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Finite Semilattices with Many Congruences

Gábor Czédli[1]

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Abstract For an integer $n \geq 2$, let $\mathrm{NCSL}(n)$ denote the set of sizes of congruence lattices of n-element semilattices. We find the four largest numbers belonging to $\mathrm{NCSL}(n)$, provided that n is large enough to ensure that $|\mathrm{NCSL}(n)| \geq 4$. Furthermore, we describe the n-element semilattices witnessing these numbers.

Keywords Number of lattice congruences · Size of the congruence lattice of a finite lattice · Lattice with many congruences

1 Introduction and Motivation

The present paper is primarily motivated by a problem on tolerance relations of lattices raised by Joanna Grygiel in her conference talk in September, 2017, which was a continuation of Górnicka, Grygiel, and Tyrala [5]. Further motivation is supplied by Czédli [1], Czédli and Mureşan [2], Kulin and Mureşan [8], and Mureşan [9], still dealing with lattices rather than semilattices.

As usual, $\mathrm{Con}(A)$ will stand for the *lattice of congruences* of an algebra A. Given a natural number $n \geq 2$ and a variety $\mathcal{V}$ of algebras, the task of

finding the *small* numbers in the set $\mathrm{NC}(\mathcal{V}, n) := \{|\mathrm{Con}(A)| : A \in \mathcal{V} \text{ and } |A| = n\}$ and *describing* the algebras $\mathcal{V}$ witnessing these numbers (1.1)

has already deserved some attention for various varieties $\mathcal{V}$, because the description of the simple n-element algebras in $\mathcal{V}$ for various varieties $\mathcal{V}$ and, in particular, even the Classification of Finite Simple Groups belong to Eq. 1.1 in some vague sense. The present paper

✉ Gábor Czédli
czedli@math.u-szeged.hu
http://www.math.u-szeged.hu/czedli/

[1] Bolyai Institute, University of Szeged, Szeged 6720, Hungary

addresses an analogous problem, which is obtained from Eq. 1.1 by changing "small" to "large". Of course, this problem is hopeless for an arbitrary variety $\mathcal{V}$. However, if $\mathcal{V}$ is the variety $\mathrm{SLat}_\wedge$ of *meet-semilattices*, see Remark 2.8 for this terminology, then we can benefit from Freese and Nation's classical description of the congruence lattices of finite members of $\mathrm{SLat}_\wedge$; see [4]. Let us fix the following notation

$$\mathrm{NCSL}(n) := \mathrm{NC}(\mathrm{SLat}_\wedge, n) = \{|\mathrm{Con}(S)| : S \in \mathrm{SLat}_\wedge \text{ and} |S| = n\}; \tag{1.2}$$

the acronym NCSL comes from "Number of Congruences of SemiLattices". Our target is to determine the four largest numbers belonging to $\mathrm{NCSL}(n)$ and, in addition, to describe the n-element semilattices witnessing these numbers.

1.1 Outline

The rest of the paper is structured as follows. In Section 2, we introduce a semilattice construction, and we use this construction in formulating the main result, Theorem 2.3, to realize our target mentioned above. This section concludes with a corollary stating that a semilattice with sufficiently many congruences is planar. Section 3 is devoted to the proof of Theorem 2.3.

2 Quasi-tree Semilattices and our Theorem

We follow the standard terminology and notation; see, for example, Grätzer [6, 7]. In particular, $a \parallel b$ means that a and b are incomparable, that is, neither $a \le b$, nor $b \le a$. Even without explicitly saying so all the time, by a *semilattice* we always mean a *finite meet* semilattice S, that is, a finite member of $\mathrm{SLat}_\wedge$. Such an $S = \langle S; \wedge\rangle$ has a least element $0 = \bigwedge S$. We always denote $S \setminus \{0\}$ by S^+. Note that $\vee$, denoting supremum with respect to the ordering inherited from $\langle S; \wedge\rangle$, is only a partial operation and $\langle S^+; \vee\rangle$ is a partial algebra in general. If no two incomparable elements of S have an upper bound, then S is called a *tree semilattice*.

Next, for a meet-semilattice S, the congruence $\boldsymbol{\tau} = \boldsymbol{\tau}(S; \wedge)$ generated by

$$\{\langle a \wedge b, a \vee b\rangle : a, b \in S^+, a \parallel b, \text{ and } a \vee b \text{ exists in} \langle S^+; \vee\rangle\} \tag{2.1}$$

will be called the *tree congruence* of $\langle S; \wedge\rangle$. Of course, we can write $a, b \in S$ instead of $a, b \in S^+$ above. Observe that for $a, b \in S^+$,

$$\{a, b\} \text{ has an upper bound in } S \text{ iff } a \vee b \text{ exists in } \langle S^+; \vee\rangle; \tag{2.2}$$

hence instead of requiring the join $a \vee b \in \langle S^+; \vee\rangle$, it suffices to require an upper bound of a and b in Eq. 2.1. The name "tree congruence" is explained by the following easy statement, which will be proved in Section 3.

Proposition 2.1 *For an arbitrary finite meet-semilattice $\langle S; \wedge\rangle$, the quotient meet-semilattice $\langle S; \wedge\rangle/\boldsymbol{\tau}$ is a tree.*

Definition 2.2 By a *quasi-tree semilattice* we mean a finite meet-semilattice $\langle S; \wedge\rangle$ such that its tree congruence $\boldsymbol{\tau} = \boldsymbol{\tau}(S; \wedge)$ has exactly one nonsingleton block. If $\langle S; \wedge\rangle$ is a quasi-tree semilattice, then the unique nonsingleton block of $\boldsymbol{\tau}$, which is a meet-semilattice, and the quotient semilattice $\langle S; \wedge\rangle/\boldsymbol{\tau}$ are called the *nucleus* and the *skeleton* of $\langle S; \wedge\rangle$.

Some quasi-tree semilattices are shown in Figs. 1, 2 and 3. In these figures, the elements of the nuclei are the black-filled ones, while the empty-filled smaller circles stand for the rest of elements. Although a quasi-tree semilattice $\langle S; \wedge\rangle$ is not determined by its skeleton and nucleus in general, the skeleton and the nucleus together carry a lot of information on $\langle S; \wedge\rangle$. In order to make the numbers occurring in the following theorem easy to compare, we give them in a redundant way as multiples of 2^{n-6}.

Theorem 2.3 *If $\langle S; \wedge\rangle$ is a finite meet-semilattice of size $n = |S| > 1$, then the following hold.*

- (i) *$\langle S; \wedge\rangle$ has at most $2^{n-1} = 32 \cdot 2^{n-6}$ many congruences. Furthermore, we have that $|\mathrm{Con}(S; \wedge)| = 2^{n-1}$ if and only if $\langle S; \wedge\rangle$ is a tree semilattice.*
- (ii) *If $\langle S; \wedge\rangle$ has less than $2^{n-1} = 32 \cdot 2^{n-6}$ congruences, then it has at most $28 \cdot 2^{n-6}$ congruences. Furthermore, $|\mathrm{Con}(S; \wedge)| = 28 \cdot 2^{n-6}$ if and only if $\langle S; \wedge\rangle$ is a quasi-tree semilattice and its nucleus is the four-element boolean lattice; see* Fig. 1 *for $n = 6$.*
- (iii) *If $\langle S; \wedge\rangle$ has less than $28\cdot 2^{n-6}$ congruences, then it has at most $26\cdot 2^{n-6}$ congruences. Furthermore, $|\mathrm{Con}(S; \wedge)| = 26 \cdot 2^{n-6}$ if and only if $\langle S; \wedge\rangle$ is a quasi-tree semilattice such that its nucleus is the pentagon N_5; see* Fig. 4 *and $S_1, \ldots, S_3$ in* Fig. 2.
- (iv) *If $\langle S; \wedge\rangle$ has less than $26\cdot 2^{n-6}$ congruences, then it has at most $25\cdot 2^{n-6}$ congruences. Furthermore, $|\mathrm{Con}(S; \wedge)| = 25 \cdot 2^{n-6}$ if and only if $\langle S; \wedge\rangle$ is a quasi-tree semilattice such that its nucleus is either F, or N_6; see* Fig. 4 *and $S_4, \ldots, S_7$ in* Fig. 3.

Remark 2.4 Although Theorem 2.3 holds for all $n \geq 2$, it neither gives the *four largest* numbers of $\mathrm{NCSL}(n)$, nor does it say too much for $n \leq 5$. For example, $25 \cdot 2^{n-6}$ is not even an integer if $n \leq 5$. Hence, we note the following facts without including their trivial proofs in the paper.

