



d -Complete Posets: Local Structural Axioms, Properties, and Equivalent Definitions

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Abstract

Although d -complete posets arose along the interface between algebraic combinatorics and Lie theory, they are defined using only requirements on their local structure. These posets are a mutual generalization of rooted trees, shapes, and shifted shapes. They possess Stanley’s hook product property for their P -partition generating functions and Schützenberger’s well-defined jeu de taquin rectification property. The original definition of d -complete poset was lengthy, but more succinct definitions were later developed. Here several definitions are shown to be equivalent. The basic properties of d -complete posets are summarized. Background and a partial bibliography for these posets is given.

Keywords d -complete poset · λ -minuscule element · Double tailed diamond · Hook length poset · Partially ordered set

1 Introduction

This paper is entirely poset-theoretic, apart from some background and motivational remarks. d -Complete posets arose in the area of overlap between combinatorics, representation theory, and algebraic geometry that is inhabited by Young tableaux, Coxeter (Weyl) groups, Kac-Moody Lie algebras, and flag varieties. Generalizing (shifted) Young diagrams, d -complete posets have been shown to possess both Stanley’s hook product property for their P -partition generating functions [29] and Schützenberger’s well-defined jeu de taquin

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rectification property [28]. Specializing these hook identities gives generalizations of the FRT hook formula for the number of standard Young tableaux to enumerations of the linear extensions of d -complete posets.

d -Complete posets can be defined with various combinations of local structural axioms. They have been classified with Dynkin diagrams [26]. We study the interplay between a number of structural axioms and indicate which combinations of these axioms produce some useful local structural properties. Both locally finite (all intervals are finite) and finite posets are considered. Sections 4 and 7 contain conjectures that graph theorists may be able to confirm by arguing that certain rank size growths are unbounded. We prove that the lengthy original published [26] definition of “ d -complete” is equivalent to the succinct most recent [29] definition. This is the first step needed in a sequence of journal papers that will provide a complete derivation of the multivariate hook product identity for colored d -complete posets [29] that was obtained with Dale Peterson. We are hoping this paper becomes a standard reference for d -complete posets: Given the growing interest in these posets, we summarize their properties, outline the foremost background material, and provide a partial bibliography.

Before [26] was written, in 1994 an earlier notion of (colored) d -complete poset was developed while combinatorializing [25] a basis theorem of Seshadri for certain representations of simple Lie algebras. There the structural axioms arose from relations within the universal enveloping algebra of the Lie algebra. When the jeu de taquin rectification algorithm was shown to be well-defined for d -complete posets, the remarkable “simultaneous” property was also obtained. It seemed likely that some algebraic phenomena related to the local structure of d -complete posets underlay this property; these phenomena would be related to the reduced decompositions of λ -minuscule elements of Weyl groups. It is hoped that the understanding of the interplay among the local structural axioms obtained here will facilitate the development of an explanation of the interplay between the order theoretic structure of d -complete posets and some of the algebraic structures in this area of mathematics.

For a full understanding of the algebraic roles of d -complete posets, the notion of colored d -complete poset is needed. The notion of colored d -complete was shown [27] to be essentially equivalent to the purely structural notion of d -complete considered here. This paper sets the stage for a sequel in which the lengthy original definition of colored d -complete poset is shown to be equivalent to the shorter current [29] colored definition. This equivalence is also needed for the journal papers version of the conference proceedings contribution [29]. To prepare for developing a notion of colored d -complete for locally finite posets, here we are careful to delineate between the finite and locally finite cases.

Ishikawa and Tagawa have developed a category of finite posets that vastly extends the category of finite colored d -complete posets. They have shown [10] that their “leaf” posets also possess hook product identities for the associated colored P -partition generating functions; these identities subsume those in [29]. Presently leaf posets are defined only by the presentation of families of Hasse diagrams that generalize the families of diagrams appearing in the classification [26] of d -complete posets. The hook property is so special and this extension of it is so nice that it is natural to expect that there exists some underlying algebraic or geometric explanation for it; currently their combinatorial generating function calculations proceed class-by-class. As a first step toward a uniform understanding, it would be desirable for someone to develop an axiomatic definition of the notion of leaf poset in the spirit of the axiomatic considerations presented here. For example, the “short intervals are small” property considered here may be useful for studying leaf posets, as well as for the study of colored d -complete posets.

In the original definition of d -complete poset, there were three local structural conditions for each positive integer $k \geq 3$. Although we show that there exist useful alternates to or hybrids of these three axioms, for each $k \geq 3$ each of the following three aspects remains present: First, if a convex subset of the poset is isomorphic to the poset formed by removing the maximum element from the fundamental “double tailed diamond” poset $d_k(1)$, then it must be completable to an interval that is isomorphic to all of $d_k(1)$. Second, the completing element must cover only elements within that interval. Third, certain kinds of overlaps between two such intervals are prohibited. It is intriguing that some of the properties obtained have alternate derivations in which adding a local structure axiom hypothesis of one of these three types allows one of the local structure axiom hypotheses of one of the other types to be weakened or omitted. Further, some of the properties obtained for locally finite posets have alternate derivations in which adding the assumption of finiteness allows one of the local structure axiom hypotheses to be weakened or omitted. Such trade-offs may parallel interactions among corresponding algebraic relations. More specific comments are made after the statement of Theorem 4.4. We also study the interplay among our axioms in Sections 3–8 so that we can compare the competing definitions of d -complete presented in Section 9, and to prepare for proving their equivalence there.

Much of the structure of a d -complete poset is already significantly constrained by the imposition of only the three $k = 3$ conditions. A poset satisfying these is called a “ d_3 -complete” poset; these posets may be interesting in their own right. In another sequel to this paper, we visually characterize the global and the local structure of finite d_3 -complete posets. This generalizes the classification of d -complete posets [26].

Section 2 presents the prototypical families of d -complete posets and gives additional background, especially for colored d -complete posets. This paper then has three parts with three sections apiece. The definitions needed for each part are presented in its first section and the proofs appear in its last section. There are notions and results that pertain only to d_3 -complete posets, and the proof details for the $k = 3$ conditions are slightly different than the details for the $k \geq 4$ conditions. Therefore Sections 3, 4, and 5 are concerned only with the axioms needed for d_3 -complete posets, while Sections 6, 7, and 8 generalize many of those results to the axioms needed for d_k -complete posets for $k \geq 3$. Section 9 presents several definitions of d -complete posets and Section 10 presents basic properties of d -complete posets. Section 12 describes appearances of d -complete posets.

2 Poset Terms, Prototypical d -Complete Posets, and More Background

A poset is *locally finite* if every closed interval is finite. ‘Poset’ will mean ‘locally finite poset’ unless ‘finite’ is assumed. Let P be a poset, and let u, v, w, x, y, z denote distinct elements of P . We write $x \rightarrow y$ when y covers x . We extend this to write $\{x, y\} \rightarrow z$ for $x \rightarrow z$ and $y \rightarrow z$, to write $w \rightarrow \{x, y\}$ for $w \rightarrow x$ and $w \rightarrow y$, and so on. Consult [32] for the notions of the (*Hasse*) *diagram* of P , *closed interval* $[w, z]$, *convex set*, *connected* poset, *connected components*, and (*shifted*) *Ferrers diagram*. The following definitions come from [4]: A (*covering*) *chain* in P is a set of elements $x_1, \dots, x_n \in P$ for $n \geq 1$ such that $x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_n$. This chain C has *length* $n - 1$; this is denoted $\ell(C) = n - 1$. A *rank function* on P is a function $\rho : P \rightarrow \mathbb{Z}$ such that $x \rightarrow y$ implies that $\rho(y) = \rho(x) + 1$. We say P is *ranked* if it has a rank function.

Definition 9.1 presents the definition of a “ d -complete” poset. Here we present the prototypical families of posets that motivated the development of that notion: A *rooted tree* is a connected poset with a unique maximal element whose diagram is acyclic; see Fig. 1a.

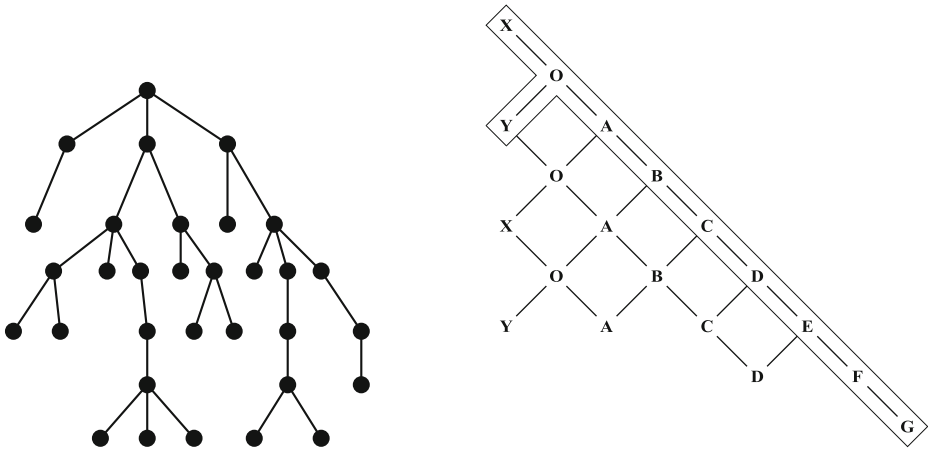


Fig. 1 Rooted tree (a) and shifted shape poset (b)

Given a Ferrers diagram for a partition of an integer, its *shape* poset is produced by rotating it 45° clockwise and drawing covering edges between adjacent dots. *Shifted shape* posets are similarly created from the shifted Ferrers diagrams for strict integer partitions. Figure 1b displays the shifted shape poset for the strict partition (9, 6, 3, 1) of 19. Let $k \geq 3$. The *double tailed diamond* poset $d_k(1)$ is defined by Fig. 2a. Any $(2k - 2)$ -element self dual poset with exactly two incomparable elements is isomorphic to $d_k(1)$. While developing his theory of P -partitions, Stanley obtained [31] hook length product identities for rooted trees, shapes, and double tailed diamonds that generalized Euler’s generating function identity for the number of integer partitions into no more than n parts, and he conjectured such an identity for shifted shapes.

