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Semi-Nelson Algebras

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Abstract Generalizing the well known and exploited relation between Heyting and Nelson algebras to semi-Heyting algebras, we introduce the variety of semi-Nelson algebras. The main tool for its study is the construction given by Vakarelov. Using it, we characterize the lattice of congruences of a semi-Nelson algebra through some of its deductive systems, use this to find the subdirectly irreducible algebras, prove that the variety is arithmetical, has equationally definable principal congruences, has the congruence extension property and describe the semisimple subvarieties.

Keywords Semi-Heyting algebras · Semi-Nelson algebras · Twist structures · Heyting algebras · Nelson algebras

1 Introduction

In this article we study the convergence of ideas coming from different varieties of algebras related to intuitionistic logics: Heyting, semi-Heyting and Nelson algebras.

Semi-Heyting algebras were introduced in [20] and their relationship with the varieties of Heyting algebras, and their expansions have been studied lately using both the algebraic [1–4, 21] and logical approaches [7–9].

Nelson algebras were defined by H. Rasiowa, [17] who also called them N-lattices and quasi-pseudo boolean algebras. They are the algebraic semantics of the intuitionistic propositional calculus with strong negation introduced by D. Nelson [15]. There is a

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close relationship between Nelson algebras and Heyting algebras, as it was investigated by D. Vakarelov [25] and A. Sendlewski [23], among others. This relationship is part of what is now known as twist structures [14, 16, 19] and date back to [12].

In this work we aim to extend the twist construction to semi-Heyting algebras, thus obtaining a new variety, which we naturally call semi-Nelson algebras. We begin by recalling the definitions of Heyting and Nelson algebras and sketching the main constructions linking them, and then present semi-Heyting algebras and introduce semi-Nelson algebras, showing that some of the features of the original constructions are preserved. We then use this to study the new variety of semi-Nelson algebras.

In the last two sections we characterize the lattice of congruences of a semi-Nelson algebra using some of its deductive systems, use this to find the subdirectly irreducible algebras, prove that the variety is arithmetical, has equationally definable principal congruences, has the congruence extension property and describe the semisimple subvarieties.

Definition 1.1 *Heyting algebras are algebras* $\mathbf{A} = \langle A; \wedge, \vee, \Rightarrow, 0, 1\rangle$ *that satisfy the conditions:*

(H1) $\langle A, \vee, \wedge, 0, 1\rangle$ *is a bounded lattice.*
(H2) $x \wedge (x \Rightarrow_H y) \approx x \wedge y$
(H3) $x \wedge (y \Rightarrow_H z) \approx x \wedge [(x \wedge y) \Rightarrow (x \wedge z)]$
(H4) $(x \wedge y) \Rightarrow_H x \approx 1.$

We denote with $\mathcal{H}$ *the variety of Heyting algebras.*

Definition 1.2 *Nelson algebras are algebras* $\mathbf{A} = \langle A; \wedge, \vee, \to_N, \sim, 1\rangle$ *that satisfy the conditions:*

(N1) $x \wedge (x \vee y) \approx x,$
(N2) $x \wedge (y \vee z) \approx (z \wedge x) \vee (y \wedge x),$
(N3) $\sim\sim x \approx x,$
(N4) $\sim (x \wedge y) \approx \sim x \vee \sim y,$
(N5) $x \wedge \sim x \approx (x \wedge \sim x) \wedge (y \vee \sim y),$
(N6) $x \to_N x \approx 1,$
(N7) $x \to_N (y \to_N z) \approx (x \wedge y) \to_N z,$
(N8) $x \wedge (x \to_N y) \approx x \wedge (\sim x \vee y).$

We denote with $\mathcal{N}$ *the variety of Nelson algebras.*

(N1) and (N2) specify that Nelson algebras are distributive lattices, while (N3) to (N5) say that they are Kleene algebras as well.

There are two key constructions that relate Heyting and Nelson algebras.

Given a Heyting algebra $\mathbf{A}$, one can define the set $V_k(A) = \{(a, b) \in A^2 : a \wedge b = 0\}$ and then endow it with the following operations:

V1) $(a, b) \sqcap (c, d) = (a \wedge c, b \vee d),$
V2) $(a, b) \sqcup (c, d) = (a \vee c, b \wedge d),$
V3) $(a, b) \to_N (c, d) = (a \Rightarrow_H c, a \wedge d),$
V4) $\sim (a, b) = (b, a),$
V5) $\top = (1, 0).$

Then $\mathbf{V}_k(\mathbf{A}) = \langle V_k(A); \sqcap, \sqcup, \to_N, \sim, \top\rangle$ is a Nelson algebra [25]. In the same article, Vakarelov proves that if A is a Nelson algebra, the relation $\equiv$ defined by $x \equiv y$ iff $x \to_N$

$y = 1$ and $y \to_N x = 1$ is an equivalence relation such that $\langle A/_{\equiv}; \cap, \cup, \Rightarrow, 0, 1\rangle$ is a Heyting algebra with the operations defined by:

- $[\![x]\!] \cap [\![y]\!] = [\![x \wedge y]\!]$,
- $[\![x]\!] \cup [\![y]\!] = [\![x \vee y]\!]$,
- $[\![x]\!] \Rightarrow_H [\![y]\!] = [\![x \to_N y]\!]$,
- $0 = [\![\sim 1]\!]$,
- $1 = [\![1]\!]$.

It is a natural question whether these constructions can be extended to semi-Heyting algebras.

Definition 1.3 [20] *Semi-Heyting algebras are algebras* $\mathbf{A} = \langle A; \wedge, \vee, \Rightarrow, 0, 1\rangle$ *that satisfy the conditions:*

(SH1) $\langle A, \vee, \wedge, 0, 1\rangle$ *is a bounded lattice,*
(SH2) $x \wedge (x \Rightarrow y) \approx x \wedge y$,
(SH3) $x \wedge (y \Rightarrow z) \approx x \wedge [(x \wedge y) \Rightarrow (x \wedge z)]$,
(SH4) $x \Rightarrow x \approx 1$.

In this work we attempt a definition of what the variety of semi-Nelson algebras should be and give the first steps in exploiting their relation to semi-Heyting algebras.

We denote by $\mathcal{SH}$ the variety of semi-Heyting algebras. On a semi-Heyting algebra $\mathbf{A}$ one can always define the term $x \Rightarrow_H y := x \Rightarrow (x \wedge y)$. With this operation, $\langle A; \wedge, \vee, \Rightarrow_H, 0, 1\rangle$ is a Heyting algebra [3]. As a consequence, we have the following results

Lemma 1.4 [7] *Let* $\mathbf{A} = \langle A; \wedge, \vee, \Rightarrow, 0, 1\rangle$ *be a semi-Heyting algebra. For every* $a, b, c \in A$ *we have:*

(a) $a \Rightarrow (a \wedge (b \Rightarrow (b \wedge c))) = (a \wedge b) \Rightarrow (a \wedge b \wedge c)$.
We can also write this as $a \Rightarrow_H (b \Rightarrow_H c) = (a \wedge b) \Rightarrow_H c$.
(b) $(a \Rightarrow_H b) \Rightarrow_H ((b \Rightarrow_H a) \Rightarrow_H ((a \Rightarrow c) \Rightarrow_H (b \Rightarrow c))) = 1$,
(c) $(a \Rightarrow_H b) \Rightarrow_H (((b \Rightarrow_H a) \Rightarrow_H ((c \Rightarrow a) \Rightarrow_H (c \Rightarrow b)))) = 1$.

2 Semi-Nelson Algebras

We proceed now to define the varieties $\mathcal{PSN}$ and $\mathcal{SN}$ of pre-semi-Nelson and semi-Nelson algebras, respectively. Later on, we present the results that justify this nomenclature and prove that the variety of Nelson algebras $\mathcal{N}$ is a proper subvariety of $\mathcal{SN}$.

Definition 2.1 *An algebra* $\mathbf{A} = \langle A; 1, \sim, \wedge, \vee, \to\rangle$ *of type* $(0, 1, 2, 2, 2)$ *is a pre-semi-Nelson algebra if the following conditions are satisfied:*

(SN1) $x \wedge (x \vee y) \approx x$,
(SN2) $x \wedge (y \vee z) \approx (z \wedge x) \vee (y \wedge x)$,
(SN3) $\sim\sim x \approx x$,
(SN4) $\sim (x \wedge y) \approx \sim x \vee \sim y$,
(SN5) $x \wedge \sim x \approx (x \wedge \sim x) \wedge (y \vee \sim y)$,
(SN6) $x \wedge (x \to_N y) \approx x \wedge (\sim x \vee y)$,
(SN7) $x \to_N (y \to_N z) \approx (x \wedge y) \to_N z$,
(SN8) $(x \to_N y) \to_N [(y \to_N x) \to_N [(x \to z) \to_N (y \to z)]] \approx 1$,

(SN9) $(x \to_N y) \to_N [(y \to_N x) \to_N [(z \to x) \to_N (z \to y)]] \approx 1$,

where $x \to_N y$ *stands for the term* $x \to (x \wedge y)$.

Definition 2.2 *A pre-semi-Nelson algebra* $\mathbf{A} = \langle A; 1, \sim, \wedge, \vee, \to_N \rangle$ *is a semi-Nelson algebra, if it also verifies the conditions:*

(SN10) $(\sim (x \to y)) \to_N (x \wedge \sim y) \approx 1$,
(SN11) $(x \wedge \sim y) \to_N (\sim (x \to y)) \approx 1$.

Example 2.3 Consider $\mathbf{B} = \langle \{a, 1\}; \to, \wedge, \vee, \sim, 1 \rangle$ with operations given by the tables below:

$\sim$	a	1
	1	a

$\to$	a	1
a	1	a
1	a	1

$\wedge$	a	1
a	a	a
1	a	1

$\vee$	a	1
a	a	1
1	1	1

$\mathbf{B}$ is in the class $\mathcal{PSN}$ but it is not a semi-Nelson algebra because the identity (SN10), does not hold: $\sim (a \to 1) \to_N (a \wedge \sim 1) = 1 \to_N a = 1 \to a = a \neq 1$.

Axioms (SN1) and (SN2) are those given by Sholander in [24] which define distributive lattices, so in what follows we will use freely the arithmetic rules of distributive lattices.

Lemma 2.4 *Let* $\mathbf{A} = \langle A; 1, \sim, \wedge, \vee, \to \rangle \in \mathcal{PSN}$. *Then the following properties hold in* $\mathbf{A}$*:*

(a) $x \wedge (x \to_N x) \approx x$,
(b) $x \to_N 1 \approx x \to_N x$,
(c) $x \to_N 1 \approx 1$,
(d) $x \wedge 1 \approx x$,
(e) $x \vee \sim 1 \approx x$,
(f) $1 \to_N x \approx x$,
(g) $1 \to x \approx x$,
(h) $x \to_N x \approx 1$.

Proof Let $a \in A$.