- (A) $\mathrm{NCSL}(2) = \{2 = 2^{2-1}\}$
- (B) $\mathrm{NCSL}(3) = \{4 = 2^{3-1}\}$
- (C) $\mathrm{NCSL}(4) = \{8 = 2^{4-1},\ 7 = 28 \cdot 2^{4-6}\}$
- (D) $\mathrm{NCSL}(5) = \{16 = 2^{5-1},\ 14 = 28 \cdot 2^{5-6},\ 13 = 26 \cdot 2^{5-6},\ 12\}$. Note that 12 is witnessed by $M_3 = \langle M_3, \wedge\rangle$; see Fig. 4.

A semilattice is *planar* if it has a planar Hasse diagram, that is a Hasse diagram in which edges can intersect only at their endpoints, that is, at vertices. Theorem 2.3 immediately implies the following statement.

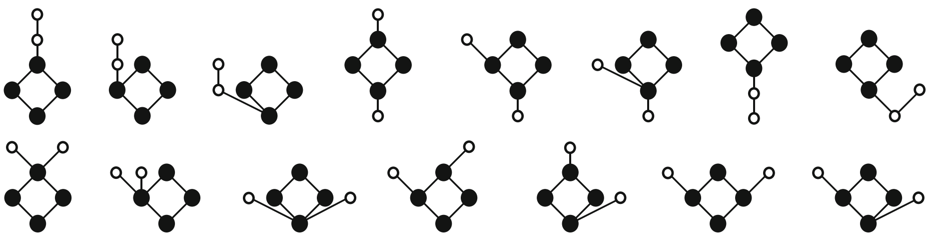

Fig. 1 The full list of 6-element meet-semilattices with exactly $28 = 28 \cdot 2^{6-6}$ many congruences

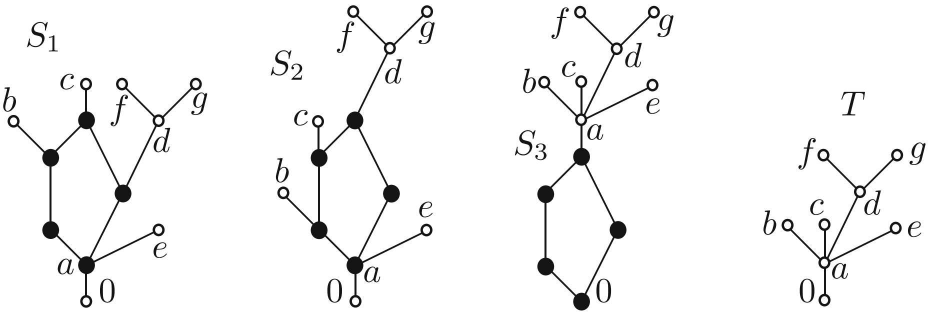

Fig. 2 Three twelve-element meet-semilattices with the same skeleton T and the same number, $26 \cdot 2^{12-6} = 1664$, of congruences

Corollary 2.5 *If an n-element meet-semilattice has at least* $25 \cdot 2^{n-6}$ *congruences, then it is planar.*

The following statement is due to Freese [3]; see also Czédli [1] for a second proof, which gives the first half of the following corollary for arbitrary finite algebras in congruence distributive varieties, not only for lattices.

Corollary 2.6 *For every n-element lattice L, we have that* $|\mathrm{Con}(L)| \le 2^{n-1}$. *Furthermore,* $|\mathrm{Con}(L)| = 2^{n-1}$ *if and only if L is a chain.*

As a preparation for a remark below, we derive this corollary from Theorem 2.3 (i) here rather than in the next section.

Proof of Corollary 2.6 The only n-element tree semilattice that is also a lattice is the n-element chain. For an equivalence relation Θ on this chain $\langle C; \le\rangle$,

$$\begin{array}{l}\Theta \ \in \ \mathrm{Con}(C; \wedge) \text{ iff } \Theta \ \in \ \mathrm{Con}(C; \vee, \wedge) \text{ iff} \\ \text{every } \Theta\text{-block is an interval of } \langle C; \le\rangle.\end{array} \tag{2.3}$$

Observe that every $\Theta \in \mathrm{Con}(L; \vee, \wedge)$ also belongs to $\mathrm{Con}(L; \wedge)$. Hence, using Theorem 2.3 (i) at $\le^*$ below, we obtain that

$$|\mathrm{Con}(L; \vee, \wedge)| \le |\mathrm{Con}(L; \wedge)| \le^* |\mathrm{Con}(C; \wedge)| = |\mathrm{Con}(C; \vee, \wedge)|,$$

proving Corollary 2.6. □

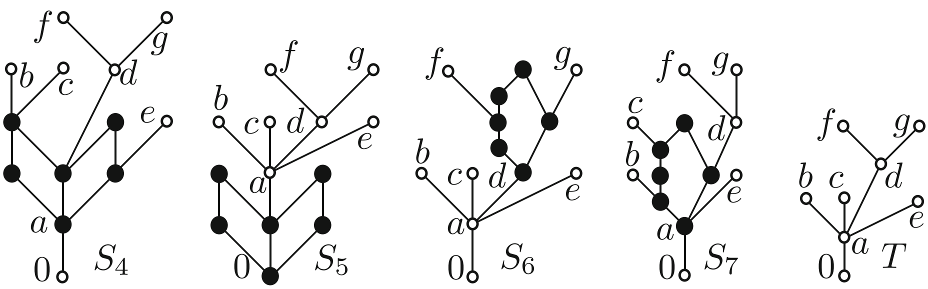

Fig. 3 Four thirteen-element meet-semilattices with the same skeleton T and the same number, $25 \cdot 2^{13-6} = 3200$, of congruences

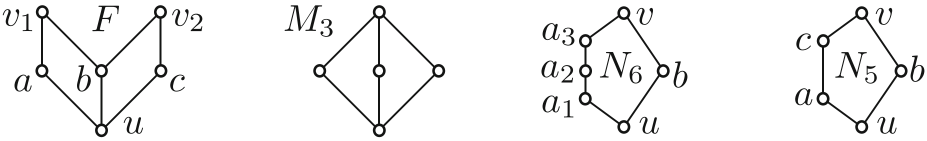

Fig. 4 F, M_3, N_6, and the pentagon, N_5

Next, we point out that Theorem 2.3 (i) plays an essential role in the proof above.

Remark 2.7 By Szpilrajn's Extension Theorem [10], every (partial) ordering on a set can be extended to a linear ordering. Hence, the second part of Eq. 2.3 might give the false feeling that this Extension Theorem in itself implies Corollary 2.6 as follows: extend the ordering relation of L to a linear ordering to obtain a chain; then we obtain more intervals and thus more equivalences whose blocks are intervals, and so more congruences by Eq. 2.3. In order to point out that this argument does not work, let $\langle L; \leq_1\rangle$ be the direct product of the two-element chain and the three-element chain. Although $\leq_1$ can be extended to a linear ordering $\leq_2$ and the chain $\langle L; \leq_2\rangle$ has more intervals than $\langle L; \leq_1\rangle$, the lattice $\langle L; \leq_1\rangle$ has 34 equivalences whose blocks are intervals but the chain $\langle L; \leq_2\rangle$ has only 32.

Remark 2.8 The concept of *meet-semilattices* $\langle S; \wedge\rangle$ and that of *semilattices* as commutative and idempotent semigroups $\langle S; \cdot\rangle$ are well known to be equivalent; see, for example, Grätzer [6, Exercises I.1.41–42 in pp. 18–19]. This paper gives preference to the former approach because of two reasons. First, as opposed to semilattices where there are two natural ways of defining an ordering, it is generally accepted that $a \leq b \iff a \wedge b = a$ for arbitrary elements a and b of a meet-semilattice $\langle S; \wedge\rangle$. Second, our figures and many arguments are order theoretical even though congruences are defined in the usual algebraic and semigroup theoretical way.

