where A is a finite set. Therefore, $\Pi_n \in \overrightarrow{\mathrm{rel}}(\mathbf{V}^{fin})$ for all $n \in \mathbb{N}$. It is now immediate from Theorems 3.2 and 3.3 that $\overrightarrow{\mathrm{rel}}(\mathbf{V}^{fin})$ has the Ramsey property as well as the ordering property.

(b)   Analogous to (a).

$\square$

4 The Weak Triangle Condition

The main idea used to prove Theorem 3.3 (adding a strategically placed triangle) can be generalized to a much larger class of structures, those which satisfy what we call the *weak triangle condition*. This is a weaker form of the *triangle condition* introduced in [11] also in connection to understanding the ordering property.

Let Θ be a first-order language and let $< \notin \Theta$ be a new binary relational symbol. Let $\Theta^* = \Theta \cup \{<\}$. As usual, given a Θ^*-structure $\mathcal{A}$ we shall always interpret $<$ in $\mathcal{A}$ as a linear order on A. Let $\mathbf{K}^*$ be a class of finite Θ^*-structures and assume that it is an order expansion of a class $\mathbf{K}$ of Θ-structures. Let $S_2(\mathbf{K}^*)$ denote the class of all the 2-generated structures in $\mathbf{K}^*$. We say that $\mathbf{K}^*$ has the *weak triangle condition* if

$(W\triangle C)$ for every nonempty finite $\Sigma \subseteq S_2(\mathbf{K}^*)$ there is a $\tau \in S_2(\mathbf{K}^*)$ such that for every $\sigma \in \Sigma$ there exists a $(\mathcal{D}, <_D) \in \mathbf{K}^*$ and $x, y, z \in D$ such that $x <_D y <_D z$, $\langle x, z \rangle_{(\mathcal{D}, <_D)} \cong \sigma$ and $\langle x, y \rangle_{(\mathcal{D}, <_D)} \cong \langle y, z \rangle_{(\mathcal{D}, <_D)} \cong \tau$.

Theorem 4.1 *Let Θ be a first-order language and let $< \notin \Theta$ be a new binary relational symbol that we interpret as linear ordering. Let $\Theta^* = \Theta \cup \{<\}$. Let $\mathbf{K}^*$ be a class of finite Θ^*-structures with (JEP), (RP) and $(W \triangle C)$. Then $\mathbf{K}^*$ has the ordering property.*

Proof It has been shown in [16] that every class with (JEP) and (RP) also has (AP). Amalgamation is the key ingredient of this proof, so before we move on let us describe a construction that the proof relies on. Take any $(\mathcal{B}, <_B) \in \mathbf{K}^*$ and let (a_i, b_i), $1 \leqslant i \leqslant n$, be an enumeration of all the pairs of elements of B satisfying $a_i <_B b_i$. Let $\sigma_i = \langle a_i, b_i \rangle_{(\mathcal{B}, <_B)}$, $1 \leqslant i \leqslant n$. Put $\Sigma = \{\sigma_i : 1 \leqslant i \leqslant n\}$. By $(W \triangle C)$ there exists a $\tau \in S_2(\mathbf{K}^*)$, and for every $i \in \{1, \ldots, n\}$ there exist $(\mathcal{D}_i, <_i) \in \mathbf{K}^*$ and $x_i, y_i, z_i \in D_i$ such that $x_i <_i y_i <_i z_i$, $\langle x_i, z_i \rangle_{(\mathcal{D}_i, <_i)} \cong \sigma_i$ and $\langle x_i, y_i \rangle_{(\mathcal{D}_i, <_i)} \cong \langle y_i, z_i \rangle_{(\mathcal{D}_i, <_i)} \cong \tau$. We now perform n amalgamations inductively as follows. Put $(\mathcal{B}_0, <_{B_0}) = (\mathcal{B}, <_B)$. Assume that $(\mathcal{B}_{i-1}, <_{B_{i-1}})$ has been constructed. In the ith step amalgamate $(\mathcal{B}_{i-1}, <_{B_{i-1}})$ with $(\mathcal{D}_i, <_i)$ over σ_i and embeddings $f_i : \sigma_i \hookrightarrow (\mathcal{B}_{i-1}, <_{B_{i-1}}) : a_i \mapsto a_i, b_i \mapsto b_i$ and $g_i : \sigma_i \hookrightarrow (\mathcal{D}_i, <_i) : a_i \mapsto x_i, b_i \mapsto z_i$, and denote the amalgam by $(\mathcal{B}_i, <_{B_i})$, Fig. 1. Without loss of generality we can assume that $(\mathcal{B}, <_B) \leqslant (\mathcal{B}_i, <_{B_i})$ so that the procedure can continue as described. Let us denote the final amalgam $(\mathcal{B}_n, <_{B_n})$ by $\nabla(\mathcal{B}, <_B)$. Note that $\nabla(\mathcal{B}, <_B) \in \mathbf{K}^*$. Without loss of generality we can assume that $(\mathcal{B}, <_B) \leqslant \nabla(\mathcal{B}, <_B)$.
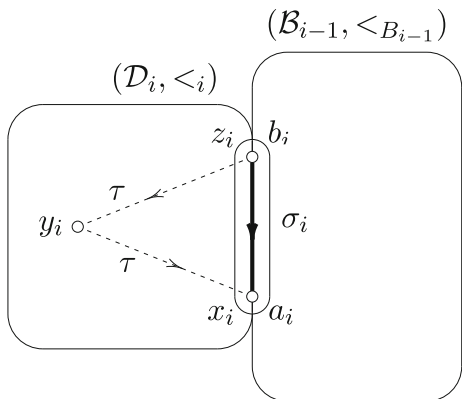
Take any $(\mathcal{A}, <_A) \in \mathbf{K}^*$.

