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Posets with Cover Graph of Pathwidth two have Bounded Dimension

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Abstract Joret, Micek, Milans, Trotter, Walczak, and Wang recently asked if there exists a constant d such that if P is a poset with cover graph of P of pathwidth at most 2, then $\dim(P) \leq d$. We answer this question in the affirmative by showing that $d = 17$ is sufficient. We also show that if P is a poset containing the standard example S_5 as a subposet, then the cover graph of P has treewidth at least 3.

Keywords Poset · Pathwidth · Cover graph · Dimension

1 Introduction

Although the dimension of a poset and the treewidth of a graph have been prominent subjects of mathematical study for many years, it is only recently that the impact of the treewidth of graphs on poset dimension has received any real attention. This new interest in connections between these topics has led to recasting an old result in terms of treewidth. It is natural to phrase the following result from 1977 in terms of treewidth, which had been defined (using a different name) by Halin in [5] a year earlier. However, the importance

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of treewidth (and the use of that name) only became widely known through the work of Robertson and Seymour [10] nearly a decade later.

Theorem 1.1 (Trotter and Moore [16]) *If P is a poset such that the cover graph of P is a tree, then* $\dim(P) \leq 3$. *Equivalently, if P is a poset such that the cover graph of P is connected and has treewidth at most 1, then* $\dim(P) \leq 3$.

Recently there have been a number of papers on the dimension of planar posets [3, 4, 13]. This work naturally led to the question of bounding a poset's dimension in terms of the treewidth of its cover graph. Over 30 years ago, Kelly showed in [8] that there are planar posets having arbitrarily large dimension by constructing a planar poset containing S_d, the standard example of dimension d, as a subposet. These examples use large height to stretch out S_d to allow a planar embedding. Joret et al. [6] point out that the pathwidth of Kelly's examples is 3 for $d \geq 5$. Thus, any bound on dimension solely in terms of pathwidth or treewidth is impossible. However, they were able to show that it suffices to add a bound on the height in order to bound the dimension. In particular, they proved the following:

Theorem 1.2 (Joret et al. [6]) *For every pair of positive integers (t, h), there exists a least positive integer $d = d(t, h)$ so that if P is a poset of height at most h and the treewidth of the cover graph of P is at most t, then* $\dim(P) \leq d$.

Motivated by the observation about the pathwidth of Kelly's examples, Joret et al. concluded their paper by asking if there is a constant d such that if P is a poset whose cover graph has pathwidth at most 2, then $\dim(P) \leq d$. They also asked this question with treewidth replacing pathwidth. (An affirmative answer to the latter question would imply an affirmative answer to the former.) In this paper, we show that the answer for pathwidth 2 is in fact "yes" with the following result:

Theorem 1.3 *Let P be a poset. If the cover graph of P has pathwidth at most 2, then* $\dim(P) \leq 17$.

In fact, the precise version of this result (Theorem 4.6) is intermediate between answering the pathwidth question and answering the treewidth question, as we only need to exclude six of the 110 forbidden minors that characterize the graphs of pathwidth at most 2. (Treewidth at most 2 is characterized simply by forbidding K_4 as a minor.)

We show in Theorem 5.2 that any poset containing the standard example S_5 has treewidth at least 3. This provides a small piece of evidence in favor of the idea that if the treewidth of a poset is at most 2, then the poset's dimension is bounded.

Before proceeding to our proofs, we provide some definitions for completeness. We then establish some essential properties of the 2-connected blocks of a graph of pathwidth at most 2. We then prove the more general version of Theorem 1.3 and conclude with the rather technical proof that posets containing S_5 have cover graphs of treewidth at least 3.

2 Definitions and Pathwidth 2 Obstructions

Let P be a poset. If $x < y$ in P and there is no $z \in P$ such that $x < z < y$ in P, we say that x is covered by y (or y covers x) and write $x <: y$. For $x \in P$, the closed down set of x is $D[x] = \{y \in P : y \leq x\}$ and the closed up set of x is $U[x] = \{y \in P : y \geq x\}$.

The cover graph of P is the graph G with the elements of P as its vertices in which x is adjacent to y in G if and only if $x <: y$ or $y <: x$. (If we view the order diagram of P as a graph, that graph is P's cover graph.) The dimension of P is the least t such that there exist t linear extensions—collectively known as a *realizer*—$L_1, \ldots, L_t$ of P with the property that $x <_P y$ if and only if $x <_{L_i} y$ for $i = 1, \ldots, t$. An incomparable pair (x, y) of P is said to be *reversed* by a linear extension L if $y <_L x$. To show that a set $\mathcal{R}$ of linear extensions of a poset P is a realizer, it suffices to show that each incomparable pair is reversed by some linear extension in $\mathcal{R}$. By the *standard example* S_n, we mean the subposet of the lattice of subsets of $\{1, 2, \ldots, n\}$ induced by the singletons and the $(n-1)$ sets. For further background on the combinatorics of partially ordered sets, refer to Trotter's monograph [14].

Let $G = (V, E)$ be a graph. A pair $(T, \mathcal{V})$, where T is a tree and $\mathcal{V} = (V_t)_{t \in T}$ with $V_t \subseteq V$ for all $t \in T$, is a *tree-decomposition* of G if

(1) $V(G)$ is the union of all the V_t;
(2) for every $e \in E$, there exists a vertex t of T such that $e \subseteq V_t$; and
(3) if t_1, t_2, t_3 are vertices of T and t_2 lies on the unique path from t_1 to t_3 in T, then $V_{t_1} \cap V_{t_3} \subseteq V_{t_2}$.

