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The Lattices of Kernel Ideals in Pseudocomplemented De Morgan Algebras

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Abstract In a pseudocomplemented de Morgan algebra, it is shown that the set of kernel ideals is a complete Heyting lattice, and a necessary and sufficient condition that the set of kernel ideals is boolean (resp. Stone) is derived. In particular, a characterization of a de Morgan Heyting algebra whose congruence lattice is boolean (resp. Stone) is given.

Keywords De Morgan algebra · Distributive p-algebra · Heyting algebra · Stone lattice · Congruence · Kernel ideal

1 Introduction

It is well-known [3] that the lattice $I(L)$ of ideals on a bounded distributive lattice L is a complete Heyting lattice. Hence $I(L)$ belongs to the equational class B_ω of all distributive pseudocomplemented algebras. We know that the lattice of equational class B_ω is the chain

$$B_0 \subset B_1 \subset \cdots \subset B_n \subset \cdots \subset B_\omega$$

of type $\omega+1$, where B_0 is the class of Boolean algebras and B_1 is the class of Stone algebras. An open question that was posed by Grätzer [8] is the following: *Give a characterization*

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of those bounded distributive lattices whose ideal lattice belongs to B_n $(n \geq 1)$. By a well-known fact, the ideal lattice of a bounded distributive lattice L is boolean if and only if L is a finite boolean lattice. In 1983, Beazer [1] showed that the ideal lattice of a bounded distributive lattice L is Stone if and only if L is a Stone lattice whose centre is complete. In particular, he also described the lattice of congruence kernels which is a Stone algebra in the context of distributive p-algebras and distributive double p-algebras, respectively. In 2011, Nimbhorkar and Rahemani [10] gave a description of lattice of ideals which is a Stone algebra on the context of pseudocomplemented modular join-semilattices with 0 and 1. In [6], Chajda et al. showed that the set of kernel ideals of the context of pseudocomplemented semilattices is a complete pseudocomplemented lattice.

We know that the notion of a de Morgan Heyting algebra was first introduced by Monteiro [9] as the algebraic counterpart of the symmetric modal propositional calculus of Moisil. Romanowska [11] initiated a study of the variety of pseudocomplemented de Morgan algebras by characterizing the finite subdirectly irreducible algebras. More contributions concerning these algebras can be found in [4, 5, 12, 13]. Therefore, an interesting problem is what are the characterizations of the lattice of kernel ideals in the context of pseudocomplemented de Morgan algebras.

The paper is organized as follows. For the sake of convenience, some notions and basic results which will be used in this note are given in Section 2. In Section 3, we show that the set $I_k(L)$ of kernel ideals in a pseudocomplemented de Morgan algebra $(L;^\circ,^*)$ is a complete Heyting lattice, $I_k(L)$ is boolean if and only if L is of finite range and $Z(L)$ is finite, and $I_k(L)$ is Stone if and only if both $\bigwedge_{n\geq 0}(a \vee a^{*\circ})^{*n(\circ *)}$ and $\bigwedge S$ exist in L for every $a \in L$ and $S \subseteq Z(L)$, where $Z(L) = \{x \in L \mid x \wedge x^\circ = 0\}$. In Section 4, we give a characterization of a de Morgan Heyting algebra whose congruence lattice is boolean and Stone, respectively. Conclusions are given in Section 5.

2 Preliminaries

In this section, certain definitions and basic results are collected and presented from [2, 3, 9, 11–13].

Definition 2.1 ([2]). A *de Morgan algebra* is an algebra $L \equiv (L; \wedge, \vee, f, 0, 1)$ of type $\langle 2, 2, 1, 0, 0\rangle$ where $(L; \wedge, \vee, 0, 1)$ is a bounded distributive lattice and f is an unary operation on L such that

(M_1) $f(0) = 1, f(1) = 0$;
(M_2) $(\forall x \in L)$ $f^2(x) = x$;
(M_3) $(\forall x, y \in L)$ $f(x \vee y) = f(x) \wedge f(y)$;
(M_4) $(\forall x, y \in L)$ $f(x \wedge y) = f(x) \vee f(y)$.

For convenience in what follows, we shall write x° for $f(x)$.

Definition 2.2 ([3]). A *(distributive) p-algebra* (or *a lattice with pseudocomplementation*) is a (distributive) lattice L with a smallest element 0 together with a mapping $^* : L \to L$ such that $x \wedge y = 0 \iff y \leq x^*$. A *Stone algebra* is a distributive p-algebra satisfying the Stone identity: $x^* \vee x^{**} = 1$.

Definition 2.3 ([3]). A *Heyting algebra* is an algebra $(L; \wedge, \vee, \rightarrow)$, in which L is a bounded distributive lattice, and $\rightarrow$ is a binary operation on L such that $x \rightarrow y$ is the relative pseudocomplement of x in y, in the sense that

$$x \wedge z \leq y \iff z \leq x \rightarrow y.$$

In particular, $x \rightarrow 0$ is the pseudocomplement of x, denoted by x^*.

Lemma 2.1 *([3]). The following identities hold in a Heyting algebra L:*
(H_1) $x \rightarrow x = 1$;
(H_2) $x \wedge (x \rightarrow y) = x \wedge y$;
(H_3) $x \wedge (y \rightarrow z) = x \wedge [(x \wedge y) \rightarrow (x \wedge z)]$.

Definition 2.4 ([9]). A *de Morgan Heyting algebra* is an algebra $(L; \wedge, \vee, {}^\circ, \rightarrow, 0, 1)$, in which $(L; {}^\circ)$ is a de Morgan algebra and $(L; \rightarrow)$ is a Heyting algebra.

Definition 2.5 ([11]). A *pesudocomplemented de Morgan* (simply, pM-algebra) is an algebra $(L; \wedge, \vee, {}^\circ, {}^*, 0, 1)$, in which $(L; {}^\circ)$ is a de Morgan algebra and $(L; {}^*)$ is a distributive p-algebra.