Case 1: $(\mathcal{A}, >_A) \notin \mathbf{K}^*$.

Let $(\mathcal{C}, <_C) = \nabla(\mathcal{A}, <_A)$. By the assumption, $(\mathcal{A}, <_A) \leqslant (\mathcal{C}, <_C)$. Since K^* has the Ramsey property there is a $(\mathcal{Q}, <_Q) \in \mathbf{K}^*$ such that $(\mathcal{Q}, <_Q) \longrightarrow (\mathcal{C}, <_C)^\tau_2$. Let us show that $\mathcal{Q}$ is the witness for the ordering property for $(\mathcal{A}, <_A)$. Take any linear order $\sqsubset$ on Q such that $(\mathcal{Q}, \sqsubset) \in \mathbf{K}^*$. Consider the following coloring $\binom{(\mathcal{Q}, <_Q)}{\tau} = \mathcal{X}_1 \cup \mathcal{X}_2$: for $q, r \in Q$ such that $q <_Q r$ and $\langle q, r \rangle_{(\mathcal{Q}, <_Q)} \cong \tau$ put $\langle q, r \rangle_{(\mathcal{Q}, <_Q)} \in \mathcal{X}_1$ if $q \sqsubset r$, and put $\langle q, r \rangle_{(\mathcal{Q}, <_Q)} \in \mathcal{X}_2$ if $q \sqsupset r$. Then there is a monochromatic copy $(\tilde{\mathcal{C}}, <_{\tilde{C}})$ of $(\mathcal{C}, <_C)$ in $(\mathcal{Q}, <_Q)$. Let $(\tilde{\mathcal{A}}, <_{\tilde{A}})$ be a copy of $(\mathcal{A}, <_A)$ in $(\tilde{\mathcal{C}}, <_{\tilde{C}})$.

Let us first show that $\binom{(\tilde{\mathcal{C}}, <_{\tilde{c}})}{\tau} \subseteq \mathcal{X}_2$ cannot happen. Assume, to the contrary, that this is the case. Then we can show that for all $\tilde{a}, \tilde{b} \in \tilde{A}$ we have that $\tilde{a} <_{\tilde{A}} \tilde{b}$ if and only if $\tilde{a} \sqsupset \tilde{b}$. Assume that $\tilde{a} <_{\tilde{A}} \tilde{b}$. Then $(\tilde{a}, \tilde{b})$ is a copy in $(\tilde{\mathcal{A}}, <_{\tilde{A}})$ of some pair (a_i, b_i) (see the beginning of the ∇ construction). By the construction, there is a $\tilde{y} \in \tilde{C}$ such that $\tilde{a} <_{\tilde{C}} \tilde{y} <_{\tilde{C}} \tilde{b}$ and $\langle \tilde{a}, \tilde{y} \rangle_{(\tilde{C}, <_{\tilde{C}})} \cong \langle \tilde{y}, \tilde{b} \rangle_{(\tilde{C}, <_{\tilde{C}})} \cong \tau$, see Fig. 1. From $\binom{(\tilde{\mathcal{C}}, <_{\tilde{c}})}{\tau} \subseteq \mathcal{X}_2$