The sets V_t are often referred to as the *bags* of the tree-decomposition. The *width* of $(T, \mathcal{V})$ is $\max_t |V_t| - 1$. The *treewidth* of G, which we denote by $\mathrm{tw}(G)$, is the minimum width of a tree-decomposition of G. A *path-decomposition* of a graph is a tree-decomposition in which the tree T is a path. The *pathwidth* of G, denoted by $\mathrm{pw}(G)$, is the minimum width of a path-decomposition of G.

Following Diestel [2], we make the following definition of a special type of path to improve the readability of parts of our argument. If G is a graph and H is a subgraph of G, we say that a path P is an H-path if P is nontrivial and intersects H precisely at its two end vertices. The length of a path is the number of edges it contains. We will also freely use terminology regarding the block structure of graphs. Readers unfamiliar with this terminology should consult Diestel's text [2], in particular Chapter 3.

By a *subdivision* of a graph G we mean a graph G' in which some edges of G are replaced by paths that are internally disjoint from each other and the vertices of G. The original vertices of G are called the *branch vertices* of G'. If a graph H contains a subdivision of G as a subgraph, then we say that G is a *topological minor* of H. An *inflation* of a graph G is a graph G' formed by replacing the vertices x of G by disjoint connected graphs G_x and the edges xy of G by nonempty sets of edges from G_x to G_y. The vertex sets $V(G_x)$ are called the *branch sets* of G'. If a graph H contains an inflation of G as a subgraph, we say that G is a *minor* of H. Equivalently, G is a minor of H if G can be obtained from H by a sequence of vertex deletions, edge deletions, and edge contractions. Note that if the maximum degree of G is at most 3, the notions of minor and topological minor are equivalent. For further information on minors and topological minors, see Diestel's text [2].

The set of graphs of pathwidth at most k is a minor closed family. Therefore, by the Graph Minor Theorem [11], this set of graphs can be characterized by forbidding a finite set of graphs as minors. For $k = 2$, Kinnersley and Langston found the entire set of 110 obstructions in [9]. The proof of this paper's main result relies on only six graphs from their list, but having the whole list at hand was critical to the development of our proof. Besides the obvious obstruction K_4, the other five we must exclude are depicted in Fig. 1. It is elementary to verify that these graphs have pathwidth 3. We will refer to these graphs in the proof by the names shown and use $\mathcal{F}$ to denote $\{K_4, T_1, \ldots, T_5\}$. If a graph G does not contain an element of $\mathcal{F}$ as a minor, we will say that G is *$\mathcal{F}$-minor free*.

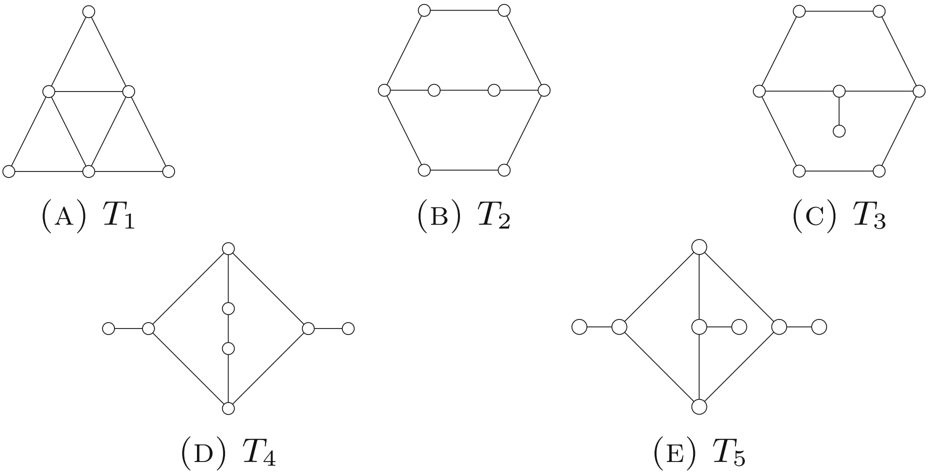


Fig. 1 Five key obstructions for pathwidth 2

3 Properties of the 2-Connected Blocks

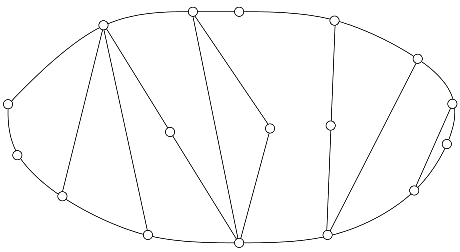
We begin without restricting our attention to only cover graphs. In this section, we consider a graph G such that $\mathrm{pw}(G) \leq 2$ and prove strong properties of the block structure. This structure is essential in the proof of our main theorem. To establish this structural result, we first make the following definition.

Definition 3.1 A *parallel nearly outerplanar graph* is a graph that consists of a longest cycle C with vertices labelled (in order) as $x_1, x_2, \ldots, x_k, y_l, y_{l-1}, \ldots, y_1$ along with some chords and chords subdivided exactly once. The chords and subdivided chords have attachment points $x_{i_1}, y_{j_1}, \ldots, x_{i_m}, y_{j_m}$ such that $i_1 \leq \cdots \leq i_m$ and $j_1 \leq \cdots \leq j_m$.

An example of a parallel nearly outerplanar graph is shown in Fig. 2. We think of the vertices along the bottom of the cycle as being the x_i and those along the top as being the y_j. Vertices to the left of the leftmost chord and to the right of the rightmost chord could be either x_i's or y_j's.

Lemma 3.2 *A graph G is a parallel nearly outerplanar graph if and only if G is 2-connected and $pw(G) \leq 2$.*

Fig. 2 A parallel nearly outerplanar graph



Proof It is easy to see that every parallel nearly outerplanar graph is 2-connected and has pathwidth at most 2. A path-decomposition of width 2 can be obtained by starting with the bag containing x_1 and y_1 and proceeding through the x_i and y_j by increasing subscript. After all edges incident with x_i have had their other attachment point included in a bag with x_i, the bag $\{x_i, x_{i+1}, y_j\}$, where y_j is the "current" vertex from the other side of the cycle, covers the edge $x_i x_{i+1}$. We can then remove x_i from the bag and continue. A symmetric process is used to move from y_j to y_{j+1} after covering all edges incident with y_j. The internal vertex of a subdivided chord appears in a bag with precisely its two attachment points.