Clearly, every de Morgan Heyting algebra is a pM-algebra. Let L be a de Morgan Heyting algebra. For each $a \in L$, $a^{*\circ}$ is defined inductively as $a^{0(*\circ)} = a$, $a^{(n+1)(*\circ)} = a^{n(*\circ)*\circ}$ for every $n \geq 0$; and $a^{n(\circ *)}$ is defined in a similar fashion. In particular, for $x \in L$, if there exists some $n \geq 0$ such that $(x \vee x^{*\circ})^{n(*\circ)} = (x \vee x^{*\circ})^{(n+1)(*\circ)}$, we say that L is of *finite range*. As shown in [11], if we define $x^+ = x^{\circ * \circ}$, then x^+ is the dual pseudocomplement of x in L. Throughout what follows, we shall denote by $\mathrm{Cen}(L)$ the sublattice of complements of L.

Lemma 2.2 *([11, 12]). Let $(L; {}^\circ, {}^*)$ be a pM-algebra. Then*

(1) $(\forall x \in L)\ x \leq x^{**}$;
(2) $(\forall x, y \in L)\ x \leq y \Rightarrow x^* \geq y^*$;
(3) $(\forall x, y \in L)\ (x \vee y)^* = x^* \wedge y^*$;
(4) $(\forall x, y \in L)\ (x \vee y)^{*\circ} = x^{*\circ} \vee y^{*\circ}$;
(5) $(\forall x \in L)\ x \vee x^{*\circ} \leq (x \vee x^{*\circ})^{*\circ} \leq \cdots \leq (x \vee x^{*\circ})^{n(*\circ)} \leq \cdots$;
(6) $Cen(L) = \{x \in L \mid x = x^{*\circ*\circ}\} = \{x \in L \mid x^{\circ *} = x^{*\circ}\}$.

Definition 2.6 ([2, 3]). A *de Morgan-congruence* on a de Morgan algebra L is a lattice congruence ϑ on L such that $(x, y) \in \vartheta \Rightarrow (x^\circ, y^\circ) \in \vartheta$; a *p-congruence* on a distributive p-algebra L is a lattice congruence ϑ on L such that $(x, y) \in \vartheta \Rightarrow (x^*, y^*) \in \vartheta$; a *Heyting-congruence* on a Heyting algebra L is a lattice congruence ϑ on L such that $(x, y), (z, w) \in \vartheta \Rightarrow (x \rightarrow z, y \rightarrow w) \in \vartheta$.

Definition 2.7 ([12, 13]). A *HM-congruence* on a de Morgan Heyting algebra $(L; {}^\circ, \rightarrow)$ is a lattice congruence ϑ on L such that ϑ is both a de Morgan-congruence and a Heyting-congruence; a *pM-congruence* on a pM-algebra $(L; {}^\circ, {}^*)$ is a lattice congruence ϑ on L such that ϑ is both a de Morgan-congruence and a p-congruence.

Definition 2.8 ([3]). For a bounded distributive lattice L, a non-empty subset I of L is said to be *an ideal* if the following conditions hold:

(1) $(\forall x \in L)\ x \leq a \in I$ implies $x \in I$;
(2) $(\forall x, y \in L)\ x, y \in I$ implies $x \vee y \in I$.

Definition 2.9 ([3]). For a subset X of a bounded distributive lattice L, we shall denote by $(X]$ the *down-set* of X, where

$$(X] = \{a \in L \mid a \leq x \ \text{ for some } \ x \in X\}.$$

Clearly, $(X]$ is an order-ideal of L. In particular, if $X = \{x\}$, we shall write $(x]$ for $(\{x\}]$, and $(x]$ is called *a principal ideal* of L.

In what follows, we shall denote by $I(L)$ the lattice of ideals in a bounded distributive lattice L in which the lattice operations $\wedge$ and $\vee$ are given by

$$I \wedge J = I \cap J; \quad I \vee J = \{x \in L | (\exists i \in I, j \in J) \ \ x \leq i \vee j\}.$$

It is well-known [3] that $I(L)$ is a complete Heyting lattice in which for $I, J \in I(L)$,

$$I \to J = \{x \in L \mid (x] \cap I \subseteq J\}.$$

Definition 2.10 ([3]). An ideal I of a lattice-ordered algebra L is called *a kernel ideal* if there exists a congruence φ on L such that

$$I = \operatorname{Ker} \varphi \overset{def}{=} \{x \in L | x \overset{\varphi}{\equiv} 0\}.$$

Through what follows, we shall denote by $I_k(L)$ the set of kernel ideals of a lattice-ordered algebra L.

3 Kernel Ideals

In this section, we shall be concerned with the structure of the set $I_k(L)$ of kernel ideals on a pseudocomplemented de Morgan algebra $(L;^\circ, ^*)$. We now begin with the following observation.

Theorem 3.1 *Let L be a pM-algebra and let I be an ideal of L. Then I is a kernel ideal if and only if $x \in I$ implies $x^{*\circ} \in I$.*

Proof ($\Rightarrow$:) Let I be a kernel ideal of L. Then from Definition 2.10 there exists a pM-congruence θ on L such that $I = \operatorname{Ker} \theta$. If $x \in I = \operatorname{Ker} \theta$, then $x^{*\circ} \in \operatorname{Ker} \theta = I$.

($\Leftarrow$:) Suppose that the stated property holds. Define the binary relation $\theta(I)$ on L as follows

$$(\star) \ \ (x, y) \in \theta(I) \iff (\exists i \in I) \ \ x \wedge i^* = y \wedge i^*.$$

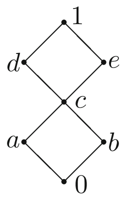
It is well-known [7] that $\theta(I)$ is a p-congruence on L such that $I = \operatorname{Ker} \theta(I)$. To show that I is a kernel ideal of L, it is enough to prove that $\theta(I)$ is a de Morgan-congruence. In fact, let $(x, y) \in \theta(I)$. Then there exists $i \in I$ such that $x \wedge i^* = y \wedge i^*$. Then $x^\circ \vee i^{*\circ} = (x \wedge i^*)^\circ = (y \wedge i^*)^\circ = y^\circ \vee i^{*\circ}$ and so by distributivity we have

$$x^\circ \wedge i^{*\circ*} = (x^\circ \vee i^{*\circ}) \wedge i^{*\circ*} = (y^\circ \vee i^{*\circ}) \wedge i^{*\circ*} = (y^\circ \wedge i^{*\circ*}) \vee (i^{*\circ} \wedge i^{*\circ*}) = y^\circ \wedge i^{*\circ*}.$$

By hypothesis, $i \in I$ implies $i^{*\circ} \in I$, from which it follows that $(x^\circ, y^\circ) \in \theta(I)$. Therefore, $\theta(I)$ is a de Morgan-congruence on L and consequently, I is a kernel ideal. □

Let L be a pM-algebra. Clearly, the trivial ideal $\{0\}$ and L are kernel ideals of L, respectively. We say a kernel ideal I of L is *non-trivial* if $I \in I_k(L)$ but $I \notin \{\{0\}, L\}$.