3 Proofs

Proof of Proposition 2.1 A subset X of $\langle S; \wedge\rangle$ is said to be *convex*, if $x < y < z$ and $x, z \in X$ imply that $y \in X$, for any $x, y, z \in S$. It is well known that

$$\text{the blocks of every congruence of } \langle S; \wedge\rangle \text{ are convex subsets of } \langle S; \wedge\rangle. \tag{3.1}$$

Indeed, if $\Theta \in \mathrm{Con}(S; \wedge)$, $x \leq y \leq z$ and $\langle x, z\rangle \in \Theta$, then $\langle x, y\rangle = \langle x \wedge y, z \wedge y\rangle \in \Theta$, whereby $y \in x/\Theta$, which shows Eq. 3.1. By Eq. 3.1, the $\boldsymbol{\tau}$-blocks are convex subsets of $\langle S; \wedge\rangle$. Next, for the sake of contradiction, suppose that $a, b \in S$ such that $a/\boldsymbol{\tau}$ and $b/\boldsymbol{\tau}$ are incomparable elements of the meet-semilattice $\langle S; \wedge\rangle/\boldsymbol{\tau}$ and they have an upper bound $c/\boldsymbol{\tau} \in \langle S; \wedge\rangle/\boldsymbol{\tau}$. Let $a' := a \wedge c$ and $b' := b \wedge c$ in $\langle S; \wedge\rangle$. Since $a/\boldsymbol{\tau} \leq c/\boldsymbol{\tau}$, we have that $a/\boldsymbol{\tau} = a/\boldsymbol{\tau} \wedge c/\boldsymbol{\tau} = (a \wedge c)/\boldsymbol{\tau} = a'/\boldsymbol{\tau}$, whence $\langle a, a'\rangle \in \boldsymbol{\tau}$. Similarly, $\langle b, b'\rangle \in \boldsymbol{\tau}$. Since $a' \leq c$ and $b' \leq c$, Eq. 2.2 implies the existence of $a' \vee b' \in \langle S^+; \vee\rangle$. Hence, by the definition of $\boldsymbol{\tau}$, we have that $\langle a' \wedge b', a' \vee b'\rangle \in \boldsymbol{\tau}$. Since the $\boldsymbol{\tau}$-block $(a' \wedge b')/\boldsymbol{\tau}$ is convex, $\langle a', b'\rangle \in \boldsymbol{\tau}$. Combining this with $\langle a, a'\rangle \in \boldsymbol{\tau}$ and $\langle b, b'\rangle \in \boldsymbol{\tau}$, we obtain that $\langle a, b\rangle \in \boldsymbol{\tau}$. Hence, $a/\boldsymbol{\tau}$ equals $b/\boldsymbol{\tau}$, which contradicts their incomparability. □

Note that, in general, $\boldsymbol{\tau} = \boldsymbol{\tau}(S; \wedge)$ is not the smallest congruence of $\langle S; \wedge\rangle$ such that $\langle S; \wedge\rangle/\boldsymbol{\tau}$ is a tree; this is exemplified by the semilattice reduct of the four-element boolean lattice.

The proof of Theorem 2.3 will be divided into several lemmas, some of them being interesting in themselves, and we are going to prove parts (i)–(iv) separately.

Remember that, for a finite meet-semilattice $S = \langle S; \wedge\rangle$, we use the notation $S^+ := S \setminus \{0\}$. Then $\langle S^+; \vee\rangle$ is a partial algebra, which we call the *partial join-semilattice* associated with S. By a *partial subalgebra* of $\langle S^+; \vee\rangle$ we mean a subset X of S^+ such that whenever $x, y \in X$ and $x \vee y$ is defined in $\langle S^+; \vee\rangle$, then $x \vee y \in X$. With respect to the set inclusion relation $\subseteq$, the set of all partial subalgebras of $\langle S^+; \vee\rangle$ turns out to be a lattice, which we denote by $\mathrm{Sub}(S^+; \vee)$. For convenience, our convention is that $\emptyset \in \mathrm{Sub}(S^+; \vee)$. The proof of Theorem 2.3 relies on the following result of Freese and Nation [4].

Lemma 3.1 (Freese and Nation [4, Lemma 1]) *For every finite meet-semilattice $\langle S; \wedge\rangle$, the lattice* $\mathrm{Con}(S; \wedge)$ *is dually isomorphic to* $\mathrm{Sub}(S^+; \vee)$. *In particular, we have that* $|\mathrm{Con}(S; \wedge)| = |\mathrm{Sub}(S^+; \vee)|$.

Note that Freese and Nation [4] uses $\mathrm{Sub}(S; \vee, 0)$, which does not contain the emptyset, but the natural isomorphism from $\mathrm{Sub}(S^+; \vee)$ onto $\mathrm{Sub}(S; \vee, 0)$, defined by $X \mapsto X \cup \{0\}$, allows us to cite their result in the above form. The following lemma is almost trivial; having no reference at hand, we are going to present a short proof. As usual, *intervals* are nonempty subsets of the form $[a, b] := \{x : a \le x \le b\}$. The *principal ideal* and the *principal filter* generated by an element $a \in S$ are denoted by $\downarrow a = \{x \in S : x \le a\}$ and $\uparrow a = \{x \in S : a \le x\}$, respectively. Meet-closed convex subsets are *convex subsemilattices*. A subsemilattice is *nontrivial* if it consists of at least two elements.

Lemma 3.2 *Let X be a nontrivial convex subsemilattice of a finite semilattice $\langle S; \wedge\rangle$, and denote the smallest element of X by $u := \bigwedge X$. Then the following two conditions are equivalent.*

(a) *The equivalence Θ on S whose only nonsingleton block is X is a congruence of $\langle S; \wedge\rangle$.*
(b) *For all $c \in S \setminus \uparrow u$ and every maximal element v of X, we have that $u \wedge c = v \wedge c$.*

Proof of Lemma 3.2 Assume (a) and let $c \notin \uparrow u$, and let v be a maximal element of X. Then $c \notin \uparrow v$, $u \not\le u \wedge c$, and $u \not\le v \wedge c$. Hence, none of $u \wedge c$ and $v \wedge c$ is in X, but these two elements are collapsed by Θ since $\langle u, v\rangle \in \Theta$. Thus, the definition of Θ gives that $u \wedge c = v \wedge c$, proving that (a) implies (b).

Next, assume (b), and let Θ be defined as in (a). First, we show that for all $x, y, z \in S$,

$$\text{if } \langle x, y\rangle \in \Theta, \text{ then } \langle x \wedge z, y \wedge z\rangle \in \Theta. \tag{3.2}$$

This is trivial for $x = y$, so we can assume that $x, y \in X$. Pick maximal elements x_1 and y_1 in X such that $x \le x_1$ and $y \le y_1$. First, let $z \in \uparrow u$. Then, using the convexity of X, $x \wedge z \in [u, x] \subseteq X$ and, similarly, $y \wedge z \in X$, whence we obtain that $\langle x \wedge z, y \wedge z\rangle \in \Theta$ by the definition of Θ. Second, let $z \in S \setminus \uparrow u$. Then $x \wedge z$ belongs to the interval $[u \wedge z, x_1 \wedge z]$, which is the singleton set $\{u \wedge z\}$ by (b). Hence, $x \wedge z = u \wedge z$. Similarly, $y \wedge z = u \wedge z$, whereby $\langle x \wedge z, y \wedge z\rangle \in \Theta$. Thus, Eq. 3.2 holds.

Finally, if $\langle x_1, y_1\rangle \in \Theta$ and $\langle x_2, y_2\rangle \in \Theta$, then we obtain from Eq. 3.2 that both $\langle x_1 \wedge x_2, y_1 \wedge x_2\rangle$ and $\langle y_1 \wedge x_2, y_1 \wedge y_2\rangle$ belong to Θ, whereby transitivity gives that $\langle x_1 \wedge x_2, y_1 \wedge y_2\rangle \in \Theta$. Consequently, Θ is a congruence and (b) implies (a). ☐

The *powerset* of a set A will be denoted by $P(A) = \{X : X \subseteq A\}$. In the rest of the paper,

$$n \geq 2 \text{ denotes a natural number, } \langle S; \wedge\rangle \text{ will stand for an } n\text{-element meet-semilattice, and we will also use the notation } k := |\mathrm{Con}(S; \wedge)| = |\mathrm{Sub}(S^+; \vee)|; \tag{3.3}$$

here the second equality is valid by Lemma 3.1.

Proof of Theorem 2.3 (i) Since $|S^+| = n - 1$, S^+ has at most 2^{n-1} subsets, whereby $|\mathrm{Con}(S; \wedge)| = k \leq |P(S^+)| = 2^{n-1}$, as required. If $\langle S; \wedge\rangle$ is a tree semilattice, then $x \vee y$ is defined only if x and y form a comparable pair of S^+, whence $x \vee y \in \{x, y\}$. Hence, every subset of S^+ belongs to $\mathrm{Sub}(S^+; \vee)$, and so $k = |\mathrm{Sub}(S^+; \vee)| = |P(S^+)| = 2^{n-1}$. If S is not a tree semilattice, then there is a pair $\langle a, b\rangle$ of incomparable elements of S^+ with an upper bound. By Eq. 2.2, $a \vee b$ is defined in $\langle S^+; \vee\rangle$. Hence, $\{a, b\} \notin \mathrm{Sub}(S^+; \vee)$ and so $k = |\mathrm{Sub}(S^+; \vee)| < |P(S^+)| = 2^{n-1}$. This completes the proof of part (i). □

By an *upper bounded two-element antichain*, abbreviated as *ubt-antichain*, we mean a two-element subset $\{x, y\}$ of a finite meet-semilattice $\langle S; \wedge\rangle$ such that $x \parallel y$ and $\uparrow x \cap \uparrow y \neq \emptyset$. By Eq. 2.2, every ubt-antichain $\{x, y\}$ has a join in S^+ but this join is outside $\{x, y\}$. Therefore,

$$\mathrm{Sub}(S^+; \vee) \text{ contains no ubt-antichain.} \tag{3.4}$$

Besides (3.4), the importance of ubt-antichains is explained by the following lemma.

