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Hindman's Theorem is only a Countable Phenomenon

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Abstract We pursue the idea of generalizing Hindman's Theorem to uncountable cardinalities, by analogy with the way in which Ramsey's Theorem can be generalized to weakly compact cardinals. But unlike Ramsey's Theorem, the outcome of this paper is that the natural generalizations of Hindman's Theorem proposed here tend to fail at all uncountable cardinals.

Keywords Ramsey-type theorem · Hindman's theorem · Uncountable cardinals · Δ-system lemma · Semigroups · Abelian groups

1 Introduction

Hindman's theorem is one of the most famous and interesting examples of a so-called *Ramsey-type theorem*, a theorem about partitions.

Theorem 1 (Hindman [7]) *For every partition $\mathbb{N} = A_0 \cup A_1$ of the set of natural numbers into two cells, there exists an infinite $X \subseteq \mathbb{N}$ such that for some $i \in 2$, $\mathrm{FS}(X) \subseteq A_i$ (where $\mathrm{FS}(X)$ denotes the set*

$$\left\{ \sum_{x \in F} x \,\middle|\, F \subseteq X \text{ is finite and nonempty} \right\}$$

of all finite sums of elements of X).

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Hindman's original proof of Theorem 1 is long and involved, but a much simpler proof, due to Baumgartner, can be found in [1]. Both proofs make extensive use of the fact that the statement of Theorem 1 is equivalent to the following statement.

Theorem 2 *For every partition* $[\mathbb{N}]^{<\aleph_0} = A_0 \cup A_1$ *of the collection* $[\mathbb{N}]^{<\aleph_0}$ *of finite subsets of* $\mathbb{N}$ *into two cells, there exists an infinite family* $X \subseteq [\mathbb{N}]^{<\aleph_0}$ *of pairwise disjoint finite subsets of* $\mathbb{N}$ *and an* $i \in 2$ *such that* $\mathrm{FU}(X) \subseteq A_i$*, where* $\mathrm{FU}(X)$ *denotes the set*

$$\left\{ \bigcup_{Y \in F} Y \middle| F \subseteq X \text{ is finite and nonempty} \right\}$$

of finite unions of elements from X.

The equivalence between Theorems 1 and 2 follows from [7, Lemma 2.2] after identifying natural numbers with the support of their binary expansion (the (finite) set of places where the corresponding digit in the binary expansion of the given number is nonzero), and this fact is explicitly pointed out in [1, p. 384]. Unlike Theorem 1, whose statement relies on a very specific semigroup operation on the set $\mathbb{N}$, the statement of Theorem 2 seems easily adaptable to higher cardinalities. So if κ, λ are cardinal numbers with $\lambda \leq \kappa$, we will denote by $\mathsf{HIND}(\kappa, \lambda)$ the statement asserting that for every partition $[\kappa]^{<\aleph_0} = A_0 \cup A_1$ of the family $[\kappa]^{<\aleph_0}$ of finite subsets of κ into two cells, there exists a family $X \subseteq [\kappa]^{<\aleph_0}$ of cardinality λ, consisting of pairwise disjoint (finite) subsets of κ, such that for some $i \in 2$ we have that $\mathrm{FU}(X) \subseteq A_i$. With this notation, the "finite-unions" version of Hindman's theorem simply states that $\mathsf{HIND}(\aleph_0, \aleph_0)$ (and hence also $\mathsf{HIND}(\kappa, \aleph_0)$ for any infinite κ) holds. This notation intends to provide an analogy with the arrow notation used for generalizations of Ramsey's theorem, where $\kappa \longrightarrow (\lambda)^2_2$ denotes that for every partition $[\kappa]^2 = A_0 \cup A_1$ of the family of unordered pairs of κ into two cells, there exists a subset $X \subseteq \kappa$ of cardinality λ such that for some $i \in 2$ we have that $[X]^2 \subseteq A_i$. Thus Ramsey's theorem states that $\aleph_0 \longrightarrow (\aleph_0)^2_2$, and for uncountable λ, the existence of a κ such that $\kappa \longrightarrow (\lambda)^2_2$ follows from the Erdős-Rado Theorem. However, the existence of an uncountable cardinal κ such that $\kappa \longrightarrow (\kappa)^2_2$ is not provable in ZFC (the usual axioms of mathematics), since such a κ would be what is known as a weakly compact cardinal (which is a notion of large cardinal and hence its existence goes beyond the ZFC axioms). Similarly, we might try to address the question of whether we can have $\mathsf{HIND}(\kappa, \lambda)$ hold for some uncountable cardinals κ, λ under certain conditions (with perhaps some large cardinal assumptions being the primary culprits).