Example 3.1 *Consider the algebra* $(L; {}^\circ, {}^*)$ *described as follows:*



	0	a	b	c	d	e	1
$\circ$	1	d	e	c	a	b	0
$*$	1	b	a	0	0	0	0

It is easy to see that $(L; {}^\circ, {}^*)$ *is a* pM*-algebra. By a simple calculation we can see that* L *has no non-trivial kernel ideals and so* $I_k(L) = \{\{0\}, L\}$.

Theorem 3.2 *Let L be a* pM*-algebra. Then* $(I_k(L); \cap, \vee, \Rightarrow)$ *is a complete Heyting lattice in which for* $I, J \in I_k(L)$*, the relative pseudocomplement of* I *in* J *is given by*

$$I \Rightarrow J = \{x \in L \mid (\forall n \geq 0) \;\; ((x \vee x^{*\circ})^{n(*\circ)}] \cap I \subseteq J\}.$$

Proof Suppose first that $I, J \in I_k(L)$. Clearly, $I \cap J \in I_k(L)$. If $x \in I \vee J$, then there exist $i \in I$ and $j \in J$ such that $x \leq i \vee j$. Hence $x^{*\circ} \leq (i \vee j)^{*\circ} = i^{*\circ} \vee j^{*\circ}$. Since I and J are kernel ideals, we have $i^{*\circ} \in I$ and $j^{*\circ} \in J$. Hence we obtain $x^{*\circ} \in I \vee J$ whence $I \vee J \in I_k(L)$. Clearly, $\{0\}$ and L are the smallest and biggest elements in $I_k(L)$, respectively. Hence $(I_k(L); \cap, \vee, \{0\}, L)$ is a distributive sublattice of the lattice $I(L)$. As for the completeness, it is not hard to see that the infimum of a family of kernel ideals of L is also a kernel ideal. Hence $I_k(L)$ is complete.

Clearly, $0 \in I \Rightarrow J$ and so $I \Rightarrow J$ is non-empty. If $x \leq y \in I \Rightarrow J$, then for each $n \geq 0$, $(x \vee x^{*\circ})^{n(*\circ)} \leq (y \vee y^{*\circ})^{n(*\circ)}$ whence $x \in I \Rightarrow J$. If now $x, y \in I \Rightarrow J$, then for every $n \geq 0$, we have $((x \vee x^{*\circ})^{n(*\circ)}] \cap I \subseteq J$ and $((y \vee y^{*\circ})^{n(*\circ)}] \cap I \subseteq J$. Observe that

$$\begin{aligned} ((x \vee y \vee (x \vee y)^{*\circ})^{n(*\circ)}] \cap I &= ((x \vee y \vee x^{*\circ} \vee y^{*\circ})^{n(*\circ)}] \cap I \\ &= (((x \vee x^{*\circ})^{n(*\circ)}] \vee ((y \vee y^{*\circ})^{n(*\circ)}]) \cap I \\ &= (((x \vee x^{*\circ})^{n(*\circ)}] \cap I) \vee (((y \vee y^{*\circ})^{n(*\circ)}] \cap I) \\ &\subseteq J \vee J \\ &= J. \end{aligned}$$

It then follows that $x \vee y \in I \Rightarrow J$. Thus $I \vee J$ is an ideal of L. Clearly, $x \in I \Rightarrow J$ implies $x^{*\circ} \in I \Rightarrow J$ and so $I \Rightarrow J$ is a kernel ideal.

If there exists a kernel ideal K such that $I \cap K \subseteq J$ and $x \in K$, then for each $n \geq 0$, we have $(x \vee x^{*\circ})^{n(*\circ)} \in K$ and so $((x \vee x^{*\circ})^{n(*\circ)}] \cap I \subseteq J$. Thus we have $x \in I \Rightarrow J$ whence $K \subseteq I \Rightarrow J$. Therefore, $I \Rightarrow J$ is the relative pseudocomplement of I in J and consequently, $I_k(L)$ is a Heyting lattice. □

In what follows for a de Morgan algebra L, we shall write $Z(L) = \{x \in L \mid x \wedge x^\circ = 0\}$. We have the following:

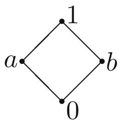
Lemma 3.1 *If L is a pM-algebra, then the following statements hold:*

(1) $(\forall x \in Z(L))\ x^\circ = x^*$;
(2) $(Z(L);^\circ)$ *is a boolean subalgebra of* $Cen(L)$.

Proof (1) For every $x \in Z(L)$, $x \wedge x^\circ = 0$ implies $x^\circ \leq x^*$; $x \vee x^\circ = 1$ implies $x^* = x^* \wedge 1 = x^* \wedge (x \vee x^\circ) = x^* \wedge x^\circ$, and so $x^* \leq x^\circ$. Thus we obtain that $x^\circ = x^*$.
(2) The argument is clear.

□

Example 3.2 *Consider the pM-algebra $(L;^\circ,^*)$ that is described as follows:*



	0	a	b	1
$\circ$	1	a	b	0
$*$	1	b	a	0

Clearly, $Z(L) = \{0, 1\}$ *and* $Cen(L) = \{0, a, b, 1\}$. *Hence* $Z(L) \neq Cen(L)$.