As product identities for bounded P -partitions on rectangular shapes and staircase shifted shapes were derived Lie theoretically by the first author with Stanley, the Bruhat orders on the quotients W^J of finite Weyl groups that are distributive lattices were classified [24]. Their posets of join irreducible elements were called “minuscule” posets; these were labelled with the root system and the dominant weight used to generate the Bruhat order. Rectangular shapes, staircase shifted shapes, and double-tailed diamonds are minuscule. Shapes, shifted shapes, the “filters” of other minuscule posets, and rooted trees are d -complete. Finite connected d -complete posets have unique maximal elements. The *top tree* of a finite connected poset with a unique maximal element is the rooted tree that consists of the elements x such that $\{y : y \geq x\}$ is a chain. The shifted shape in Fig. 1b is a filter of the staircase minuscule poset denoted $d_{10}(\omega_9)$; its top tree is circled. The top tree of a filter of a minuscule poset is the Dynkin diagram for the Weyl group from which its Bruhat order was formed. In the shifted shape the top tree is the Dynkin diagram D_{10} . For $n \geq 3$ the double tailed diamond $d_n(1)$ is the minuscule poset $d_n(\omega_1)$, and its top tree is the Dynkin diagram D_n . Hence for $k \geq 3$ the subscript ‘ k ’ in $d_k(1)$ is the number of generators for the associated Weyl (Coxeter) group of type D_k .

A finite connected poset with a unique maximal element is *simply colored* [27] if its elements have been colored such that the elements in its top tree are distinctly colored, every other element receives one of those colors, and no two elements in a chain interval receive the same color. Inspired by a formulation [33] of Stembridge, we now [29] define it to be a (*simply*) *colored d-complete* poset if: equichromatic elements are comparable,

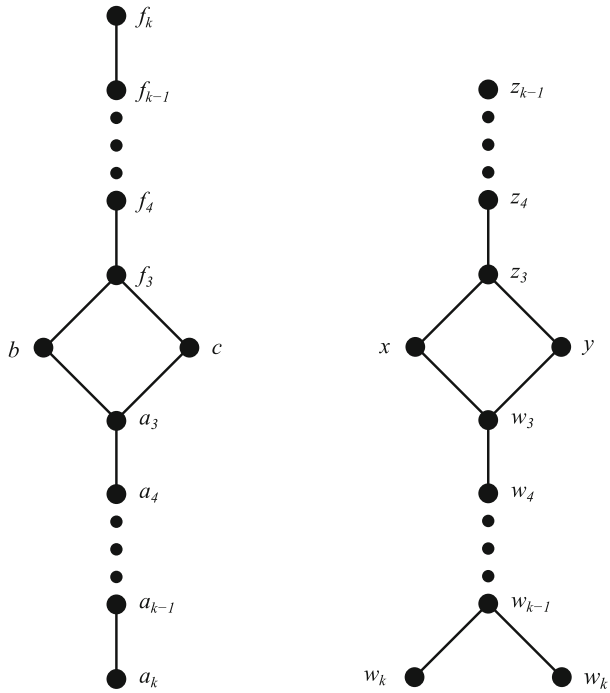


Fig. 2 Double tailed diamond poset $d_k(1)$ in (a) and overlapping d_k^- -intervals (b)

any two elements with colors that are adjacent in the top tree are comparable, the colors of two elements that are adjacent in the Hasse diagram are adjacent in the top tree, and in the open interval between two consecutive equichromatic elements there are exactly two elements whose colors are adjacent to that color in the top tree. Figures 1b and 3a display such colored posets; the latter one is the exceptional minuscule colored d -complete poset $e_7(7)$. The notion of colored d -complete poset was developed [25] when a combinatorial linear algebra version of the geometric representation basis theorem of Seshadri that had been used [24] to prove m -bounded P -partition identities was developed. For finite posets, the notions of d -complete poset and of colored d -complete poset are essentially equivalent [27]: Ignoring the colors of one of the latter produces one of the former. Given one of the former, its elements may be colored in essentially only one way to produce one of the latter. (The (colored) d -complete definition used in [27] was the order dual of the definition used here and elsewhere.)

Dale Peterson introduced [2] the notion of a “ λ -minuscule” element w of a Kac-Moody Weyl group W . When W is simply laced, it was shown [27] for such an element that the “ideal” (w) of the Bruhat order on W is a distributive lattice. It was further shown that the order dual of a poset P is colored d -complete if and only if P arises as the poset of join irreducible elements of such a distributive lattice (w) for some dominant λ . Here $(w) \cong J(P)$ in the language of [32]. Then, as the “heap” of w , the colored poset contains much information [33] concerning the reduced decompositions of w .

The definitions of “ d -complete poset” require that the intervals in the poset that are isomorphic to $d_k(1)$ (or are nearly isomorphic to $d_k(1)$) for $k \geq 3$ are well behaved in certain

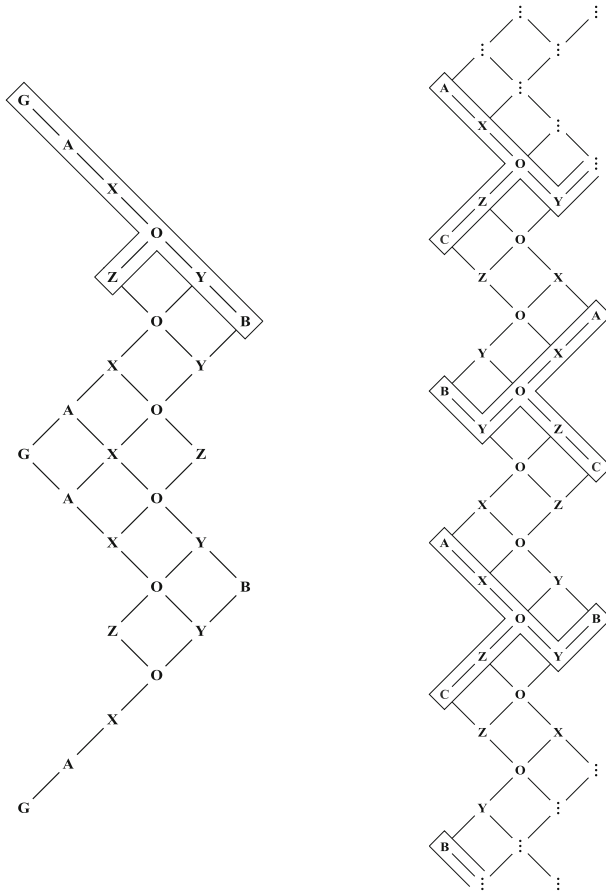


Fig. 3 Minuscule poset $e_7(7)$ in (a) and the poset for $\tilde{E}_6(\omega_A - \omega_B)$ in (b)

respects. So one may view a d -complete poset as consisting of double tailed diamonds that have been carefully “woven” together. The doubly infinite colored poset displayed in Fig. 3b is generated from the weight $\lambda = \omega_A - \omega_B$ for the exceptional affine Weyl group \tilde{E}_6 in the “numbers game” manner implicitly used in the proofs of Lemma 3.2 and Proposition 3.1 in [24]. In addition to satisfying our definition of d -complete for locally finite uncolored posets, this colored poset also satisfies the requirements above for finite simply colored d -complete posets, once the role of top tree has been taken by an embedded copy of the Dynkin diagram. The intervals formed from the consecutive occurrences of a color in Figs. 1b and 3 are double tailed diamonds. In this paper both of the infinite uncolored posets in Fig. 4 qualify to be d -complete posets; see the Added Note.

Roughly speaking, the overall global structure of a finite connected d -complete poset is that of a rooted tree, but with interspersed “slant irreducible” components [26]. These irreducible components fall into 15 classes, of which 14 are indexed by top trees which are Dynkin diagrams of general type E. The sole member of the 15th class is $e_7(7)$.