For the converse, let C be a longest cycle in G. A C-path will be called an *ear*. We first note that C cannot have crossing ears. More precisely, if P and Q are ears, $V(C) \cap V(P) = \{p_1, p_2\}$, and $V(C) \cap V(Q) = \{q_1, q_2\}$, then the order of these intersection vertices on C must be $p_i, p_{3-i}, q_j, q_{3-j}$ for some $i, j \in \{1, 2\}$. If this were not the case, then G would have a K_4-minor, forcing $\mathrm{pw}(G) \geq 3$.

Next we show that no ear may have more than one internal vertex. Indeed, if P is an ear with at least two internal vertices and $V(C) \cap P = \{v_1, v_2\}$, then both paths between v_1 and v_2 on C must contain at least two internal vertices, for otherwise C is not the longest cycle. If this occurs, then G has a T_2-minor.

We now show that the internal vertex, if one exists, of any ear is of degree 2. Let v be the internal vertex of the ear xvy, and suppose that the degree of v is at least 3. Let H be the subgraph induced by the vertices of C and the vertex v. If v has degree at least 3 in H, then H contains a K_4-minor. Otherwise, there is a $v' \in V(G)$ such that $v'v \in E(G)$, but $v' \notin V(H)$. Let H' be the subgraph of G formed from H by adding the vertex v' and edge vv'. Since G is 2-connected and H' is not, there is an H'-path P (possibly just a single edge) with one endpoint being v'. The other endpoint may only be x or y, since otherwise we have a K_4-minor. Without loss of generality, the other endpoint is x, which implies that $xPv'vy$ is an ear with at least two internal vertices, a contradiction.

We have now shown that G contains a (longest) cycle and some non-crossing ears with at most one inner vertex which must have degree two. The only thing that remains to be shown is that the vertices of the cycle may be labeled as in the definition, effectively placing an ordered structure on the ears. If this were not true, there would be three ears with attachment points a_1, b_1, a_2, b_2, and a_3, b_3 that appear around the longest cycle of G ordered as $a_1, b_1, a_2, b_2, a_3, b_3$ around C, with the possibility that $b_i = a_{i+1}$ for any i (cyclically). In this case, G contains the forbidden minor T_1, which gives our final contradiction. □

We observe that our proof of the "if" direction of Lemma 3.2 only requires that G is 2-connected and not contain K_4, T_1, or T_2 as a minor. Furthermore, the cycle bounding the infinite face may be chosen to be *any* longest cycle of the graph, a fact which we will use in the proof of Lemma 3.3.

We note that after proving Lemma 3.2, we discovered that Barát et al. [1] had previously proved this fact while working to simplify the characterization of graphs of pathwidth 2. They used the name *track* for what we call a parallel nearly outerplanar graph. We use the latter name because it is more evocative of the aspects of the structure that are important in our proof and include the proof of Lemma 3.2 for completeness.

By Lemma 3.2, each 2-connected block of a graph of pathwidth two is a parallel nearly outerplanar graph. Our next lemma establishes that the vertices where these blocks join together lie on the parallel nearly outerplanar graphs' longest cycles.

Lemma 3.3 *Let G be a connected graph that does not contain an element of $\mathcal{F}$ as a minor. Let B be a 2-connected block of G. There exists a longest cycle C of B such that if there is a vertex v of B adjacent to a vertex v' not in B, then $v \in V(C)$.*

Proof Let C be a longest cycle of B that minimizes the number of internal vertices of ears adjacent to vertices outside B. Let v be an internal vertex of an ear xvy, i.e., $v \notin V(C)$, and suppose v is adjacent a vertex v' not in B. Deletion of x and y from the cycle C leaves two paths, which we will call C_1 and C_2. If both C_1 and C_2 contain at least two vertices, then G has a T_3-minor, since we have assumed that v has a neighbor v' not in B. Thus, suppose C_1 contains a single vertex u. If the degree of u in G is two, then the cycle formed from C by replacing u by v is also a longest cycle of B, and has fewer internal vertices of ears adjacent to vertices outside B. If the degree of u in B were 3, then there would be an ear uzx or uzy. In either case, C would not be a longest cycle, as the edge ux (or uy) could be replaced by the path uzx (or uzy). Therefore, we may assume that u is adjacent to a vertex u' not in B. Furthermore, $u' \neq v'$, and there is no path from u' to v' in G that does not go through B, as otherwise G would contain a K_4-minor. If C_2 contains at least two vertices, then G contains a T_4-minor. If C_2 is a single vertex w, then it must have degree 2 in G to avoid having T_5 as a minor. But then $\{x, v, y, u\}$ is a longest cycle of B with fewer internal vertices of ears adjacent to vertices outside B than C. □

In light of Lemma 3.3, we see that every $\mathcal{F}$-minor free graph G has a planar embedding in which each 2-connected block B is embedded such that the vertices of B lying on the unbounded face form a longest cycle of B. We call such an embedding a *canonical embedding* of G.

4 Posets with Cover Graphs of Pathwidth 2

Definition 4.1 Let P be a poset. A *subdivision* of the cover relation $x <: y$ in P is the addition of new points $z_1, z_2, \dots, z_l$ such that $x < z_1 < \cdots < z_l < y$ and the new points z_i are incomparable with all points of P that are not greater than y or less than x. We say that Q is a *subdivision* of P if Q can be constructed from P by subdividing some of its cover relations.