For each $I \in I_k(L)$ of a pM-algebra L, since $I_k(L)$ is a Heyting lattice by Theorem 3.2, the pseudocomplement I^* of I is

$$I^* = \{x \in L \mid (\forall n \geq 0)\ \ ((x \vee x^{*\circ})^{n(*\circ)}] \cap I = \{0\}\}$$

from which it follows that

$$I^* = \{x \in L \mid (\forall n \geq 0, \forall i \in I)\ \ ((x \vee x^{*\circ})^{n(*\circ)} \wedge i = 0\}.$$

Using this fact, we have the following

Theorem 3.3 *Let L be a pM-algebra and let $I \in I_k(L)$. Then I is complemented if and only if $I = (z]$ for some $z \in Z(L)$.*

Proof ($\Rightarrow$:) Suppose that I is a complemented kernel ideal of L. Then $I \vee I^* = L$. Then there exist $z \in I$ and $w \in I^*$ such that $z \vee w = 1$. Since $w \in I^*$, we have $(w \vee w^{*\circ}) \wedge z = 0$ and so $w \wedge z = 0$ and $w^{*\circ} \wedge z = 0$. Thus z and w are complementary. Note that $w \in \mathrm{Cen}(L)$ implies w^* is the complement of w, and since a complement is unique in a distributive lattice, so we obtain $z = w^*$. Since $w^{*\circ} \wedge z = 0$, we have that $w^{*\circ} \leq z^* = w^{**} = w$. Hence $z \wedge z^\circ = w^* \wedge w^{*\circ} \leq w^* \wedge w = 0$ and consequently, $z \in Z(L)$. For every $x \in I$, since $w \in I^*$, we have $w \wedge x = 0$ whence $x \leq w^* = z$. Then $I \subseteq (z]$. Since $z \in I$, we have $(z] \subseteq I$ and therefore, $I = (z]$ where $z \in Z(L)$.

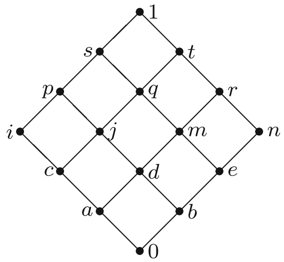
($\Leftarrow$:) If $I = (z]$ for some $z \in Z(L)$, then I is clearly an ideal of L. By Lemma 3.1 (1), $z \in Z(L)$ implies $z^{*\circ} = z^{\circ\circ} = z$, then I is a kernel ideal. Consider another kernel ideal $J = (z^\circ]$. Clearly,

$$I \vee J = (z] \vee (z^\circ] = (z \vee z^\circ] = (1] = L;$$

$$I \cap J = (z] \cap (z^\circ] = (z \wedge z^\circ] = (0] = \{0\}.$$

Therefore, I and J are complementary and consequently, I is complemented. □

Example 3.3 *Given a* pM*-algebra* $(L; ^\circ, ^*)$ *that is depicted as follows:*



	0	a	b	c	d	e	i	j	m	n	p	q	r	s	t	1
$\circ$	1	t	s	r	q	p	n	m	j	i	e	d	c	b	a	0
$*$	1	n	i	n	0	i	n	0	0	i	0	0	0	0	0	0

Consider the ideals $I = (i]$ *and* $J = (n]$*. It is not hard to verify that* I *and* J *are non-trivial kernel ideals and* $I \vee J = L$ *and* $I \cap J = \{0\}$*. Hence* I *and* J *are complementary.*

In order to further characterize the structure of the lattice of kernel ideals in a pseudocomplemented de Morgan algebra, we require the following lemmas.

Lemma 3.2 *If* L *is a* pM*-algebra, then every kernel ideal of* $Z(L)$ *is of the form* $I \cap Z(L)$ *where* $I \in I_k(L)$*.*

Proof Since for every $x \in Z(L)$, by Lemma 3.1(1), we have $x^{*\circ} = x$. Thus every ideal of $Z(L)$ is a kernel ideal. If $I \in I_k(L)$ then obviously, $I \cap Z(L)$ is a kernel ideal of $Z(L)$. Conversely, if I' is an ideal of $Z(L)$, let

$$I \overset{def}{=} (I'] = \{x \in L \mid (\exists a \in I')x \leq a\}.$$

Then clearly, I is a kernel ideal of L with $I' = I \cap Z(L)$. $\square$

For a non-empty subset A of a lattice L, in what follows we shall denote by $K(A)$ the kernel ideal that is generated by A. In particular, if $A = \{a\}$, we shall write $K(a)$ for $K(\{a\})$.

Lemma 3.3 *If* L *is a* pM*-algebra and let* A *be a non-empty subset of* L*, then*

$$(\dagger)\quad K(A) = \{x \in L \mid x \leq (a_1 \vee a_1^{*\circ} \vee \cdots \vee a_k \vee a_k^{*\circ})^{n(*\circ)} \text{ for some } n \geq 0, a_i \in A\}.$$

In particular, if $A = \{a\}$*, then*

$$K(a) = \{x \in L \mid x \leq (a \vee a^{*\circ})^{n(*\circ)} \text{ for some } n \geq 0\}.$$

Proof Let H be the right side of the stated subset of $(\dagger)$. Clearly, $0 \in H$ and so H is non-empty. If $x \leq y \in H$, it is clear that $x \in H$. If now $x, y \in H$, then there exist $a_i, b_i \in A$ $(i = 0, 1, \cdots, k)$ such that

$$x \leq (a_1 \vee a_1^{*\circ} \vee \cdots \vee a_k \vee a_k^{*\circ})^{n(*\circ)} \text{ and } y \leq \left(b_1 \vee b_1^{*\circ} \vee \cdots \vee b_k \vee b_k^{*\circ}\right)^{n(*\circ)}.$$

Then we have

$$\begin{aligned} x \vee y &\leq \left(a_1 \vee a_1^{*\circ} \vee \cdots \vee a_k \vee a_k^{*\circ}\right)^{n(*\circ)} \vee \left(b_1 \vee b_1^{*\circ} \vee \cdots \vee b_k \vee b_k^{*\circ}\right)^{n(*\circ)} \\ &= \left(a_1 \vee b_1 \vee a_1^{*\circ} \vee b_1^{*\circ} \vee \cdots \vee a_k \vee b_k \vee a_k^{*\circ} \vee b_k^{*\circ}\right)^{n(*\circ)} \\ &= \left(a_1 \vee b_1 \vee \cdots \vee a_k \vee b_k \vee a_1^{*\circ} \vee b_1^{*\circ} \vee \cdots \vee a_k^{*\circ} \vee b_k^{*\circ}\right)^{n(*\circ)}. \end{aligned}$$