Fig. 1 The ∇ construction

$(\mathcal{B}_{i-1}, <_{B_{i-1}})$

$(\mathcal{D}_i, <_i)$

$z_i \quad b_i$

τ

$y_i \circ$

σ_i

τ

$x_i \quad a_i$

it now follows that $\tilde{a} \sqsubset \tilde{y} \sqsubset \tilde{b}$. But, then we have that $(\tilde{\mathcal{A}}, >_{\tilde{A}}) \cong \langle \tilde{A} \rangle_{(\mathcal{Q}, \sqsubset)}$ whence $(\mathcal{A}, >_A) \hookrightarrow (\mathcal{Q}, \sqsubset)$. Since $(\mathcal{Q}, \sqsubset) \in \mathbf{K}^*$ we conclude $(\mathcal{A}, >_A) \in \mathbf{K}^*$. Contradiction.

Therefore, $\binom{(\tilde{\mathcal{C}}, <_{\tilde{C}})}{\tau} \subseteq \mathcal{X}_1$. We can now repeat the same argument to show that $(\tilde{\mathcal{A}}, <_{\tilde{A}}) \cong \langle \tilde{A} \rangle_{(\mathcal{Q}, \sqsubset)}$ whence $(\mathcal{A}, <_A) \hookrightarrow (\mathcal{Q}, \sqsubset)$.

Case 2: $(\mathcal{A}, >_A) \in \mathbf{K}^*$.

Let $(\mathcal{B}, <_B)$ be a structure which embeds both $(\mathcal{A}, <_A)$ and $(\mathcal{A}, >_A)$. Then $(\mathcal{A}, <_A) \hookrightarrow (\mathcal{B}, <_B)$ and $(\mathcal{A}, <_A) \hookrightarrow (\mathcal{B}, >_B)$. Let $(\mathcal{C}, <_C) = \nabla(\mathcal{B}, <_B)$. By the Ramsey property there is a $(\mathcal{Q}, <_Q) \in \mathbf{K}^*$ such that $(\mathcal{Q}, <_Q) \longrightarrow (\mathcal{C}, <_C)_2^\tau$. Let us show that $\mathcal{Q}$ is the witness for the ordering property for $(\mathcal{A}, <_A)$. Take any linear order $\sqsubset$ on Q such that $(\mathcal{Q}, \sqsubset) \in \mathbf{K}^*$ and construct the coloring $\binom{(\mathcal{Q}, <_Q)}{\tau} = \mathcal{X}_1 \cup \mathcal{X}_2$ as in Case 1. Then there is a monochromatic copy $(\tilde{\mathcal{C}}, <_{\tilde{C}})$ of $(\mathcal{C}, <_C)$ in $(\mathcal{Q}, <_Q)$. Let $(\tilde{\mathcal{B}}, <_{\tilde{B}})$ be a copy of $(\mathcal{B}, <_B)$ in $(\tilde{\mathcal{C}}, <_{\tilde{C}})$.

If $\binom{(\tilde{\mathcal{C}}, <_{\tilde{C}})}{\tau} \subseteq \mathcal{X}_1$ we can repeat the argument from Case 1 to show that $(\tilde{\mathcal{B}}, <_{\tilde{B}}) \cong \langle \tilde{B} \rangle_{(\mathcal{Q}, \sqsubset)}$ whence $(\mathcal{A}, <_A) \hookrightarrow (\tilde{\mathcal{B}}, <_{\tilde{B}}) \hookrightarrow (\mathcal{Q}, \sqsubset)$. If, however, $\binom{(\tilde{\mathcal{C}}, <_{\tilde{C}})}{\tau} \subseteq \mathcal{X}_2$ then $(\tilde{\mathcal{B}}, >_{\tilde{B}}) \cong \langle \tilde{B} \rangle_{(\mathcal{Q}, \sqsubset)}$ whence $(\mathcal{A}, <_A) \hookrightarrow (\tilde{\mathcal{B}}, >_{\tilde{B}}) \hookrightarrow (\mathcal{Q}, \sqsubset)$. □