(a) By (SN6) we have that $a \wedge (a \to_N a) = a \wedge (\sim a \vee a) = a$.
(b)

$$
\begin{aligned}
a \to_N 1 &= a \to_N [(a \to_N a) \to_N [(a \to_N a) \to_N [(a \to a) \to_N (a \to a)]]] && \text{by (SN8)}\\
&= [a \wedge (a \to_N a)] \to_N [(a \to_N a) \to_N [(a \to a) \to_N (a \to a)]] && \text{by (SN7)}\\
&= a \to_N [(a \to_N a) \to_N [(a \to a) \to_N (a \to a)]] && \text{by part (a)}\\
&= [a \wedge (a \to_N a)] \to_N [(a \to a) \to_N (a \to a)] && \text{by (SN7)}\\
&= a \to_N [(a \to a) \to_N (a \to a)] && \text{by part (a)}\\
&= a \to_N [(a \to (a \wedge a)) \to_N (a \to (a \wedge a))] &&\\
&= a \to_N [(a \to_N a) \to_N (a \to_N a)] && \text{by the definition of } \to_N\\
&= [a \wedge (a \to_N a)] \to_N (a \to_N a) && \text{by (SN7)}\\
&= a \to_N (a \to_N a) && \text{by part (a)}\\
&= (a \wedge a) \to_N a && \text{by (SN7)}\\
&= a \to_N a.
\end{aligned}
$$

(c)

$$
\begin{aligned}
1 &= (a \to_N a) \to_N [(a \to_N a) \to_N [(a \to (a \wedge a)) \to_N (a \to (a \wedge a))]] && \text{by (SN8)}\\
&= (a \to_N a) \to_N [(a \to_N a) \to_N [(a \to_N a) \to_N (a \to_N a)]] && \text{by the definition of } \to_N\\
&= [(a \to_N a) \wedge (a \to_N a)] \to_N [(a \to_N a) \to_N (a \to_N a)] && \text{by (SN7)}\\
&= (a \to_N a) \to_N [(a \to_N a) \to_N (a \to_N a)] &&\\
&= [(a \to_N a) \wedge (a \to_N a)] \to_N (a \to_N a) && \text{by (SN7)}\\
&= (a \to_N a) \to_N (a \to_N a) &&\\
&= (a \to_N a) \to_N (a \to_N 1) && \text{by part (b)}\\
&= [(a \to_N a) \wedge a] \to_N 1 && \text{by (SN7)}\\
&= [a \wedge (a \to_N a)] \to_N 1 &&\\
&= a \to_N 1 && \text{by part (a).}
\end{aligned}
$$

(d)

$$
\begin{aligned}
a \wedge 1 &= a \wedge (a \to_N 1) && \text{by part (c)}\\
&= a \wedge (a \to_N a) && \text{by part (b)}\\
&= a && \text{by part (a).}
\end{aligned}
$$

(e) From part (d) we have that $\sim a = \sim a \wedge 1$. Then $a = \sim\sim a = \sim (\sim a \wedge 1) = \sim\sim a \vee \sim 1 = a \vee \sim 1$.

(f) Let $a \to_N A$, so using (SN6) and part (d), $1 \to_N a = 1 \wedge (1 \to_N a) = 1 \wedge (\sim 1 \vee a) = \sim 1 \vee a = a$ by the previous item.

(g) Using parts (d) and (f) we have that $1 \to a = 1 \to (1 \wedge a) = 1 \to_N a = a$.

(h) By axiom (SN8) we have that $(1 \to_N 1) \to_N [(1 \to_N 1) \to_N [(1 \to a) \to_N (1 \to a)]] = 1$. From part (f) we can conclude $(1 \to a) \to_N (1 \to a) = 1$. Then by part (g), $a \to_N a = 1$.

□

As a result, we have proven the following theorem, which is similar in spirit to Lemma 4.1 of [3].

Theorem 2.5 *If* $\langle A; 1, \sim, \wedge, \vee, \to\rangle \in \mathcal{PSN}$ *then* $\langle A; 1, \sim, \wedge, \vee, \to_N\rangle \in \mathcal{N}$.

Proof The Theorem follows from the identities (SN1), (SN2), (SN3), (SN4), (SN5), (SN6) and (SN7), and from Lemma 2.4 (h). □

The proofs of the following properties and calculation rules valid in Nelson algebras can be found in [26], and their validity in pre semi-Nelson algebras is a direct consequence of Theorem 2.5.

Lemma 2.6 *Let* $\mathbf{A} = \langle A; 1, \sim, \wedge, \vee, \to\rangle \in \mathcal{PSN}$ *and* $a, b, c \in A$. *Then*

(a) $a \to_N (b \wedge c) = (a \to_N b) \wedge (a \to_N c)$

(b) $(a \to_N b) \to_N ((b \to_N c) \to_N (a \to_N c) = 1$,

(c) *If* $a \leq b$ *then* $a \to_N b = 1$,

(d) $a \to_N (b \to_N c) = (a \to_N b) \to_N (a \to_N c)$,

(e) $a \to_N (b \to_N c) = b \to_N (a \to_N c)$,

(f) $(\sim (a \to_N b)) \to_N (a \wedge \sim b) = 1$,

(g) $(a \wedge \sim b) \to_N (\sim (a \to_N b)) = 1$,

(h) $(a \vee b) \to_N c = (a \to_N c) \wedge (b \to_N c)$,

(i) $(\sim a \wedge \sim b) \to_N (\sim (a \vee b)) = 1$,

(j) $(\sim (a \vee b)) \to_N (\sim a \wedge \sim b) = 1$,

(k) $a \leq b$ *if and only if* $a \to_N b = 1$ *and* $\sim b \to_N \sim a = 1$,
(l) *If* $a \to_N b = b \to_N c = 1$ *then* $a \to_N c = 1$,
(m) $\sim 1 \to_N a = 1$,
(n) $(a \wedge b) \to_N b = 1$,
(o) $a \to_N b = 1$ *if and only if* $a = a \wedge (\sim a \vee b)$,
(p) *If* $a \to_N b = 1$ *and* $b \to_N c = 1$ *then* $a \to_N c = 1$,
(q) $(a \wedge \sim a) \to_N b = 1$.

From now on, we denote by $x \rightarrowtail y$ the term $(x \to y) \wedge (\sim y \to \sim x)$.

Lemma 2.7 *Let* $\mathbf{A} = \langle A; 1, \sim, \wedge, \vee, \to \rangle \in \mathcal{PSN}$ *and* $a, b, c \in A$. *Then:*

(a) $a \to a = 1$,
(b) $(a \to_N \sim 1) \to_N (a \to \sim 1) = 1$,
(c) $a \to_N b = a \to_N (a \wedge b)$,
(d) $a \to_N (a \wedge (b \to_N a)) = 1$,
(e) $[a \wedge ((a \wedge b) \to (a \wedge c))] \to_N (b \to c) = 1$,
(f) $(a \wedge (b \to c)) \to_N ((a \wedge b) \to (a \wedge c)) = 1$,
(g) $(a \to b) \to_N (a \to_N b) = 1$,
(h) $a \to_N b = 1$ *and* $b \to_N a = 1$ *if and only if* $a \to b = 1$ *and* $b \to a = 1$,
(i) $[a \wedge (a \to_N b)] \to_N c = (a \wedge b) \to_N c$,
(j) $(a \wedge b) \to_N (a \to b) = 1$,
(k) $a \leq (a \rightarrowtail b) \to_N b$.

Proof (a) By part (h) of Lemma 2.4, we have that $a \to a = a \to (a \wedge a) = a \to_N a = 1$.
(b) By axiom (SN9),

$$((a \wedge \sim 1) \to_N \sim 1) \to_N [(\sim 1 \to_N (a \wedge \sim 1)) \to_N [(a \to (a \wedge \sim 1)) \to_N (a \to \sim 1)]] = 1.$$

By Lemma 2.6, (n), $(a \wedge \sim 1) \to_N \sim 1 = 1$. By part (m) of Lemma 2.6, $\sim 1 \to_N (a \wedge \sim 1) = 1$, so $1 \to_N [1 \to_N [(a \to (a \wedge \sim 1)) \to_N (a \to \sim 1)]] = 1$. Using Lemma 2.4, (f) we conclude $(a \to (a \wedge \sim 1)) \to_N (a \to \sim 1) = 1$. Therefore, $(a \to_N \sim 1) \to_N (a \to \sim 1) = 1$.
(c) $a \to_N b = a \to (a \wedge b) = a \to (a \wedge (a \wedge b)) = a \to_N (a \wedge b)$.
(d) By Lemma 2.6 (n) we have $(a \wedge b) \to_N a = 1$. Then, using axiom (SN7) and part (c), $a \to_N (a \wedge (b \to_N a)) = a \to_N (b \to_N a) = (a \wedge b) \to_N a = 1$.
(e) By part (d),

$$a \to_N (a \wedge (c \to_N a)) = 1. \tag{1}$$

By axiom (SN9) we have that

$$1 = ((a \wedge c) \to_N c) \to_N [(c \to_N (a \wedge c)) \to_N [((a \wedge b) \to (a \wedge c)) \to_N ((a \wedge b) \to c)]].$$

By Lemma 2.6, (n), $(a \wedge c) \to_N c = 1$. Therefore, using part (f) of Lemma 2.4,

$$1 = (c \to_N (a \wedge c)) \to_N [((a \wedge b) \to (a \wedge c)) \to_N ((a \wedge b) \to c)].$$

This is equivalent, by part (c) to

$$1 = (c \to_N a) \to_N [((a \wedge b) \to (a \wedge c)) \to_N ((a \wedge b) \to c)].$$

By part (c) of Lemma 2.4, $1 = a \to_N 1 = a \to_N [(c \to_N a) \to_N [((a \wedge b) \to (a \wedge c)) \to_N ((a \wedge b) \to c)]]$ and, by (SN7),

$$1 = [a \wedge (c \to_N a)] \to_N [((a \wedge b) \to (a \wedge c)) \to_N ((a \wedge b) \to c)]. \tag{2}$$

From the Eqs. 1 and 2 and part (l) of Lemma 2.6 we can conclude that

$$a \to_N [((a \wedge b) \to (a \wedge c)) \to_N ((a \wedge b) \to c)] = 1. \tag{3}$$

By (SN7) and part (c), we can write $a \to_N [((a \wedge b) \to (a \wedge c)) \to_N ((a \wedge b) \to c)] = [a \wedge ((a \wedge b) \to (a \wedge c))] \to_N [(a \wedge b) \to c] = [(a \wedge b) \to (a \wedge c)] \to_N [a \to_N [(a \wedge b) \to c]] = [(a \wedge b) \to (a \wedge c)] \to_N [a \to_N [a \wedge ((a \wedge b) \to c)]] = [a \wedge ((a \wedge b) \to (a \wedge c))] \to_N [a \wedge ((a \wedge b) \to c)]$.