In light of what we know from the previous section about the structure of graphs of pathwidth at most 2, it is tempting to consider the effect of subdivision on dimension. Since such an approach would allow us to deal with some of the subdivided chords preventing the cover graph from being outerplanar, we might be inclined to hope that if Q is a subdivision of P, then $\dim(Q) \leq c \dim(P)$ for some absolute constant c. (Perhaps even $c = 2$.) However, this is not the case. In fact, Spinrad showed in [12] that this construction can increase dimension by an arbitrarily large factor. Fortunately, as we show in Lemma 4.2, there is a subdivision-like operation on the graphs of relevance to our result that has a small effect on the poset's dimension.

Our proof requires that we first introduce some additional terminology. Let G be a parallel nearly outerplanar graph that is the cover graph of a poset P, and let C be a longest cycle provided by Lemma 3.3. An ear with no inner vertex is simply called a *chord*. We call an ear xzy *unidirected* if $x < z < y$ or $y < z < x$ in P. Otherwise we call the ear a *beak*. An *upbeak* is an ear with $x < z > y$ in P, and a *downbeak* is an ear with $x > z < y$ in P.

(In either case, $x \parallel y$.) We call the internal point of a beak a *beak peak*. Our first step will be to address unidirected ears. We will then turn our attention to the issue of beaks.

Lemma 4.2 *Let P be a poset with cover graph G. Suppose that G is $\mathcal{F}$-minor free and fix a canonical embedding of G in the plane. If Z is the collection of points that are not on the unbounded face of G and are neither minimal nor maximal in P, then* $\dim(P) \leq 2\dim(P - Z) + 1$.

Proof First notice that in a canonical embedding of G, our definition of Z means that every element of Z is the internal vertex of a unidirected ear $\ell < z < u$ in P. If the relation $\ell < u$ in $P - Z$ is a cover, then z is a subdividing point of the cover relation $\ell <: u$ in P. Note, however, that P is not necessarily a subdivision of $P - Z$, as some of the unidirected ears may not correspond to cover relations in $P - Z$. Nevertheless, we will refer to Z as the set of subdividing points of P and an element of Z will be called a subdividing point of P even if the comparability involved is not a cover of P. When $\ell z u$ is a unidirected ear of P with $\ell < z < u$ in P, we will refer to ℓ as the lower element of z. Similarly, u will be called the upper element of z.

Let $\{L_1, \ldots, L_d\}$ be a realizer of $P - Z$ with $d = \dim(P - Z)$. For each L_i, we will construct two linear extensions L_i' and L_i'' of P by inserting the subdividing elements appropriately, and we will show that most incomparable pairs will be reversed in one of these linear extensions. We will create one extra linear extension to reverse the rest of the incomparable pairs.

To construct L_i', we place each subdividing point of P immediately above its lower element in L_i. We form L_i'' by placing each subdividing point immediately below its upper element in L_i. There may be some ambiguity in this definition if subdividing points share upper or lower elements. To deal with such situations, let $z_1, \ldots, z_k$ be subdividing points of P that share the lower element ℓ. For $j = 1, \ldots, k$, let the upper element of z_j be u_j. We may assume that these upper elements are distinct, since the removal of one point of a pair of points with duplicated holdings does not impact dimension (other than in the irrelevant case of a two-element antichain). Let σ be a permutation of $\{1, \ldots, k\}$ such that $u_{\sigma(1)} < \cdots < u_{\sigma(k)}$ in L_i. In L_i' we insert the subdividing points so that $\ell < z_{\sigma(k)} < \cdots < z_{\sigma(1)}$. For L_i'', our concern is with subdividing points $z_1, \ldots, z_k$ sharing the upper element u. Let ℓ_j be the lower element of z_j, and let σ be a permutation of $\{1, \ldots, k\}$ such that $\ell_{\sigma(1)} < \cdots < \ell_{\sigma(k)}$ in L_i. To form L_i'', we insert the subdividing elements so that $z_{\sigma(k)} < \cdots < z_{\sigma(1)} < u$ in L_i''.

Consider an incomparable pair (a, b). If $a, b \in P - Z$, then obviously there is a linear extension L_i' (and an L_i'') with $a > b$. Suppose $a \in P - Z$ and $b \in Z$ and let ℓ be the lower element of b. Then $a \not< \ell$ in $P - Z$ implies that there in an L_i in which $a > \ell$, and hence $a > b$ in L_i'. Similarly, if $a \in Z$ and $b \in P - Z$, there exists an L_i'' with $a > b$.

If $a, b \in Z$ have the same lower element, then their order in L_i' will be opposite to their order in L_i''. Hence, one of L_i' and L_i'' has $a > b$. A similar argument works when a and b have the same upper element.

Next we assume that $a, b \in Z$ have distinct upper and lower elements. Specifically, let ℓ_a and u_a be the lower and upper elments of a and let ℓ_b and u_b be the lower and upper elements of b. If $\ell_a \not< \ell_b$, then $\ell_a > \ell_b$ in some L_i, and hence $a > b$ in L_i'. Similarly, if $u_a \not< u_b$, then $a > b$ in some L_i''.

At this stage, we have shown that the incomparable pair (a, b) will be reversed, unless all of the following conditions are satisfied:

(1) $a, b \in Z$;

(2) a and b have distinct lower elements ℓ_a and ℓ_b, respectively, and distinct upper elements u_a and u_b, respectively; and
(3) $\ell_a < \ell_b$ and $u_a < u_b$.

We say such a pair (a, b) is *in a bad diamond*. We will prove that there exists a single linear extension that reverses all such pairs.

We do this by viewing the poset P as an acyclic directed graph D, with directed edges corresponding to covers and pointing from smaller elements to larger elements. For each incomparable pair (a, b) in a bad diamond, we introduce a new directed edge ba. We call these *new edges*, and the directed graph formed from D by adding these new edges is denoted by D'. Note that a and b must lie in the same 2-connected block, so the new edge ba will be added to within that block.