Lemma 3.3 *Let X be a convex subsemilattice of a finite semilattice $\langle S; \wedge\rangle$ such that $|X| \geq 2$ and $X \times X \subseteq \boldsymbol{\tau}$; see* Eq. 2.1. *If X contains all ubt-antichains $\{p, q\}$ of $\langle S; \wedge\rangle$ together with their joins $p \vee q$, then $\langle S; \wedge\rangle$ is a quasi-tree semilattice and its nucleus is X.*

Proof of Lemma 3.3 Denote the smallest element of X by $u := \bigwedge X$. Let Θ be the equivalence relation on S with X as the only nonsingleton block of Θ. In order to prove that $\Theta \in \mathrm{Con}(S; \wedge)$, assume that $c \in S \setminus \uparrow u$ and v is a maximal element of X. For the sake of contradiction, suppose that $u \wedge c \neq v \wedge c$, which means that $u \wedge c < v \wedge c$. If we had that $v \wedge c \leq u$, then $v \wedge c = u \wedge (v \wedge c) = (u \wedge v) \wedge c = u \wedge c$ would be a contradiction. Thus, $v \wedge c \nleq u$. On the other hand, $u \nleq v \wedge c$ since $u \nleq c$, whereby $u \parallel v \wedge c$. Since v is a common upper bound of u and $v \wedge c$, we obtain that $\{u, v \wedge c\}$ is a ubt-antichain. This is a contradiction since $c \notin \uparrow u$ implies that $u \nleq v \wedge c$, whence the ubt-antichain $\{u, v \wedge c\}$ is not a subset of X. Hence, $u \wedge c = v \wedge c$, and it follows from Lemma 3.2 that $\Theta \in \mathrm{Con}(S; \wedge)$.

Next, in order to show that $\langle S; \wedge\rangle/\Theta$ is a tree, suppose the contrary. Then there are two incomparable Θ-blocks x/Θ and y/Θ that have an upper bound z/Θ. Since $u \in X$ and all other Θ-blocks are singletons, every Θ-block has a smallest element. This fact allows us to assume that each of x, y, and z is the least element of its Θ-block. Since $x/\Theta \leq z/\Theta$, we have that $x/\Theta = x/\Theta \wedge z/\Theta = (x \wedge z)/\Theta$, that is, $\langle x, x \wedge z\rangle \in \Theta$. But the least element of x/Θ is x, whence $x = x \wedge z$, that is, $x \leq z$. We obtain similarly that $y \leq z$, that is, $\{x, y\}$ has an upper bound, z. Since $x \wedge y = x$ would imply that $x/\Theta \wedge y/\Theta = (x \wedge y)/\Theta = x/\Theta$, contradicting that $\{x/\Theta, y/\Theta\}$ is an antichain, we obtain that $x \nleq y$. We obtain $y \nleq x$ similarly. Thus, $\{x, y\}$ is a ubt-antichain, whereby $\{x, y\} \subseteq X$. But then $x/\Theta = X = y/\Theta$, contradicting the initial assumption that these two Θ-blocks are incomparable. Therefore, $\langle S; \wedge\rangle/\Theta$ is a tree. Hence, in order to complete the proof, we need to show that $\Theta = \boldsymbol{\tau}$. Since $X \times X \subseteq \boldsymbol{\tau}$, the inclusion $\Theta \subseteq \boldsymbol{\tau}$ is clear. In order to see the converse inclusion, let $\langle a \wedge b, a \vee b\rangle$ be a pair occurring in Eq. 2.1. Then $\{a, b\}$ is a ubt-antichain, so $\{a, b\} \subseteq X$ and, by the assumptions of the lemma, both $a \vee b$ and $a \wedge b$ belong to X. Hence, the pairs in Eq. 2.1 are collapsed by Θ and we conclude that $\boldsymbol{\tau} \subseteq \Theta$. Consequently, $\Theta = \boldsymbol{\tau}$, and the proof of Lemma 3.3 is complete. □

Lemma 3.4 *If* $\langle S; \wedge\rangle$ *from* Eq. 3.3 *contains exactly one ubt-antichain, then* $\langle S; \wedge\rangle$ *is a quasi-tree semilattice and its nucleus is the four-element boolean lattice.*

Proof of Lemma 3.4 Let us denote by $\{a, b\}$ the unique ubt-antichain of $\langle S; \wedge\rangle$. Let $v := a \vee b$, which exists by Eq. 2.2, and let $u := a \wedge b$. Then $L := [u, v]$ contains every ubt-antichain. Since $\langle u, v\rangle \in \boldsymbol{\tau}$ by Eq. 2.1 and the $\boldsymbol{\tau}$-blocks are convex, $L \times L \subseteq \boldsymbol{\tau}$. So, with reference to Lemma 3.3, it suffices to show that L is the four-element boolean lattice. In fact, it suffices to show that $L \subseteq \{u, a, b, v\}$ since the converse inclusion is evident. Suppose the contrary, and let $x \in L \setminus \{u, a, b, v\}$. If $x \parallel a$, then $\{a, x\}$ is a ubt-antichain (with upper bound v) but it is distinct from $\{a, b\}$, which contradicts the fact that $\{a, b\}$ is the only ubt-antichain. Hence, a and x are comparable. We obtain similarly that b and x are comparable. If $x \le a$ and $x \le b$, then $u \le x \le a \wedge b = u$ leads to $x = u \in L$, which is not the case. We obtain dually that the conjunction of $x \ge a$ and $x \ge b$ is impossible. Hence, $a \le x \le b$ or $b \le x \le a$, contradicting that $\{a, b\}$ is an antichain. This shows that $L \subseteq \{u, a, b, v\}$, completing the proof of Lemma 3.4. □

Proof of Theorem 2.3(ii) Assume that $k < 2^{n-1}$; see Eq. 3.3. By Theorem 2.3(i), $\langle S; \wedge\rangle$ is not a tree. Hence, $n = |S| \ge 4$. Since $|\mathrm{Sub}(S^+; \vee)| = k < 2^{n-1} = |P(S^+)|$, not every subset of S^+ is $\vee$-closed. Thus, we can pick $a, b \in S^+$ such that $a \parallel b$ and $a \vee b$ exists in $\langle S^+; \vee\rangle$. Since $|S^+ \setminus \{a, b, a \vee b\}| = n - 4$, there are 2^{n-4} subsets of S^+ that contain a, b, but not $a \vee b$; these subsets do not belong to $\mathrm{Sub}(S^+; \vee)$. Thus, $k \le 2^{n-1} - 2^{n-4} = 32 \cdot 2^{n-6} - 4 \cdot 2^{n-6} = 28 \cdot 2^{n-6}$, proving the first half of (ii).

Next, assume that $k = 28 \cdot 2^{n-6}$ and choose a and b as above. There are $2^{n-4} = 4 \cdot 2^{n-6}$ subsets of S^+ containing a and b, but not containing $a \vee b$; these subsets are not in $\langle S^+; \vee\rangle$. Thus, all the remaining $32 \cdot 2^{n-6} - 4 \cdot 2^{n-6} = 28 \cdot 2^{n-6}$ subsets belong to $\langle S^+; \vee\rangle$ since $k = 28 \cdot 2^{n-6}$. In particular, for every ubt-antichain $\{x, y\}$, we have that $\{x, y\} \ne \{a, b\} \Rightarrow \{x, y\} \in \mathrm{Sub}(S^+; \vee)$. This implication and Eq. 3.4 yield that $\{a, b\}$ is the only ubt-antichain in $\langle S; \wedge\rangle$. Thus, it follows from Lemma 3.4 that $\langle S; \wedge\rangle$ is a quasi-tree semilattice of the required form.

Conversely, assume that $\langle S; \wedge\rangle$ is of the form described in Theorem 2.3(ii). Choosing the notation so that its nucleus is $\{a \wedge b, a, b, a \vee b\}$, the only ubt-antichain is $\{a, b\}$, whence a subset X of S^+ is not in $\mathrm{Sub}(S^+; \vee)$ iff $a, b \in X$ but $a \vee b \notin X$. There are $2^{n-4} = 4 \cdot 2^{n-6}$ such subsets X, and we obtain that $k = |\mathrm{Sub}(S^+; \vee)| = |P(S^+)| - 4 \cdot 2^{n-6} = 32 \cdot 2^{n-6} - 4 \cdot 2^{n-6} = 28 \cdot 2^{n-6}$, as required. This completes the proof of Theorem 2.3(ii). □

Lemma 3.5 *If* $\langle S; \wedge\rangle$ *from* Eq. 3.3 *contains exactly two ubt-antichains,* $\{a, b\}$ *and* $\{c, b\}$ *such that* $a < c$*, then* $\langle S; \wedge\rangle$ *is a quasi-tree semilattice and its nucleus is the pentagon lattice* N_5.