Then using the Eq. 3,

$$[a \wedge ((a \wedge b) \to (a \wedge c))] \to_N [a \wedge ((a \wedge b) \to c)] = 1. \tag{4}$$

By part (d), we have

$$a \to_N (a \wedge (b \to_N a)) = 1. \tag{5}$$

By axiom (SN8) we obtain

$$1 = ((a \wedge b) \to_N b) \to_N [(b \to_N (a \wedge b)) \to_N [((a \wedge b) \to c) \to_N (b \to c)]],$$

and by Lemma 2.6 (n), $(a \wedge b) \to_N b = 1$. Then, using part (f) of Lemma 2.4,

$$1 = (b \to_N (a \wedge b)) \to_N [((a \wedge b) \to c) \to_N (b \to c)].$$

By part (c), this is equivalent to

$$1 = (b \to_N a) \to_N [((a \wedge b) \to c) \to_N (b \to c)].$$

From part (c) of Lemma 2.4, $1 = a \to_N 1 = a \to_N [(b \to_N a) \to_N [((a \wedge b) \to c) \to_N (b \to c)]]$ and, by (SN7),

$$1 = [a \wedge (b \to_N a)] \to_N [((a \wedge b) \to c) \to_N (b \to c)]. \tag{6}$$

From the Eqs. 5 and 6 and part (l) of Lemma 2.6 we can conclude that

$$a \to_N [((a \wedge b) \to c) \to_N (b \to c)] = 1. \tag{7}$$

From (SN7),

$$[a \wedge ((a \wedge b) \to c)] \to_N (b \to c) = 1. \tag{8}$$

By the Eqs. 4 and 8 and part (l) of Lemma 2.6 we conclude

$$[a \wedge ((a \wedge b) \to (a \wedge c))] \to_N (b \to c) = 1.$$

(f) It follows from part (d) that

$$a \to_N (a \wedge (c \to_N a)) = 1. \tag{9}$$

By axiom (SN9) we have

$$1 = (c \to_N (a \wedge c)) \to_N [((a \wedge c) \to_N c) \to_N [((a \wedge b) \to c) \to_N ((a \wedge b) \to (a \wedge c))]].$$

By Lemma 2.6 (n), $(a \wedge c) \to_N c = 1$. So using part (f) of Lemma 2.4,

$$1 = (c \to_N (a \wedge c)) \to_N [((a \wedge b) \to c) \to_N ((a \wedge b) \to (a \wedge c))].$$

This is equivalent, by part (c) to

$$1 = (c \to_N a) \to_N [((a \wedge b) \to c) \to_N ((a \wedge b) \to (a \wedge c))].$$

By part (c) of Lemma 2.4, $1 = a \to_N 1 = a \to_N [(c \to_N a) \to_N [((a \wedge b) \to c) \to_N ((a \wedge b) \to (a \wedge c))]]$ and, by (SN7),

$$1 = [a \wedge (c \to_N a)] \to_N [((a \wedge b) \to c) \to_N ((a \wedge b) \to (a \wedge c))]. \tag{10}$$

From the Eqs. 9 and 10 and by part (l) of Lemma 2.6 we infer that

$$a \to_N [((a \wedge b) \to c) \to_N ((a \wedge b) \to (a \wedge c))] = 1. \tag{11}$$

By (SN7) and the Eq. 11,

$$[a \wedge ((a \wedge b) \to c)] \to_N [(a \wedge b) \to (a \wedge c)] = 1. \tag{12}$$

By part (d)

$$a \to_N (a \wedge (b \to_N a)) = 1. \tag{13}$$

By axiom (SN8),

$$1 = (b \to_N (a \wedge b)) \to_N [((a \wedge b) \to_N b) \to_N [(b \to c) \to_N ((a \wedge b) \to c)]].$$

By Lemma 2.6 (n), $(a \wedge b) \to_N b = 1$. Therefore, using part (f) of Lemma 2.4,

$$1 = (b \to_N (a \wedge b)) \to_N [(b \to c) \to_N ((a \wedge b) \to c)],$$

which is equivalent, by part (c) to

$$1 = (b \to_N a) \to_N [(b \to c) \to_N ((a \wedge b) \to c)].$$

By part (c) of Lemma 2.4, $1 = a \to_N 1 = a \to_N [(b \to_N a) \to_N [(b \to c) \to_N ((a \wedge b) \to c)]]$ and by (SN7),

$$1 = [a \wedge (b \to_N a)] \to_N [(b \to c) \to_N ((a \wedge b) \to c)]. \tag{14}$$

From the Eqs. 13 and 14 and part (l) of Lemma 2.6 we can conclude that

$$a \to_N [(b \to c) \to_N ((a \wedge b) \to c)] = 1.$$

By (SN7),

$$[a \wedge (b \to c)] \to_N ((a \wedge b) \to c) = 1.$$

Then, by (SN7) and part (c) , $1 = [a \wedge (b \to c)] \to_N ((a \wedge b) \to c) = (b \to c) \to_N [a \to_N [(a \wedge b) \to c]] = (b \to c) \to_N [a \to_N [a \wedge ((a \wedge b) \to c)]] = [a \wedge (b \to c)] \to_N [a \wedge ((a \wedge b) \to c)]$. So

$$[a \wedge (b \to c)] \to_N [a \wedge ((a \wedge b) \to c)] = 1. \tag{15}$$

It follows from the Eqs. 12 and 15 and part (l) of Lemma 2.6 that

$$[a \wedge (b \to c)] \to_N [(a \wedge b) \to (a \wedge c)] = 1.$$

(g) By part (f) we have that $(a \wedge (a \to b)) \to_N ((a \wedge a) \to (a \wedge b)) = 1$, so $(a \wedge (a \to b)) \to_N (a \to_N b) = 1$. As a consequence, using (SN7), $1 = (a \wedge (a \to b)) \to_N (a \to_N b) = (a \to b) \to_N (a \to_N (a \to_N b)) = (a \to b) \to_N ((a \wedge a) \to_N b) = (a \to b) \to_N (a \to_N b)$.

(h) Assume that $a \to_N b = 1$ and $b \to_N a = 1$. From axiom (SN9) it follows that

$$(a \to_N b) \to_N [(b \to_N a) \to_N [(a \to a) \to_N (a \to b)]] = 1.$$

By parts (f) of Lemma 2.4 and (a) of this one, $a \to b = 1$. In a similar manner, using axiom (SN8), $b \to a = 1$ is verified.

The converse follows immediately by part (f) of Lemma 2.4 and part (g).

(i) By axiom (SN6) we have that $[a \wedge (a \to_N b)] \to_N c = [a \wedge (\sim a \vee b)] \to_N c = [(a \wedge \sim a) \vee (a \wedge b)] \to_N c$. From Lemma 2.6 (h), $[a \wedge (a \to_N b)] \to_N c = ((a \wedge \sim a) \to_N c) \wedge ((a \wedge b) \to_N c)$. Therefore, by Lemma 2.6 (q) we can conclude that $[a \wedge (a \to_N b)] \to_N c = (a \wedge b) \to_N c$.

(j) Using part (e), $[a \wedge ((a \wedge a) \to (a \wedge b))] \to_N (a \to b) = 1$. Therefore, by part (i), $1 = [a \wedge ((a \wedge a) \to (a \wedge b))] \to_N (a \to b) = [a \wedge (a \to (a \wedge b))] \to_N (a \to b) = [a \wedge (a \to_N b)] \to_N (a \to b) = (a \wedge b) \to_N (a \to b)$.

(k) We calculate:

$$\begin{aligned}
[(a \rightarrowtail b) \wedge \sim b] \rightarrow_N \sim a &= [(a \to b) \wedge (\sim b \to \sim a) \wedge \sim b] \rightarrow_N \sim a && \text{by definición of } \rightarrowtail \\
&= [(a \to b) \wedge (\sim b \to \sim a)] \rightarrow_N (\sim b \rightarrow_N \sim a) && \text{by (SN7)} \\
&= (a \to b) \rightarrow_N [(\sim b \to \sim a) \rightarrow_N (\sim b \rightarrow_N \sim a)] && \text{by (SN7)} \\
&= (a \to b) \rightarrow_N 1 && \text{por lemma 2.7 (g)} \\
&= 1. && \text{by Lemma 2.4 (c)}
\end{aligned}$$

In consequence,

$$[(a \rightarrowtail b) \wedge \sim b] \rightarrow_N \sim a = 1. \tag{16}$$

By Lemma 2.6 (f), we have that

$$\sim ((a \rightarrowtail b) \rightarrow_N b) \rightarrow_N ((a \rightarrowtail b) \wedge \sim b) = 1. \tag{17}$$

So, from Eqs. 16 and 17, by Lemma 2.6 (l),

$$\sim ((a \rightarrowtail b) \rightarrow_N b) \rightarrow_N (\sim a) = 1. \tag{18}$$

On the other hand,

$$\begin{aligned}
a \rightarrow_N [(a \rightarrowtail b) \rightarrow_N b] &= [a \wedge (a \rightarrowtail b)] \rightarrow_N b && \text{by (SN7)} \\
&= [a \wedge (a \to b) \wedge (\sim b \to \sim a)] \rightarrow_N b && \text{by definition of } \rightarrowtail \\
&= [(a \to b) \wedge (\sim b \to \sim a)] \rightarrow_N (a \rightarrow_N b) && \text{by (SN7)} \\
&= (\sim b \to \sim a) \rightarrow_N [(a \to b) \rightarrow_N (a \rightarrow_N b)] && \text{by (SN7)} \\
&= (\sim b \to \sim a) \rightarrow_N 1 && \text{by Lemma 2.7 (g)} \\
&= 1. && \text{by Lemma 2.4 (c)}
\end{aligned}$$

Therefore

$$a \rightarrow_N [(a \rightarrowtail b) \rightarrow_N b] = 1. \tag{19}$$

By Lemma 2.6 (k) and from the Eqs. 18 and 19 we conclude that

$$a \leq (a \rightarrowtail b) \rightarrow_N b.$$

□

We can prove the following theorem.

Theorem 2.8 *The class $\mathcal{N}$ is the subvariety of $\mathcal{PSN}$ defined by the identity $x \rightarrow_N y \approx x \to y$.*

Proof Let $\mathcal{V}$ be the subvariety of $\mathcal{PSN}$ defined by the identity $x \rightarrow_N y \approx x \to y$ and consider $\mathbf{A} = \langle A; \sim, \wedge, \vee, \to, 1\rangle \in \mathcal{V}$. We check that $\mathbf{A} \in \mathcal{N}$. Observe first that $\mathbf{A}$ verifies trivially axioms (N1) to (N5). Since $x \rightarrow_N y \approx x \to y$ holds in $\mathbf{A}$, by axioms (SN6) and (SN7), $\mathbf{A}$ satisfies (N8) and (N7). By Lemma 2.7 (a) axiom (N6) holds as well and therefore $\mathbf{A} \in \mathcal{N}$.

Consider now $\mathbf{A} = \langle A; 1, \sim, \wedge, \vee, \to\rangle \in \mathcal{N}$ and let us check that $\mathbf{A} \in \mathcal{V}$. $\mathbf{A}$ verifies axioms (SN1) to (SN5) trivially. Now let $a, b \in A$. Then, by Lemma 2.6(a) and Lemma 2.7 (a), we have that $a \rightarrow_N b = a \to (a \wedge b) = (a \to a) \wedge (a \to b) = a \to b$. Then $\mathbf{A} \models x \rightarrow_N y \approx x \to y$ and thus the algebra verifies axioms (SN6) and (SN7). To see that axioms (SN8) and (SN9) hold, let $a, b, c \in A$.