Although this question seems (at least to the author) to be a very natural one, there seems to be very few earlier results along these lines. A theorem [11, Th. 9] of Milliken is easily seen to establish that $\mathsf{HIND}(\kappa^+, \kappa^+)$ fails whenever κ is an infinite cardinal such that $2^\kappa = \kappa^+$ (in particular, the Continuum Hypothesis CH implies that $\mathsf{HIND}(\aleph_1, \aleph_1)$ fails). And the author was recently made aware [6] of an argument of Moore that shows that for every uncountable cardinal κ, the statement $\mathsf{HIND}(\kappa, \kappa)$ implies that $\kappa \longrightarrow (\kappa)^2_2$ and therefore $\mathsf{HIND}(\kappa, \kappa)$ fails unless κ is a weakly compact cardinal (in particular, $\mathsf{HIND}(\aleph_1, \aleph_1)$ fails even without assuming CH). The main result from this paper is that the statement $\mathsf{HIND}(\kappa, \lambda)$ fails whenever $\aleph_0 < \lambda \leq \kappa$, thus banishing every possible attempt of generalizing Hindman's theorem, at least in its finite-unions version, to uncountable cardinalities. We also state other possible generalizations of Hindman's theorem to the uncountable realm, and show that they all fail as well.

It is worth noting that the generalizations considered in this paper are of a quantitative nature, meaning that they deal only with cardinality, hence the results obtained do not preclude the possibility of obtaining other sorts of generalizations of Hindman's Theorem. For example, Tsaban [13] has shown that Hindman's Theorem may be viewed as a colouring theorem dealing with open covers of a certain countable topological space, and he proved a generalization of this theorem to arbitrary Menger spaces (which can have any arbitrary cardinality, although the objects coloured in this result are countable families of open sets). As another example, Zheng [14] has built on Todorčević's theory of Ramsey Spaces [12] to obtain results where finite subsets of ω together with real numbers are coloured, and monochromatic combinations of FU-sets and perfect subsets of $\mathbb{R}$ are obtained. Thus, it is possible to improve Hindman's theorem in terms of the richness of structure of the obtained monochromatic structure, but not in terms of its size.

The second section of this paper contains the proof that $\mathsf{HIND}(\kappa, \lambda)$ fails for every uncountable $\lambda \leq \kappa$. In the third and fourth sections, we consider other possible generalizations of Hindman's Theorem to groups and semigroups, and we explain that most of these fail as well.[1]

2 The Uncountable Hindman Statement Fails

The following argument (which, quite surprisingly, is reasonably simple) establishes the main result of this paper.

Theorem 3 *Let κ be any infinite cardinal. Then there is a partition of $[\kappa]^{<\omega}$ into two cells, none of which can contain* $\mathrm{FU}(X)$ *for any uncountable pairwise disjoint family $X \subseteq [\kappa]^{<\omega}$. Thus, for any uncountable λ and any $\kappa \geq \lambda$, the statement* $\mathsf{HIND}(\kappa, \lambda)$ *fails.*

Proof Partition $[\kappa]^{<\omega}$ as $A_1 \cup A_0$, where

$$A_i = \{x \in [\kappa]^{<\omega} \big| \lfloor \log_2 |x| \rfloor \equiv i \mod 2\},$$

this is, $x \in A_i$ if and only if the unique $k \in \omega$ such that $2^k \leq |x| < 2^{k+1}$ has the same parity as i. Note that, if x and y are disjoint, both in the same A_i, and $\lfloor \log_2 |x| \rfloor = \lfloor \log_2 |y| \rfloor = k$, then (since $|x \cup y| = |x| + |y|$) we have that $\lfloor \log_2 |x \cup y| \rfloor = k + 1$ and hence $x \cup y \in A_{1-i}$. Thus if $X \subseteq [\kappa]^{<\omega}$ is a pairwise disjoint family such that $\mathrm{FU}(X) \subseteq A_i$, we must have that $\lfloor \log_2 |\cdot| \rfloor : X \longrightarrow \omega$ is an injective function, and hence X must be countable. □

An easy consequence of Theorem 3 is that many other Ramsey-theoretic results will not have an uncountable analog either. Let us consider two of these. For instance, Gower's c_0 theorem and Carlson's theorem on sequences of variable words[2] are extensions of results that have Hindman's theorem as a straightforward consequence. Thus, any reasonable statement constituting an analog of these results to the uncountable realm should have $\mathsf{HIND}(\kappa, \lambda)$ as an easy consequence, for some uncountable λ and a $\kappa \geq \lambda$; and will therefore fail because of Theorem 3.