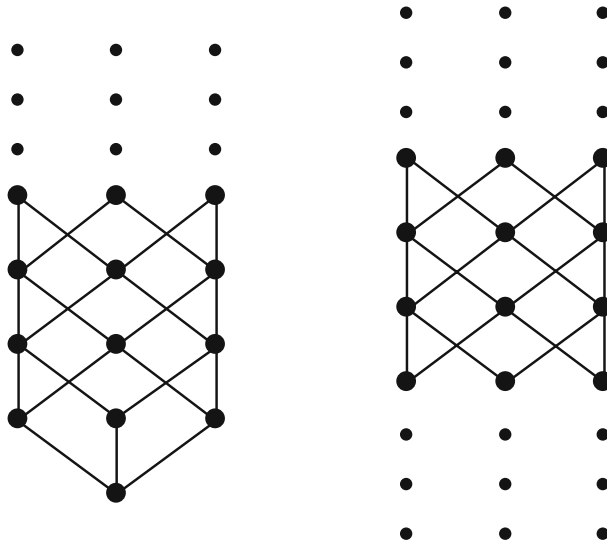


Fig. 4 Locally finite posets

3 Definitions, Axioms, and Properties for $k = 3$

Let P be a poset with distinct elements u, w, w', x, y, z . A *diamond* is a subset $\{w; x, y; z\}$ of P such that $w \rightarrow \{x, y\}$ and $\{x, y\} \rightarrow z$. The *bottom* and *top* of the diamond are w and z respectively, and the *elbows* are x and y . We say the top z is *free* if it covers only x and y . An interval $[w, z]$ is a d_3^- -interval if it is a diamond $\{w; x, y; z\}$ for some $x, y \in P$. Once it is known that a diamond $\{w; x, y; z\}$ forms all of the interval $[w, z]$, we refer to z as the maximum element. A subset $\{w; x, y\}$ of P is a *vee* or a d_3^- -set if $w \rightarrow \{x, y\}$. Note that a d_3^- -set is convex. A d_3^- -set $\{w; x, y\}$ is *completed* if there exists a *completing element* z such that $\{w; x, y; z\}$ is a d_3^- -interval. Two d_3^- -sets $\{w; x, y\}$ and $\{w'; x, y\}$ are said to *overlap*. Two d_3^- -intervals (or diamonds) are *distinct* if their set symmetric difference is non-empty. An interval $[w, z]$ is *short* if there exists u such that $w \rightarrow u \rightarrow z$.

Fact 3.1 Let $S = \{w; x, y\}$ be a d_3^- -set and let $z \in P$.

- (a) If $S \cup \{z\}$ is a d_3^- -interval, then $S \cup \{z\} = [w, z]$ and z covers x and y .
- (b) If z covers x and y and no other elements, then $S \cup \{z\}$ is the d_3^- -interval $[w, z]$.

We call a structural property an “axiom” if we use it as part of a definition of the d -complete property in Section 9 or near the end of Section 4.

Axioms A poset satisfies the

[Class I: Completion Axioms]

- (VT) “Vees have Tops” axiom if $w \rightarrow \{x, y\}$ implies that there exists z such that $\{x, y\} \rightarrow z$, the

- (D3⁻C) “ d_3^- -sets are Completed” axiom if for each d_3^- -set S there exists z such that $S \cup \{z\}$ is a d_3 -interval, the

[Class II: Freeness Axioms]

- (FT) “diamonds have Free Tops” axiom if the top element of each diamond covers only the elbows of the diamond, the
- (D3MF) “ d_3 -interval Maxs are Free” axiom if the maximum element of each d_3 -interval covers only the elbows of the interval, the

[Classes I/II: Completion/Freeness Axiom]

- (D3⁻CF) “ d_3^- -sets are Completed Freely” axiom if for each d_3^- -set S there exists z such that $S \cup \{z\}$ is a d_3 -interval with maximum element z such that z covers only elements from S , the

[Class III: Forbidden Structure Axioms]

- (NCC) “No Criss Cross” axiom if there do not exist overlapping d_3^- -sets, and the
- (D3MD) “ d_3 -interval Maxs are Distinct” axiom if the maximum elements of distinct d_3 -intervals are distinct.

At times, Axioms VT and FT together are referred to as Diamond I+II and Axioms D3⁻C and D3MF together are referred to as Classic I+II.

Remark 3.2 Axioms D3⁻C, D3⁻CF, and FT obviously respectively imply VT, D3⁻C, and D3MF. Also, Axioms D3⁻C and D3MF together imply D3⁻CF.

Properties A poset has the

- (UPUE) “Upward Propagation of Up Edges” property if $w \leq y$, $w \rightarrow x$, and $x \not\leq y$ imply there exists z such that $y \rightarrow z$ and $x \leq z$, the
- (UM) “Unique Maximal element” property if it has a unique maximal element, the
- (CLMEE) “Chain Lengths to Maximal Element are Equal” property if it has UM and if every chain from an element w to the unique maximal element z has the same length, the
- (CLE) “Chain Lengths are Equal” property if whenever $x < y$ all chains from x to y have equal length, the
- (SIS) “Short Intervals are Small” property if whenever $[w, z]$ is a short interval, then $|[w, z]| \in \{3, 4\}$, the
- (DAI) “Diamonds Are Intervals” property if each diamond is an interval, the
- (NTC) “No Triply Covered” property if no element is covered by three elements, the
- (UT) “Unique Top” property if each vee has exactly one top, and the
- (UC3) “Unique Completion” property if each d_3^- -set has exactly one completing element.

The ranked and connected properties are indicated with the labels (Rank) and (Conn).

Remark 3.3 If Property NTC were to be regarded as an axiom, it would belong to Class III since it could be viewed as prohibiting two more kinds of overlap between two d_3^- -sets that are not prohibited by Axiom NCC: Let $\{w; x, y\}$ and $\{w'; x', y'\}$ be two distinct d_3^- -sets. Suppose these sets have a coincidence between their minimal elements and/or a coincidence among their maximal elements. If $\{x, y\} = \{x', y'\}$, then $w \neq w'$ and this is prohibited by NCC. If $\{x, y\} \neq \{x', y'\}$ and $w = w'$, then Property NTC prohibits $w = w'$ from being covered by three or four distinct elements from $\{x, y, x', y'\}$. (Axiom NCC and Property NTC together do not prohibit the one remaining possibility, the “W”: here $|\{x, y\} \cap \{x', y'\}| = 1$ and $w \neq w'$.)

It is easy to see that:

Fact 3.4 (DAI #1) *If a poset P is FT or has No Triply Covereds or has Short Intervals are Small, then it has Diamonds Are Intervals.*

4 Results for $k = 3$

The implications in Theorem 4.1 below are displayed in tabular form, with the first four columns displaying hypotheses that are Class I, II, III axioms or a property. For example, Parts (k), (l), and (m) have the acronym $D3^-CF$ entered midway between the columns for Class I and Class II axioms. This indicates that the Class I/II hybrid axiom “ d_3^- -sets are Completed Freely” is being assumed in these parts. Parts (l) and (m) together indicate that the Class III axioms “No Criss Cross” and “ d_3^- -intervals Maxs are Distinct” are equivalent in the presence of $D3^-CF$. Parts (b)-(e) describe ways in which the easy-to-check “Vees have Tops” axiom may be strengthened to the “ d_3^- -sets are Completed” axiom needed for a d -complete poset. Parts (e), (i), (l), (m), and (n) are concerned only with axioms; this illustrates the interplay among the axioms mentioned in the introduction. For example, part of Part (l) strengthens the weak Class II requirement contained in the Class I/II hybrid axiom $D3^-CF$ to the full-strength Class II axiom $D3MF$ when the Class III axiom NCC is present. And without a Class I axiom being present, in Part (i) the Class III axiom NCC implies the other Class III axiom $D3MD$ when the Class II axiom $D3MF$ is present. An entry of “etc.” in the last column indicates that further conclusions may be drawn using one or both of the listed conclusions to satisfy an earlier line in the table. The last part propagates an edge in a vee upwardly along a chain when VT is present.

Theorem 4.1 *The implications in Table 1 hold in a poset.*

In Section 9 we see that the hypotheses of Parts (l), (m), and (n) satisfy the definition of “ d_3^- -complete” poset. Posets satisfying these axioms satisfy all of the $k = 3$ axioms and have the DAI, UT, and UC3 properties. Given this remark, it can be seen that Part (n) is closely related to Part (l). We have included Part (n) because it gives a partial converse to Part (e), and because it clarifies the misworded statement “We have just required . . .” on pp. 65 and 283 of [26] and [27]; that statement should have instead begun “It can be shown that . . .”.

Combining Remark 3.2, Fact 3.4, and Part (j) of Theorem 4.1, we have:

Corollary 4.2 *A poset is FT if and only if it is $D3MF$ and has Diamonds Are Intervals.*

Table 1 Implications for Theorem 4.1

	I	II	III	Property	Conclusion(s)
(a)	VT	—	NCC	—	UT
(b)	VT	—	—	DAI	$D3^-C$
(c)	VT	—	—	NTC	$D3^-C$
(d)	VT	—	—	SIS	$D3^-C$
(e)	VT	FT	—	—	$D3^-C + D3MF$
(f)	$D3^-C$	—	—	UT	DAI
(g)	$D3^-C$	—	NCC	—	UT, DAI, UC3
(h)	$D3^-C$	—	D3MD	NTC	NCC, etc.
(i)	—	D3MF	NCC	—	D3MD
(j)	—	D3MF	—	DAI	FT
(k)		$D3^-CF$	—	UC3	$D3^-C + D3MF$
(l)		$D3^-CF$	NCC	—	D3MF, D3MD, etc.
(m)		$D3^-CF$	D3MD	—	NCC, D3MF, etc.
(n)	$D3^-C$	D3MF	NCC	—	VT + FT, etc.
(o)	VT	—	—	—	UPUE

Now we consider finite posets. We believe that the converse of Part (d) above holds here:

Conjecture 4.3 *If a finite poset is $D3^-C$, then it has Short Intervals are Small (and is VT).*

We propose a proof for this conjecture just before stating Conjecture 4.5. The next theorem presents some implications that may be deduced when the poset is finite. Parts (a) and (b) present four fundamental structural properties that follow from the Upward Propagation of Up Edges part above. Part (e) says that the converse of Theorem 4.1(e) is known to hold when the poset is finite. The first part of Part (f) says that the $D3^-CF$ hypothesis for the first part of Theorem 4.1(m) may be weakened to $D3^-C$ in the finite case.