The goal of the rest of the argument is to prove that D' contains no oriented cycles. Recall that we have fixed a canonical embedding of D in the plane, which defines (up to duality) a natural linear order on the subdividing points. We fix one of these orders and use the terms "left" and "right" to refer to directions in this linear order. For upper and lower elements of the subdivided chords there is also a natural notion of two sides of the outer cycle defined by the embedding, depending on whether they are x_i's or y_j's. (This notion is well-defined, since we are concerned only with attachment points of subdivided chords.)

Claim 1 *Let ba be a new edge. Then there is a directed path P_ℓ from ℓ_a to ℓ_b, and a directed path P_u from u_a to u_b in D, and for any such directed paths we have $P_\ell \cap P_u = \emptyset$, and in particular, $u_a, u_b \notin P_\ell$ and $\ell_a, \ell_b \notin P_u$.* □

Proof The existence of the paths follows from condition (3) of the definition of bad diamonds. If there exists $x \in P_\ell \cap P_u$, then we have that $a < u_a \leq x \leq \ell_b < b$, a contradiction. □

Claim 2 *Let ba be a new edge. Then ℓ_a and ℓ_b are on the same side of the outer cycle, and u_a and u_b are also on the same side. Furthermore, P_ℓ and P_u are on the outer cycle.*

Proof This is direct consequence of Claim 1. If any part of the statement is not true, then P_ℓ topologically separates u_a from u_b or P_u topologically separates ℓ_a from ℓ_b. □

Claim 3 *Let cb and ba be two new edges. Then they both go left, or both go right.*

Proof Without loss of generality assume for a contradiction that ba goes left, and cb goes right. By Claim 2, all of ℓ_a, ℓ_b, ℓ_c are on the same side, and u_a, u_b, u_c are on the same side. Furthermore, every directed path P_{ab} from ℓ_a to ℓ_b goes on the outer cycle; a similar statement holds for paths P_{bc} from ℓ_b to ℓ_c. However, one of these is a subpath of the other, and they are directed contradictorily. □

Now we are ready to show that D' does not contain a directed cycle. Suppose for a contradiction that it does, and let C be a directed cycle in D' that contains as few new edges as possible. Notice that C must contain at least one new edge and at least one old edge by Claim 3. Let P_1 be a maximal path in C that consists entirely of new edges. Suppose that P_1's initial point is b and its terminal point is a. Notice that C must lie entirely within a 2-connected block of D, and this block is parallel nearly outerplanar. Also notice that C must include the edges au_a and $\ell_b b$, and a directed path P_2 from u_a to ℓ_b that is disjoint from

P_1. For any $x, y \in P_2$ denote by xP_2y the subpath of P_2 starting with x and terminating with y. If $\ell_a \in P_2$, then the directed cycle $\ell_a a(u_a P_2 \ell_a)$ contains fewer new edges than C; if $u_b \in P_2$, then $\ell_b b(u_b P_2 \ell_b)$ is such a cycle.

Therefore P_2 connects the unidirected ears $\ell_a a u_a$ and $\ell_b b u_b$. Hence P_2 must cross from the side of u_a to the side of ℓ_b. This must occur via a chord or a unidirected ear. Let u_0 be the attachment point for the chord or unidirected ear on the same side as u_a and let ℓ_0 be the attachment point on the same side as ℓ_a. As all the new edges which form P_1 are all consistently oriented, this crossing occurs between some a' and b' which are consecutive vertices on P_1. Since $b'a'$ is a new edge, we have that (a', b') is in a bad diamond and in particular, a' is incomparable to b'. However, by Claim 2 and the definition of a bad diamond, we have that $a' < u_{a'} \le u_0 < \ell_0 \le \ell_{b'} < b'$, a contradiction.

Since D' is acyclic, there is a total order L_0 on its vertices that respects the orientation of its edges. By construction, L_0 is then a linear extension of P that reverses all incomparable pairs that are in bad diamonds. Therefore, we can conclude that $\{L_0, L'_1, \ldots, L'_d, L''_1, \ldots, L''_d\}$ is a realizer of P and $\dim(P) \le 2\dim(P - Z) + 1$.

To address the case of beaks in the cover graph, we will form two extensions of the poset and show that their intersection is $P - Z$. (Recall that Z is the set of vertices that, in a canonical embedding of G, are not on the unbounded face and are not beak peaks.) We will then apply Lemma 4.2 to P and use what we know about the extensions of $P - Z$ to bound its dimension. Note that in the remainder of this section, we often view the poset as a directed graph and refer to a chain of covers as a directed path.

Lemma 4.3 *Let P be a poset with cover graph G. If G is $\mathcal{F}$-minor free, then P has extensions Υ and Δ with cover graphs G_Υ and G_Δ that are outerplanar except for some chords replaced by directed paths of length 2.*

Proof Fix a canonical embedding of G. To construct Υ and Δ, we consider the 2-connected blocks of the cover graph of P one at a time. In each block, we consider the beaks xzy and introduce a comparability between x and y. It is clear that if we are able to do this, beaks in G will become edges in G_Υ and G_Δ and a pendant vertex (corresponding to the beak peak) will be added to one of the beak attachment points. Thus, the only obstruction to G_Υ and G_Δ being outerplanar will come from unidirected ears, corresponding to replacing chords of an outerplanar graph by directed paths of length 2.

We introduce comparabilities between beak attachment points for all beaks in such a way that we maintain consistency of these new comparabilities. Since two blocks intersect in at most one point on their longest cycles, introducing a new comparability within one block cannot force two incomparable beak attachment points in another block to become comparable by transitivity. Therefore, we may define the extensions on the blocks independently.