It then follows that $x \vee y \in H$, and so H is an ideal of L. If now $z \in H$, then there exists $c_i \in A$ $(i = 0, 1, \cdots, k)$ such that $z \leq (c_1 \vee c_1^{*\circ} \vee \cdots \vee c_k \vee c_k^{*\circ})^{n(*\circ)}$. Hence we have

$$z^{*\circ} \leq \left(c_1 \vee c_1^{*\circ} \vee \cdots \vee c_k \vee c_k^{*\circ}\right)^{n(*\circ)*\circ} = \left(c_1 \vee c_1^{*\circ} \vee \cdots \vee c_k \vee c_k^{*\circ}\right)^{(n+1)(*\circ)}$$

from which it follows that $z^{*\circ} \in H$. Thus H is a kernel ideal. If there exists some kernel ideal I of L such that $A \subseteq I$, then for every $a \in A$, we have $a^{n(*\circ)} \in I$ for each $n \geq 0$. Thus we have $H \subseteq I$. Hence H is the kernel ideal of L that is generated by A and consequently, $K(A) = H$. □

Lemma 3.4 *If L is a pM-algebra and $a \in L$, then the pseudocomplement of $K(a)$ is given by*

$$K(a)^* = \bigcap_{n \geq 0} ((a \vee a^{*\circ})^{*n(\circ*)}].$$

Proof Let $J = \bigcap_{n \geq 0} ((a \vee a^{*\circ})^{*n(\circ*)}]$. Clearly, J is an ideal of L. Let $x \in J$. Then for each $k \geq 0$ we have $x \leq (a \vee a^{*\circ})^{*(k+1)(\circ*)}$ whence

$$x^* \geq (a \vee a^{*\circ})^{*(k+1)(\circ*)*} = (a \vee a^{*\circ})^{*k(\circ*)\circ**} \geq (a \vee a^{*\circ})^{*k(\circ*)\circ},$$

this follows

$$x^{*\circ} \leq (a \vee a^{*\circ})^{*k(\circ*)\circ\circ} = (a \vee a^{*\circ})^{*k(\circ*)}.$$

Hence $x^{*\circ} \in J$ and so J is a kernel ideal. If now $x \in K(a) \cap J$, then by Lemma 3.3, there exists some $m \geq 0$ such that $x \leq (a \vee a^{*\circ})^{m(*\circ)}$. Since $x \in J$, we have $x \leq (a \vee a^{*\circ})^{*m(\circ*)} = (a \vee a^{*\circ})^{m(*\circ)*}$. Thus

$$x \leq (a \vee a^{*\circ})^{m(*\circ)} \wedge (a \vee a^{*\circ})^{m(*\circ)*} = 0.$$

It then follows that $K(a) \cap J = \{0\}$. If there exists some kernel ideal K such that $K(a) \cap K = \{0\}$, then for each $n \geq 0$, $a \in K(a)$ gives $(a \vee a^{*\circ})^{n(*\circ)} \in K(a)$. Hence for every $y \in K$, $y \wedge (a \vee a^{*\circ})^{n(*\circ)} = 0$. Thus we have $y \leq (a \vee a^{*\circ})^{n(*\circ)*} = (a \vee a^{*\circ})^{*n(\circ*)}$. It then follows that $y \in J$ and so $K \subseteq J$. Therefore, J is the pseudocomplement of $K(a)$. □

Lemma 3.5 *If L is a pM-algebra and $S \subseteq Z(L)$, then*

$$K(S^\circ) = \{x \in L \mid x \leq s_1^\circ \vee s_2^\circ \vee \cdots \vee s_k^\circ \text{ for some } s_i \in S, 1 \leq i \leq k\}$$

where $S^\circ = \{s^\circ \mid s \in S\}$.

Proof Let $s^\circ \in S^\circ$ for some $s \in S$. It follows by Lemma 3.1(1) that $s^{\circ n(*\circ)} = s^\circ$. Then we can obtain by Lemma 3.3 that

$$K(S^\circ) = \{x \in L \mid x \leq s_1^\circ \vee s_2^\circ \vee \cdots \vee s_k^\circ \text{ for some } s_i \in S, 1 \leq i \leq k\}.$$

□

It is well-known [8] that the lattice $I(B)$ of ideals of a Boolean algebra B is boolean if and only if B is finite. Using this fact and above lemmas, the main conclusions in this section can be established as follows.

Theorem 3.4 *Let L be a pM-algebra. Then $I_k(L)$ is a boolean lattice if and only if L is of finite range and $Z(L)$ is finite.*

Proof Suppose that $I_k(L)$ is boolean. Then for every $x \in L$, the kernel ideal $K(x)$ is complemented. Thus there exists some kernel ideal I of L such that I and $K(x)$ are complementary. Since $x \in K(x)$ and $1 \in L = K(x) \vee I$, there exists $i \in I$ such that $x \vee i = 1$. Then we have by Lemma 2.2(5) that $x \vee i \leq (x \vee x^{*\circ})^{n(*\circ)} \vee (i \vee i^{*\circ})^{n(*\circ)}$, which means $(x \vee x^{*\circ})^{n(*\circ)} \vee (i \vee i^{*\circ})^{n(*\circ)} = 1$. Note that $x \in K(x)$ and $i \in I$, $K(x)$ and I are kernel ideals. Then by Theorem 3.1 we have that $(x \vee x^{*\circ})^{n(*\circ)} \in K(x)$ and $(i \vee i^{*\circ})^{n(*\circ)} \in I$. Hence we have $(x \vee x^{*\circ})^{n(*\circ)} \wedge (i \vee i^{*\circ})^{n(*\circ)} = 0$. It then follows that $(x \vee x^{*\circ})^{n(*\circ)}$ and $(i \vee i^{*\circ})^{n(*\circ)}$ are complementary. Thus we obtain that $(x \vee x^{*\circ})^{n(*\circ)} \in \mathrm{Cen}(L)$. Hence by Lemma 2.2(6) we have

$$(x \vee x^{*\circ})^{n(*\circ)} = (x \vee x^{*\circ})^{n(*\circ)*\circ*\circ} = (x \vee x^{*\circ})^{(n+2)(*\circ)}.$$