Notice that the properties of Nelson algebras indicated in Lemma 2.6 are valid in $\mathbf{A}$ although we have not established yet it is in $\mathcal{PSN}$.

$$\begin{aligned}
&(a \rightarrow_N b) \rightarrow_N [(b \rightarrow_N a) \rightarrow_N [(a \to c) \rightarrow_N (b \to c)]] \\
&\quad = (a \to b) \to [(b \to a) \to [(a \to c) \to (b \to c)]] && \text{by the identity } x \rightarrow_N y \approx x \to y \\
&\quad = (a \to b) \to 1 = 1 && \text{by Lemma 2.6(b) and (c).}
\end{aligned}$$

On the other hand,
$(a \to_N b) \to_N [(b \to_N a) \to_N [(c \to a) \to_N (c \to b)]]$
$= (a \to b) \to [(b \to a) \to [(c \to a) \to (c \to b)]]$ by the identity $x \to_N y \approx x \to y$
$= (a \to b) \to [(b \to a) \to [c \to (a \to b)]]$ by Lemma 2.6(d)
$= (b \to a) \to [(a \to b) \to [c \to (a \to b)]]$ by Lemma 2.6(e)
$= (b \to a) \to [((a \to b) \wedge c) \to (a \to b)]$ by axiom (N7)
$= (b \to a) \to 1 = 1$ by Lemma 2.6(c).

□

The next example shows that $\mathcal{N}$ is a proper subvariety of $\mathcal{SN}$ and therefore, of $\mathcal{PSN}$. Let $\mathbf{B} = \langle \{0, a, 1\}; \to, \wedge, \vee, \sim, 1\rangle$ where

$\sim$	0	a	1
	1	a	0

$\to$	0	a	1
0	1	1	a
a	1	1	a
1	0	a	1

$\wedge$	0	a	1
0	0	0	0
a	0	a	a
1	0	a	1

$\vee$	0	a	1
0	0	a	1
a	a	a	1
1	1	1	1

In this algebra $a = 0 \to 1 \neq 0 \to_N 1 = 1$.

3 The Quotient Algebra

We describe next one of the constructions that realize the connection between Semi-Nelson and semi-Heyting algebras.

Lemma 3.1 *Let* $\mathbf{A} \in \mathcal{PSN}$. *The binary relation* $\equiv$ *defined on* A *by*

$$x \equiv y \text{ if and only if } x \to y = 1 \text{ and } y \to x = 1$$

is an equivalence relation compatible with the operations $\wedge$, $\vee$ *and* $\to$.

Proof By Lemma 2.7 (h) and the results in [25], $\equiv$ is an equivalence relation compatible with the operations 1, $\wedge$ and $\vee$. Let $a, b, c, d \to_N A$ be such that $a \equiv b$ and $c \equiv d$.

Since $a \equiv b$, $a \to_N b = b \to_N a = 1$. By Lemma 2.4(f) and axiom (SN8) we have that $1 = (a \to_N b) \to_N [(b \to_N a) \to_N [(a \to c) \to_N (b \to c)]] = 1 \to_N [1 \to_N [(a \to c) \to_N (b \to c)]] = (a \to c) \to_N (b \to c)$. In a similar fashion, $(b \to c) \to_N (a \to c) = 1$. Therefore, by Lemma 2.7 (h), $a \to c \equiv b \to c$.

Since $c \equiv d$, $c \to_N d = d \to_N c = 1$. By Lemma 2.4(f) and axiom (SN9) we have as before that $(c \to_N d) \to_N [(d \to_N c) \to_N [(b \to c) \to_N (b \to d)]] = (b \to c) \to_N (b \to d) = 1$. In a similar way we can check that $(b \to d) \to_N (b \to c) = 1$. Therefore, by Lemma 2.7 (h), $b \to c \equiv b \to d$ and we can conclude that $a \to c \equiv b \to d$. □

Let $\mathbf{A} \in \mathcal{PSN}$. We denote by $\mathbf{sH}(\mathbf{A})$ the algebra $\langle A/_{\equiv}; \cap, \cup, \Rightarrow, 0, 1\rangle$ where the operations are defined by:

- $0 = [\![\sim 1]\!]$,
- $1 = [\![1]\!]$,
- $[\![x]\!] \cap [\![y]\!] = [\![x \wedge y]\!]$,
- $[\![x]\!] \cup [\![y]\!] = [\![x \wedge y]\!]$,
- $[\![x]\!] \Rightarrow [\![y]\!] = [\![x \wedge y]\!]$.

Observe that by Lemma 3.1 the operations are well defined. As in [25], by Theorem 2.5, the next result follows.

Lemma 3.2 *Let* $\mathbf{A} \in \mathcal{PSN}$. *The algebra* $\langle A/_{\equiv}; \cap, \cup, 0, 1\rangle$ *is a bounded distributive lattice.*

Lemma 3.3 *Let* $\mathbf{A} \in \mathcal{PSN}$. *The algebra* $\mathbf{sH}(\mathbf{A})$ *satisfies the condition*

$$[\![a]\!] \leq [\![b]\!] \text{ if and only if } [\![a \to_N b]\!] = [\![1]\!].$$

where $\leq$ *is the natural order relation in the lattice.*

Proof Assume that $[\![a]\!] = [\![a]\!] \cap [\![b]\!]$. Then $[\![a \to_N b]\!] = [\![a \to (a \wedge b)]\!] = [\![a]\!] \Rightarrow ([\![a]\!] \cap [\![b]\!]) = ([\![a]\!] \cap [\![b]\!]) \Rightarrow ([\![a]\!] \cap [\![b]\!]) = [\![(a \wedge b) \to (a \wedge b)]\!] = [\![1]\!]$.

For the converse observe that $[\![(a \wedge b) \to_N a]\!] = [\![(b \wedge a) \to_N a]\!] = [\![b \to_N]\!](a \to_N a) = [\![b \to_N 1]\!] = [\![1]\!]$ by Lemma 2.4 (h), (SN7) and Lemma 2.4 (c). Therefore, by the definition of $\equiv$, $1 \to ((a \wedge b) \to_N a) = 1$. By Lemma 2.4(g),

$$(a \wedge b) \to_N a = 1. \tag{20}$$

Furthermore, by Lemma 2.7, (c) $[\![a \to_N (a \wedge b)]\!] = [\![a \to_N b]\!] = [\![1]\!]$. As a consequence, $1 \to (a \to_N (a \wedge b)) = 1$. By Lemma 2.4 (g),

$$a \to_N (a \wedge b) = 1. \tag{21}$$

From Eqs. 20 and 21 it follows, using Lemma 2.7 (h), that $[\![a]\!] = [\![a \wedge b]\!] = [\![a]\!] \cap [\![b]\!]$. □

Theorem 3.4 *Let* $\mathbf{A} \in \mathcal{PSN}$. *Then* $\mathbf{sH}(\mathbf{A})$ *is a semi-Heyting algebra.*

Proof By Lemma 3.2, it will be enough to verify that axioms (SH2), (SH3) and (SH4) hold. Let $a, b, c \in A$.

- $[\![(a \wedge (a \to b)) \to_N b]\!] = [\![((a \to b) \wedge a) \to_N b]\!] = [\![(a \to b) \to_N (a \to_N b)]\!] = [\![1]\!]$ by (SN7) and Lemma 2.7 (g). By Lemma 3.3 we have that $[\![a \wedge (a \to b)]\!] \leq [\![b]\!]$. Then
$$[\![a]\!] \cap ([\![a]\!] \Rightarrow [\![b]\!]) \leq [\![a]\!] \cap [\![b]\!] \tag{22}$$
By Lemma 2.7 (j) we have that $(a \wedge b) \to_N (a \to b) = 1$. Therefore $[\![(a \wedge b) \to_N (a \to b)]\!] = [\![1]\!]$. By Lemma 3.3, $[\![a \wedge b]\!] \leq [\![a \to b]\!]$. Then
$$[\![a]\!] \cap [\![b]\!] \leq [\![a]\!] \cap ([\![a]\!] \Rightarrow [\![b]\!]). \tag{23}$$
As a consequence, by Eqs. 22 and 23 we can deduce that $[\![a]\!] \cap ([\![a]\!] \Rightarrow [\![b) = [\![a]\!] \cap [\![b]\!]$, so (SH2) holds.
- By Lemma 2.7 Eq. 2.7, $[\![(a \wedge (b \to c)) \to_N ((a \wedge b) \to (a \wedge c))]\!] = [\![1]\!]$. By Lemma 3.3, $[\![a]\!] \cap ([\![b]\!] \Rightarrow [\![c]\!]) \leq ([\![a]\!] \cap [\![b]\!]) \Rightarrow ([\![a]\!] \cap [\![c]\!])$. As a consequence,
$$[\![a]\!] \cap ([\![b]\!] \Rightarrow [\![c]\!]) \leq [\![a]\!] \cap (([\![a]\!] \cap [\![b]\!]) \Rightarrow ([\![a]\!] \cap [\![z]\!])) \tag{24}$$
in a similar fashion, using Lemma 2.7 (e) we can deduce that
$$[\![a]\!] \cap (([\![a]\!] \cap [\![b]\!]) \Rightarrow ([\![a]\!] \cap [\![c]\!])) \leq [\![a]\!] \cap ([\![b]\!] \Rightarrow [\![c]\!]). \tag{25}$$
In consequence, by Eqs. 24 and 25 we conclude that $[\![a]\!] \cap (([\![a]\!] \cap [\![b]\!]) \Rightarrow ([\![a]\!] \cap [\![c]\!])) = [\![a]\!] \cap ([\![b]\!] \Rightarrow [\![c]\!])$. Therefore, (SH3) holds.
- From Lemma 2.7 (a) it follows that $[\![a]\!] \Rightarrow [\![a]\!] = [\![a]\!] \to a = [\![1]\!]$. So (SH4) holds as well.

□

4 Vakarelov's Construction

We check now that Vakarelov's construction on Heyting algebras works as well for semi-Heyting algebras.

Let $\mathbf{A} = \langle A; \wedge, \vee, \Rightarrow, 0, 1\rangle$ be a semi-Heyting algebra. We denote with

$$V_k(A) = \{(a, b) \in A^2 : \ a \wedge b = 0\}$$

and $\mathbf{V}_k(\mathbf{A})$ the algebra $\langle V_k(A); \sqcap, \sqcup, \rightarrow, \sim, \top\rangle$ where the operations are given by:

V1) $(a, b) \sqcap (c, d) = (a \wedge c, b \vee d)$,
V2) $(a, b) \sqcup (c, d) = (a \vee c, b \wedge d)$,
V3) $(a, b) \rightarrow (c, d) = (a \Rightarrow c, a \wedge d)$,
V4) $\sim (a, b) = (b, a)$,
V5) $\top = (1, 0)$.

We can derive also the rule:

V6) $(a, b) \rightarrow_N (c, d) = (a, b) \rightarrow ((a, b) \sqcap (c, d)) = (a, b) \rightarrow ((a \wedge c, b \vee d)) = (a \Rightarrow (a \wedge c), a \wedge (b \vee d)) = (a \Rightarrow c, (a \wedge b) \vee (a \wedge d)) = (a \Rightarrow c, a \wedge d)$.