Consider a 2-connected block B. Since B is parallel nearly outerplanar, a fixed plane embedding provides (up to duality) a natural left-to-right ordering on its beaks as suggested in Fig. 2. Fix one of these orders and number the k beaks of B accordingly from 1 to k. Denote the attachment points for beak i by x_i and y_i, with the x_i all lying on the same side of the outer cycle of B and the y_i lying on the other.

We now show that there exists an extension of the subposet induced by the vertices of B in which $x_i < y_i$ for all $i = 1, \ldots, k$. Let D be the digraph defined by the subposet induced by the vertices of B. More specifically, $V(D) = V(B)$ and there is a directed edge uv in D if and only if (u, v) is a cover in P. To prove that such an extension exists, it suffices to show that if we construct D' by adding the directed edges $x_i y_i$ to D, then D' contains no directed cycle. By a slight abuse of terminology, we will call the added directed edge $x_i y_i$ a beak.

Suppose for a contradiction that D' contains a directed cycle C'. Notice that C' must contain at least one beak, because D is an acyclic graph. In fact, C' has to contain at least two beaks, for if the only beak it contains were $x_i y_i$, then $y_i < x_i$ in P, which would contradict the fact that $x_i y_i$ is a beak. Therefore, C' contains the beaks $x_i y_i$ and $x_j y_j$. As a consequence, C' must contain a directed path between y_i and x_j. This path forces x_i and y_j to belong to different (topological) regions, contradicting the existence of C' as a directed cycle.

By a symmetric argument, there exists an extension of the subposet induced by the vertices of B in which $y_i < x_i$ for all $i = 1, \ldots, k$.

Now we can define the extensions Υ and Δ of P. In a given embedding with left-right orientations of the 2-connected blocks, construct Υ by adding, for each block, the relations $x_i < y_i$ for all i. Similarly, construct Δ by adding the relations $y_i < x_i$ for all i in each block. □

The final major step in our argument is to prove that $P = \Upsilon \cap \Delta$, as then we may use realizers of Υ and Δ to construct a realizer of P, thereby bounding the dimension.

Lemma 4.4 *Let P be a poset with $\mathcal{F}$-minor-free cover graph. If Υ and Δ are extensions of P as defined in the proof of Lemma 4.3, then $P = \Upsilon \cap \Delta$.*

Proof It is sufficient to show that if $w \not< w'$ in P, then one of the extensions preserves this (non)relation. We begin by considering the situation where w and w' are in the same 2-connected block of the cover graph. We first address the case where w and w' are both on the outer cycle of a 2-connected block and then reduce the remaining cases to this one. We conclude by addressing what happens when w and w' are in different blocks.

Case I Suppose w and w' are both on the outer cycle C of a 2-connected block B and that $w < w'$ in both Υ and Δ. There are directed paths (chains) from w to w' in both Υ and Δ. We consider the shortest of these paths in the sense of containing the fewest beaks. Let $x_i y_i$ be the last beak on the path in Υ, and $y_j x_j$ be the last beak on the path in Δ. If $y_i = y_j$, then since $y_i < w'$ in P, there is a shorter path in Δ that skips $y_j x_j$. Thus $y_i \neq y_j$. For a similar reason, $x_i \neq x_j$.

Without loss of generality, assume that $i < j$. Suppose w' is right of $y_j x_j$ (allowing $w' = x_j$) and consider a path in P from y_i to w'. By minimality, this path cannot pass through y_j, because then $y_j x_j$ could be skipped. Hence, the path separates x_i from y_j. Notice that w is not on the path from y_i to w', as this would imply $w < w'$ in P. Therefore, w would have to be in both (topological) regions, which is a contradiction. A similar contradiction can be derived if w' is left of $x_i y_i$ or $w' = y_i$. In that case the path from x_j to w' in P would separate x_i from y_j.

This leaves only the possibility that w' is between the two beaks. If w' is on the $x_i x_j$ arc of the outer cycle, then the path from y_i to w' separates x_i from y_j, and if w' is on the path from y_i to y_j, then the $x_j w'$ path performs the separation. Therefore, we may conclude that $w \not< w'$ in Υ or Δ.

Case II Still assuming w and w' are in the same 2-connected block, we now suppose that exactly one of them is on the outer cycle C. Specifically, we will consider the case when w is on C and w' is not, and the ear conatining w' is right of w. This is just for convenience of discussion; the other three possibilities have identical proofs.

Suppose there is a directed path from w to w' in both Υ and Δ; consider one of these that goes through the minimum number of (newly-directed) beaks. First note that w' cannot be a peak of a downbeak, since that would make w' minimal in P and thus in Υ and Δ. If w' is a subdividing point of a unidirected ear, then let $u < w'$ be its attachment point. We have $w \not< u$ in P, so by Case I, we maintain this in one of Υ or Δ. That extension preserves $w \not< w'$.

The remaining possibility in this case is that w' is the peak of an upbeak. By the minimality of the path from w to w', the path uses no beaks right of the beak containing w'. For the purpose of the argument, we may ignore all ears, chords, and points of C strictly right from the beak of w'. By so doing, w' becomes a point on the outer cycle, and by Case I, one of Υ or Δ will preserve $w \not< w'$.

Case III To conclude the scenario where both w and w' are in the same 2-connected block, it remains only to address the case when neither of them is on C. Without loss of generality assume that w is left of w'. Considering a path from w to w' in Υ or Δ through the fewest number of beaks, we may assume that this path does not touch any part of the block left of w and right of w'. (If either w or w' is part of a unidirected ear, using these portions would imply the existence of a directed cycle, and for beak peaks the path can be shortened by going via the other attachment point.) By ignoring the parts of the block left of w and right of w', we place w and w' on an outer cycle, and thus Case I guarantees one of Υ and Δ preserves $w \not< w'$.