Further, by Lemma 2.2(5) we have $(x \vee x^{*\circ})^{n(*\circ)} = (x \vee x^{*\circ})^{(n+1)(*\circ)}$. Consequently, L is of finite range. Moreover, since

$$\begin{aligned}(x \vee x^{*\circ})^{n(*\circ)} \wedge (x \vee x^{*\circ})^{n(*\circ)\circ} &= (x \vee x^{*\circ})^{n(*\circ)} \wedge (x \vee x^{*\circ})^{(n+1)(*\circ)\circ} \\ &= (x \vee x^{*\circ})^{n(*\circ)} \wedge (x \vee x^{*\circ})^{n(*\circ)*} \\ &= 0,\end{aligned}$$

we have $(x \vee x^{*\circ})^{n(*\circ)} \in Z(L)$.

We now show that $I_k(L) \simeq I(Z(L))$, where $I(Z(L))$ is the lattice of ideals of $Z(L)$. Consider the mapping: $I \mapsto I \cap Z(L)$ from $I_k(L)$ to $I(Z(L))$. It follows from Lemma 3.2 that the mapping is well-defined. If $I' \in I(Z(L))$, then $K(I')$ is the kernel ideal of L generated by I'. It then follows by Lemma 3.5 that

$$K(I') = \{x \in L \mid x \leq i_1 \vee i_2 \vee \cdots \vee i_k \text{ for some } i_j \in I', 1 \leq j \leq k\}.$$

Clearly, $I' = K(I') \cap Z(L)$. Hence, $I \mapsto I \cap Z(L)$ is surjective. To see that $I \mapsto I \cap Z(L)$ is injective, let $I, J \in I_k(L)$ such that $I \cap Z(L) = J \cap Z(L)$. Then if $x \in I$, we have from the above observations that $(x \vee x^{*\circ})^{n(*\circ)} \in I \cap Z(L) = J \cap Z(L)$, and so $x \leq (x \vee x^{*\circ})^{n(*\circ)} \in J$. Then $x \in J$ whence $I \subseteq J$. Similarly, $J \subseteq I$. Thus we have $I = J$. Hence $I \mapsto I \cap Z(L)$ is injective. Therefore, we conclude that $I_k(L) \simeq I(Z(L))$. Thus $I_k(L)$ is boolean implies that $I(Z(L))$ is boolean. Hence $Z(L)$ is finite.

Conversely, suppose that the stated properties hold. It then follows from the above observations that $I_k(L) \simeq I(Z(L))$. Thus, $I_k(L)$ is boolean if and only if $I(Z(L))$ is boolean, which is the case, if and only if $Z(L)$ is finite. □

Theorem 3.5 *Let L be a pM-algebra. Then $I_k(L)$ is a Stone lattice if and only if both $\bigwedge_{n\geq 0} (a \vee a^{*\circ})^{*n(\circ*)}$ and $\bigwedge S$ exist in L for each $a \in L$ and $S \subseteq Z(L)$.*

Proof (⇒:) Suppose that $I_k(L)$ is a Stone lattice. Then for every $a \in L$, $K(a)^*$ is complemented. It follows by Theorem 3.3 that there exists $z \in Z(L)$ such that $K(a)^* = (z]$. Then we have by Lemma 3.4 that $\bigcap_{n\geq 0} ((a \vee a^{*\circ})^{*n(\circ*)}] = (z]$, from which it follows that $\bigwedge_{n\geq 0} (a \vee a^{*\circ})^{*n(\circ*)}$ exists and equals z. For the subset $S \subseteq Z(L)$, we have by Lemma 3.5

that $K(S^\circ)$ is a kernel ideal. By distributivity and Lemma 3.5 again, for $I \in I_k(L)$, we can see that

$$\begin{aligned} I \cap K(S^\circ) = \{0\} &\iff (\forall i \in I, \forall s \in S)\ i \wedge s^\circ = 0 \\ &\iff i \leq s^{\circ *} = s \\ &\iff I \subseteq \bigcap_{s \in S} (s]. \end{aligned}$$

Consequently, $K(S^\circ)^* = \bigcap_{s\in S}(s]$. Since, by hypothesis, $K(S^\circ)^* = (t]$ for some $t \in Z(L)$, from which it follows that $\bigwedge S$ exists and equals t.

($\Leftarrow$:) Suppose that the stated properties hold. For each $a \in L$, we show first that $m(a) \overset{def}{=} \bigwedge_{n\geq 0} (a \vee a^{*\circ})^{*n(\circ *)} \in Z(L)$. For every $k \geq 0$, since $m(a) \leq (a \vee a^{*\circ})^{*(k+1)(\circ *)}$, we have

$$m(a)^* \geq (a \vee a^{*\circ})^{*(k+1)(\circ *)*} = (a \vee a^{*\circ})^{*k(\circ *)\circ **} \geq (a \vee a^{*\circ})^{*k(\circ *)\circ}$$

from which it follows $m(a)^{*\circ} \leq (a \vee a^{*\circ})^{*k(\circ *)\circ\circ} = (a \vee a^{*\circ})^{*k(\circ *)}$. Hence we have $m(a)^{*\circ} \leq m(a)$, and so $m(a)^* \geq m(a)^\circ$. It then follows that $m(a) \wedge m(a)^\circ \leq m(a) \wedge m(a)^* = 0$. Thus we have $m(a) \in Z(L)$. Secondly, we show that, for a subset $S \subseteq Z(L)$, $m(S) \overset{def}{=} \bigwedge S \in Z(L)$. Clearly, for every $s \in S$, $m(S) \leq s$. It follows by Lemma 3.1(1) that $m(S)^{*\circ} \leq s^{*\circ} = s$. Hence we have $m(S)^{*\circ} \leq m(S)$ and so $m(S)^* \geq m(S)^\circ$. Therefore, $m(S) \wedge m(S)^\circ \leq m(S) \wedge m(S)^* = 0$. Thus we have $m(S) \in Z(L)$.