Theorem 4.4 *The implications in Table 2 hold in a finite poset.*

The poset on the left in Fig. 4 gives a counterexample to dropping the assumption of finiteness from Part (d) and from the second part of Part (f) of this theorem. Requiring Axioms $D3^-C$ or $D3^-CF$ without requiring the completions of vees to be unique can lead to messy situations if insufficient requirements have been imposed with Class II or Class III axioms or with finiteness. But we have not considered Property UC3 as an axiom in this paper due to space limitations. We believe that Parts (c) and (e) and the first part of Part (f) of this theorem also do not hold when finiteness is dropped. The diagram on the left in Fig. 5 presents a “seed” for a proposed counterexample to Part (c) and the diagram on the right does so for proposed counterexamples to Part (e) and the first part of Part (f). The failure of the diagrams generated from these seeds to “close up” in a neat finite fashion may correspond to some kind of messy infinite algebraic quotient that has resulted from insufficient relations having been imposed. In a similar vein, attempting to prove that the contrapositive “ $\neg \text{SIS} \Rightarrow \neg D3^-C \vee \neg \text{finite}$ ” of Conjecture 4.3 is true when the diagram on the right in Fig. 5 is present as a subdiagram also seems to generate an infinite poset.

Table 2 Implications for Theorem 4.4

	I	II	III	Property	Conclusion(s)
(a)	VT	—	—	Conn	UM, CLMEE
(b)	VT	—	—	—	Ranked, CLE
(c)	VT	—	—	NTC	SIS
(d)		D3 ⁻ CF	—	—	NTC
(e)	D3 ⁻ C	D3MF	—	—	VT + FT
(f)	D3 ⁻ C	—	D3MD	—	NCC, NTC, UC3, etc.

Conjecture 4.5 *There exist infinite locally finite posets that are counterexamples to dropping the assumption of finiteness for Parts (c) and (e) and the first part of Part (f) of Theorem 4.4.*

Remark 4.6 Since VT and FT imply D3⁻C and D3MF, and D3⁻C and D3MF obviously imply D3⁻CF, Theorem 4.4(d) implies that a finite poset has NTC whenever it is Diamond I+II or Classic I+II.

Several definitions of d_3 -complete will be given for locally finite posets in Section 9. For this paragraph, let us use that Classic definition to say that a poset is d_3 -complete if it is D3⁻C, D3MF, and NCC. This provides a context to discuss the interplay among the axioms and between the axioms and the assumption of finiteness, especially in regard to forming other combinations of axioms that are equivalent to the Classic definition. Within Class I, Axiom D3⁻C is stronger than VT. Within Class II, Axiom FT is stronger than D3MF. It is interesting that in the diamond point of view, using the stronger FT compensates for using the weaker VT in Theorem 4.1(e) so that one can still obtain the combination D3⁻C plus D3MF needed in the d_3 -interval point of view for the Classic definition. Conversely, in the d_3 -interval point of view when NCC is present (Theorem 4.1(n)) or the poset is finite (Theorem 4.4(e)), using the stronger D3⁻C compensates for using the weaker D3MF so that one can still obtain the more convenient combination of VT plus FT in the diamond point of view. What happens if the weaker Class I axiom VT is paired with the weaker Class II axiom D3MF? By Theorem 4.1(c), strengthening the Class III axiom of NCC by also assuming NTC to prohibit two more same-rank overlaps between two d_3^- -sets allows one to satisfy the Classic definition of d_3 -complete with the combination of VT,

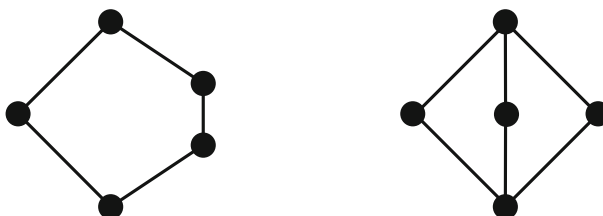


Fig. 5 Seeds for proposed counterexamples

D3MF, NCC, and NTC. (This was done in some earlier editions of [28].) When the poset is finite, Remark 3.2 and Theorem 4.4(d) indicate that this strengthening did not go too far: here a finite poset that is d_3 -complete by the Classic definition already has the NTC property.

5 Proofs for $k = 3$

Proof of Theorem 4.1 Parts (a), (b), (j), and (k) follow quickly from the definitions; Part (i) also follows directly with a bit of thought. For Parts (c) and (d), note that: If VT is present, then $D3^-C$ can fail at a d_3^- -set only if there is an “extra” chain from the minimum element to the diamond top required by VT. Such a chain is ruled out by either NTC or SIS. For Part (e), recall from Section 3 the implications $FT \Rightarrow D3MF$ and $FT \Rightarrow DAI$. Then by Part (b) we get $D3^-C$.

For Part (f), given a diamond $\{w; x, y; z\}$, for $\{w; x, y\}$ by $D3^-C$ there is a z' such that $[w, z']$ is a d_3 -interval. So UT implies $z' = z$, implying that the diamond is an interval. For Part (g), follow Remark 3.2 and Part (a) by Part (f).

For Part (h), suppose $\{w, w'\} \rightarrow \{x, y\}$. Applying $D3^-C$ to $\{w; x, y\}$ gives z such that $[w, z]$ is a d_3 -interval. Here D3MD implies that $[w', z]$ is not a d_3 -interval. So there exists some $u \in [w', z]$ such that $u \notin \{w'; x, y; z\}$. Let v be the minimal such element; we have $w' \rightarrow v$. Since $v \notin \{x, y\}$, this would violate NTC. So NCC holds.

For Part (l), first note $D3^-CF \Rightarrow D3^-C$; add NCC with Part (g) to obtain UC3, which via Part (k) implies D3MF. Then Part (i) gives D3MD. For Part (m), suppose $\{w, w'\} \rightarrow \{x, y\}$. Applying $D3^-CF$ to $\{w; x, y\}$ gives z such that $[w, z]$ is a d_3 -interval with z free. This z covers exactly x and y in $\{w'; x, y\}$. So $[w', z]$ is a d_3 -interval. This contradicts D3MD at z . For Part (n), first note $D3^-C \Rightarrow VT$; add NCC with Part (a), which via Part (f) gives DAI. Then Part (j) gives FT.

Part (o) was Proposition F1 of [26], which did not actually need finiteness. \square

Proof of Theorem 4.4 Part (a) was Propositions F2 and F3 of [26]. For Part (b), each component has a unique maximal element. Then CLMEE can be used to construct a well-defined rank function on each component. Property CLE follows.

For Part (c), a short interval $[w, z]$ will be contained in a connected component. Let u be such that $w \rightarrow u \rightarrow z$. Suppose $|[w, z]| \geq 5$. So there exists $x, y \in [w, z]$ such that w, u, x, y, z are distinct. There exist chains from w to z that pass through x and through y . Here CLE implies that these chains are of length 2. So $w \rightarrow \{u, x, y\}$, contradicting NTC.

For Part (d), suppose $w \rightarrow \{x_1, y_1, z_1\}$. Applying $D3^-CF$ three times yields three distinct free completing elements x_2, y_2, z_2 . This axiom can be repeatedly applied three times in this fashion ad infinitum, contradicting finiteness. For Part (e), note that $D3^-C$ implies VT, and adding in D3MF gives $D3^-CF$. Part (d) provides NTC, which was used in Fact 3.4 to get DAI. Then Theorem 4.1(j) gives FT.

To prove the first part of Part (f), suppose $\{x_0, y_0\} \rightarrow \{x_1, y_1\}$. Applying axioms $D3^-C$ and D3MD together twice implies that there exist distinct completing elements x_2 and y_2 . These axioms can be repeatedly applied twice in this fashion ad infinitum, contradicting finiteness. The proof of the second part of Part (f) is the same as the proof of Part (d), except now $D3^-C$ and D3MD are used instead of $D3^-CF$ to construct the infinite succession of completing elements three at a time. For the third part use Theorem 4.1(g). \square

6 Definitions, Axioms, and Properties for $k \geq 3$

Let P be a poset. Here and below u, v, w, x, y denote arbitrary elements. We define convex sets $[w; x, y] := [w, x] \cup [w, y]$ and $[u, v; x, y] := [u; x, y] \cup [v; x, y]$.

Let $k \geq 3$. Consider the double tailed diamond (DTD) poset $d_k(1)$ of Fig. 2a. The two incomparable elements b and c are its *elbows*. Its *neck elements* are f_3, f_4, \dots, f_k and its *tail elements* are a_3, a_4, \dots, a_k . When $k \geq 4$, all but the lowest of the neck elements are its *strict neck elements* and all but the highest of the tail elements are its *strict tail elements*. A Y_k -set $[w; x, y] \subseteq P$ is a convex set such that $[w; x, y] \cong [a_k; b, c] \subseteq d_k(1)$. Note that a Y_3 -set is a *vee*. Suppose the elements of this set are $w =: w_k \rightarrow w_{k-1} \rightarrow \dots \rightarrow w_3 \rightarrow \{x, y\}$. Here w_k, \dots, w_3 are the *stem elements* of Y_k . A ΛY_k -set $[u, v; x, y] \subseteq P$ is a convex set of the following form: we require $\{u, v\} \rightarrow w_k$ and $[u, v; x, y] = \{u, v\} \cup [w_k; x, y]$, where $[w_k; x, y]$ is a Y_k -set.