[1] Further results along these lines are obtained in a recent follow-up joint paper of the author and Rinot [4].

[2] This result of Carlson [2], also discovered independently by Furstenberg and Katznelson [5], is sometimes (e.g. in [12, Theorem 2.35]) referred to as *the infinite Hales-Jewett theorem.*

As a final observation, I would like to anticipate that in a forthcoming joint paper between Chodounský, Krautzberger and the author, the partition from the proof of Theorem 3 is used (with $\kappa = \omega$) to show that the core of a union ultrafilter is not a rapid filter (where an ultrafilter u on $[\mathbb{N}]^{<\aleph_0}$ is said to be a union ultrafilter if it has a base of sets of the form $\mathrm{FU}(X)$ for an infinite pairwise disjoint $X \subseteq [\mathbb{N}]^{<\aleph_0}$, and the core of u is $\{\bigcup A \big| A \in u\}$, which is a filter on $\mathbb{N}$).

3 Generalizations in terms of Abelian Groups

If one proves Hindman's Theorem 1 via the argument of Galvin and Glazer (see e.g. [8, Th. 5.8]), one can see right away that the scope of this theorem goes way beyond the realm of the natural numbers $\mathbb{N}$. This is, that same argument yields the following much more general statement, which is therefore commonly known as the Galvin-Glazer-Hindman Theorem.

Theorem 4 *Let S be any semigroup, and suppose that we partition $S = A_0 \cup A_1$ into two cells. Then there exists a sequence $\mathbf{x} = \langle x_n \big| n < \omega\rangle$ (which in most cases of interest can be found to be injective) and an $i \in 2$ such that $\mathrm{FP}(\mathbf{x}) \subseteq A_i$, where $\mathrm{FP}(\mathbf{x})$ denotes the set*

$$\left\{x_{k_0} * x_{k_1} * \cdots * x_{k_l} \big| l < \omega \text{ and } k_0 < k_1 < \cdots < k_l < \omega\right\}$$

of all finite products of elements from the sequence $\mathbf{x}$.

Thus, given a semigroup S and an ordinal α, we introduce the symbol $\mathsf{HIND}(S, \alpha)$ to denote the statement that whenever we partition $S = A_0 \cup A_1$ into two cells, there exists an α-sequence $\mathbf{x} = \langle x_\xi \big| \xi < \alpha\rangle$ and an $i \in 2$ such that $\mathrm{FP}(\mathbf{x}) \subseteq A_i$ (where $\mathrm{FP}(\mathbf{x})$ consists of all finite products $x_{\xi_0} * x_{\xi_1} * \cdots * x_{\xi_l}$ such that $l < \omega$ and $\xi_0 < \xi_1 < \cdots < \xi_l < \alpha$). If our semigroup S is commutative, chances are that we will be using additive notation and so we will use the symbol $\mathrm{FS}(\mathbf{x})$ rather than $\mathrm{FP}(\mathbf{x})$ (finite sums instead of finite products). Also, in this case the order in which the sums are taken is not important and so we will only consider the statements $\mathsf{HIND}(S, \lambda)$ where λ is a cardinal (since in this case $\mathsf{HIND}(S, \alpha)$ holds if and only if $\mathsf{HIND}(S, |\alpha|)$ holds). The main result of this section is that for every abelian group (in fact, for every commutative cancellative semigroup) G, and for every uncountable cardinal λ, the statement $\mathsf{HIND}(G, \lambda)$ fails.[3]

Theorem 5 *Let G be a commutative cancellative semigroup. Then there exists a partition $G = A_0 \cup A_1$ of G into two cells such that for no uncountable $X \subseteq G$ and no $i \in 2$ do we have that $\mathrm{FS}(X) \subseteq A_i$. This is, the statement $\mathsf{HIND}(G, \aleph_1)$ (and hence also $\mathsf{HIND}(G, \lambda)$ for every uncountable λ) fails.*

