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Two Applications of a Generalization of an Asymptotic Fixed Point Theorem

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Abstract We present a variant of an asymptotic version of the Abian-Brown Fixed Point Theorem, and applications to recursively defined sequences and Hammerstein integral equations.

Keywords Partially ordered sets · Fixed points · Monotone increasing functions

1 Asymptotic Fixed Point Theorems

Let $(\Omega, \leq)$ be a nonempty partially ordered set and for $a \in \Omega$, $M \subseteq \Omega$ let $M_a := \{x \in M \mid x \geq a\}$. A *chain* is a nonempty totally ordered subset of Ω, and $f : \Omega \to \Omega$ is called *increasing* if $x, y \in \Omega, x \leq y \Rightarrow f(x) \leq f(y)$. For $f : \Omega \to \Omega$ let $\mathrm{Fix}(f) := \{x \in \Omega \mid f(x) = x\}$. In particular $\mathrm{Fix}(f)_a = \{x \in \Omega \mid a \leq x, f(x) = x\}$.

In [4, Corollary 2.1] Heikkilä proved the following asymptotic variant of the Abian-Brown Fixed Point Theorem [1], see also [3] and [6].

Theorem 1 *Let $f : \Omega \to \Omega$ be increasing, let $a \leq f(a)$ for some $a \in \Omega$, and for some $n \in \mathbb{N}$ let each chain in $f^{(n)}(\Omega_a)$ have a supremum in Ω. Then* $\min(\mathrm{Fix}(f)_a)$ *exists, i.e. f has a unique smallest fixed point above a.*

Related fixed point theorems were found by Lemmert [5] (for the case $n = 1$) and Baranyai [2] (for the case that $f^{(n)}(\Omega)$ is contained in a complete lattice).

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In view of applications one would like to make a further generalization. Consider for example the following problem. Let $\mathbb{R}^d$ be partially ordered by the coordinatewise ordering, let $f : \mathbb{R}^d \to \mathbb{R}^d$ be increasing and let $f^{(n)}(\mathbb{R}^d)$ be bounded in norm for some $n \in \mathbb{N}$. Does f has a fixed point? Theorem 1 can be applied if there is some $a \in \mathbb{R}^d$ such that $a \leq f(a)$. This is not obvious in case $n \geq 2$. But it *is* obvious that $a \leq f^{(n)}(a)$ for some $a \in \mathbb{R}^d$. However replacing "$a \leq f(a)$" by "$a \leq f^{(n)}(a)$" in Theorem 1 does not work. Simply consider the ordered set $(\Omega, \leq) = (\{0, 1\}, =)$ and $f : \Omega \to \Omega$ defined by $f(0) = 1$, $f(1) = 0$, and set $n = 2$. On the other hand we will prove:

Theorem 2 *Let $f : \Omega \to \Omega$ be increasing, and let $n \in \mathbb{N}$ and $a \in \Omega$ be so that $a \leq f^{(n)}(a)$, and so that each chain in $f^{(n)}(\Omega_a)$ has a supremum in Ω. Moreover let each finite nonempty subset of* $\mathrm{Fix}(f^{(n)})$ *have a supremum in Ω. Then* $\min(\mathrm{Fix}(f)_a)$ *exists.*

Thus the answer to the question above is yes (see also Theorem 3 below).

Proof We first apply Theorem 1 to the function $f^{(n)} : \Omega \to \Omega$ and 1 (instead of f and n) and obtain $z_0 := \min(\mathrm{Fix}(f^{(n)})_a)$. Now we set

$$Q := \{z_0, f(z_0), \dots, f^{(n-1)}(z_0)\} \subseteq \mathrm{Fix}(f^{(n)}).$$

By assumption $b := \sup(Q)$ exists in Ω. From $a \leq z_0$ we get $a \leq b$. Moreover $f(Q) = Q$ and $f^{(k)}(z_0) \leq f(b)$ $(k = 1, \dots, n)$, and therefore $b \leq f(b)$. Again, by Theorem 1, we obtain $z_1 := \min(\mathrm{Fix}(f)_b)$. Now let $z \in \mathrm{Fix}(f)_a$. Then $z \in \mathrm{Fix}(f^{(n)})_a$, thus $z_0 \leq z$. Hence $f^{(k)}(z_0) \leq f^{(k)}(z) = z$ $(k = 1, \dots, n)$, and therefore $b \leq z$. Thus $\mathrm{Fix}(f)_a \subseteq \mathrm{Fix}(f)_b$. Since $a \leq b$ we also have $\mathrm{Fix}(f)_b \subseteq \mathrm{Fix}(f)_a$. This proves $z_1 = \min(\mathrm{Fix}(f)_a)$. □

2 Two Applications

Let T be any nonempty set, and let $\Omega = \mathbb{R}^T$ be the lattice of all functions $x : T \to \mathbb{R}$ ordered by $x \leq y :\Leftrightarrow x(t) \leq y(t)$ $(t \in T)$. It is obvious that each nonempty set in $\mathbb{R}^T$ which is pointwise bounded has a supremum and an infimum: if M is such a set, then the function $t \mapsto s(t) := \sup_{x \in M} x(t)$ is the supremum of M. As a consequence of Theorem 2 we have the following fixed point theorem.

Theorem 3 *Let the function $f : \mathbb{R}^T \to \mathbb{R}^T$ be increasing and let $f^{(n)}(\mathbb{R}^T)$ be pointwise bounded for some $n \in \mathbb{N}$. Then* $\mathrm{Fix}(f) \neq \emptyset$.