An interval $[w, z]$ in P is a d_k -interval if $[w, z] \cong d_k(1) = \{a_k, \dots, a_3, b, c, f_3, \dots, f_k\}$. We say that $[w, z]$ is a *DTD interval* if we do not want to mention k . Note that if $[w, z] = \{w =: w_k, \dots, w_3; x, y; z_3, \dots, z_k := z\}$ is a d_k -interval, then $[w_h, z_h]$ is a d_h -interval for $3 \leq h \leq k$. A neck element u in a d_k -interval $[w, z]$ is *free* if u covers only (an) element(s) in $[w, z]$. A convex set is a d_k^- -set if it is isomorphic to $d_k(1) - \{f_k\}$. When $k \geq 4$, a d_k^- -set is an interval and thus may be referred to as a d_k^- -interval. Here a d_k^- -interval $[w, z']$ may be described as $\{w =: w_k, w_{k-1}, \dots, w_3; x, y; z_3, \dots, z_{k-2}, z_{k-1} := z'\}$. Returning to $k \geq 3$, such a d_k^- -set is *completed* if there exists a *completing element* z_k such that $\{w_k, \dots, w_3; x, y; z_3, \dots, z_{k-1}, z_k\}$ is a d_k -interval. Two d_k -intervals are *distinct* if their set symmetric difference is non-empty. Let $k \geq 4$. Suppose $[w, z']$ is a d_k^- -interval in which u is the unique element covering w . If there exists $w' \neq w$ also covered by u such that $[w', z']$ is also a d_k^- -interval, then the d_k^- -intervals $[w, z']$ and $[w', z']$ *overlap*. Overlapping d_k^- -intervals are shown in Fig. 2b: They differ only in their minimal elements [22].

The following statement is the analog of Fact 3.1 for $k \geq 4$:

Fact 6.1 *Let $k \geq 4$. Let $S = \{w_k, \dots, w_3; x, y; z_3, \dots, z_{k-1}\}$ be a d_k^- -set and let $z_k \in P$.*

- (a) *If $S \cup \{z_k\}$ is a d_k -interval, then $S \cup \{z_k\} = [w_k, z_k]$ and z_k covers z_{k-1} .*
- (b) *If z_k covers z_{k-1} and no other elements, then $S \cup \{z_k\}$ is the d_k -interval $[w_k, z_k]$.*

Axioms Let $k \geq 3$. A poset satisfies the

[Class I: Completion Axiom]

- (Dk⁻C) “ d_k^- -sets are Completed” axiom if for each d_k^- -set S there exists an element z such that $S \cup \{z\}$ is a d_k -interval, the

[Class II: Freeness Axiom]

- (DkMF) “ d_k -interval Maxs are Free” axiom if the maximum element of each d_k -interval covers only (an) element(s) in that interval, the

[Classes I/ II: Completion/Freeness Axiom]

- (Dk⁻CF) “ d_k^- -sets are Completed Freely” axiom if for each d_k^- -set S there exists an element z such that $S \cup \{z\}$ is a d_k -interval with maximum element z such that z covers only (an) element(s) from S , the

[Class III: Forbidden Structure Axioms]

- (NOD k^-) “No Overlapping d_k^- -sets” axiom if there do not exist overlapping d_k^- -sets, and the
- (DkMD) “ d_k -interval Maxs are Distinct” axiom if the maximum elements of distinct d_k -intervals are distinct.

Remark 6.2 Each of these axioms subsumes the corresponding $k = 3$ axiom, with NCC having been renamed NOD 3^- . Axiom Dk $^-$ CF obviously implies Dk $^-$ C. Axioms Dk $^-$ C and DkMF together imply Dk $^-$ CF.

Properties Let $k \geq 3$. A poset has the

- (YECOI) “Y-stem Elements Covered Only Internally” property if each stem element of a Y_k -set is covered only by element(s) from that set, the
- ($N\Delta Y_k$) “No ΔY_k -sets” property if there do not exist ΔY_k -sets, and the
- (UCK) “Unique Completion” property if each d_k^- -set has exactly one completing element.

7 Results for $k \geq 3$

Let $k \geq 3$ throughout this section.

Proposition 7.1 (YECOI) *If a VT poset has No Triply Covered, then it has Y-stem Elements Covered Only Internally.*

Here is how we can extend a d_k -interval to a d_{k+1}^- -interval:

Lemma 7.2 (Add To Tail #1 (ATTk #1)) *Consider any poset. Let $\{w_k, \dots, w_3; x, y; z_3, \dots, z_k\}$ be a d_k -interval and let w_{k+1} be such that $w_{k+1} \rightarrow w_k$. If $[w_{k+1}; x, y]$ is a Y_{k+1} -set and the neck elements of $[w_k, z_k]$ are free, then $[w_{k+1}, z_k]$ is a d_{k+1}^- -interval.*

The hybrid axioms Dh $^-$ CF for $3 \leq h \leq k$ enable us to extend Y_k -sets to nice d_k -intervals:

Lemma 7.3 (Y_k -sets are Completed Freely (YkCF)) *Suppose a poset is Dh $^-$ CF for $3 \leq h \leq k$. If $\{w_k, w_{k-1}, \dots, w_3; x, y\}$ is a Y_k -set, then there exist elements $z_3 \rightarrow z_4 \rightarrow \dots \rightarrow z_k$ such that $[w_h, z_h]$ is a d_h -interval and z_h is free for $3 \leq h \leq k$.*

If in addition we assume the Class III axiom at the next index, we can rule out a forbidden structure:

Proposition 7.4 ($N\Delta Y_k$) *Suppose a poset is NOD $(k + 1)^-$ and Dh $^-$ CF for $3 \leq h \leq k$. Then there cannot exist ΔY_k -sets.*

In the following theorem, Parts (a), (b) (c), (d), (e), and the second part of (f) respectively generalize Part (h), Part (k), the first part of Part (m), the second part of Part (l), Part (i), and the first part of Part (l) of Theorem 4.1. The first part of Part (f) generalizes the third part of Theorem 4.1(g), once D3 $^-$ C has been strengthened to Dh $^-$ CF for $3 \leq h \leq k$.

Table 3 Implications for Theorem 7.5

	I	II	III	Property	Conclusion(s)
(a)	VT, Dk ⁻ C	—	DkMD	NTC	NODk ⁻
(b)		Dk ⁻ CF	—	UCk	Dk ⁻ C + DkMF
(c)		Dk ⁻ CF	DkMD	—	NODk ⁻
(d)		Dh ⁻ CF	NODh ⁻	—	DkMD
(e)	—	DhMF	NODh ⁻	—	DkMD
(f)		Dh ⁻ CF	NODk ⁻	—	UCk, DkMF

Theorem 7.5 *The implications in Table 3 hold in a poset; here the letter “h” indicates that the axiom is to be assumed for $3 \leq h \leq k$.*

The hypotheses for part (c) and (d) will be used in Section 9 to define d_k -complete and d_h -complete posets respectively.

The remaining results assume the No Triply Covered property. In practice these results may most often be applied to finite posets via the following: By the second part of Theorem 4.4(f) (or Theorem 4.4(d)), a D3⁻C poset has the No Triply Covered property if it is finite and is D3MD (or is D3⁻CF).

Lemma 7.6 (Add To Tail #2 (ATTk #2)) *Consider a VT poset that has No Triply Covered. Let $\{w_k, \dots, w_3; x, y; z_3, \dots, z_k\}$ be a d_k -interval and let w_{k+1} be such that $w_{k+1} \rightarrow w_k$. If $[w_{k+1}; x, y]$ is a Y_{k+1} -set, then $[w_{k+1}, z_k]$ is a d_{k+1}^- -interval.*

Lemma 7.7 (Y_k -sets are Completed (YkC)) *Consider a poset that is Dh⁻C for $3 \leq h \leq k$ and has No Triply Covered. If $\{w_k, w_{k-1}, \dots, w_3; x, y\}$ is a Y_k -set, then there exist elements $z_3 \rightarrow z_4 \rightarrow \dots \rightarrow z_k$ such that $[w_h, z_h]$ is a d_h -interval for $3 \leq h \leq k$.*

The next result, which obtains the unique completion property a second time, generalizes the result of following Theorem 4.1(h) by the last part of Theorem 4.1(g):

Proposition 7.8 (UCk #2) *Consider a poset that is Dh⁻C for $3 \leq h \leq k$ and has No Triply Covered. If it is DkMD, then it has Unique Completion at k.*

8 Proofs for $k \geq 3$

Proof of Proposition 7.1 Suppose some stem element w_i of a Y_k -set $[w_k; x, y]$ is covered by some $u \notin [w_k; x, y]$. Apply UPUE from Theorem 4.1(o) to $w_i \leq w_3, w_i \rightarrow u$, and $u \not\leq w_3$ to produce $z \notin \{x, y\}$ such that $w_3 \rightarrow z$. □

Proof of Lemma 7.2 Let $u \in [w_{k+1}, z_k]$ be maximal such that $u \notin \{w_{k+1}\} \cup [w_k, z_k]$. Here maximality implies that u is covered by some element v of $[w_k, z_k]$. This v cannot be a neck element, since those are free. But $v \in \{w_k, \dots, w_3, x, y\}$ with $u \geq w_{k+1}$ would violate $[w_{k+1}; x, y]$ being a Y_{k+1} -set. So there is no such u . Hence $[w_{k+1}, z_k] \cong d_{k+1}(1) - \{f_{k+1}\}$. □

Proof of Lemma 7.3 Here D3⁻CF says that $\{w_3; x, y\}$ is freely completed with a z_3 . And $\{w_4, w_3; x, y\}$ is a Y_4 -set. So Lemma ATT3#1 says that $[w_4, z_3]$ is a d_4^- -interval. Now

repeatedly alternate the application of Dh^-CF and then $ATTh\#1$ for $4 \leq h < k$ to construct the d_h^- -intervals $[w_{h+1}, z_h]$. Finish with Dk^-CF . \square

Proof of Proposition 7.4 Suppose $[u, v; x, y]$ is a ΛY_k -set with $\{u, v\} \rightarrow w_k \in [u, v; x, y]$. Here $[w_k; x, y]$ is a Y_k -set. Apply Lemma $YkCF$ to construct a d_k -interval $[w_k, z_k]$ with free neck elements z_3, \dots, z_k . Note that $[u; x, y]$ and $[v; x, y]$ are Y_{k+1} -sets. So Lemma $ATTh\#1$ says that $[u, z_k]$ and $[v, z_k]$ are d_{k+1}^- -intervals. But they are overlapping. \square

Proof of Theorem 7.5 Part (b) follows from the definitions for $k \geq 3$. Since it has been noted how the other parts reduce to parts of Theorem 4.1 when $k = 3$, suppose $k \geq 4$.