Proof Since G is commutative and cancellative, it is possible to embed G into $\bigoplus_{\alpha<\kappa} \mathbb{T}$ for some cardinal κ, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is the unit circle group. Moreover, it is possible to do this embedding in such a way that the α-th projection $\pi_\alpha[G]$ is either (isomorphic to) $\mathbb{Q}$ or a quasicyclic group, so in either case the projection is a countable set (this is all explained with detail in [3, p. 123]). Thus throughout this proof, every element $x \in G$ will be thought of as a member of $\bigoplus_{\alpha<\kappa} \mathbb{T}$, with α-th projections denoted by $\pi_\alpha(x) \in \mathbb{T} = \mathbb{R}/\mathbb{Q}$ and finite support $\mathrm{supp}(x) = \{\alpha < \kappa \big| \pi_\alpha(x) \neq 0\}$.

[3] In the notation of our recent follow-up paper [4], the failure of this statement is denoted by $G \not\to [\lambda]^{\mathrm{FS}}_2$.

In a way totally similar to what we did in the proof of Theorem 3, for $i \in 2$ we define

$$A_i = \left\{x \in G \middle| \lfloor \log_2 |\operatorname{supp}(x)| \rfloor \equiv i \mod 2\right\}$$

and claim that $G = A_0 \cup A_1$ is the partition that makes the theorem work. So by way of contradiction, we start by assuming that $X \subseteq G$ is an uncountable subset such that $\mathrm{FS}(X) \subseteq A_i$ for some $i \in 2$. We first notice that for any given finite $F \subseteq \kappa$, there can only be countably many elements $x \in G$ such that $\operatorname{supp}(x) = F$ (because we assumed that each $\pi_\alpha[G]$ is countable) and so by thinning out X we can assume that the supports of elements of X are pairwise distinct. Furthermore, by the Δ-system lemma (see e.g. [10, Th. 1.5] or [9, Th. 16.1]), there exists an uncountable $Y \subseteq X$ such that the supports of elements from Y form a Δ-system, this is, there is a fixed finite $R \subseteq \kappa$ (called the **root** of the Δ-system) such that for every two distinct $x, y \in Y$, we have that $\operatorname{supp}(x) \cap \operatorname{supp}(y) = R$.

Ideally, we would like to have that $\operatorname{supp}\left(\sum_{k<l} x_k\right) = \bigcup_{k<l} \operatorname{supp}(x_k)$ whenever $x_0, x_1, \dots, x_l \in Y$, but in order to ensure that we still need to process Y a bit more.

Claim 1 There exists an uncountable Z with $\mathrm{FS}(Z) \subseteq A_i$ such that the supports of its elements form a Δ-system and moreover, whenever $x_0, x_1, \dots, x_l \in Z$ we have that $\operatorname{supp}\left(\sum_{k<l} x_k\right) = \bigcup_{k<l} \operatorname{supp}(x_k)$.

Proof of Claim Let $n = |R|$ with $R = \{\alpha_1, \dots, \alpha_n\}$. We will recursively construct a sequence of uncountable sets $Y_0, Y_1, \dots, Y_{n+1}$ such that $Y = Y_0$ and each Y_{k+1} is a *sum subsystem* of Y_k (this is, for each $x \in Y_{k+1}$ there is a finite $F_x \subseteq Y_k$ such that $x = \sum_{y \in F_x} y$ and moreover whenever $x, y \in Y_{k+1}$ we have that $F_x \cap F_y = \varnothing$), and satisfying that either $\alpha_k \in \operatorname{supp}(x)$ for all $x \in \mathrm{FS}(Y_k)$, or $\alpha_k \notin \operatorname{supp}(x)$ for all $x \in \mathrm{FS}(Y_k)$. In the end we will let $Z = Y_{n+1}$ and $r = \{\alpha_k | \alpha_k \in \operatorname{supp}(x) \text{ for all } x \in \mathrm{FS}(Y)\} \subseteq R$. This will ensure that the supports of elements from Z form a Δ-system with root r, and moreover we will have that Z is a sum subsystem of Y, hence $\mathrm{FS}(Z) \subseteq \mathrm{FS}(Y) \subseteq A_i$. Finally, the fact that $r \subseteq \operatorname{supp}(x)$ for every $x \in \mathrm{FS}(Z)$ will imply that $\operatorname{supp}\left(\sum_{k<l} x_k\right) = \bigcup_{k<l} \operatorname{supp}(x_k)$ whenever $x_0, x_1, \dots, x_l \in Z$.