Theorem 4.1 *If* $\mathbf{A} = \langle A; \wedge, \vee, \Rightarrow, 0, 1\rangle \in \mathcal{SH}$, *then* $\mathbf{V}_k(\mathbf{A})$ *is a semi-Nelson algebra.*

Proof As proved in [25], the operations of $\mathbf{V}_k(\mathbf{A})$ are well defined and the identities (SN1) to (SN5) hold. Let us check the remaining identities. Let $(a, b), (c, d), (e, f) \in V_k(A)$.

We prove first the identity (SN6):

$$\begin{aligned}(a, b) \sqcap ((a, b) \rightarrow_N (c, d)) &= (a, b) \sqcap (a \Rightarrow c, a \wedge d)\\ &= (a \wedge (a \Rightarrow c), b \vee (a \wedge d))\\ &= (a \wedge c, b \vee (a \wedge d))\end{aligned}$$

and

$$\begin{aligned}(a, b) \sqcap (\sim (a, b) \sqcup (c, d)) &= (a, b) \sqcap ((b, a) \sqcup (c, d))\\ &= (a, b) \sqcap (b \vee c, a \wedge d)\\ &= (a \wedge (b \vee c), b \vee (a \wedge d))\\ &= (a \wedge c, b \vee (a \wedge d)).\end{aligned}$$

To prove (SN7) we calculate:

$$\begin{aligned}(a, b) \rightarrow_N ((c, d) \rightarrow_N (e, f)) &= (a, b) \rightarrow_N (c \Rightarrow e, c \wedge f)\\ &= (a \Rightarrow (c \Rightarrow e), a \wedge c \wedge f).\end{aligned}$$

and

$$\begin{aligned}((a, b) \sqcap (c, d)) \rightarrow_N (e, f) &= (a \wedge c, b \vee d) \rightarrow_N (e, f)\\ &= ((a \wedge c) \Rightarrow e, a \wedge c \wedge f).\end{aligned}$$

By Lemma 1.4(a) the identity is proven.

We prove next (SN8):

$$\begin{aligned}&((a, b) \rightarrow_N (c, d)) \rightarrow_N [((c, d) \rightarrow_N (a, b)) \rightarrow_N [((a, b) \rightarrow (e, f)) \rightarrow_N ((c, d) \rightarrow (e, f))]]\\ &= (a \Rightarrow c, a \wedge d) \rightarrow_N [(c \Rightarrow a, c \wedge b) \rightarrow_N [(a \Rightarrow e, a \wedge f) \rightarrow_N (c \Rightarrow e, c \wedge f)]]\\ &= (a \Rightarrow c, a \wedge d) \rightarrow_N [(c \Rightarrow a, c \wedge b) \rightarrow_N ((a \Rightarrow e) \Rightarrow (c \Rightarrow e), (a \Rightarrow e) \wedge c \wedge f)]]\\ &= (a \Rightarrow c, a \wedge d) \rightarrow_N ((c \Rightarrow a) \Rightarrow ((a \Rightarrow e) \Rightarrow (c \Rightarrow e)), (c \Rightarrow a) \wedge (a \Rightarrow e) \wedge c \wedge f)]]\\ &= ((a \Rightarrow c) \Rightarrow ((c \Rightarrow a) \Rightarrow ((a \Rightarrow e) \Rightarrow (c \Rightarrow e))), (a \Rightarrow c) \wedge (c \Rightarrow a) \wedge (a \Rightarrow e) \wedge c \wedge f)]]\end{aligned}$$

By Lemma 1.4(b) the first coordinate is 1 so

$$((a,b) \to_N (c,d)) \to_N [((c,d) \to_N (a,b)) \to_N [((a,b) \to (e,f)) \to_N ((c,d) \to (e,f))]]$$
$$= (1, (a \Rightarrow c) \wedge (a \Rightarrow e) \wedge c \wedge a \wedge f)]]$$
$$= (1, (a \Rightarrow c) \wedge e \wedge c \wedge a \wedge f)]] = (1,0).$$

We proceed with (SN9):

$$((a,b) \to_N (c,d)) \to_N [((c,d) \to_N (a,b)) \to_N [((e,f) \to (a,b)) \to_N ((e,f) \to (c,d))]] =$$
$$(a \Rightarrow_H c, a \wedge d) \to_N [(c \Rightarrow_H a, c \wedge b)) \to_N [(e \Rightarrow a, e \wedge b) \to_N (e \Rightarrow c, e \wedge d)]] =$$
$$(a \Rightarrow_H c, a \wedge d) \to_N [(c \Rightarrow_H a, c \wedge b) \to_N ((e \Rightarrow a) \Rightarrow_H (e \Rightarrow c), (e \Rightarrow a) \wedge e \wedge d)] =$$
$$(a \Rightarrow_H c, a \wedge d) \to_N [((c \Rightarrow_H a) \Rightarrow_H ((e \Rightarrow a) \Rightarrow_H (e \Rightarrow c)), (c \Rightarrow_H a) \wedge a \wedge e \wedge d)] =$$
$$((a \Rightarrow_H c) \Rightarrow_H ((c \Rightarrow_H a) \Rightarrow_H ((e \Rightarrow a) \Rightarrow_H (e \Rightarrow c))), (a \Rightarrow_H c) \wedge (c \Rightarrow_H a) \wedge a \wedge e \wedge d)] =$$
$$(1, c \wedge (c \Rightarrow_H a) \wedge a \wedge e \wedge d)] = (1,0).$$

Note that in the next to last step we used Lemma 1.4(c).

We prove now (SN10):

$$(\sim ((a,b) \to (c,d))) \to_N ((a,b) \sqcap \sim (c,d))$$
$$= (\sim (a \Rightarrow c, a \wedge d)) \to_N ((a,b) \sqcap (d,c))$$
$$= (a \wedge d, a \Rightarrow c) \to_N (a \wedge d, b \vee c)$$
$$= ((a \wedge d) \Rightarrow_H (a \wedge d), a \wedge d \wedge (b \vee c))$$
$$= (1, (a \wedge d \wedge b) \vee (a \wedge d \wedge c))$$
$$= (1, 0 \vee 0) = \top.$$

Finally we check that (SN11) holds:

$$((a,b) \sqcap \sim (c,d)) \to_N (\sim ((a,b) \to (c,d)))$$
$$= ((a,b) \sqcap \sim (c,d)) \to_N (\sim ((a,b) \to (c,d)))$$
$$= ((a,b) \sqcap (d,c)) \to_N (\sim (a \Rightarrow c, a \wedge d))$$
$$= (a \wedge d, b \vee c) \to_N (a \wedge d, a \Rightarrow c)$$
$$= ((a \wedge d) \Rightarrow (a \wedge d), a \wedge d \wedge (a \Rightarrow c))$$
$$= (1, a \wedge d \wedge c) = (1,0) = \top.$$

□

5 Representations

In this section we show that there exists a representation of semi-Nelson algebras as subalgebras of the Vakarelov construction applied to a suitable semi-Heyting algebra, much in the same manner as in the case of Nelson and Heyting algebras. In the other direction, we can also represent every semi-Heyting algebra as a quotient of a semi-Nelson one.

Kalman proves in [12] the next theorem:

Theorem 5.1 *Let* $\mathbf{A} \in \mathcal{PSN}$*. Then the* $\{\vee, \wedge, \sim, 1\}$*-reduct of* $\mathbf{A}$*, which is a Kleene algebra, is isomorphic to a subalgebra of the* $\{\vee, \wedge, \sim, 1\}$*-reduct of* $\mathbf{V}_k(\mathbf{sH}(\mathbf{A}))$.

Corollary 5.2 *If* $\mathbf{A} \in \mathcal{SN}$*, then* $\mathbf{A}$ *is isomorphic to a subalgebra of* $\mathbf{V}_k(\mathbf{sH}(\mathbf{A}))$.

Proof By Theorem 5.1, it is enough to check that the function $h : A \to V_k(A/_\equiv)$ defined by $h(a) = ([\![a]\!], [\![\sim a]\!])$ for all $a \in A$ preserves the implication.

By axioms (SN10) and (SN11), and Lemma 2.7 (h), $[\![\sim (a \to b)]\!] = [\![a \wedge \sim b]\!]$. Then $h(a) \to h(b) = ([\![a]\!], [\![\sim a]\!]) \to ([\![b]\!], [\![\sim b]\!]) = ([\![a]\!] \Rightarrow [\![b]\!], [\![a \cap [\![\sim b]\!]) = ([\![a \to b]\!], [\![a \wedge \sim b]\!]) = ([\![a \to b]\!], [\![\sim (a \to b)]\!]) = h(a \to b)$. □

In semi-Heyting algebras we define $a^* = a \Rightarrow 0$ and this always equals $a \Rightarrow 0$. Thus, by well known properties of the pseudocomplement in Heyting algebras, $(x \vee y)^* = x^* \wedge y^*$.

Theorem 5.3 *If* $\mathbf{A} = \langle A; \Rightarrow, \wedge, \vee, 0, 1\rangle$ *be a semi-Heyting algebra, then* $\mathbf{A}$ *is isomorphic to* $\mathbf{sH}(\mathbf{V}_k(\mathbf{A}))$.

Proof Consider the algebras $\mathbf{V}_k(\mathbf{A}) = \langle V_k(A); \sqcap, \sqcup, \rightarrow, \sim, \top\rangle$ and $\mathbf{sH}(\mathbf{V}_k(\mathbf{A})) = \langle V_k(A)/\equiv, \cap, \cup, \Rightarrow, 0, 1\rangle$.

We check that the function $h : A \to V_k(A)/_{\equiv}$ defined by $h(a) = [\![(a, a^*)]\!]$ for every $a \in A$ is an isomorphism. We observe first that h is well defined since $a \wedge a^* = 0$ and therefore $(a, a^*) \in V_k(A)$.

We start by calculating $(a \wedge b, (a \wedge b)^*) \rightarrow_N (a \wedge b, (a^* \vee b^*)) = ((a \wedge b) \Rightarrow_H (a \wedge b), a \wedge b \wedge (a^* \vee b^*)) = (1, 0) = \top$ and $(a \wedge b, (a^* \vee b^*)) \rightarrow_N (a \wedge b, (a \wedge b)^*) = ((a \wedge b) \Rightarrow_H (a \wedge b), (a \wedge b) \wedge (a \wedge b)^*)) = (1, 0) = \top$. Therefore, by Lemma 2.7 (h), $[\![(a \wedge b, (a \wedge b)^*)]\!] = [\![(a \wedge b, (a^* \vee b^*)]\!]$, so $h(a \wedge b) = [\![(a \wedge b, (a \wedge b)^*)]\!] = [\![(a \wedge b, a^* \vee b^*)]\!] = [\![(a, a^*) \sqcap (b, b^*)]\!] = [\![(a, a^*) \cap [\![(b, b^*)]\!] = h(a) \cap h(b)$.