- (a) Let $\{w_k, w_{k-1}, \dots, w_3; x, y; z_3, \dots, z_{k-1}\}$ and $\{w'_k, w_{k-1}, \dots, w_3; x, y; z_3, \dots, z_{k-1}\}$ be overlapping d_k^- -intervals. Apply Dk^-C to obtain z_k such that $[w_k, z_k]$ is a d_k -interval. By $DkMD$, there exists $u \in [w'_k, z_k]$ with $u \notin \{w_k, w'_k\} \cup [w_{k-1}, z_k]$. Let v be the minimal such u . Since $w_{k-1} < v$ is not possible, it must be that $w'_k \rightarrow v$. Since $[w'_k; x, y]$ is a Y_k -set, this violates Proposition $YECOI$.
- (c) Let $[w_k, z_{k-1}]$ and $[w'_k, z_{k-1}]$ be overlapping d_k^- -intervals. Apply Dk^-CF to obtain z_k such that $[w_k, z_k]$ is a d_k -interval with z_k free. Since z_k covers only z_{k-1} , we see that $[w'_k, z_k]$ is a d_k -interval. This contradicts $DkMD$.
- (d) Let $\{w_k, \dots, w_3; x, y; z_3, \dots, z_k\}$ and $\{a_k, \dots, a_3; b, c; f_3, \dots, f_k\}$ be two d_k -intervals with $z_k = f_k$. Apply Dk^-CF to $[w_k, z_{k-1}]$ to produce z'_k such that $[w_k, z'_k]$ is a d_k -interval with z'_k free. Suppose $z'_k \neq z_k$. Here $[x, y; z_k, z'_k]$ is a ΛY_{k-1} -set. This would contradict Proposition $\Lambda Y(k-1)$, and so $z'_k = z_k$. Hence z_k is free, which implies $z_{k-1} = f_{k-1}$. This argument can be repeated to conclude that $z_h = f_h$ for $k \geq h \geq 3$. From Theorem 4.1(l) we have $D3MD$. So $\{x, y\} = \{b, c\}$ and $w_3 = a_3$. By $NODh^-$ for $4 \leq h \leq k$ we have $w_h = a_h$. Hence $[w_k, z_k] = [a_k, f_h]$.
- (e) Let $\{w_k, \dots, w_3; x, y; z_3, \dots, z_k\}$ and $\{a_k, \dots, a_3; b, c; f_3, \dots, f_k\}$ be two d_k -intervals with $z_k = f_k$. By $DhMF$ for $k \geq h \geq 4$ we have $z_{h-1} = f_{h-1}$. By $D3MF$ we have $\{x, y\} = \{b, c\}$. Now NCC requires $w_3 = a_3$. Finish as in the proof of Part (d).
- (f) We note that for UCk we will not need the freeness of Dh^-CF at $h = k$, but only for $3 \leq h \leq k - 1$. Let $\{w_k, \dots, w_3; x, y; z_3, \dots, z_{k-1}\}$ be a d_k^- -interval. Then Dk^-C gives a z_k so that $[w_k, z_k]$ is a d_k -interval. Suppose that $z'_k \neq z_k$ also completes $[w_k, z_{k-1}]$, now to a d_k -interval $[w_k, z'_k]$. Here $[x, y; z_k, z'_k]$ is a ΛY_{k-1} -set. Since this would contradict Proposition $\Lambda Y(k-1)$, Property UCk holds. For $DkMF$, let $\{w_k, \dots, w_3; x, y; z_3, \dots, z_k\}$ be a d_k -interval. Then Dk^-CF gives a free completing element z'_k for the d_k^- -interval $[w_k, z_{k-1}]$. Here UCk implies that $z'_k = z_k$, and so z_k is free. \square

Proof of Lemma 7.6 Let $u \in [w_{k+1}, z_k]$ be minimal such that $u \notin \{w_{k+1}\} \cup [w_k, z_k]$. Here minimality implies that u covers some element v of $\{w_{k+1}\} \cup [w_k, z_{k-1}]$. Since $u \leq z_k$, we cannot have $v \in [w_k, z_{k-1}]$. So $v = w_{k+1}$, and $w_{k+1} \rightarrow u$ violates Proposition $YECOI$. \square

Proof of Lemma 7.7 In the proof of Lemma 7.3, replace each instance of ‘ Dh^-CF ’ with ‘ Dh^-C ’, replace each instance of ‘ $ATTh\#1$ ’ with ‘ $ATTh\#2$ ’, and delete all references to ‘free’ completions. \square

Proof of Proposition 7.8 Given the generalization remark, suppose $k \geq 4$. Let $\{w_k, \dots, w_3; x, y; z_3, \dots, z_{k-1}\}$ be a d_k^- -interval. Then Dk^-C gives a z_k so that $[w_k, z_k]$ is a d_k^- -interval. Suppose that $z'_k \neq z_k$ also completes $[w_k, z_{k-1}]$, now to a d_k^- -interval $[w_k, z'_k]$. Here $[z_3; z_k, z'_k]$ is a Y_{k-1} -set. Lemma $Y(k-1)C$ constructs $u_{k-1} \rightarrow u_{k-2} \rightarrow \dots \rightarrow u_3$ such that $[z_h, u_h]$ is a d_{k+2-h} -interval for $k-1 \geq h \geq 3$. So $[z_3, u_3]$ is a d_{k-1} -interval. Here $[x; z_k, z'_k]$ and $[y; z_k, z'_k]$ are Y_k -sets. Use Lemma $ATT(k-1)\#2$ on each of them to see that $[x, u_3]$ and $[y, u_3]$ are d_k^- -intervals. They are overlapping, which contradicts Theorem 7.5(a). \square

9 Definitions for d -Complete Posets and Their Equivalences

We continue to consider locally finite posets. Our currently preferred (and shortest) definitions of d_k -complete, $d_{\leq k}$ -complete, and d -complete posets appeared in the paper [29] published in the RIMS Kôkyûroku series. Invoke the parenthetical parts when $k = 3$:

Definition 9.1 (Kôkyûroku definitions) A poset is d_k -complete if for every d_k^- -set S there exists an element that covers exactly the maximal element(s) of S and that does not cover (both of) the maximal element(s) of any other d_k^- -set. It is $d_{\leq k}$ -complete if it is d_h -complete for every $3 \leq h \leq k$ and it is d -complete if it is d_k -complete for every $k \geq 3$.

Alternatively, one could define these three notions using any one of the four combinations of axioms from Sections 4 and 7 that are presented in Table 4. Once the remark in Section 4 concerning the inadvertent double-defining of d_3 -complete in [26, 27] is taken into account, Combination (a) of Table 4 at $h = k$ was essentially used in those papers to define d_k -complete and d -complete finite posets. A nicely worded version of that combination appeared in [22]. We refer to it as the *Classic* definition.

Now we relate the Kôkyûroku definitions of d_k -complete and $d_{\leq k}$ -complete to the four combinations of axioms. For the notion of d_k -complete, some of the $(4+1) \times [(4+1) - 1] = 20$ possible implications among these five criteria do not hold true at $h = k$ alone. But some do hold true. See Section 11 for further remarks. For the notion of $d_{\leq k}$ -complete, all of these 20 implications hold true when one or more of the two or three hypothesis axioms is assumed for $3 \leq h \leq k$:

Theorem 9.2 Let $k \geq 3$. A poset is $d_{\leq k}$ -complete if and only if it satisfies any one of the Combinations (a) - (d) of axioms displayed in Table 4 for $3 \leq h \leq k$. Hence it is d -complete if and only if it satisfies any one of these combinations of axioms for $k \geq 3$.

Table 4 Combinations (a) - (d) of axioms for $h \geq 3$

	I	II	III
(a)	Dh ⁻ C	DhMF	NODh ⁻
(b)	Dh ⁻ C	DhMF	DhMD
(c)		Dh ⁻ CF	NODh ⁻
(d)		Dh ⁻ CF	DhMD

10 Properties of d -Complete Posets

We take note of two important facts that are not used in this paper:

Fact 10.1 Any “filter” of a d -complete poset is d -complete, as is the disjoint union of two d -complete posets.

For the first statement, note that removing an “ideal” of elements to produce a filter of P does not adversely affect the satisfaction of the d -complete requirements for P .