Proof of Lemma 3.5 By Eq. 2.2, we can let $v := a \vee b$. Since $v \le c$ would lead to $b \le c$, we have that $v \not\le c$. In particular, $v \ne c$, and we also have that $v \notin \{a, b\}$ since $\{a, b\}$ is an antichain. Thus, $\{c, v\}$ is a two-element subset of S and it is distinct from $\{a, b\}$ and $\{a, c\}$. Hence, $\{c, v\}$ is not a ubt-antichain. Since $b \vee c$, which exists by Eq. 2.2, is clearly an upper bound of $\{c, v\}$, it follows that $\{c, v\}$ is not an antichain. This fact and $v \not\le c$ yield that $c \le v$. Thus, $v = a \vee b \le c \vee b \le v$, that is, $v = a \vee b = c \vee b$. Next, let $u := b \wedge c$; clearly, $u \notin \{b, c\}$. If we had that $a \parallel u$, then $\{a, u\}$ would be a third ubt-antichain (with upper bound c), whence a and u are comparable elements. Since $a \le u$ would lead to $a \le b$ by transitivity, we have that $u \le a$. Hence, $u \le a \wedge b \le c \wedge b = u$, and so $a \wedge b = u$. The equalities established so far show that $L := \{u, a, b, c, v\}$ is a sublattice isomorphic to N_5. In order to show that L is the interval $[u, v]$, suppose the contrary, and let $x \in [u, v] \setminus L$. If $x \parallel b$, then $\{b, x\}$ would be a third ubt-antichain (with upper bound v), which would be a contradiction. If we had that $b < x < v$, then $\{c, x\}$ would be a ubt-antichain, a contradiction. Similarly, $u < x < b$ gives that $\{a, x\}$ is a ubt-antichain, a contradiction again. Thus, $L = [u, v]$ is an interval of S. By Eq. 2.1, $\langle u, v\rangle = \langle a \wedge b, a \vee b\rangle \in \boldsymbol{\tau}$. Using that the $\boldsymbol{\tau}$-blocks are convex subsets, we obtain that $L \times L = [u, v] \times [u, v] \subseteq \boldsymbol{\tau}$. Thus, Lemma 3.5 follows from Lemma 3.3. □

Proof of Theorem 2.3(iii) Assume that $k < 28 \cdot 2^{n-6}$; see Eq. 3.3.

Note at this point that *no equality* will be assumed for k before Eq. 3.24. Therefore the numbered equations, equalities, and statements *before* Eq. 3.24 can be used later in the proof of Theorem 2.3(iv).

We introduce the following notation. For a ubt-antichain $\{a, b\}$, let

$$U(a, b) := \{X \in P(S^+) : a \in X, b \in X, \text{ but } a \vee b \notin X\}; \tag{3.5}$$

it is a subset of $P(S^+)$; note that the existence of $a \vee b$ above follows from Eq. 2.2. By Theorem 2.3(i), $\langle S; \wedge\rangle$ is not a tree, whereby it has at least one ubt-antichain. If it had only one ubt-antichain, then Lemma 3.4 and Theorem 2.3(ii) would imply that $k = 28 \cdot 2^{n-6}$. Hence, $\langle S; \wedge\rangle$ has at least two ubt-antichains. Let $\{a_1, b_1\}, \{a_2, b_2\}, \ldots, \{a_t, b_t\}$ be a repetition-free list of all ubt-antichains of $\langle S; \wedge\rangle$; note that $t \ge 2$. Let $v_i := a_i \vee b_i$ and $U_i := U(a_i, b_i)$, see Eq. 3.5, for $i = 1, \ldots, t$. That is, U_i is the set of all those $X \in P(S^+)$ that contain a_i and b_i but not v_i. Observe that, for $1 \le i < j \le t$,

$$\text{if } |\{a_i, b_i, v_i, a_j, b_j, v_j\}| = \ell, \text{ then } |U_i \cap U_j| \text{ is either } 2^{5-\ell} \cdot 2^{n-6}, \text{ or } 0. \tag{3.6}$$

Indeed, when we choose elements from the $(n-1)$-element $P(S^+)$ in order to form a set $X \in U_i \cap U_j$, then we can dispose only over $(n-1)-\ell = (5-\ell)+(n-6)$ elements, because the containment $X \in U_i \cap U_j$ determines what to do with ℓ elements. If the stipulations for these ℓ elements are contradictory, then $|U_i \cap U_j|$ equals 0; this can happen only if $\ell < 6$. Otherwise, $|U_i \cap U_j| = 2^{5-\ell} \cdot 2^{n-6}$, showing the validity of Eq. 3.6.

Next, we show that for any $1 \le i < j \le t$,

$$\text{if } |\{a_i, b_i, v_i, a_j, b_j, v_j\}| = 6, \text{ then } k \le 24.5 \cdot 2^{n-6}, \tag{3.7}$$

$$\text{if } |\{a_i, b_i, v_i, a_j, b_j, v_j\}| = 5, \text{ then } k \le 25 \cdot 2^{n-6}, \text{ and} \tag{3.8}$$

$$\text{if } |\{a_i, b_i, v_i, a_j, b_j, v_j\}| = 4, \text{ then } k \le 26 \cdot 2^{n-6}. \tag{3.9}$$

As a stronger form of Eq. 3.4 for the present situation, it is clear that

$$\mathrm{Sub}(S^+; \vee) = P(S^+) \setminus (U_1 \cup \cdots \cup U_t). \tag{3.10}$$

In particular, $U_i \cup U_j$ is disjoint from $\mathrm{Sub}(S^+; \vee)$. Hence, the Inclusion-Exclusion Principle, $k = |\mathrm{Sub}(S^+; \vee)|$, $|P(S^+)| = 32 \cdot 2^{n-6}$, and $|U_i| = |U_j| = 4 \cdot 2^{n-6}$ give that

$$\mathrm{Sub}(S^+; \vee) \subseteq P(S^+) \setminus (U_i \cup U_j), \text{ and so} \tag{3.11}$$

$$k \le 2^{n-6} \cdot (32 - 4 - 4) + |U_i \cap U_j| = 24 \cdot 2^{n-6} + |U_i \cap U_j|, \tag{3.12}$$

$$\text{and if (3.11) holds with equality in it, then so does (3.12).} \tag{3.13}$$

Clearly, Eqs. 3.7, 3.8 and 3.9 follow from Eqs. 3.6 and 3.12. Furthermore, it is also clear from this argument that strict inequalities lead to strict inequalities. For later reference, we formulate this as follows.

If $|U_i \cap U_j|$ is *strictly* less than 2^{n-7}, 2^{n-6}, and $2 \cdot 2^{n-6}$, then k is strictly less than $24.5 \cdot 2^{n-6}$, $25 \cdot 2^{n-6}$, and $26 \cdot 2^{n-6}$, respectively. (3.14)

Next, we claim that for $1 \le i < j \le t$,

$$\text{if } v_i \neq v_j, \text{ then } |\{a_i, b_i, v_i, a_j, b_j, v_j\}| \ge 5. \tag{3.15}$$

In order to show this, first we deal with the case where $v_j \in \{a_i, b_i\}$ or $v_i \in \{a_j, b_j\}$. Let, say, $v_1 = a_2$. Then $v_2 > a_2 = v_1 > a_1$ and $v_2 > a_2 = v_1 > b_1$ yield that $|\{a_1, b_1, v_1, v_2\}| = 4$. Clearly, $b_2 \notin \{a_2 = v_1, v_2\}$. If we had that $b_2 \in \{a_1, b_1\}$, then $b_2 < v_1 = a_2$ would contradict $a_2 \parallel b_2$. Hence, the inequality in Eq. 3.15 holds in this case. Second, assume that $v_j \notin \{a_i, b_i\}$ and $v_i \notin \{a_j, b_j\}$. Using also that $v_i \neq v_j$, we have that $|\{a_i, b_i, v_i, v_j\}| = 4$. Since $v_i \notin \{a_j, b_j\}$, $\{a_i, b_i\} \neq \{a_j, b_j\}$, and, of course, $v_j \notin \{a_j, b_j\}$, at least one of a_j and b_j is not in $\{a_i, b_i, v_i, v_j\}$, and the required inequality in Eq. 3.15 holds again. This proves (3.15). Clearly,

$$\text{if } v_i = v_j \text{ but } i \neq j, \text{ then } |\{a_i, b_i, v_i, a_j, b_j, v_j\}| \ge 4, \tag{3.16}$$

because $\{a_i, b_i, a_j, b_j\}$ has at least three elements and does not contain $v_i = v_j$, which is strictly larger than every element of $\{a_i, b_i, a_j, b_j\}$. Observe that the inequality $k \le 26 \cdot 2^{n-6}$, which is the first half of Theorem 2.3(iii), follows from Eqs. 3.7, 3.8, 3.9, 3.15 and 3.16, because $t \ge 2$ implies the existence of a pair $\langle i, j\rangle$ such that $1 \le i < j \le t$.