Now for the construction, suppose that we have already constructed Y_k satisfying the imposed requirements. To simplify notation let $\alpha = \alpha_{k+1}$. Since we assumed that $\pi_\alpha[G]$ is countable, by the pigeonhole principle there is an uncountable $Y' \subseteq Y_k$ such that all of the $\pi_\alpha(y)$, for $y \in Y'$, equal some fixed $t \in \mathbb{T}$. If this t is of infinite order, then we simply make $Y_{k+1} = Y'$ and notice that $\alpha \in \operatorname{supp}(x)$ for every $x \in \mathrm{FS}(Y_{k+1})$. If, on the other hand, t is of finite order (say, of order n) then we partition $Y' = \bigcup_{\xi<|Y'|} F_\xi$ into uncountably many cells F_ξ of cardinality n, and let Y_{k+1} consist of the elements $\sum_{x \in F_\xi} x$ for $\xi < |Y'|$. Then we will have that $\alpha \notin \operatorname{supp}(y)$ for every $y \in Y_{k+1}$ and subsequently, $\alpha \notin \operatorname{supp}(x)$ for every $x \in \mathrm{FS}(Y_{k+1})$. This finishes the construction. □

We now use the Z given by the claim in order to reach a contradiction. We will argue that the function $|\operatorname{supp}(\cdot)| \restriction Z : Z \longrightarrow \omega$ is finite-to-one, which will imply that Z must be countable, contrary to its construction. So assume that there is an infinite family $\{z_k | k < \omega\} \subseteq Z$ such that all of the $|\operatorname{supp}(z_k)|$ are equal to some l, and let $m < \omega$ be such that $2^m \leq l < 2^{m+1}$ (then by assumption, $i \equiv m \mod 2$). Also, let n be such that $2^n \leq l - |r| < 2^{n+1}$ (note that $n \leq m$). We then let $z = \sum_{k \leq 2^{m-n}} z_k \in A_i$ and notice that, since by the claim we have that $\operatorname{supp}(z) = \bigcup_{k \leq 2^{m-n}} \operatorname{supp}(z_k)$ and the $\operatorname{supp}(z_k)$ form a Δ-system with root r, we can conclude that

$$\begin{aligned} |\operatorname{supp}(z)| &= \left(\sum_{k \leq 2^{m-n}} |\operatorname{supp}(x_k)| \right) - 2^{m-n}|r| = (2^{m-n}+1)l - 2^{m-n}|r| \\ &= 2^{m-n}(l - |r|) + l, \end{aligned}$$

and since $2^m \leq 2^{m-n}(l - |r|) < 2^{m+1}$, we conclude that $2^{m+1} \leq |\operatorname{supp}(x)| < 2^{m+2}$, meaning that $x \in A_{1-i}$, contrary to the assumption. □

It might be argued that the statement $\mathsf{HIND}(\kappa, \lambda)$ as in the previous section is the "wrong" way of generalizing the finite-union version of Hindman's theorem, and that one should consider partitions of $[\kappa]^{<\kappa}$ instead of $[\kappa]^{<\omega}$. However, if we equip $[\kappa]^{<\kappa}$ with the symmetric difference $\triangle$ as a group operation, we obtain an abelian group (in fact, a Boolean group) with the peculiarity that taking a finite union of pairwise disjoint elements of $[\kappa]^{<\kappa}$ coincides with taking its finite sum according to this group operation. Hence Theorem 5 implies that this purported finite-union generalization of Hindman's theorem also fails at all uncountable cardinals.

4 The Noncommutative Case

The next natural question is whether the previous results can be generalized to noncommutative semigroups. This is, is it true that $\mathsf{HIND}(S, \lambda)$ fails whenever λ is uncountable and S is any semigroup? As test cases for this question, the first two that come to mind are the free semigroup and the free group. If we let S_κ be the free semigroup on κ generators, and consider the partition $S_\kappa = A_0 \cup A_1$ with

$$A_i = \{x \in S_\kappa \big| \lfloor \log_2 \ell(x) \rfloor \equiv i \mod 2\}$$

(where $\ell(x)$ denotes the length of x), it is easy to see (arguing as in the proof of Theorem 3) that $\mathsf{HIND}(S_\kappa, \lambda)$ fails for every uncountable λ. In the case of the free group, a slightly more complicated argument is needed.