The following theorem applies the results of Sections 4 and 7 to d -complete posets:

Theorem 10.2 Let P be a d -complete poset.

- (a) The poset P satisfies all of the axioms defined in Sections 3 and 6 and possesses the following properties defined there: UPUE, DAI, UT, and UCK.
- (b) If P is finite, it also possesses all of the properties defined in Sections 3 and 6, except for UM and CLMEE when P is not connected.

Here Part (a) follows from the observation that by Theorem 9.2 the hypotheses of all of the Propositions and Theorems in Sections 4 and 7 that do not assume finiteness, NTC, or SIS are satisfied. For Part (b), note that if P is finite, then the hypotheses of all of the Propositions and Theorems in Sections 4 and 7 are satisfied, apart from Theorem 4.4(a).

We now study the necks and tails of DTD intervals in d -complete posets. In the following statement, Part (a) restates Axiom DhMF for $3 \leq h \leq k$ and Part (b) follows immediately from Proposition YECOI of Section 7:

Fact 10.3 Let P be a d -complete poset. Let $k \geq 3$. Let $\{w_k, \dots, w_3; x, y; z_3, \dots, z_k\}$ be a d_k -interval.

- (a) A neck element z_i of $[w_k, z_k]$ can cover only element(s) in $[w_k, z_k]$: If z_i is strict (i.e. $i \geq 4$), then it covers only z_{i-1} . Otherwise (i.e. $i = 3$) it covers only x and y .
- (b) If P has the No Triply Covered property, then a tail element w_i of $[w_k, z_k]$ can be covered by only element(s) in $[w_k, z_k]$: If w_i is strict (i.e. $i \geq 4$), then it is covered only by w_{i-1} . Otherwise (i.e. $i = 3$) it is covered only by x and y .

The next result says that the necks (tails) of two DTD intervals may intersect only in a particular way. For Part (b), keep in mind that every finite d -complete poset has the NTC property.

Proposition 10.4 Let P be a d -complete poset. Let $3 \leq k \leq k'$.

- (a) If there exists an element that is both a neck element for a d_k -interval $[w_k, z_k]$ and a neck element for a $d_{k'}$ -interval $[a_{k'}, f_{k'}]$, then $[w_k, z_k] \subseteq [a_{k'}, f_{k'}]$.
- (b) If P has the No Triply Covered property and there exists an element that is both a tail element for a d_k -interval $[w_k, z_k]$ and a tail element for a $d_{k'}$ -interval $[a_{k'}, f_{k'}]$, then $[w_k, z_k] \subseteq [a_{k'}, f_{k'}]$.

Corollary 10.5 Let P be a d -complete poset. Let $k \geq 3$. Let $\{w_k, \dots, w_3; x, y; z_3, \dots, z_k\}$ be a d_k -interval. For each $k' \geq k$ there is at most one $d_{k'}$ -interval that contains $[w_k, z_k]$, and if such an interval exists it must be of the form $\{w_{k'}, \dots, w_k, \dots, w_3; x, y;$

$z_3, \dots, z_k, \dots, z_{k'}\}$. A neck element z_i of $[w_k, z_k]$ can be a neck element of only those $d_{k'}$ -intervals $\{w_{k'}, \dots, w_k, \dots, w_3; x, y; z_3, \dots, z_k, \dots, z_{k'}\}$. If P has the No Triply Covered property, then a tail element w_i of $[w_k, z_k]$ can be a tail element of only those $d_{k'}$ -intervals $\{w_{k'}, \dots, w_k, \dots, w_3; x, y; z_3, \dots, z_k, \dots, z_{k'}\}$.

11 Proofs of Equivalences and Properties

Proof of Theorem 9.2. Table 5 presents six implications for parts of the $d_{\leq k}$ -complete statement. The first five come from Sections 6 and 7. Here the ‘ $h = k$ ’ and ‘ $h \leq k$ ’ entries under “Realm” indicate whether the hypothesis of the implication needs to assume that the axioms at hand hold merely at k or it needs to assume that the axioms hold for all $3 \leq h \leq k$. The last implication is verified by composing two of the earlier implications and then remembering one of its hypotheses. So we can finish this proof by relating the Kôkyûroku definition to any of these combinations of axioms. We show that Combination (d) \Rightarrow Kôkyûroku and that Kôkyûroku \Rightarrow Combination (c), both within the $h = k$ realm. Let $k \geq 3$.

Suppose Combination (d) holds at $h = k$. Let S be a d_k^- -set with minimum element w_k and maximum element z_{k-1} (when $k \geq 4$) or maximum elements $\{x, y\}$ (when $k = 3$). Then Dk^-CF gives some z_k such that $S \cup \{z_k\}$ is the d_k -interval $[w_k, z_k]$ and z_k covers only element(s) from S . Facts 6.1(a) and 3.1(a) imply that z_k covers exactly these maximum element(s) of S . Suppose that z_k covers the maximum element(s) f_{k-1} (or $\{b, c\}$) of some d_k^- -set T , whose minimum element is denoted a_k . Since z_k covers z_{k-1} (or $\{x, y\}$) exactly, we have $z_{k-1} = f_{k-1}$ or $\{x, y\} = \{b, c\}$. Since z_k covers f_{k-1} (or $\{b, c\}$) exactly, Facts 6.1(b) and 3.1(b) say that $T \cup \{z_k\}$ is the d_k -interval $[a_k, z_k]$. Here $DkMD$ says that $[w_k, z_k] = [a_k, z_k]$, and so T must coincide with S . Hence Kôkyûroku holds.

Suppose the Kôkyûroku definition of d_k -complete holds at $k \geq 3$. Let S be a d_k^- -set with minimum element w_k and maximum element z_{k-1} (when $k \geq 4$) or maximum elements $\{x, y\}$ (when $k = 3$). Then by Kôkyûroku there exists some z_k that covers these maximum element(s) of S exactly and that does not cover the maximum element(s) of any other d_k^- -set. Facts 6.1(b) and 3.1(b) say that $S \cup \{z_k\}$ is the d_k -interval $[w_k, z_k]$, and so Dk^-CF is satisfied. Suppose $\{w_k, w_{k-1}, \dots, w_3; x, y; z_3, \dots, z_{k-1}\}$ and $\{w_{k'}, w_{k-1}, \dots, w_3; x, y; z_3, \dots, z_{k-1}\}$ are overlapping d_k^- -sets. There exists some z_k that covers z_{k-1} (or $\{x, y\}$) exactly and that does not cover the maximal element(s) of any other d_k^- -set. Since $w_{k'} \neq w_k$ and the maximal element(s) of

Table 5 Implications in Theorem 9.2

Implication	Realm	Citation/Justification
(a) \Rightarrow (c)	$h = k$	Remark 6.2
(c) \Rightarrow (a)	$h \leq k$	Remark 6.2 + Theorem 7.5(f)
(d) \Rightarrow (c)	$h = k$	Theorem 7.5(c)
(c) \Rightarrow (d)	$h \leq k$	Theorem 7.5(d)
(b) \Rightarrow (d)	$h = k$	Remark 6.2
(a) \Rightarrow (b)	$h \leq k$	(a) \Rightarrow (c) \Rightarrow (d); + (a)

$\{w_{k'}, w_{k-1}, \dots, w_3; x, y; z_3, \dots, z_{k-1}\}$ is (are) z_{k-1} (or $\{x, y\}$), we see that z_k covers the maximal element(s) of another d_k^- -set. This contradiction implies that NODk^- holds. So Combination (c) is satisfied at $h = k$. \square

Proof of Proposition 10.4 Let $3 \leq k \leq k'$. Let $\{w_k, \dots, w_3; x, y; z_3, \dots, z_k\}$ be a d_k -interval and $\{a_{k'}, \dots, a_3; b, c; f_3, \dots, f_{k'}\}$ be a $d_{k'}$ -interval.

(a) Let $3 \leq j \leq k'$ be maximal such that there exists $3 \leq i \leq k$ with $z_i = f_j$. Let $m := \min\{i - 3, j - 3\}$. Since $[w_k, z_k]$ is a d_k -interval, if $m \geq 1$ use Fact 10.3(a) twice to get $z_{i-1} = f_{j-1}$. Similarly, if $m \geq 2$ then $z_{i-2} = f_{j-2}$. Continuing downward, we conclude that $z_{i-l} = f_{j-l}$ for $l \in \{0, \dots, m\}$. This includes the case $m = 0$. If $i \neq j$, then $\{i - m, j - m\} = \{3, h\}$ with $h > 3$. So one of z_{i-m} and f_{j-m} is a diamond top and thus covers two distinct elements while the other is the maximum element of a d_h -interval and thus by Fact 10.3(a) can only cover one element. This contradicts $z_{i-m} = f_{j-m}$. Thus $i = j$ and $z_{i-l} = f_{i-l}$ for $l \in \{0, \dots, i - 3\}$. Since $z_3 = f_3$, Axiom D3MD implies that $[w_3, z_3] = [a_3, f_3]$. So $\{x, y\} = \{b, c\}$ and $w_3 = a_3$.

Suppose $i < k$. Then the element z_{i+1} exists and the choice of j implies that $z_{i+1} \neq f_{i+1}$. Since $[w_k, z_k]$ and $[a_{k'}, f_{k'}]$ are DTD intervals, we see that $[x, y; z_{i+1}, f_{i+1}]$ is a ΔY_i -set. This contradicts Proposition $\text{N}\Delta Y_i$. Thus it must be that $i = k$. So $z_t = f_t$ for $t \in \{3, \dots, k\}$. Here $[w_k, z_k]$ and $[a_k, f_k]$ are both d_k -intervals with $z_k = f_k$. Hence DkMD implies $[w_k, z_k] = [a_k, f_k]$. Therefore $[w_k, z_k] \subseteq [a_{k'}, f_{k'}]$.