The identity $h(a \vee b) = h(a) \cup h(b)$ is easily verified since $h(a \vee b) = [\![(a \vee b, (a \vee b)^*)]\!] = [\![(a \vee b, a^* \wedge b^*)]\!] = [\![(a, a^*) \sqcup (b, b^*)]\!] = [\![(a, a^*)]\!] \cup [\![(b, b^*)]\!] = h(a) \cup h(b)$.

Since $(a \Rightarrow b, (a \Rightarrow b)^*) \rightarrow_N (a \Rightarrow b, a \wedge b^*) = ((a \Rightarrow b) \Rightarrow_H (a \Rightarrow b), (a \Rightarrow b) \wedge a \wedge b^*) = (1, a \wedge b \wedge b^*) = (1, 0) = \top$ and $(a \Rightarrow b, a \wedge b^*) \rightarrow_N (a \Rightarrow b, (a \Rightarrow b)^*) = ((a \Rightarrow b) \rightarrow_N (a \Rightarrow b), (a \Rightarrow b) \wedge (a \Rightarrow b)^*) = (1, 0) = \top$ we have by Lemma 2.7 (h) that $h(a \Rightarrow b) = [\![(a \Rightarrow b, (a \Rightarrow b)^*) = [\![(a \Rightarrow b, a \wedge b^*)]\!] = [\![(a, a^*) \rightarrow (b, b^*)]\!] = [\![(a, a^*)]\!] \Rightarrow [\![(b, b^*)]\!] = h(a) \Rightarrow h(b)$.

Finally, h also preserves the constants: $h(0) = [\![(0, 0^*)]\!] = [\![(0, 1)]\!] = [\![\sim (1, 0)]\!] = [\![\sim \top]\!] = 0$ and $h(1) = [\![(1, 1^*)]\!] = [\![(1, 0)]\!] = [\![\top]\!] = 1$.

Assume now that for $a, b \in A$ we have that $h(a) = h(b)$. This means that $[\![(a, a^*)]\!] = [\![(b, b^*)]\!]$ thus by Lemma 2.7 (h), $(a, a^*) \rightarrow_N (b, b^*) = \top$ and $(b, b^*) \rightarrow_N (a, a^*) = \top$. Then $\top = (1, 0) = (a, a^*) \rightarrow_N (b, b^*) = (a \Rightarrow_H b, a \wedge b^*)$. Therefore $a \Rightarrow_H b = 1$. In a similar manner, $b \Rightarrow_H a = 1$, so $a = b$ and h es injective.

Let $[\![(a, b)]\!] \in V_k(A)/_{\equiv}$. Since $(a, a^*) \rightarrow_N (a, b) = (a \Rightarrow_H a, a \wedge b) = (1, 0) = \top$ and $(a, b) \rightarrow_N (a, a^*) = (a \Rightarrow_H a, a \wedge a^*) = (1, 0) = \top$, by Lemma 2.7 (h), it follows that $[\![(a, a^*)]\!] = [\![(a, b)]\!]$. Therefore $h(a) = [\![(a, b)]\!]$ so h is surjective. □

6 Deductive Systems and Congruences in $\mathcal{SN}$

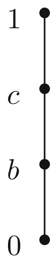
In this section we characterize the congruences in the variety $\mathcal{SN}$ in terms of a subclass of its deductive systems. The basic definitions and first results, however, are valid in the more general variety $\mathcal{PSN}$.

Definition 6.1 *Let* $\mathbf{A} = \langle A; 1, \sim, \wedge, \vee, \rightarrow\rangle \in \mathcal{PSN}$. *A subset* $D \subseteq A$ *is a deductive system of* A *if for all* $a, b \in A$*, the following conditions hold:*

*Ds*1) $1 \in D$.
*Ds*2) *If* $a, a \rightarrow b \in D$ *then* $b \in D$.

In the case of Nelson algebras, deductive systems are lattice filters. In the variety $\mathcal{SN}$, however, there are deductive systems that are not filters, as the following example shows.

$\mathbf{A}_1$:



→	0	1	b	c
0	1	1	1	c
1	0	1	b	c
b	1	1	1	c
c	b	1	b	1

∼	0	1	b	c
	1	0	c	b

The subset $D = \{0, b, 1\}$ of $\mathbf{A}_1$ is a deductive system but not a filter.

In order to characterize congruences for an algebra $\mathbf{A} \in \mathcal{SN}$, through some of its filters, we need the following definition:

Definition 6.2 *Let* $\mathbf{A} = \langle A; 1, \sim, \wedge, \vee, \rightarrow\rangle \in \mathcal{PSN}$. *A subset* $D \subseteq A$ *is called an* N*-deductive system of* A *if for all* $a, b \rightarrow_N A$*:*

$NDs1)$ $1 \in D$.
$NDs2)$ *If* $a, a \in b \in D$, *then* $b \in D$.

Since $\mathbf{A}$ with the implication $\rightarrow_N$ is a Nelson algebra, we know that the N-deductive systems of an algebra $\mathbf{A} \in \mathcal{PSN}$, ordered by inclusion, form a lattice, which from now on will be denoted by $\mathbf{Ded_N}(\mathbf{A})$. If $a \in A$ we denote by $D_N(a)$ the N-deductive system generated by a. By [26, Lemma 2.3],

$$D_N(a) = \{x \in A : \ a \rightarrow_N x = 1\}.$$

Notice that using Lemma 2.7 (g) it is easy to verify that every N-deductive system is a deductive system. The previous example also shows that the converse is not true.

Lemma 6.3 *Let* $\mathbf{A} = \langle A; 1, \sim, \wedge, \vee, \rightarrow\rangle \in \mathcal{PSN}$. *If* D *is an* N*-deductive system of* A *then* D *is a filter.*

Proof In Lemma 2.1 of [26] it is proved that every deductive system of a Nelson algebra is a filter. Therefore, by Theorem 2.5, the result follows. □

Also by the example above, since the chosen set D is not a filter, by Lemma 6.3, not every deductive system of a semi-Nelson algebra is a N-deductive system.

Let us now check that every congruence on a pre-semi-Nelson algebra determines an N-deductive system. From now on we denote by $\mathbf{Con}(\mathbf{A})$ the congruence lattice of an algebra $\mathbf{A}$. If $\Theta \rightarrow_N \mathbf{Con}(\mathbf{A})$ we denote by $[\![a]\!]_\Theta$ the class of an element $a \in A$ modulo Θ.

Lemma 6.4 *Let* $\mathbf{A} = \langle A; 1, \sim, \wedge, \vee, \rightarrow\rangle \in \mathcal{PSN}$. *If* $\Theta \in \mathbf{Con}(\mathbf{A})$, *then* $[\![1]\!]_\Theta$ *is an* N*-deductive system.*

Proof Clearly $1 \in [\![1]\!]_\Theta$. Let $a, b \in A$ be such that $a, a \rightarrow_N b \rightarrow_N [\![1]\!]_\Theta$. We check that $b \in [\![1]\!]_\Theta$. Since $a, a \rightarrow_N b \in [\![1]\!]_\Theta$, $[\![1]\!]_\Theta = [\![a]\!]_\Theta$ y $[\![1]\!]_\Theta = [\![a \rightarrow_N b]\!]_\Theta$. Then we get that $[\![a \wedge (a \rightarrow_N b)]\!]_\Theta = [\![1]\!]_\Theta$. By axiom (SN6),

$$[\![a \wedge (\sim a \vee b)]\!]_\Theta = [\![1]\!]_\Theta.$$

Furthermore, by $[\![1]\!]_\Theta = [\![a]\!]_\Theta$, we get that $[\![\sim 1]\!]_\Theta = [\![\sim a]\!]_\Theta$. Then, using Lemma 2.4 (d), $[\![\sim a \vee b]\!]_\Theta = [\![1 \wedge (\sim a \vee b)]\!]_\Theta = [\![a \wedge (\sim a \vee b)]\!]_\Theta = [\![1]\!]_\Theta$. Therefore $[\![b]\!]_\Theta = [\![\sim 1 \vee b]\!]_\Theta = [\![\sim a \vee b]\!]_\Theta = [\![1]\!]_\Theta$. □

The following is a well known result on Nelson algebras. See for example, [26], Lemma 2.12.

Lemma 6.5 *Let D be an N-deductive system. If $x \to_N y \in D$ and $y \to_N z \in D$, then $x \to_N z \to_N D$.*

Let us show that every N-deductive system determines a congruence on $A \in \mathcal{SN}$.

Lemma 6.6 *Let $\mathbf{A} = \langle A; 1, \sim, \wedge, \vee, \to \rangle \in \mathcal{PSN}$ and D be an N-deductive system of A. If $a, b \in A$, then*

$$a \to b \in D \textit{ and } b \to a \in D \textit{ if and only if } a \to_N b \in D \textit{ and } b \to_N a \in D.$$

Proof Assume that $a \to b \in D$ and $b \to a \in D$. By Lemma 2.7(g), $1 = (a \to b) \to_N (a \to_N b)$. Since $1 \in D$, $(a \to b) \to_N (a \to_N b) \in D$. From $a \to b \in D$ we conclude that $a \to_N b \in D$. In a similar manner, $b \to_N a \in D$.

For the converse, assume that $a \to_N b \in D$ and $b \to_N a \in D$. By axiom (SN9) we have that $1 = (a \to_N b) \to_N ((b \to_N a) \to_N ((a \to a) \to_N (a \to b)))$. Since $1 \in D$, $(a \to_N b) \to_N ((b \to_N a) \to_N ((a \to a) \to_N (a \to b))) \in D$. Therefore $(a \to a) \to_N (a \to b) \to_N D$ given that $a \to_N b \in D$ and $b \to_N a \in D$. By Lemma 2.7 (a), $a \to b \in D$. Similarly, one can prove that $b \to a \in D$. □

Lemma 6.7 *Let $\mathbf{A} = \langle A; 1, \sim, \wedge, \vee, \to \rangle \in \mathcal{SN}$ and D be an N-deductive system of A. The binary relation defined on A by $a \equiv_D b$ if and only if*

$$a \to b, b \to a, \sim a \to \sim b, \sim b \to \sim a \in D$$

is a congruence.

Proof Note that by Lemma 6.6, the definition of $\equiv_D$ is equivalent to

$$a \to_N b, b \to_N a, \sim a \to_N \sim b, \sim b \to_N \sim a \in D.$$

Since $\langle A; 1, \sim, \wedge, \vee, \to_N \rangle \in \mathcal{N}$, by [26], Lemma 2.13, $\equiv_D$ is compatible with the operations $\sim$, $\wedge$ and $\vee$. Let us check it is also compatible with $\to$. Let $a, a', b, b', c \in A$.

- If $a \equiv_D a'$ then $c \to a \equiv_D c \to a'$.