As an application of the above theorem let us consider structures with several poset relations (cf. [3]). Let $\mathcal{T} = (\{1, \ldots, n\}, \preccurlyeq)$ be a poset, $n \geqslant 1$, which we refer to as a *template*. A structure $(A, \sqsubseteq_1, \ldots, \sqsubseteq_n)$ consisting of n partial orders *conforms to the template* $\mathcal{T}$ if $(\sqsubseteq_i) \subseteq (\sqsubseteq_j)$ whenever $i \preccurlyeq j$ in $\mathcal{T}$; and is *consistent* if there is a linear order on A which extends each $\sqsubseteq_i$, $1 \leqslant i \leqslant n$. Given a template $\mathcal{T}$, let $\mathbf{P}_{\mathcal{T}}$ denote the class of all the structures $(A, \sqsubseteq_1, \ldots, \sqsubseteq_n)$ (of all cardinalities) consisting of n partial orders which conform to $\mathcal{T}$ and which are consistent; and let $\overrightarrow{\mathbf{P}}_{\mathcal{T}}$ denote the class of all the structures $(A, \sqsubseteq_1, \ldots, \sqsubseteq_n, <)$ such that $(A, \sqsubseteq_1, \ldots, \sqsubseteq_n) \in \mathbf{P}_{\mathcal{T}}$ and $<$ is a linear order which extends each $\sqsubseteq_i$, $1 \leqslant i \leqslant n$. (Note that $\mathbf{P}(1) = \mathbf{P}$ and $\overrightarrow{\mathbf{P}}(1) = \overrightarrow{\mathbf{P}}$, where 1 denotes the trivial one-element template.)

Corollary 4.2 *For every finite poset $\mathcal{T}$ with $T = \{1, \ldots, n\}$ the class $\overrightarrow{\mathbf{P}}_{\mathcal{T}}^{fin}$ has the ordering property.*

Proof Let $\mathcal{T}$ be a finite poset with $T = \{1, \ldots, n\}$. It was shown in [3] that the class $\overrightarrow{\mathbf{P}}_{\mathcal{T}}^{fin}$ has the Ramsey property (note that the template of structures in $\overrightarrow{\mathbf{P}}_{\mathcal{T}}^{fin}$ is *not* $\mathcal{T}$, but $\mathcal{T}$ + the new top element), so Theorem 4.1 implies that it suffices to show that the class $\overrightarrow{\mathbf{P}}_{\mathcal{T}}$ has the $(W \triangle C)$. But this is straightforward since for any $\Sigma \subseteq S_2(\overrightarrow{\mathbf{P}}_{\mathcal{T}}^{fin})$ we can always take $\tau = (\{0, 1\}, \sqsubseteq_1, \ldots, \sqsubseteq_n, <)$ where $0 < 1$ and all the $\sqsubseteq_i$'s are the trivial partial orders $\{(0, 0), (1, 1)\}$. □

Classes of structures having the Ramsey property and the ordering property are particularly intriguing in the context of Kechris-Pestov-Todorčević correspondence [11]. Let G be a topological group. Its *action* on X is a mapping $\cdot : G \times X \to X$ such that $1 \cdot x = x$ and $g \cdot (f \cdot x) = (gf) \cdot x$. We also say that G *acts on* X. A G-*flow* is a continuous action of a topological group G on a topological space X. A *subflow* of a G-flow $\cdot : G \times X \to X$ is a continuous map $* : G \times Y \to Y$ where $Y \subseteq X$ is a closed subspace of X and $g * y = g \cdot y$ for all $g \in G$ and $y \in Y$. A G-flow $G \times X \to X$ is *minimal* if it has no proper closed subflows. A G-flow $u : G \times X \to X$ is *universal* if every compact minimal G-flow $G \times Z \to Z$ is a factor of u. It is a well-known fact that for each Hausdorff topological group G there

is, up to isomorphism of G-flows, a unique universal minimal G-flow, usually denoted by $G \curvearrowright M(G)$.

A topological group G is *extremely amenable* if every G-flow $\cdot : G \times X \to X$ on a compact Hausdroff space X has a fixed point, that is, there is an $x_0 \in X$ such that $g \cdot x_0 = x_0$ for all $g \in G$. Since $\mathrm{Sym}(A)$, the group of all the permutations on a set A, carries naturally the topology of pointwise convergence, permutation groups can be thought of as topological groups. In [11] the authors show the following.