Case IV It remains only to consider the case where no 2-connected block contains both w and w'. If w and w' lie in different components of the cover graph, both Υ and Δ preserve $w \not< w'$. Hence, we may assume there exists a path in the cover graph from w to w'. (Since $w \not< w'$ in P, this path is *not* a directed path.) Let the 2-connected blocks containing an edge of the path be called $B_1, B_2, \ldots, B_l$. Note that we allow $l = 0$ if the path does not pass through any 2-connected blocks, in which case Υ and Δ do not introduce comparabilities that could make w and w' comparable. Let a_i and b_i be the (uniquely-determined) entry and exit vertices of the path into and out of B_i; if $w \in B_1$, then let $a_1 = w$, and if $w' \in B_l$, then let $b_l = w'$.

If $a_i \leq b_i$ in P for all $i = 1, 2, \ldots, l$, then since the path from w to w' in the cover graph of P is not directed, $w \not< w'$ must be forced by consecutive edges of the path that are oppositely-oriented and do not both lie in the same 2-connected block. Therefore, Υ and Δ preserve $w \not< w'$. On the other hand, if there exists an i_0 such that $a_{i_0} \not< b_{i_0}$, then this (non)relation is preserved in one of Υ or Δ. That extension preserves $w \not< w'$, since any directed path from w to w' would have to pass through the points a_i and b_i, but there is no directed path between them in that extension. Therefore, we have shown $w \not< w'$ in at least of Υ and Δ. □

As we combine the three preceding lemmas to prove our main theorem, we will reduce to a poset with an outerplanar cover graph. The following result guarantees that such posets have small dimension.

Theorem 4.5 (Felsner, Trotter, and Wiechert [4]) *If a poset P has an outerplanar cover graph, then* $\dim(P) \leq 4$.

We are finally ready to state the full version of our main theorem.

Theorem 4.6 *Let P be a poset with cover graph G. If G is $\mathcal{F}$-minor free, then* $\dim(P) \leq 17$.

Proof Begin by fixing a canonical embedding of G in the plane and, as in Lemma 4.2, let Z be the collection of points that are not on the unbounded face of G and are neither minimal nor maximal in P. By Lemma 4.2, we know that $\dim(P) \leq 2\dim(P - Z) + 1$. We now claim that $\dim(P - Z) \leq 8$, which will prove the theorem.

Applying Lemmas 4.3 and 4.4 to $P - Z$, we find that $P - Z$ has two extensions Υ and Δ for which $P - Z = \Upsilon \cap \Delta$. Furthermore, since $P - Z$ does not contain any unidirected ears, the process of constructing Υ and Δ cannot introduce unidirected ears, and the comparabilities added to form Υ and Δ turn beak peaks into vertices of degree 1 in the cover graphs, we have that Υ and Δ have outerplanar cover graphs. Therefore, by Theorem 4.5, there are realizers $\mathcal{R}_\Upsilon$ and $\mathcal{R}_\Delta$ of Υ and Δ, respectively, with $|\mathcal{R}_\Upsilon|, |\mathcal{R}_\Delta| \leq 4$. Since $P - Z = \Upsilon \cap \Delta$, we know that $\mathcal{R}_\Upsilon \cup \mathcal{R}_\Delta$ is a realizer of $P - Z$. Therefore, $\dim(P - Z) \leq 8$ and $\dim(P) \leq 17$. □

To obtain Theorem 1.3, we now note that if P is a poset with cover graph G of pathwidth at most 2, then G is $\mathcal{F}$-minor free, so Theorem 4.6 implies $\dim(P) \leq 17$. It is natural to wonder whether the bound of Theorem 4.6 is best possible. We have no reason to believe the result is optimal and suspect it may be possible to reduce the bound to 4 with more work. That would be best possible, as Felsner, Trotter, and Wiechert give a 4-dimensional poset having cover graph with pathwidth 2 in [4].

We also note that Trotter [15] has subsequently made an observation regarding the relationship between dimension and the block structure of the cover graph, making it possible to drop T_3, T_4, and T_5 from the list of forbidden minors. However, that approach leads to a weaker bound on the dimension than the one we offer here.

5 Standard Examples and Treewidth

A second question posed in [6] remains open.

Question 5.1 Is there a constant d such that if P is a poset with cover graph G and $\text{tw}(G) \leq 2$, then $\dim(P) \leq d$?

The following theorem provides some weak evidence for an affirmative answer to this question, since the theorem implies that if the answer to Question 5.1 is "no", a counterexample cannot be constructed using large standard examples.

Theorem 5.2 *If P is a poset that contains the standard example S_5 as a subposet, then the cover graph of P has treewidth at least 3.*

Proof Since $\text{tw}(K_4) = 3$, it will suffice to show that the cover graph of P has a K_4-minor. (In fact, more is true, in that K_4 is the only forbidden minor required to characterize graphs of treewidth 2.) Since the notions of containing a K_4-minor and containing K_4 as a topological minor are equivalent, we use an approach that blends both techinques by seeking branch sets of a K_4 minor and joining them by internally disjoint paths. To aid in exposition, we will not fully specify the branch sets. Instead, we will refer to vertices or sets of vertices as being *corners* of the K_4 minor if they lie in distinct branch sets. We denote a path between any two comparable elements x and y such that the path represents a maximal chain between x and y in P by $P(x, y)$.