Next, strengthening Eq. 3.8, we are going to show that for any $1 \le i < j \le t$,

$$\text{if } |\{a_i, b_i, v_i, a_j, b_j, v_j\}| = 5 \text{ and } t \ge 3, \text{ then } k < 25 \cdot 2^{n-6}. \tag{3.17}$$

Assume the premise of Eq. 3.17. Since $t \ge 3$, we can pick an $m \in \{1, \dots, t\} \setminus \{i, j\}$. For the sake of contradiction,

$$\text{suppose that } |\{a_i, b_i, v_i, a_j, b_j, v_j\}| = 5 \text{ but } k \ge 25 \cdot 2^{n-6}. \tag{3.18}$$

By Eqs. 3.6 and 3.18, $|U_i \cap U_j|$ is either 0 or 2^{n-6}, but the first alternative is ruled out by Eqs. 3.14 and 3.18. Hence

$$|U_i \cap U_j| = 2^{n-6}. \tag{3.19}$$

By Eqs. 3.7 and 3.18, none of $\{a_i, b_i, v_i, a_m, b_m, v_m\}$ and $\{a_j, b_j, v_j, a_m, b_m, v_m\}$ consists of six elements. Hence, it follows from Eqs. 3.15 and 3.16, that each of these two sets consists of four or five elements. Thus, Eq. 3.6 gives that

$$|U_i \cap U_m| \le 2 \cdot 2^{n-6} \text{ and } |U_j \cap U_m| \le 2 \cdot 2^{n-6}. \tag{3.20}$$

We also need the following observation.

If $U_i \cap U_j \neq \emptyset$, $U_i \cap U_m \neq \emptyset$, and $U_j \cap U_m \neq \emptyset$, then $U_i \cap U_j \cap U_m \neq \emptyset$. (3.21)

To show Eq. 3.21, assume that its premise holds. If $\{a_i, b_i, a_j, b_j, a_m, b_m\}$ is disjoint from $\{v_i, v_j, v_m\}$, then $U_i \cap U_j \cap U_m$ contains $\{a_i, b_i, a_j, b_j, a_m, b_m\}$ and so it is nonempty. Otherwise, by *a*–*b* symmetry and since the subscripts in Eq. 3.21 play symmetric roles, we can assume that $a_i = v_j$. However, then $U_i \cap U_j = \emptyset$, contradicting the premise of Eq. 3.21. Consequently, Eq. 3.21 holds. Based on the Inclusion-Exclusion Principle, as in

Eqs. 3.11–3.13, and using Eqs. 3.19 and 3.20, we can compute as follows; the overline and the underlines below will serve as reference points.

$$k \le 2^{n-6} \cdot \big(32 - (4+4+4)\big) + (|U_i \cap U_j| + |U_i \cap U_m| + |U_j \cap U_j|) \\ -|U_i \cap U_j \cap U_m|, \quad \text{and so} \tag{3.22}$$

$$k \le 2^{n-6} \cdot (20 + \overline{1} + \underline{2} + \underline{2}) - |U_i \cap U_j \cap U_m| \\ = 25 \cdot 2^{n-6} - |U_i \cap U_j \cap U_m|. \tag{3.23}$$

We know from Eq. 3.19 that $U_i \cap U_j \neq \emptyset$. Both underlined numbers in Eq. 3.23 come from Eq. 3.20. So if at least one the intersections $U_i \cap U_m$ and $U_j \cap U_m$ is empty, then at least one of the underlined numbers can be replaced by 0 and Eq. 3.23 gives that $k < 25 \cdot 2^{n-6}$. Otherwise the subtrahend at the end of Eq. 3.23 is positive by Eq. 3.21, and we obtain again that $k < 25 \cdot 2^{n-6}$.

This contradicts (3.18) and proves the validity of Eq. 3.17. Next, we assume that

$$k = 26 \cdot 2^{n-6}. \tag{3.24}$$

It follows from Eqs. 3.7, 3.8, 3.15 and 3.24 that

$$\text{all the } v_i \text{ are the same, so we can let } v := v_1 = \cdots = v_t. \tag{3.25}$$

Hence, we get from Eqs. 3.6, 3.7, 3.8, 3.16 and 3.24 that, for any $1 \le i < j \le t$,

$$\begin{aligned} &|\{a_i, b_i, a_j, b_j, v\}| = 4 \text{ and so } |U_i \cap U_j| \le 2 \cdot 2^{n-6}, \\ &|\{a_i, b_i, a_j, b_j\}| = 3, \text{ and } |\{a_i, b_i\} \cap \{a_j, b_j\}| = 1. \end{aligned} \tag{3.26}$$

Next, we are going to prove that t, the number of ubt-antichains, equals 2. Suppose the contrary. Since now we have Eq. 3.26 instead of Eq. 3.19, $\overline{1}$ and 25 in Eq. 3.23 turn into $\underline{2}$ and 26, respectively. These two modifications do not influence the paragraph following Eq. 3.23, and we conclude that the inequality in the modified Eq. 3.23 is a strict one, that is, $k < 26 \cdot 2^{n-6}$. This contradicts Eq. 3.24, whence we conclude that there are exactly $t = 2$ ubt-antichains. We know from Eq. 3.26 that they are not disjoint. So we can denote them by $\{a, b\}$ and $\{c, b\}$ where $|\{a, b, c\}| = 3$. By Eq. 3.25, $v = a \vee b = c \vee b$. Since $t = 2$, the set $\{a, c\}$ is not a ubt-antichain, whence a and c are comparable. So we can assume that $a < c$, and it follows from Lemma 3.5 that $\langle S; \wedge\rangle$ is a quasi-tree semilattice of the required form.

Finally, assume that $\langle S; \wedge\rangle$ is a quasi-tree semilattice and its nucleus is the pentagon $N_5 = \{u, a, b, c, v\}$ with bottom u, top v, and $a < c$. Let $U_1 := U(a, b)$ and $U_2 := U(c, b)$; see Eq. 3.5. Since $\mathrm{Sub}(S^+; \vee) = P(S^+) \setminus (U_1 \cup U_2)$ by Eq. 3.10,

$$k = |P(S^+)| - |U_1| - |U_2| + |U_1 \cap U_2| = (32 - 4 - 4 + 2) \cdot 2^{n-6} = 26 \cdot 2^{n-6},$$

as required. This completes the proof of Theorem 2.3(iii). □

Lemma 3.6 *If* $\langle S; \wedge\rangle$ *from* Eq. 3.3 *contains exactly two ubt-antichains,* $\{a, b\}$ *and* $\{b, c\}$ *such that* $v_1 := a \vee b$ *and* $v_2 := b \vee c$ *are incomparable, then* $\langle S; \wedge\rangle$ *is a quasi-tree semilattice and its nucleus is* $F = \{u := a \wedge b \wedge c, a, b, c, v_1, v_2\}$ *given in* Fig. 4.

Proof of Lemma 3.6 Let $u := a \wedge b$. It is not in $\{a, b\}$. Since $b \not\geq c$, we have that $u \not\geq c$. Using that v_2 is an upper bound of $\{u, c\}$ and $\{u, c\}$ is not a ubt-antichain, it follows that $\{u, c\}$ is not an antichain. Hence, $u \le c$, whence $u = a \wedge b \wedge c$. Since $a \parallel b$ and $v_1 \parallel v_2$ implies that $a \parallel c$, we obtain that $a \wedge c \notin \{a, b, c\}$. Hence, $\{b, a \wedge c\}$ is a two-element set and it is distinct from $\{a, b\}$ and $\{b, c\}$. Using that v_1 is an upper bound of $\{b, a \wedge c\}$, we obtain that $\{b, a \wedge c\}$ is not an antichain. Since $b \not\le c$, we have that $b \not\le a \wedge c$. Hence, $a \wedge c \le b$, implying that $a \wedge c = a \wedge c \wedge b$. Summarizing the facts above and taking into account that a and c play a symmetric role, we have that

$$u = a \wedge b = a \wedge b \wedge c = b \wedge c = a \wedge c. \tag{3.27}$$

Let $M := \{a, b, c, u, v_1, v_2\}$; we claim that

$$M \text{ is a convex meet-subsemilattice of } \langle S; \wedge\rangle. \tag{3.28}$$

First, we show that M is a convex subset of $\langle S; \wedge\rangle$. For the sake of contradiction, suppose that $x \in S \setminus M$ such that $u < x < v_1$; the case $u < x < v_2$ would be similar since a and c play symmetric roles. Both $\{a, x\}$ and $\{x, b\}$ have an upper bound, v_1. Hence, none of them is an ubt-antichain since $x \notin M$. Thus, $a \le x \le b$, or $b \le x \le a$, or $a, b \in {\downarrow}x$, or $a, b \in {\uparrow}x$. The first two alternatives are ruled out by $a \parallel b$. The third alternative leads to $v_1 = a \vee b \le x \le v_1$, contradicting $x \notin M$. We obtain a contradiction from the fourth alternative dually by using u instead of v_1. Thus, M is a convex subset of $\langle S; \wedge\rangle$. Since M is convex and $b \le v_1 \wedge v_2 \le v_1$, we have that $v_1 \wedge v_2 \in \{b, v_1\}$. Similarly, $v_1 \wedge v_2 \in \{b, v_2\}$. So $v_1 \wedge v_2 \in \{b, v_1\} \cap \{b, v_2\} = \{b\}$, that is, $v_1 \wedge v_2 = b$. This equality together with Eq. 3.27 give easily that M is a meet-subsemilattice of $\langle S; \wedge\rangle$, whence (3.28) holds. It is clear by Eq. 3.27 that $M \cong F$.