Let I be a kernel ideal of L. Then for every $i \in I$, there follows from the above observations that

$$m(i) = \bigwedge_{n\geq 0} (i \vee i^{*\circ})^{*n(\circ *)} \in Z(L) \text{ and } z = \bigwedge_{i \in I} m(i) \in Z(L).$$

We now show that $I^* = (z]$. If $x \in I^*$. Then for each $i \in I$ and every $n \geq 0$, we have by Theorem 3.1 that $(i \vee i^{*\circ})^{n(*\circ)} \in I$. Thus $x \wedge (i \vee i^{*\circ})^{n(*\circ)} = 0$, and whence $x \leq (i \vee i^{*\circ})^{n(*\circ)*} = (i \vee i^{*\circ})^{*n(\circ *)}$. It then follows that $x \leq m(i)$ and so $x \leq z$. Thus we have $I^* \subseteq (z]$. If now $x \leq z$, then by Lemma 2.2(5) and Lemma 3.1(1) we have $(x \vee x^{*\circ})^{n(*\circ)} \leq (z \vee z^{*\circ})^{n(*\circ)} = z$ for each $n \geq 0$. Since $z \leq m(i)$, we have $z \leq (i \vee i^{*\circ})^{*n(\circ *)}$ for each $n \geq 0$. Thus we can obtain by Lemma 2.2(5) again that

$$i \leq i \vee i^{*\circ} \leq (i \vee i^{*\circ})^{n(*\circ)} \leq (i \vee i^{*\circ})^{n(*\circ)**} = (i \vee i^{*\circ})^{*n(\circ *)*} \leq z^* \leq (x \vee x^{*\circ})^{n(*\circ)*}.$$

It follows, now, that $K(x) \cap I = \{0\}$; because $i \in K(x) \cap I$ implies for some $m \geq 0$, $i \leq (x \vee x^{*\circ})^{m(*\circ)}$, we have

$$i \leq (x \vee x^{*\circ})^{m(*\circ)*} \wedge (x \vee x^{*\circ})^{m(*\circ)} = 0.$$

Thus $x \in K(x) \subseteq I^*$. Hence $(z] \subseteq I^*$ and consequently, $I^* = (z]$. Therefore, it follows by Theorem 3.3 that I^* is complemented, and consequently, $I_k(L)$ is a Stone lattice. □

4 De Morgan Heyting Algebras

We now turn our attention to the class of de Morgan Heyting algebras. In what follows we shall denote by Con L the lattice of HM-congruences on the de Morgan Heyting algebra L. The following result is crucial to describe the structure of lattice of congruences on a de Morgan Heyting algebra.

Theorem 4.1 *Let L be a de Morgan Heyting algebra. Then $\mathrm{Con}\, L \simeq I_k(L)$.*

Proof We now show first that an ideal I of L is a kernel ideal if and only if $x^{*\circ} \in I$ whenever $x \in I$. In order to do so, it is enough to verify that $\theta(I)$ in $(\star)$ of the proof in Theorem 3.1 is a Heyting-congruence. Let $(x, y), (z, w) \in \theta(I)$. Then there exist $i, j \in I$ such that $x \wedge i^* = y \wedge i^*$ and $z \wedge j^* = w \wedge j^*$. Then we have $i \vee j \in I$ and

$$\begin{aligned}(x \to z) \wedge (i \vee j)^* &= (x \to z) \wedge i^* \wedge j^* \\ &\overset{H_3}{=} [(x \wedge i^* \wedge j^*) \to (z \wedge i^* \wedge j^*)] \wedge i^* \wedge j^* \\ &= [(y \wedge i^* \wedge j^*) \to (w \wedge i^* \wedge j^*)] \wedge i^* \wedge j^* \\ &\overset{H_3}{=} (y \to w) \wedge i^* \wedge j^* \\ &= (y \to w) \wedge (i \vee j)^*.\end{aligned}$$

It then follows that $(x \to z, y \to w) \in \theta(I)$. Hence $\theta(I)$ is a Heyting-congruence.

We now consider the mapping $I \mapsto \theta(I)$ from $I_k(L)$ to $\mathrm{Con}\, L$. It is readily seen that for $I, J \in I_k(L)$,

$$I \subseteq J \iff \theta(I) \leq \theta(J).$$

Then the mapping $I \mapsto \theta(I)$ is injective, and we have $\theta(I \cap J) \leq \theta(I) \wedge \theta(J)$ and $\theta(I) \vee \theta(J) \leq \theta(I \vee J)$. To see the converse inequality, we let $(x, y) \in \theta(I) \wedge \theta(J)$. Then there exist $i \in I$ and $j \in J$ such that $x \wedge i^* = y \wedge i^*$ and $x \wedge j^* = y \wedge j^*$. Note that for $z \in \{i, j\}$, $z^+ \overset{def}{\equiv} z^{\circ * \circ}$ is the dual pseudocomplement of z, then $z \vee z^+ = 1$. By distributivity we have

$$\begin{aligned}x \vee i^{*\circ*\circ} &= (x \vee i^{*\circ*\circ}) \wedge (i^* \vee i^{*\circ*\circ}) \\ &= (x \wedge i^*) \vee i^{*\circ*\circ} \\ &= (y \wedge i^*) \vee i^{*\circ*\circ} \\ &= (y \vee i^{*\circ*\circ}) \wedge (i^* \vee i^{*\circ*\circ}) \\ &= y \vee i^{*\circ*\circ}.\end{aligned}$$

Similarly, we also obtain that $x \vee j^{*\circ*\circ} = y \vee j^{*\circ*\circ}$. Since I and J are kernel ideals of L, we have $i^{*\circ*\circ} \in I$ and $j^{*\circ*\circ} \in J$. Then $i^{*\circ*\circ} \wedge j^{*\circ*\circ} \in I \cap J$. Observe that

$$\begin{aligned}x \vee (i^{*\circ*\circ} \wedge j^{*\circ*\circ}) &= (x \vee i^{*\circ*\circ}) \wedge (x \vee j^{*\circ*\circ}) \\ &= (y \vee i^{*\circ*\circ}) \wedge (y \vee j^{*\circ*\circ}) \\ &= y \vee (i^{*\circ*\circ} \wedge j^{*\circ*\circ}).\end{aligned}$$