  By the hypothesis and Lemma 6.6, it follows that $a \to_N a' \in D$ and $a' \to_N a \in D$. By (SN9),

  $$(a \to_N a') \to_N [(a' \to_N a) \to_N [(c \to a) \to_N (c \to a')]] = 1 \in D,$$

  so

  $$(c \to a) \to_N (c \to a') \in D.$$

  In a similar way, one can prove

  $$(c \to a') \to_N (c \to a) \in D.$$

By (SN10), $(\sim (c \to a)) \to_N (c \wedge \sim a) = 1 \in D$. From the hypothesis it follows that $(c \wedge \sim a) \equiv_D (c \wedge \sim a')$. Using Lemma 6.5, and (SN11) $(c \wedge \sim a') \to_N (\sim (c \to a')) = 1 \in D$, we get

$$\sim (c \to a) \to_N (\sim (c \to a')) \in D.$$

Finally,

$$\sim (c \to a') \to_N \sim (c \to a) \in D$$

is obtained in a similar fashion.

- If $a \equiv_D a'$ then $a \to c \equiv_D a' \to c$.
  From (SN8), $(a \to_N a') \to_N [(a' \to_N a) \to_N [(a \to c) \to_N (a' \to c)]] = 1 \in D$, so by the hypotesis,

$$(a \to c) \to_N (a' \to c) \in D.$$

Similarly,

$$(a' \to c) \to_N (a \to c) \in D$$

obtains. Finally, from the hypothesis $a \wedge \sim c \equiv_D a' \wedge \sim c$, it follows as in the first part that

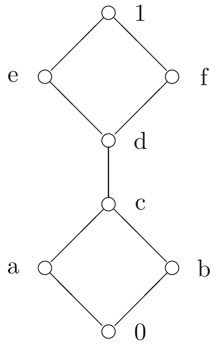
$$\sim (a \to c) \to_N \sim (a' \to c) \in D.$$

and

$$\sim (a' \to c) \to_N \sim (a \to c) \in D.$$

From the calculations above, we can conclude that if $a \equiv_D a'$ and $b \equiv_D b'$ then $a \to b \equiv_D a \to b' \equiv_D a' \to b'$. □

The next example shows that N-deductive systems do not determine congruences in pre semi-Nelson algebras in this fashion.



→	0	a	b	c	d	e	f	1	∼
0	1	1	1	1	0	0	0	0	1
a	1	1	1	1	a	0	0	0	f
b	1	1	1	1	0	0	0	0	e
c	1	1	1	1	0	0	0	0	d
d	c	c	c	c	1	1	1	1	c
e	b	c	b	c	f	1	f	1	b
f	a	a	c	c	e	e	1	1	a
1	0	a	b	c	d	e	f	1	0

Here we have that setting $D = \{e, 1\}$, $a \equiv_D c$ and $d \equiv_D f$, but $a = a \to d \not\equiv_D c \to f = 0$, since $\sim 0 \to \sim a = 1 \to f = f \notin D$.

Lemma 6.8 *Let* $\mathbf{A} = \langle A; 1, \sim, \wedge, \vee, \to \rangle \in \mathcal{SN}$ *and let D be an N-deductive system of A. Then* $[\![1]\!]_{\equiv_D} = D$.

Proof Let $a \in [\![1]\!]_{\equiv_D}$. Then $a \equiv_D 1$, so $1 \to a = a \in D$.

To check that the converse holds, consider $a \in D$. Notice that by Lemma 2.4 (f) and (c), $a \to_N 1 = 1 \in D$ and $1 \to_N a = a \in D$. Furthermore, by Lemma 2.6 (m) we have that $\sim 1 \to_N \sim a = 1 \in D$. By Lemma 6.6 it remains to check that $\sim a \to_N \sim 1 \in D$. By Lemma 2.6 (q), $(a \wedge \sim a) \to_N \sim 1 = 1$. Thus, by axiom (SN7), $a \to_N (\sim a \to_N \sim 1) = 1 \in D$. Since $a \in D$ and D is an N-deductive system, $\sim a \to_N \sim 1 \in D$. □

Given a congruence Θ we write $a\ \Theta\ b$ to denote $\langle a, b\rangle \in \Theta$.

Lemma 6.9 *Let* $\mathbf{A} = \langle A; 1, \sim, \wedge, \vee, \rightarrow\rangle \in \mathcal{SN}$ *and* $\Theta \in \mathbf{Con}(\mathbf{A})$. *If*

$$a \rightarrow_N b\ \Theta\ \sim b \rightarrow_N \sim a\ \Theta\ b \rightarrow_N a \Theta \sim a \rightarrow_N \sim b \Theta 1$$

then $a\ \Theta\ b$.

Proof Note that

$$a = a \wedge 1\ \Theta\ a \wedge (a \rightarrow_N b) = a \wedge (\sim a \vee b)$$

and

$$\sim b = \sim b \wedge 1 \Theta \sim b \wedge (\sim b \rightarrow_N \sim a) = \sim b \wedge (\sim\sim b \vee \sim a) = \sim b \wedge (b \vee \sim a).$$

Since $\mathbf{A} = \langle A; 1, \sim, \wedge, \vee, \rightarrow_N\rangle \in \mathcal{N}$, $\langle A; 1, \sim, \wedge, \vee\rangle$ is a de Morgan algebra [18]. Therefore

$$b = \sim\sim b\ \Theta\ \sim(\sim b \wedge (b \vee \sim a)) = \sim\sim b \vee \sim(b \vee \sim a) = b \vee (\sim b \wedge \sim\sim a) = b \vee (\sim b \wedge a).$$

Now we can calculate: $a \wedge b\ \Theta\ a \wedge (\sim a \vee b) \wedge (b \vee (\sim b \wedge a)) = [(a \wedge \sim a) \vee (a \wedge b)] \wedge [(b \vee \sim b) \wedge (b \vee a)] = [(a \wedge \sim a) \wedge (b \vee \sim b) \wedge (b \vee a)] \vee [(a \wedge b) \wedge (b \vee \sim b) \wedge (b \vee a)] = [(a \wedge \sim a) \wedge (b \vee a)] \vee [(a \wedge b) \wedge (b \vee \sim b)] = (a \wedge \sim a) \vee (a \wedge b) = a \wedge (\sim a \vee b) \Theta a$. Then $(a \wedge b)\ \Theta\ a$. In a similar way, using that $1\ \Theta\ b \rightarrow_N a\ \Theta\ \sim a \rightarrow_N \sim b$, we can prove that $(a \wedge b) \Theta b$. Thus, $a\ \Theta\ b$. □

Lemma 6.10 *Let* $\mathbf{A} = \langle A; 1, \sim, \wedge, \vee, \rightarrow\rangle \in \mathcal{SN}$ *and* $\Theta \in \mathbf{Con}(\mathbf{A})$. *Then*

$$\equiv_{[\![1]\!]_\Theta} = \Theta.$$

Proof We shall prove first that $\equiv_{[\![1]\!]_\Theta} \subseteq \Theta$. Let $a, b \in A$ be such that $a \equiv_{[\![1]\!]_\Theta} b$. Then by Lemmas 6.4 and 6.6, $a \rightarrow_N b, \sim b \rightarrow_N \sim a, b \rightarrow_N a, \sim a \rightarrow_N \sim b \rightarrow_N [\![1]\!]_\Theta$. Then $a \rightarrow_N b\ \Theta\ \sim b \rightarrow_N \sim a\ \Theta\ b \rightarrow_N a\ \Theta\ \sim a \rightarrow_N \sim b\ \Theta\ 1$. By Lemma 6.9, then $a\ \Theta\ b$.

We prove now that $\Theta \subseteq \equiv_{[\![1]\!]_\Theta}$. Let $a, b \rightarrow_N A$ be such that $a \Theta b$. Then $a \rightarrow_N b\ \Theta\ b \rightarrow_N b = 1$ and $\sim b \rightarrow_N \sim a\ \Theta\ \sim a \rightarrow_N \sim a = 1$. In a similar manner, $b \rightarrow_N a\ \Theta\ 1$ and $\sim a \rightarrow_N \sim b\ \Theta\ 1$. By Lemma 6.6 we have that $\langle a, b\rangle \in \equiv_{[\![1]\!]_\Theta}$. □

From the previous results, the next theorem follows.

Theorem 6.11 *Let* $\mathbf{A} = \langle A; 1, \sim, \wedge, \vee, \rightarrow\rangle \in \mathcal{SN}$. *Then the lattices* $\mathbf{Ded_N}(\mathbf{A})$ *and* $\mathbf{Con}(\mathbf{A})$ *are isomorphic.*

Proof By lemmas 6.4, 6.7, 6.8 and 6.10, the application $\Phi : \mathbf{Con}(\mathbf{A}) \rightarrow \mathbf{Ded_N}(\mathbf{A})$ defined by $\Phi(\Theta) = [\![1]\!]_\Theta$ is a bijection. It is easy to check that $D \subseteq D'$ if and only if $\equiv_D \subseteq \equiv_{D'}$. □

From Theorem 6.11 and [22, Theorem 2.1] the following theorem obtains.

Theorem 6.12 *Let* $\mathbf{A} \in \mathcal{SN}$. *Then the following conditions are equivalent:*

(a) **A** *is subdirectly irreducible.*
(b) **A** *has a single atom.*
(c) **A** *has a single coatom.*

Theorem 6.13 *The variety* $\mathcal{PSN}$ *is congruence-permutable.*

Proof Consider the term

$$p(x, y, z) = [(x \rightarrowtail y) \to_N z] \wedge [(z \rightarrowtail y) \to_N x] \wedge (x \vee z).$$

Recall that $x \rightarrowtail y$ is the term $(x \to y) \wedge (\sim y \to \sim x)$. Notice that

$$\begin{aligned} p(x, y, y) &= [(x \rightarrowtail y) \to_N y] \wedge [(y \rightarrowtail y) \to_N x] \wedge (x \vee y) && \\ &= [(x \rightarrowtail y) \to_N y] \wedge [1 \to_N x] \wedge (x \vee y) && \text{by Lemma 2.7 a} \\ &= [(x \rightarrowtail y) \to_N y] \wedge x \wedge (x \vee y) && \text{by Lemma 2.4 (f)} \\ &= x \wedge (x \vee y) && \text{by Lemma 2.7 (k)} \\ &= x. \end{aligned}$$

In a similar manner one checks that $p(x, x, y) = y$. Then the term $p(x, y, z)$ is a Mal'cev's term so the variety $\mathcal{PSN}$ is congruence-permutable. □

Since the algebras of the variety $\mathcal{PSN}$ have lattices as reducts, $\mathcal{PSN}$ is congruence-distributive. Therefore, from the theorem above it follows:

Corollary 6.14 *The variety $\mathcal{PSN}$ is arithmetical.*

We prove next two important properties of the variety of semi-Nelson algebras. From now on, $\Theta(a, b)$ will denote the congruence generated by $\langle a, b\rangle$.

Lemma 6.15 *Let $\mathbf{A} \in \mathcal{SN}$. Then $\Theta(c, d)$ coincides with the congruence associated to the N-deductive system $D_N(e)$ from Lemma 6.7, where $e = (c \to_N d) \wedge (d \to_N c) \wedge (\sim c \to_N \sim d) \wedge (\sim d \to_N \sim c)$.*

Proof We want to prove that $\Theta(c, d)$ is the congruence $\equiv_{D_N(e)}$. We shall follow the next steps to verify that $\equiv_{D_N(e)}$ is the least congruence containing the pair $\langle c, d\rangle$.