Since $\langle u, v_1\rangle = \langle a \wedge b, a \vee b\rangle$ occurs in Eq. 2.1 and the $\boldsymbol{\tau}$-blocks are convex subsets, $\{a, b, v_1, u\} \subseteq u/\boldsymbol{\tau}$. We obtain similarly that $\{b, c, v_2, u\} \subseteq u/\boldsymbol{\tau}$, whence we have that $M \times M \subseteq \boldsymbol{\tau}$. Therefore, since M contains both ubt-antichains and their joins, Lemma 3.3 implies the validity of Lemma 3.6. □

Lemma 3.7 *If* $\langle S; \wedge\rangle$ *from* Eq. 3.3 *contains exactly three ubt-antichains,* $\{a_1, b\}$*,* $\{a_2, b\}$*, and* $\{a_3, b\}$ *such that* $v := a_1 \vee b = a_2 \vee b = a_3 \vee b$ *and* $a_1 < a_2 < a_3$*, then* $\langle S; \wedge\rangle$ *is a quasi-tree semilattice and its nucleus is* $N_6 = \{u := a_1 \wedge b = a_2 \wedge b = a_3 \wedge b, a_1, a_2, a_3, v\}$ *given in* Fig. 4.

Proof of Lemma 3.7 Let $u := a_3 \wedge b$. Since $a_3 \parallel b$, $u \ne b$. We are going to show that $M := \{u, a_1, a_2, a_3, v\}$ is a subsemilattice isomorphic to N_6. Let $i \in \{1, 2\}$. Since v is an upper bound of the set $\{a_i, u\}$, this set is not an antichain. Since $a_i \not\le b$, we have that $a_i \not\le u$. Hence, $u < a_i$, and we obtain that $u \le a_i \wedge b \le a_3 \wedge b = u$. Thus, the meets in M are what they are required to be, and we conclude that $M \cong N_6$. Next, for the sake of contradiction, suppose that M is not a convex subset of $\langle S; \wedge\rangle$, and pick an element $x \in S \setminus M$ such that $u \le x \le v$. Since no more ubt-antichain is possible, none of a_1, a_2, a_3, and b is incomparable with x. If we had that $x \le a_j$ for some $j \in \{1, 2, 3\}$, then $b \le x$ would contradict $b \not\le a_j$ while $x \le b$ would lead to $u \le x \le a_j \wedge b \le u$, a contradiction since $x \ne u \in M$. A dual argument, with v instead of u, would lead to a contradiction if $a_j \le x$. Hence, M is a convex subsemilattice of $\langle S; \wedge\rangle$. Since $\langle u, v\rangle = \langle a_1 \wedge b, a_1 \vee b\rangle$ occurs in Eq. 2.1 and the $\boldsymbol{\tau}$-blocks are convex subsets, $M \times M \subseteq \boldsymbol{\tau}$. Therefore, since M contains all the three ubt-antichains and their common join, Lemma 3.7 follows from Lemma 3.3. □

Proof of Theorem 2.3 (iv) We assume that $k = |\mathrm{Con}(S; \wedge)| < 26 \cdot 2^{n-6}$. In the first part of the proof, we are going to focus on the required inequality, $k \le 25 \cdot 2^{n-6}$.

As it has been mentioned in the proof of Theorem 2.3(iii), any part of that proof before Eq. 3.24 is applicable here, including the notation. If $|\{a_i, b_i, v_i, a_j, b_j, v_j\}| \ge 5$ or $v_i \ne v_j$ for some $1 \le i < j \le t$, then the required $k \le 25 \cdot 2^{n-6}$ follows from Eqs. 3.7, 3.8 and 3.15. Hence, we can assume that $v := v_1 = v_2 = \cdots = v_t$. By Eqs. 3.8 and 3.16, we can assume also that $|\{a_i, b_i, a_j, b_j, v\}| = 4$ for all $1 \le i < j \le t$. For later reference, we summarize our assumptions as

$$v := v_1 = v_2 = \cdots = v_t \text{ and } |\{a_i, b_i, a_j, b_j, v\}| = 4, \text{ whereby } |\{a_i, b_i, a_j, b_j\}| = 3 \text{ and } |\{a_i, b_i\} \cap \{a_j, b_j\}| = 1, \text{ for all } 1 \le i < j \le t. \tag{3.29}$$

We claim that

$$\text{if } t \ge 3, \text{ Eq. 3.29, and } \{a_1, b_1\} \cap \{a_2, b_2\} \cap \{a_3, b_3\} = \emptyset, \text{ then } k \le 24 \cdot 2^{n-6}. \tag{3.30}$$

The pairwise intersections in Eq. 3.29 are singletons, whereby the only way that the intersection in Eq. 3.30 is empty is that $|\{a_1, b_1, a_2, b_2, a_3, b_3\}| = 3$. Hence, for all $1 \le i < j \le t$, we have that $U_i \cap U_j = U_1 \cap U_2 \cap U_3$ and $|U_1 \cap U_2 \cap U_3| = |U_i \cap U_j| = 2 \cdot 2^{n-6}$, and Eq. 3.30 follows from Eq. 3.22. We also claim that

$$\text{if } t \ge 3, \text{ Eq. 3.29, and } \{a_1, b_1\} \cap \{a_2, b_2\} \cap \{a_3, b_3\} \ne \emptyset, \text{ then } k \le 25 \cdot 2^{n-6}. \tag{3.31}$$

With the assumption made in Eq. 3.31, if we consider the same intersections as in the argument right after Eq. 3.30, then we obtain that $|\{a_1, b_1, a_2, b_2, a_3, b_3\}| = 4$. Hence, $|U_i \cap U_j| = 2 \cdot 2^{n-6}$ and $|U_1 \cap U_2 \cap U_3| = 1 \cdot 2^{n-6}$, and Eq. 3.31 follows from Eq. 3.22. Our next observation is that

$$\text{if } t \le 2 \text{ and Eq. 3.29, then } k \ge 26 \cdot 2^{n-6}. \tag{3.32}$$

For $t \le 1$, this is clear from Theorem 2.3(i), Lemma 3.4, and Theorem 2.3(ii); so let $t = 2$. Since the intersection in Eq. 3.29 is a singleton, the two ubt-antichains are of the form $\{a, b\}$ and $\{c, b\}$. Since $\{a, c\}$ cannot be a third ubt-antichain, the elements a and c are comparable, whereby Lemma 3.5, and Theorem 2.3(iii) imply that $k = 26 \cdot 2^{n-6}$. Thus, Eq. 3.32 holds. Now, the required $k \le 25 \cdot 2^{n-6}$ follows from Eqs. 3.30, 3.31 and 3.32, and the paragraph above Eq. 3.29; completing the first part of the proof.