It then follows by distributivity that

$$\begin{aligned}x \wedge (i^{*\circ*\circ} \wedge j^{*\circ*\circ})^* &= (x \wedge (i^{*\circ*\circ} \wedge j^{*\circ*\circ})^*) \vee ((i^{*\circ*\circ} \wedge j^{*\circ*\circ}) \wedge (i^{*\circ*\circ} \wedge j^{*\circ*\circ})^*) \\ &= (x \vee (i^{*\circ*\circ} \wedge j^{*\circ*\circ})) \wedge (i^{*\circ*\circ} \wedge j^{*\circ*\circ})^* \\ &= (y \vee (i^{*\circ*\circ} \wedge j^{*\circ*\circ})) \wedge (i^{*\circ*\circ} \wedge j^{*\circ*\circ})^* \\ &= (y \wedge (i^{*\circ*\circ} \wedge j^{*\circ*\circ})^*) \vee ((i^{*\circ*\circ} \wedge j^{*\circ*\circ}) \wedge (i^{*\circ*\circ} \wedge j^{*\circ*\circ})^*) \\ &= y \wedge (i^{*\circ*\circ} \wedge j^{*\circ*\circ})^*.\end{aligned}$$

Thus we obtain that $(x, y) \in \theta(I \cap J)$ whence $\theta(I) \wedge \theta(J) \leq \theta(I \cap J)$. Consequently, we have $\theta(I \cap J) = \theta(I) \wedge \theta(J)$. If now $(x, y) \in \theta(I \vee J)$, then there exists some $a \in I \vee J$ such that $x \wedge a^* = y \wedge a^*$. Since $a \in I \vee J$, there exist $i \in I$ and $j \in J$ such that $a \leq i \vee j$. Hence $i^* \wedge j^* = (i \vee j)^* \leq a^*$ and so $x \wedge i^* \wedge j^* = y \wedge i^* \wedge j^*$. Note that $i \in I = \mathrm{Ker}\, \theta(I)$

and $j \in J = \mathrm{Ker}\,\theta(J)$. Then $i \overset{\theta(I)}{\equiv} 0$ and $j \overset{\theta(J)}{\equiv} 0$. Hence we have $i^* \overset{\theta(I)}{\equiv} 1$ and $j^* \overset{\theta(J)}{\equiv} 1$. Observe that

$$x = x \wedge 1 \overset{\theta(I)}{\equiv} x \wedge i^* = x \wedge i^* \wedge 1 \overset{\theta(J)}{\equiv} x \wedge i^* \wedge j^*;$$

$$y = y \wedge 1 \overset{\theta(I)}{\equiv} y \wedge i^* = y \wedge i^* \wedge 1 \overset{\theta(J)}{\equiv} y \wedge i^* \wedge j^*.$$

It then follows that $(x, y) \in \theta(I) \vee \theta(J)$. Hence $\theta(I \vee J) \leq \theta(I) \vee \theta(J)$ and consequently, we conclude that $\theta(I \vee J) = \theta(I) \vee \theta(J)$. Therefore, the mapping $I \mapsto \theta(I)$ is a lattice homomorphism.

To show the mapping $I \mapsto \theta(I)$ is surjective, it suffices to verify that $\alpha = \theta(\mathrm{Ker}\,\alpha)$ for every $\alpha \in \mathrm{Con}\,L$. Clearly, $\theta(\mathrm{Ker}\,\alpha) \leq \alpha$. Let now $(x, y) \in \alpha$. Then $(x^\circ, y^\circ) \in \alpha$. Hence we have $(x^\circ \to y^\circ, 1) \in \alpha$ and $(y^\circ \to x^\circ, 1) \in \alpha$, which follow that $i \overset{def}{=} (x^\circ \to y^\circ)^\circ \vee (y^\circ \to x^\circ)^\circ \in \mathrm{Ker}\,\alpha$. Observe that

$$\begin{aligned} x \vee i &= x^{\circ\circ} \vee (x^\circ \to y^\circ)^\circ \vee (y^\circ \to x^\circ)^\circ \\ &= [x^\circ \wedge (x^\circ \to y^\circ) \wedge (y^\circ \to x^\circ)]^\circ \\ &\overset{H_2}{=} [x^\circ \wedge y^\circ \wedge (y^\circ \to x^\circ)]^\circ \\ &= (x^\circ \wedge y^\circ)^\circ \\ &= x \vee y. \end{aligned}$$

Similarly, we can obtain that $y \vee i = x \vee y$, and whence $x \vee i = y \vee i$. It then follows that

$$x \wedge i^* = (x \vee i) \wedge i^* = (y \vee i) \wedge i^* = y \wedge i^*.$$

Hence we have $(x, y) \in \theta(\mathrm{Ker}\,\alpha)$. Thus $\alpha \leq \theta(\mathrm{Ker}\,\alpha)$ and consequently, $\alpha = \theta(\mathrm{Ker}\,\alpha)$.

It therefore follows from the above observations that $I_k(L)$ is lattice isomorphic with $\mathrm{Con}\,L$. □

Theorems 3.4, 3.5 and 4.1 come together in establishing the following result.

Theorem 4.2 *If L is a de Morgan Heyting algebra, then*

(1) *Con L is boolean if and only if L is of finite range and $Z(L)$ is finite;*
(2) *Con L is Stone if and only if both $\bigwedge_{n\geq 0} (a \vee a^{*\circ})^{*n(\circ *)}$ and $\bigwedge S$ exist in L for each $a \in L$ and $S \subseteq Z(L)$.*

5 Conclusions

In this contribution, we gave some necessary and sufficient conditions that the lattice of kernel ideals is boolean and Stone, respectively, on a pseudocomplemented de Morgan algebra. In particular, we also obtained the conditions that the lattice of congruences on a de Morgan Heyting algebra is boolean and Stone, respectively. We generalized the results obtained in [1] to the class of pseudocomplemented de Morgan algebras. However, the problem of a (kernel) ideal lattice which belongs to B_n $(n \geq 2)$ on a bounded distributive lattice (resp. a lattice-ordered algebra) is still open.

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