Theorem 4.3 [11, Theorem 4.7] *Let G be a closed subgroup of $\mathrm{Sym}(F)$ for a countable set F. Then G is extremely amenable if and only if $G = \mathrm{Aut}(\mathcal{F})$ for a countable ultrahomogeneous structure $\mathcal{F}$ whose age has the Ramsey property.*

If G is not extremely amenable [11] provides us with a means to compute its universal minimal flow. Let $\mathrm{LO}(A)$ be the set of all linear orders on A and let G be a closed subgroup of $\mathrm{Sym}(A)$. The set $\mathrm{LO}(A)$ with the standard product topology is a compact Hausdorff space and the action of G on $\mathrm{LO}(A)$ given by $x <^g y$ if and only if $g^{-1}(x) < g^{-1}(y)$ is continuous. This action is usually referred to as the *logical action of G on $\mathrm{LO}(A)$*.

Note that if $\mathbf{K}^*$ is a reasonable order expansion of $\mathbf{K}$ and $\mathbf{K}^*$ has (HP), resp. (JEP) or (AP), then $\mathbf{K}$ has (HP), resp. (JEP) or (AP) [11]; consequently if $\mathbf{K}^*$ is a Fraïssé age, then so is $\mathbf{K}$. Moreover, assume that $\mathbf{K}^*$ is a Fraïssé age of Θ^*-structures, let $\mathcal{F}_* = (\mathcal{F}, <)$ be the Fraïssé limit of $\mathbf{K}^*$ and let $\mathbf{K} = \mathbf{K}^* \restriction_\Theta$. Then $\mathbf{K}^*$ is a reasonable expansion of $\mathbf{K}$ if and only if $\mathbf{K}$ is a Fraïssé age and $\mathcal{F}$ is the Fraïssé limit of $\mathbf{K}$ [11].

Theorem 4.4 [11, Theorem 10.8] *Let $\mathbf{K}^*$ be a Fraïssé age which is a reasonable order expansion of a Fraïssé age $\mathbf{K}$. Let $\mathcal{F}$ be the Fraïssé limit of $\mathbf{K}$, let $\mathcal{F}^* = (\mathcal{F}, \sqsubset)$ be the Fraïssé limit of $\mathbf{K}^*$, let $G = \mathrm{Aut}(\mathcal{F})$ and $X^* = \overline{G \cdot \sqsubset}$ (in the logical action of G on $\mathrm{LO}(F)$). Then the logical action of G on X^* is the universal minimal flow of G if and only if $\mathbf{K}^*$ has the Ramsey property and the ordering property.*

We can now present an infinite family of topological groups whose universal minimal flows can be computed using the Kechris-Pestov-Todorčević correspondence.

Corollary 4.5 *Let $\mathcal{T}$ be a finite poset with $T = \{1, \ldots, n\}$. Let $\mathcal{P}_\mathcal{T}$ be the Fraïssé limit of $\mathbf{P}_\mathcal{T}^{fin}$, let $\overrightarrow{\mathcal{P}}_\mathcal{T} = (\mathcal{P}_\mathcal{T}, \sqsubset)$ be the Fraïssé limit of $\overrightarrow{\mathbf{P}}_\mathcal{T}^{fin}$, let $G = \mathrm{Aut}(\mathcal{P}_\mathcal{T})$ and $X^* = \overline{G \cdot \sqsubset}$ (in the logical action of G on $\mathrm{LO}(P_\mathcal{T})$). Then the logical action of G on X^* is the universal minimal flow of G.*

Proof It is easy to see that for every template $\mathcal{T}$ with $T = \{1, \ldots, n\}$ both $\mathbf{P}_\mathcal{T}$ and $\overrightarrow{\mathbf{P}}_\mathcal{T}$ are Fraïssé ages. It was shown in [3] that the class $\overrightarrow{\mathbf{P}}_\mathcal{T}^{fin}$ has the Ramsey property, while Corollary 4.2 establishes the ordering property for the class. The rest is now an immediate consequence of Theorem 4.4. $\qquad\square$

Let us conclude the paper with another immediate consequence of Theorems 4.1 and 4.4:

Corollary 4.6 *Let $\mathbf{K}^*$ be a Fraïssé age satisfying the weak triangle condition which is a reasonable order expansion of a Fraïssé age $\mathbf{K}$. Let $\mathcal{F}$ be the Fraïssé limit of $\mathbf{K}$, let $\mathcal{F}^* = (\mathcal{F}, \sqsubset)$ be the Fraïssé limit of $\mathbf{K}^*$, let $G = \mathrm{Aut}(\mathcal{F})$ and $X^* = \overline{G \cdot \sqsubset}$ (in the logical*

action of G on LO(*F*)*). Then the logical action of G on X* is the universal minimal flow of G if and only if* **K*** *has the Ramsey property.*

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