In the rest of the proof, we will always assume that $k = 25 \cdot 2^{n-6}$, even if this is not emphasized all the time. We claim that

$$\text{if } k = 25 \cdot 2^{n-6} \text{ and } t \ge 3, \text{ then } t = 3, v := v_1 = \cdots = v_t, \text{ and Eq. 3.26 holds for all } 1 \le i < j \le t. \tag{3.33}$$

Assuming the premise of Eq. 3.33, we obtain from Eq. 3.7 that the size of the set $\{a_i, b_i, v_i, a_j, b_j, v_j\}$ is not 6. We obtain from Eq. 3.17 that it is neither 5, whereby this size is 4 since $\{a_i, b_i\} \ne \{a_j, b_j\}$. Thus, Eq. 3.15 implies $v := v_1 = \cdots = v_t$ as well as the validity of Eq. 3.26. The $|\{a_i, b_i\} \cap \{a_j, b_j\}| = 1$ part of Eq. 3.26 implies that apart from notation (that is, modulo permutations of the sets $\{i, j, m\}$, $\{a_i, b_i\}$, $\{a_j, b_j\}$, $\{a_m, b_m\}$),

$$\text{whenever } 1 \le i < j < m \le t, \text{ then either } b_i = a_j, b_j = a_m, \text{ and } b_m = a_i, \text{ or } b := b_i = b_j = b_m \text{ and } |\{a_i, a_j, a_m\}| = 3. \tag{3.34}$$

It follows similarly to Eqs. 3.22 and 3.23 that

$$\text{if the first alternative of Eq. 3.34 holds, then } |U_i \cup U_j \cup U_m| = \big((4+4+4) - (2+2+2) + 2\big) \cdot 2^{n-6}, \text{ whereby } k \le (32-8) \cdot 2^{n-6}, \text{ which contradicts } k = 25 \cdot 2^{n-6}, \tag{3.35}$$

since $U_i \cap U_j \cap U_m = U_i \cap U_j$. Thus, Eq. 3.35 excludes the first alternative of Eq. 3.34. Hence we have the second alternative $|U_i \cap U_j \cap U_m| = 2^{n-6}$, and it follows similarly to Eqs. 3.22 and 3.23 that

$$|U_i \cup U_j \cup U_m| = \big((4+4+4) - (2+2+2) + 1\big) \cdot 2^{n-6} = 7 \cdot 2^{n-6}. \tag{3.36}$$

Now, for the sake of contradiction, suppose that $t \geq 4$. Then we can pick an index $s \in \{1, \dots, t\} \setminus \{i, j, m\}$. The ubt-antichain $\{a_s, b_s\}$ belongs to U_s but it does not belong to U_i since the members of U_i contain both a_i and b_i but $\{a_s, b_s\} \neq \{a_i, b_i\}$. Similarly, $\{a_s, b_s\}$ belongs neither to U_j, nor to U_m, whence it is not in $U_i \cup U_j \cup U_m$. Hence, $U_i \cup U_j \cup U_m$ is a proper subset of $U_i \cup U_j \cup U_m \cup U_s$, which is disjoint from $\mathrm{Sub}(S^+; \vee)$ by Eq. 3.10. Thus, by Eq. 3.36, strictly more than $7 \cdot 2^{n-6}$ subsets of S^+ are *not* in $\mathrm{Sub}(S^+; \vee)$, and we obtain that $k = |\mathrm{Sub}(S^+; \vee)| < (32 - 7) \cdot 2^{n-6}$. This contradicts $k = 25 \cdot 2^{n-6}$ and excludes that $t \geq 4$. Thus, $t = 3$ and we have proved Eq. 3.33.

Next, assume that $t \geq 3$. We know from Eq. 3.33 that $t = 3$. Furthermore, we have by Eqs. 3.33, 3.34 and 3.35 that $\{a_1, b\}$, $\{a_2, b\}$, and $\{a_3, b\}$ is the list of all ubt-antichains of $\langle S; \wedge\rangle$ and they have a common join v. No two of a_1, a_2, and a_3 are incomparable, since otherwise those two would form a ubt-antichain (with upper bound v). Hence, we can assume that $a_1 < a_2 < a_3$. Thus, it follows from Lemma 3.7 that $\langle S; \wedge\rangle$ is a quasi-tree semilattice with nucleus N_6.

Finally, assume that $t \ngeq 3$. By Theorem 2.3(i)–(ii) and Lemma 3.4, $t \notin \{0, 1\}$, whence $t = 2$. There are several cases to consider.

Case 1 (we assume that $v_1 = v_2$ and $\{a_1, b_1\} \cap \{a_2, b_2\} \neq \emptyset$) By the *a*–*b* symmetry, we can choose the notation so that $a := a_1$, $b := b_1 = b_2$, and $c := a_2$. If $a \parallel c$, then $\{a, c\}$ is a third ubt-antichain (with upper bound $v_1 = v_2$), contradicting $t = 2$. Hence, we can assume that $a < c$. But then, by Lemma 3.5, $\langle S; \wedge\rangle$ is a quasi-tree semilattice with nucleus N_5, and so Theorem 2.3(iii) gives that $k = 26 \cdot 2^{n-6}$, a contradiction again since $k = 25 \cdot 2^{n-6}$ has been assumed. So Case 1 cannot occur.

Case 2 (we assume that $v_1 = v_2$ and $\{a_1, b_1\} \cap \{a_2, b_2\} = \emptyset$) Observe that for every $X \subseteq \{a_1, b_1, a_2, b_2\}$ such that $|X| = 2$,

$$\text{if } \{a_1, b_1\} \neq X \neq \{a_2, b_2\}, \text{ then } X \text{ is not an antichain,} \tag{3.37}$$

since otherwise X would be a third ubt-antichain with upper bound $v_1 = v_2$. By the 1–2 symmetry, we can assume that $a_1 < a_2$. By Eq. 3.37, a_2 and b_1 are comparable elements. If we had that $a_2 \leq b_1$, then we would obtain $a_1 \leq b_1$ by transitivity, contradicting that $\{a_1, b_1\}$ is a ubt-antichain. Hence, $b_1 < a_2$. But then the inequality $v_1 = a_1 \vee b_1 \leq a_2 < v_2 = v_1$ is a contradiction. Therefore, Case 2 cannot occur either.

Cases 1 and 2 make it clear that now, when $t = 2$, we have that $v_1 \neq v_2$. We obtain from Eqs. 3.7 and 3.15 that

$$|\{a_1, b_1, v_1, a_2, b_2, v_2\}| = 5. \tag{3.38}$$

The following two cases have to be considered.

Case 3 (we assume that $v_1 \neq v_2$ and $\{a_1, b_1, a_2, b_2\} \cap \{v_1, v_2\} = \emptyset$) This assumption and Eq. 3.38 allow us to assume that $\{a_1, b_1\} = \{a, b\}$ and $\{a_2, b_2\} = \{c, b\}$. So $v_1 = a \vee b$ and $v_2 = c \vee b$. For the sake of contradiction, suppose that a and c are comparable. Let, say, $a < c$; then $v_1 = a \vee b \leq c \vee b = v_2$. But $v_1 \neq v_2$, so $v_1 < v_2$. If we had that $c \leq v_1$, then $v_2 = b \vee c \leq v_1$ would contradict $v_1 < v_2$. If we had that $v_1 \leq c$, then this would lead to the contradiction $b \leq c$ by transitivity. Hence, $c \parallel v_1$. So $\{c, v_1\}$ is an additional ubt-antichain (with upper bound v_2), which is a contradiction showing that $a \parallel c$. If v_1 and v_2 were comparable, then the larger one of them would be an upper bound of $\{a, c\}$, and so $\{a, c\}$ would be a third ubt-antichain. Thus, $v_1 \parallel v_2$, and Lemma 3.6 gives that $\langle S; \wedge\rangle$ is a quasi-tree semilattice with nucleus F, as required.

Case 4 (we assume that $v_1 \neq v_2$ and $\{a_1, b_1, a_2, b_2\} \cap \{v_1, v_2\} \neq \emptyset$) Since a and b play symmetric roles and so do the subscripts 1 and 2, we can assume that $v_1 = a_2$. We have that $|\{a_1, b_1, a_2, b_2\}| = 4$ since $b_2 \nleq a_2 = v_1$ excludes the possibility that $b_2 \in \{a_1, b_1, a_2\}$. None of the sets $\{a_1, b_2\}$ and $\{b_1, b_2\}$ is an antichain, since otherwise the set in question would be a new ubt-antichain with upper bound v_2, which would be a contradiction. Hence, a_1 and b_2 are comparable elements, and so are b_1 and b_2. If we had that $a_1 \geq b_2$ or $b_1 \geq b_2$, then transitivity would lead to $a_2 = v_1 \geq b_2$, a contradiction. Thus, $a_1 \leq b_2$ and $b_1 \leq b_2$. But then $a_2 = v_1 = a_1 \vee b_1 \leq b_2$ is a contradiction. This shows that Case 4 cannot occur.

Now that all cases have been considered, we have shown that if $k = 25 \cdot 2^{n-6}$, then $\langle S; \wedge\rangle$ is of the required form.

Finally, if $\langle S; \wedge\rangle$ is a quasi-tree semilattice with nucleus N_6, then using the Inclusion-Exclusion Principle and Eq. 3.10, a computation similar to Eqs. 3.22 and 3.23 yields that

$$|\mathrm{Con}(S; \wedge)| = 2^{n-6}\big(32 - (4+4+4) + (2+2+2) - 1\big) = 25 \cdot 2^{n-6},$$

as required. Also, if the nucleus is F, then a computation similar to Eqs. 3.11–3.13 derives from Eq. 3.10 and the Inclusion-Exclusion Principle that

$$|\mathrm{Con}(S; \wedge)| = 2^{n-6}\big(32 - (4+4) + 1\big) = 25 \cdot 2^{n-6}.$$

This completes the proof of Theorem 2.3(iv). □

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