Proof of Theorem 3 Let $a := \inf(f^{(n)}(\mathbb{R}^T))$ and apply Theorem 2. □

Note that Theorem 3 also can be proved by Baranyai's Fixed Point Theorem [2], since $f^{(n)}(\mathbb{R}^T)$ is contained in the order interval $\{x \in \mathbb{R}^T \mid \inf(f^{(n)}(\mathbb{R}^T)) \leq x \leq \sup(f^{(n)}(\mathbb{R}^T))\}$, which is a complete lattice.

Theorem 3 can be applied to recursively defined sequences on $T = \mathbb{N}$. Let $r = (r_k)_{k=1}^{\infty}$ be a sequence in $(0, \infty)$. We claim that there exists $x_1 \in \mathbb{R}$ such that the recursion

$$x_{k+1} = \frac{(-1)^{k+1}}{r_k} \log((-1)^{k+1} x_k) \quad (k \in \mathbb{N}) \tag{1}$$

defines a bounded sequence of real numbers. Indeed, consider $f : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ defined by

$$f(x) = (\exp(r_1 x_2), -\exp(-r_2 x_3), \exp(r_3 x_4), -\exp(-r_4 x_5), \dots).$$

The function f is increasing and

$$f^{(2)}(\mathbb{R}^{\mathbb{N}}) \subseteq [0,1] \times [-1,0] \times [0,1] \times [-1,0] \times \dots .$$

In particular $f^{(2)}(\mathbb{R}^{\mathbb{N}})$ is pointwise bounded. According to Theorem 3 f has a fixed point, which is easily checked to be a solution of (1).

As a second application consider a compact interval $I \subseteq \mathbb{R}$ of length $|I|$, $\Omega = C(I, \mathbb{R}^d)$ the Banach lattice of all continuous functions $x : I \to \mathbb{R}^d$ endowed with the maximum norm $\|x\|_\infty = \max_{t\in I} \|x(t)\|$ (where $\|\cdot\|$ is the maximum norm on $\mathbb{R}^d$) and ordered by $x \leq y :\Leftrightarrow x_k(t) \leq y_k(t)$ $(t \in I, k = 1, \dots, d)$. It is easy to check that each nonempty, relatively compact set in $C(I, \mathbb{R}^d)$ has a supremum (and an infimum). In particular, each nonempty and finite set has a supremum. As a consequence of Theorem 2 we have the following fixed point theorem.

Theorem 4 *Let the function* $f : C(I, \mathbb{R}^d) \to C(I, \mathbb{R}^d)$ *be increasing and let* $f^{(n)}(C(I, \mathbb{R}^d))$ *be relatively compact for some* $n \in \mathbb{N}$. *Then* $\mathrm{Fix}(f) \neq \emptyset$.

Proof of Theorem 4 Let $a := \inf(f^{(n)}(C(I, \mathbb{R}^d)))$ and apply Theorem 2. □

For an application of Theorem 4 let the functions $g_k : \mathbb{R} \to \mathbb{R}$ $(k = 1, \dots, d)$ be increasing, and assume that there exist $k_1, k_2 \in \{1, \dots, d\}$ such that g_{k_1} is bounded below and g_{k_2} is bounded above. Let $K : I^2 \to [0, \infty)$ be continuous and consider $f : C(I, \mathbb{R}^d) \to C(I, \mathbb{R}^d)$ defined by

$$f(x)(t) = \int_I K(t,s)(g_1(x_2(s)), g_2(x_3(s)), \dots, g_{d-1}(x_d(s)), g_d(x_1(s)))ds.$$

Note that the integral is understood in the sense of Lebesgue, since the conjunction of a monotone and a continuous function is not Riemann-integrable, in general.

Clearly f is increasing. From the boundedness assumptions on g_{k_1}, g_{k_2} we get that $f^{(d)}(x)(t)$ is of the form $\int_I K(t,s)h(x)(s)ds$ with $h(x) : I \to \mathbb{R}^d$ measurable and uniformly bounded in x and s, i.e. $\|h(x)(s)\| \leq c$ $(x \in C(I, \mathbb{R}^d), s \in I)$ for some $c > 0$. Thus $f^{(d)}(C(I, \mathbb{R}^d))$ is bounded. If $\varepsilon > 0$ we find some $\delta > 0$ such that $|K(t_1, s) - K(t_2, s)| \leq \varepsilon/(c|I|)$ if $|t_1 - t_2| \leq \delta$ and $s \in I$. For each $y \in f^{(d)}(C(I, \mathbb{R}^d))$ we get $\|y(t_1) - y(t_2)\| \leq \varepsilon$ if $|t_1 - t_2| \leq \delta$. Thus $f^{(d)}(C(I, \mathbb{R}^d))$ is equicontinuous. According to Arzelà-Ascoli's Theorem $f^{(d)}(C(I, \mathbb{R}^d))$ is relatively compact.

Application of Theorem 4 with $n = d$ proves $\mathrm{Fix}(f) \neq \emptyset$, and each $x = (x_1, \dots, x_d) \in \mathrm{Fix}(f)$ is a continuous solution of the Hammerstein integral equation

$$x(t) = \int_I K(t,s)(g_1(x_2(s)), g_2(x_3(s)), \dots, g_{d-1}(x_d(s)), g_d(x_1(s)))ds \quad (t \in I).$$

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