Theorem 6 *Let κ be a cardinal and let F_κ be the free group on κ generators. Then for every uncountable ordinal λ, the statement* $\mathsf{HIND}(F_\kappa, \lambda)$ *fails.*

Proof Following the general theme of this paper, we will consider the partition $F_\kappa = A_0 \cup A_1$, where (letting $\ell(x)$ denote the length of a reduced word $x \in F_\kappa$)

$$A_i = \{x \in F_\kappa \big| \lfloor \log_2 \ell(x) \rfloor \equiv i \mod 2\},$$

and we will argue that no sequence $\mathbf{x} = \langle x_\alpha \big| \alpha < \omega_1 \rangle$ of elements of a given A_i can be such that $\mathrm{FP}(\mathbf{x}) \subseteq A_i$, so we assume by way of contradiction that we have such a sequence $\mathbf{x}$ with $\mathrm{FP}(\mathbf{x}) \subseteq A_i$. Let L (with $|L| = \kappa$) be the alphabet that generates F_κ, and for a reduced word $x \in L_\kappa$ we define its support by

$$\operatorname{supp}(x) = \{a \in L \big| \text{either } a \text{ or } a^{-1} \text{ occur in } x\}$$

Since there are only countably many reduced words having the same fixed support, it is possible to thin out the sequence $\mathbf{x}$ so that the supports of its members are pairwise distinct, and by the Δ-system lemma we can also assume that said supports form a Δ-system, whose root we will denote by r. Now, any member x_α of the sequence $\mathbf{x}$ can be written as $x_\alpha = z_\alpha y_\alpha w_\alpha$ with $\text{supp}(z_\alpha), \text{supp}(w_\alpha) \subseteq r$ and such that the first and last letters of y_α do not belong to r (it is possible that z_α or w_α are empty, but y_α has to be nonempty). A couple of applications of the pigeonhole principle will thin out the sequence $\mathbf{x}$ in such a way that all of the z_α equal some fixed z and all of the w_α equal some fixed w. Let v be the reduced word that results from multiplying $w \cdot z$ (this is, after performing all of the needed cancellations). Thus for $\alpha < \beta < \omega_1$ we have that $x_\alpha \cdot x_\beta = z y_\alpha v y_\beta w$, where the expression on the right has no cancellations, and so $\ell(x_\alpha \cdot x_\beta) = \ell(x_\alpha) + \ell(x_\beta) - n$ where $n = \ell(w) + \ell(z) - \ell(v)$, and similarly

$$l\left(x_{\alpha_0} \cdots x_{\alpha_t}\right) = \left(\sum_{j=0}^{t} \ell(x_{\alpha_j})\right) - tn.$$

Now, the pigeonhole principle implies that there is an ω_1 sequence $\alpha_0 < \alpha_1 < \cdots < \alpha_\xi < \cdots$, for $\xi < \omega_1$, such that all of the lengths $\ell(x_{\alpha_j})$ equal some fixed number l. However, if we let $k = \lfloor \log_2(l) \rfloor$ and $m = \lfloor \log_2(l-n) \rfloor$ (notice that $l > \ell(w) + \ell(z) \geq n$ so it makes sense to take the latter logarithm), then we would have that (letting $x = \prod_{j=0}^{2^{k-m}} x_j = x_0 \cdots x_{2^{k-m}}$)

$$\ell(x) = \left(\sum_{j=0}^{2^{k-m}} \ell(x_{\alpha_j})\right) - 2^{k-m} n = (2^{k-m}+1)l - 2^{k-m} n = 2^{k-m}(l-n) + l,$$

thus

$$2^{k+1} = 2^{k-m} 2^m + 2^k \leq \ell(x) < 2^{k-m} 2^{m+1} + 2^{k+1} = 2^{k+2};$$

so that $k+1 = \lfloor \log_2 \ell(x) \rfloor$ and hence $x \in A_{1-i}$, contrary to the assumption that $\text{FP}(\mathbf{x}) \subseteq A_i$. Finding this contradiction finishes the proof. □

It is, however, possible to find noncommutative semigroups that behave differently to the ones considered so far.