- Notice that $e \leq c \to_N d$. By Lemma 2.6 (k) $e \to_N (c \to_N d) = 1 \in D_N(e)$. Since $e \in D_N(e)$ and $D_N(e)$ is an N-deductive system, $c \to_N d \in D_N(e)$. In a similar manner, $d \to_N c, \sim c \to_N \sim d, \sim d \to_N \sim c \in D_N(e)$. By Lemma 6.6, $c \to d, d \to c, \sim c \to \sim d, \sim d \to \sim c \to_N D_N(e)$. Therefore $c \equiv_{D_N(e)} d$.
- Let $\Psi \in \mathbf{Con}(\mathbf{A})$ be such that $c\Psi d$. We want to check that $\equiv_{D_N(e)} \subseteq \Psi$. Let $\langle a, b\rangle \in \equiv_{D_N(e)}$. Then $a \to b, b \to a, \sim a \to \sim b, \sim b \to \sim a \in D_N(e)$. By Lemma 6.6, $a \to_N b, b \to_N a, \sim a \to_N \sim b, \sim b \to_N \sim a \in D_N(e)$. Since $D_N(e)$ is a lattice filter, $f = (a \to_N b) \wedge (b \to_N a) \wedge (\sim a \to_N \sim b) \wedge (\sim b \to_N \sim a) \in D_N(e) = \{x : e \to_N x = 1\}$. Therefore $e \to_N f = 1$. Since $c\Psi d$, we have that $c \to_N d \;\Psi\; \sim d \to_N \sim c \;\Psi\; d \to_N c \;\Psi\; \sim c \to_N \sim d \;\Psi\; 1$. Thus $e\Psi 1$ and therefore $e \in [\![1]\!]_\Psi$. Since $e \to_N f = 1 \in [\![1]\!]_\Psi$, then $f \in [\![1]\!]_\Psi$. Since $[\![1]\!]_\Psi$ is a lattice filter, we can conclude that $a \to_N b \;\Psi\; \sim b \to_N \sim a \;\Psi\; b \to_N a \;\Psi\; \sim a \to_N \sim b \;\Psi\; 1$. Then by Lemma 6.9, $a\Psi b$.

□

Definition 6.16 [13] *A variety $\mathcal{V}$ is said to have equationally definable principal congruences (EDPC) if there are 4-ary terms, $\sigma_i(x, y, z, w), \tau_i(x, y, z, w), i = 0, \ldots, n-1$ for some natural number n, such that for every algebra $\mathbf{A} \to_N \mathcal{V}$, and for all $a, b, c, d \to_N A$,*

$$c \equiv d(\Theta(a, b)) \text{ iff } \mathbf{A} \models \sigma_i(a, b, c, d) \approx \tau_i(a, b, c, d), 0 \leq i < n.$$

Theorem 6.17 *The variety $\mathcal{SN}$ has equationally definable principal congruences.*

Proof Let $\langle a, b\rangle \in \Theta(c, d)$. Notice that the previous condition is equivalent to

$$\langle (a \to_N b) \wedge (b \to_N a) \wedge ({\sim} a \to_N {\sim} b) \wedge ({\sim} b \to_N {\sim} a), 1\rangle \in \Theta(c, d).$$

By Lemma 6.15,

$$\Theta(c, d) = \equiv_{D_N(e)}$$

with $e = (c \to_N d) \wedge (d \to_N c) \wedge ({\sim} c \to_N {\sim} d) \wedge ({\sim} d \to_N {\sim} c)$. Thus, $\langle (a \to_N b) \wedge (b \to_N a) \wedge ({\sim} a \to_N {\sim} b) \wedge ({\sim} b \to_N {\sim} a), 1\rangle \in \Theta(c, d)$ if and only if $\langle (a \to_N b) \wedge (b \to_N a) \wedge ({\sim} a \to_N {\sim} b) \wedge ({\sim} b \to_N {\sim} a), 1\rangle \in \equiv_{D_N(e)}$ or, equivalently,

$$e \to_N [(a \to_N b) \wedge (b \to_N a) \wedge ({\sim} a \to_N {\sim} b) \wedge ({\sim} b \to_N {\sim} a)] = 1.$$

□

Using a result of Day [10], by Theorem 6.17, we get the following result.

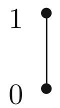
Corollary 6.18 *The variety $\mathcal{SN}$ has the CEP.*

We have already shown that the variety $\mathcal{PSN}$ is congruence distributive, but for the subvariety $\mathcal{SN}$, this is also a consequence of having equationally definable principal congruences, [13].

7 Semisimple Subvarieties of Semi-Nelson Algebras

Let us consider the following algebras:

$\mathbf{B}_1$:



$\to$	0	1
0	1	1
1	0	1

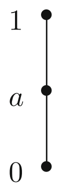
$\sim$	0	1
	1	0

$\mathbf{T}_1$:

$\to$	0	a	1
0	1	1	1
a	1	1	1
1	0	a	1

$\sim$	0	a	1
	1	a	0

$\mathbf{T}_2$:



$\to$	0	a	1
0	1	1	a
a	1	1	a
1	0	a	1

$\sim$	0	a	1
	1	a	0

It is easy to verify that the algebras $\mathbf{B}_1$, $\mathbf{T}_1$ and $\mathbf{T}_2$ are simple semi-Nelson algebras. Furthermore, $\mathbf{T}_1$ is quasiprimal by Theorem 10.7 of the book [6], page 173. Also, $\mathbf{T}_2$ is primal by Corollary 10.8 of Chapter IV [6]. We shall prove these are the only simple algebras in the variety $\mathcal{SN}$ and obtain as a consequence a complete description of the lattice of the semi-simple subvarieties of $\mathcal{SN}$.

Note that in both $\mathbf{B}_1$ and $\mathbf{T}_1$, $x \to y \approx x \to_N y$ holds and therefore, by Theorem 2.8, they are Nelson algebras.

Theorem 7.1 *Let* $\mathbf{A} = \langle A; \to, \wedge, \vee, \sim, 1\rangle$ *be a simple semi-Nelson algebra. Then* $\mathbf{A}$ *is (isomorphic to) one of the algebras* $\mathbf{B}_1$*,* $\mathbf{T}_1$ *or* $\mathbf{T}_2$*.*

Proof Let $\mathbf{A} = \langle A; \to, \wedge, \vee, \sim, 1\rangle$ be a simple semi-Nelson algebra. By Lemma 2.5 $\langle A; \to_N, \wedge, \vee, \sim, 1\rangle$ is a Nelson algebra. By Theorem 6.11 , $\langle A; \to_N, \wedge, \vee, \sim, 1\rangle$ is a simple Nelson algebra. By [26, Corollary 2.5], $\langle A; \to_N, \wedge, \vee, \sim, 1\rangle$ is isomorphic to $\mathbf{B}_1$ or $\mathbf{T}_1$. Next we consider cases based on the number of elements of A.

- Suppose $|A| = 2$. If $x \geq y$ we know that que $x \to y = x \to (x \wedge y) = x \to_N y$. So we know that the table of the operation $\to$ is of the form:

$\to$	0	1
0	1	?
1	0	1

  If $0 \to 1 = 0$ we have that $\sim (0 \to 1) \to_N (0 \wedge \sim 1) = \sim 0 \to_N 0 = 1 \to_N 0 = 0 \neq 1$. This calculation shows that axiom (SN10) does not hold in $\mathbf{A}$, a contradiction. Therefore, $0 \to 1 = 1$ and $\mathbf{A}$ is isomorphic to $\mathbf{B}_1$.
- Assume now that $|A| = 3$. If $x \geq y$ we can calculate $x \to y = x \to (x \wedge y) = x \to_N y$, so we know that the table of $\to$ has the form:

$\to$	0	a	1
0	1	?	?
a	1	1	?
1	0	a	1

  We note that $0 \to_N a = a \to_N 0 = 1$. Then by (SN9) we have that $(0 \to_N a) \to_N [(a \to_N 0) \to_N [(0 \to 0) \to_N (0 \to a)]] = 1$. Therefore, by Lemma 2.7 (a), $1 \to_N [1 \to_N [1 \to_N (0 \to a)]] = 1$. As a consequence, $0 \to a = 1$ and the table is now of the form:

$\to$	0	a	1
0	1	1	?
a	1	1	?
1	0	a	1

  as in the previous case, $0 \to 1 \neq 0$. In a similar manner, if $a \to 1 = 0$, we have that $\sim (a \to 1) \to_N (a \wedge \sim 1) = \sim 0 \to_N 0 = 1 \to_N 0 = 0 \neq 1$. Therefore $a \to 1 \neq 0$.

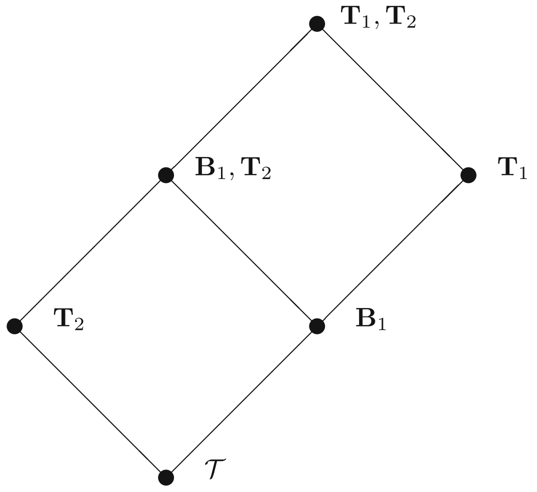
  If $a \to 1 = a$ then by (SN8), $(0 \to_N a) \to_N [(a \to_N 0) \to_N [(0 \to 1) \to_N (a \to 1)]] = 1$. As a consequence, $(0 \to 1) \to_N a = 1$. Then $0 \to 1 \neq 1$. Therefore, $0 \to 1 = a$ and, thus $\mathbf{A}$ is isomorphic to $\mathbf{T}_2$.

  On the other hand, if $a \to 1 = 1$ then since by axiom (SN8), $(a \to_N 0) \to_N [(0 \to_N a) \to_N [(a \to 1) \to_N (0 \to 1)]] = 1$ it follows that $0 \to 1 = 1$ and $\mathbf{A}$ is isomorphics to $\mathbf{T}_1$.

□

Theorem 7.2 *A subvariety $\mathcal{V}$ of $\mathcal{SN}$ is semi-simple if and only if $\mathcal{V} \subseteq \mathbb{V}(\mathbf{B}_1, \mathbf{T}_1, \mathbf{T}_2)$. Therefore:*

(a) *these are the filtral subvarieties of $\mathcal{SN}$ (*[11, Theorem 5.7]*).*
(b) *these are also the discriminator subvarieties of $\mathcal{SN}$ (*[5, Corollary 3.4]*).*
(c) *The lattice of semisimple subvarieties of $\mathcal{SN}$ is the one depicted in the following figure, where $\mathbf{A}_1, \ldots, \mathbf{A}_n$ denotes $\mathbb{V}(\mathbf{A}_1, \ldots, \mathbf{A}_n)$.*



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