(b) Let $3 \leq j \leq k'$ be such that there exists $3 \leq i \leq k$ with $w_i = a_j$. Let $m := \min\{i - 3, j - 3\}$. The argument above can be “reflected” to move up toward a diamond: Use Fact 10.3(b) instead of Fact 10.3(a) to soon obtain $w_{i-1} = a_{j-1}$. After analogizing five more sentences (again using Fact 10.3(b) instead of Fact 10.3(a)), we arrive at contradicting $w_{i-m} = a_{j-m}$. Thus $i = j$ and $w_{i-l} = a_{i-l}$ for $l \in \{0, \dots, i - 3\}$. Since $w_3 = a_3$, to avoid contradicting NTC it must be that $\{x, y\} = \{b, c\}$. Then $z_3 = f_3$ to avoid contradicting NCC. Now note that z_3 is both a neck element for the d_k -interval $[w_k, z_k]$ and a neck element for the $d_{k'}$ -interval $[a_{k'}, f_{k'}]$. Then Part (a) implies that $[w_k, z_k] \subseteq [a_{k'}, f_{k'}]$. \square

Proof of Corollary 10.5 Let $k' \geq k \geq 3$ and let $\{w_k, \dots, w_3; x, y; z_3, \dots, z_k\}$ be a d_k -interval. Suppose two $d_{k'}$ -intervals contain $[w_k, z_k]$. Their elbows must pairwise coincide with $\{x, y\}$ and z_k is a neck element of both $d_{k'}$ -intervals. Using Proposition 10.4(a) for two containments, we find that the two $d_{k'}$ -intervals must be equal and have the claimed form. Let z_i (or w_i) be a neck (respectively tail) element of $[w_k, z_k]$. If z_i (or w_i) is also a neck (respectively tail) element of a $d_{k'}$ -interval, then using Proposition 10.4 and the first statement we find that that $d_{k'}$ -interval must be the unique $d_{k'}$ -interval that contains $[w_k, z_k]$. \square

12 Other Work on d -Complete Posets

For the most part, we list only papers that work in a substantive fashion with d -complete posets that are more general than filters of minuscule posets or rooted trees. We include structures that are closely related to d -complete posets: λ -minuscule elements of Kac-Moody Weyl groups, their heaps, and Nakada’s “generalized Young diagrams”.

Colors play no role in some appearances of d -complete posets, beginning with their classification [26] and continuing with the jeu de taquin result of [28]. Ishikawa and Tagawa used determinants and Pfaffians [9] to prove that many classes of slant irreducible d -complete posets possess Stanley’s hook product property. For standard shifted Young tableaux Konvalinka gave [15] a bijective proof of the branching recursion that implies the hook product enumeration formula, and he began to develop this approach for proving the hook product formula for counting linear extensions of d -complete posets. Riegler and Neumann [30] use jeu de taquin slides with respect to a fixed linear extension of a poset P to sort any labelling into an linear extension of P . They began to study this for d -complete posets, showing that the linear extensions produced are uniformly distributed when P is a filter of $d_n(1)$ (and hence d -complete), but not when P is a non-chain proper ideal of $d_n(1)$ (and hence not d -complete). The website [5] has lists of connected d -complete posets with up to 9 elements and a Mathematica procedure that determines whether a poset is d -complete.

Some appearances of d -complete posets have initial statements that refer to uncolored structures from pre-existing combinatorial problems, but at the same time have fuller colored statements or have proofs that refer to a colored version of the d -complete poset. In addition to the hook product identity [29] found with Peterson for d -complete posets, this remark also applies to the generalizations of that identity found by Ishikawa and Tagawa for leaf posets [10]. Okamura referred to the classification of d -complete posets to give a case-by-case probabilistic proof [23] of the hook product formula for counting the number of linear extensions of a d -complete poset. Nakada’s results concern “generalized Young diagram” posets that are formed from Kac-Moody roots: In [17] his fractional “colored hook formula” was a multivariate generalization of the formula used by Greene, Nijenhuis, and Wilf for their probabilistic proof of the hook product formula for counting standard Young tableaux. In [18] he presented his version of the multivariate hook product identity of [29]. Nakada and Okamura noted [20] that a uniform probability algorithm proof of a product formula for counting linear extensions of these posets that was analogous to that of [23] could be deduced in this context from [17]. After proving (q, t) -generalizations of multivariate hook product identities for reverse plane partitions on shapes and shifted shapes, Okada conjectured [22] an extension of it that would (q, t) -generalize the hook product identity of [29] for d -complete posets. He confirmed this for rooted trees, and Ishikawa confirmed [8] it for two of the simpler classes of slant irreducible d -complete posets. Kawanaka extended [11] the Sato-Welter-Sprague-Grundy winning strategy for nim from shapes to d -complete posets. Later he introduced [12] “finitely branching principal plain algorithm” games; it can be seen using [27] that portions of the digraphs of these games arise in his Theorem 6.3 as the Hasse diagrams of lattices of filters of d -complete posets. Uncolored d -complete posets can serve as “boards” on which jeu de taquin rectification procedures are performed during the computation of cohomology products for some Schubert varieties in some flag varieties. The “squares” of these boards do not need to be colored for the sliding mechanics, but they need to be colored when one labels the Schubert varieties with elements of the Weyl group. Some such results of Chaput and Perrin [3] for Kac-Moody flag varieties use the well-defined jeu de taquin rectification result of [28] for some d -complete posets that are not filters of minuscule posets. The K -theoretic Littlewood-Richardson results of Buch and Samuel [1] refer only to minuscule posets, as do several cohomology computation references of [1].

Colors play a central role in some appearances of d -complete posets, beginning with their first formulation in [25]. Earlier, the product formula on p. 348 of [24] for the number of linear extensions of a minuscule poset did not refer to colors. However, Theorem 11 there

described a minuscule poset as a poset of certain colored coroots for its associated Weyl group. Combining the remark on pp. 345–346 with Theorem 11, in hindsight that product formula also expressed the number of reduced decompositions of a minuscule element of a finite Weyl group (a colored problem) as a product over a poset of colored coroots. Peterson extended this product-over-roots formula [2] for reduced decompositions to λ -minuscule elements of Kac-Moody Weyl groups. For further information on Peterson’s work and the development of the notion of d -complete in [25] from the work in [24], which later led to [29], see Section 13 of [29]. Nakada’s overview [19] of [17, 18], and [20] notes that Peterson’s formula can be deduced from the main result of any of those papers. Given the connection between the linear extensions of colored d -complete posets and such reduced decompositions that was described in [27] for the simply laced cases, a closely related hook product formula for this number can be deduced from [23] or [29]. Also via this connection, the classification of d -complete posets in [26] gave a classification of the λ -minuscule elements of simply laced Kac-Moody Weyl groups. Stembridge extended [33] this classification to all symmetrizable Kac-Moody Weyl groups. There Theorem 5.5 extended Theorem 11 of [24] to use posets of coroots to describe the heaps of the λ -minuscule elements in all symmetrizable Kac-Moody Weyl groups. Kleshchev’s and Ram’s Theorem 3.10 of [14] can be seen to be saying that the dimensions of certain homogenous irreducible modules of Khovanov-Lauda-Rouquier algebras are equal to the number of linear extensions of associated d -complete posets. When the hook product expression of [23] or [29] is applied here, this theorem generalizes the fact that the dimensions of the irreducible representations of the symmetric group are given by the FRT hook formula for enumerating standard Young tableaux.

Green’s “full heaps” [6] are candidates to be regarded as locally finite colored d -complete posets once that definition is finalized; they play a central role in that book. Our Fig. 3b appears as the full heap of his Figure 6.13. Lax refers to several of the axioms for d -complete and colored d -complete posets when he uses minuscule posets to give uniform derivations [16] of the “extreme” Plücker relations for the embeddings of minuscule flag varieties. Michael Strayer has shown (personal communication) that a finite poset P can be colored in such a way that the lattice $J(P)$ carries a representation of a simply laced Kac-Moody Borel derived subalgebra in a certain natural “minuscule” fashion exactly when P is a simply colored d -complete poset.

Added Notes Before this paper, the notion of “ d -complete” was considered only for finite posets. As this paper was being written, it was observed that most of the axioms and definitions for finite d -complete posets continued to work well for locally finite posets, without additions or modifications. However, recent work by Michael Strayer and the first author indicates that it will be useful in the future to require that the No Triply Covered property holds as an axiom for an infinite locally finite poset to be called d -complete. Then the poset on the left in Fig. 4 will no longer qualify to be d -complete. More recently, while revisiting and extending Theorem 4.1.6(i) of [6], Strayer has obtained [34] a definition of locally finite colored d -complete posets that is motivated by the consideration of representations of Kac-Moody algebras.

Kim and Yoo evaluate integrals of the q -Selberg kind to give a new (class-by-class) proof [13] of Stanley’s hook product property for all d -complete posets. Naruse and Okada use formulas for products in the equivariant K -theory of Kac-Moody flag varieties to re-prove and generalize [21] the multivariate hook product identity of [29] for d -complete posets. The recent paper [7] by Ilango, Pechenik, and Zlatin is closely related to [3].

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