Example 1 Let S be a linearly ordered set and turn it into a semigroup by making $x * y = \max\{x, y\}$ (everything we say for this example also holds if we consider $x * y = \min\{x, y\}$). This is a commutative semigroup with the property that for every $X \subseteq S$, $\text{FS}(X) = X$. Thus the statement $\mathsf{HIND}(S, |S|)$ holds (as an easy instance of the pigeonhole principle), regardless of whether $|S|$ is countable or uncountable.

Example 2 For an ordinal α we let S_α be the semigroup of ordinals smaller than α with ordinal addition as the semigroup operation (this semigroup is not commutative). The key observation that for every infinite ordinal α there exists a $\beta \geq \alpha$ such that $|\beta| = |\alpha|$ and $\gamma + \delta = \delta$ whenever $\delta \geq \beta$ and $\gamma \leq \alpha$ (which follows by just taking $\beta = \alpha \cdot \omega$ where $\cdot$ is ordinal multiplication) allows us to conclude that for every infinite ordinal α, the statement $\mathsf{HIND}(S_\alpha, \text{cf}(|\alpha|))$ holds. For if we have a partition $S_\alpha = A_0 \cup A_1$ into two cells, by the pigeonhole principle we must have that $|A_i| = |\alpha|$ for some $i \in 2$, and consequently we can recursively build a sequence $\gamma = \langle \gamma_\xi \big| \xi < \text{cf}(|\alpha|) \rangle$ by picking a $\gamma_\xi > \left(\sup\{\gamma_\eta \big| \eta < \xi\}\right) \cdot \omega$ with $\gamma_\xi \in A_i$ and $\gamma_\xi < |\alpha|$. Thus the observation at the beginning of this example implies that $\gamma_{\xi_1} + \cdots + \gamma_{\xi_n} = \gamma_{\xi_n}$ whenever $\xi_1 < \cdots < \xi_n < \text{cf}(|\alpha|)$ and so $\text{FP}(\gamma) = \{\gamma_\xi \big| \xi < \text{cf}(|\alpha|)\} \subseteq A_i$. (If one is more careful, it is possible to construct this sequence with length $|\alpha|$, but that is not so relevant since we only wanted to show that $\mathsf{HIND}(S_\alpha, \kappa)$ holds for some α and uncountable κ.)

5 Conclusions

The main result that we have proved in this paper, is that the uncountable analog of Hindman's theorem in the realm of commutative cancellative semigroups fails, in the sense that any such semigroup S can be partitioned in two cells, in such a way that for no uncountable $X \subseteq S$ is it possible for the set $\mathrm{FS}(X)$ to be contained within one single cell of the partition. An analogous result holds for the symmetric group as well (with the standard required changes in the definition of $\mathrm{FS}(X)$ to account for the non-commutativity of this group). As a consequence of this, when considering uncountable analogs of the Ramsey-theoretic results that have Hindman's theorem as a particular case (such as Gowers's theorem, or the infinitary version of the Hales-Jewett theorem), we have that these analogs fail as well.

When we drop cancellativity, we are able to obtain two examples of (non-commutative) semigroups for which the uncountable analog of Hindman's theorem holds. Something that both of these examples have in common is that the semigroups S under consideration contain elements $x \in S$ that can "absorb" many $y \in S$ in the sense that $y * x = x$. Thus, it is conceivable to conjecture that $\mathsf{HIND}(S, \lambda)$ fails for uncountable λ provided that we are dealing with an S that does not involve the aforementioned phenomenon, which naturally leads to the following question.

Question 1 Does there exist a weakly right cancellative (or a cancellative) semigroup S and an uncountable ordinal α such that $\mathsf{HIND}(S, \alpha)$ holds? We can also restrict our attention to groups: Does there exist a (non-abelian) group G and an uncountable α such that $\mathsf{HIND}(G, \alpha)$ holds?

A partial (negative) answer to the previous question (in the context of groups) was provided by Milliken, who showed [11, Th. 9] that $\mathsf{HIND}(G, |G|)$ fails whenever G is a group satisfying that $|G| = \kappa^+ = 2^\kappa$ for some infinite cardinal κ (in particular, assuming CH we get that $\mathsf{HIND}(\mathbb{R}, \aleph_\Delta)$ fails, which is [11, Cor. 11]). Theorem 6, and also the remark in the paragraph prior to that theorem, constitute partial negative answers to this question as well.

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