Let $\{a_1, \ldots, a_5\}$ and $\{b_1, \ldots, b_5\}$ be elements of the subposet of P isomorphic to S_5 with the standard ordering, that is, $a_i < b_j$ if and only if $i \neq j$. We first restrict our attention to the copy of S_3 determined by $\{a_1, a_2, a_3, b_1, b_2, b_3\}$. In this context, fix c_i as one of the maximal elements in $U[a_i] \cap D[b_{i+1}] \cap D[b_{i+2}]$ where the subscripts are interpreted cyclically among $\{1, 2, 3\}$. Notice that $\{c_1, c_2, c_3\}$ is an antichain in P since a_i is incomparable to b_i for all i. In a similar manner, fix d_i as a minimal element in $U[c_{i+1}] \cap U[c_{i+2}] \cap D[b_i]$. Thus the poset P contains four (not necessarily disjoint) antichains $\{a_1, a_2, a_3\}$, $\{b_1, b_2, b_3\}$, $\{c_1, c_2, c_3\}$, and $\{d_1, d_2, d_3\}$ together with paths $P(a_i, c_i)$ and $P(d_i, b_i)$ for $i \in \{1, 2, 3\}$ and paths $P(c_i, d_j)$ for $i, j \in \{1, 2, 3\}$ with $i \neq j$. See Fig. 3. It is a straightforward, but tedious argument, to verify that these paths are all internally disjoint. We call the subposet on these elements S.

After noting that $P(c_i, d_j)$ is internally disjoint from $P(c_{i'}, d_{j'})$ when $(i, j) \neq (i', j')$, it is easy to see that

$$P(c_1, d_2),\ P(d_2, c_3),\ P(c_3, d_1),\ P(d_1, c_2),\ P(c_2, d_3),\ P(d_3, c_1)$$

is a cycle in the cover graph of P. We denote this cycle by C. Thus, if any element x of the poset is connected to this cycle by three paths intersecting only at x, then the cover graph contains a K_4-minor, as desired. Noting that $a_4 < b_1, b_2, b_3$ we now consider the relationship between a_4 and S. Suppose first that a_4 is not less than any element of $\{c_1, c_2, c_3\}$. By our definitions, every element of $C - \{c_1, c_2, c_3\}$ is less than precisely one element of $\{b_1, b_2, b_3\}$. Hence, there exist three paths P_1, P_2, P_3 in the cover graph from a_4 to C. (Note that these paths may use the paths $P(b_i, d_i)$ if a_4 is not less than some of the d_i.) Each P_i enters C at a distinct point, creating a K_4-minor.

Therefore, we may assume that a_4 is less than one element of $\{c_1, c_2, c_3\}$, say c_1. By a similar argument, we may assume b_4 is greater than an element of $\{d_1, d_2, d_3\}$. Furthermore, since b_4 is incomparable to a_4 while d_2 and d_3 are comparable to c_1, our assumption that $a_4 < c_1$ forces d_1 to be the element of $\{d_1, d_2, d_3\}$ that is less than b_4. Note that the incomparability between a_4 and b_4 implies that a_4 is incomparable to c_2 and c_3 and b_4 is incomparable to d_2 and d_3. Additionally, there is a vertex β_4 on $P(d_1, b_1)$ such that $\beta_4 < b_4$ and a vertex α'_4 on $P(a_1, c_1)$ such that $a_4 < \alpha'_4$. Since $a_4 < b_1$ and a_4 is incomparable to b_4, there is some element α_4 on $P(d_1, b_1)$ with $\alpha_4 > \beta_4$ and $a_4 < \alpha_4$. Similarly, there is an element β'_4 on $P(a_1, c_1)$ with $\beta'_4 < \alpha'_4$ and $\beta'_4 < b_4$. See Fig. 4 for an illustration of the relationship between these points. In a similar manner, we can find a $j \in \{1, 2, 3\}$ and

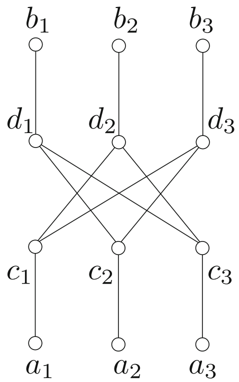


Fig. 3 The subposet S with vertices internal to chains/paths not shown

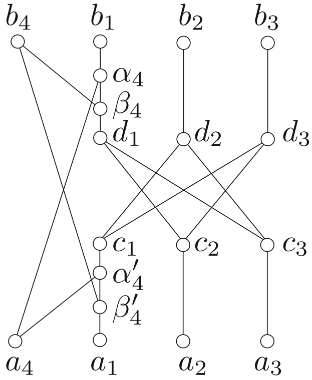


Fig. 4 Expanding S by adding $a_4, b_4, \alpha_4, \beta_4, \alpha'_4, \beta'_4$

elements β_5 on $P(d_j, b_j)$ and β'_5 on $P(a_j, c_j)$ such that $\beta_5, \beta'_5 < b_5$. There are also elements $\alpha_5, \alpha'_5 > a_5$ such that $\alpha_5 > \beta_5$ on $P(d_j, b_j)$ and $\alpha'_5 > \beta'_5$ on $P(a_j, c_j)$. If there are multiple choices for β_i, β'_i, α_i, and α'_i that satisfy all these requirements, we choose β_i and β'_i to be maximal and α_i and α'_i to be minimal among the possible choices. By our definitions of the c_i and d_j, it is straightforward, but tedious, to verify that $P(a_4, \alpha_4)$, $P(a_4, \alpha'_4)$, and $P(\beta_4, b_4)$ are internally disjoint from S. Further, $P(\beta'_4, b_4)$ is internally disjoint from S except for possibly $P(c_2, d_3)$ and $P(c_3, d_2)$.

Suppose then that $P(\beta'_4, b_4)$ intersects both $P(c_2, d_3)$ and $P(c_3, d_2)$. Let $K_4 - e$ denote the graph that results from deleting any edge from K_4. It is easy to see that there is a $(K_4 - e)$-minor with corners c_1, d_1, and the two intersection points of the path $P(\beta'_4, b_4)$ with $P(c_2, d_3)$ and $P(c_3, d_2)$. (Note that this minor can be formed using only C and the part of $P(\beta'_4, b_4)$ between $P(c_2, d_3)$ and $P(c_3, d_2)$.) The missing connection to complete the K_4-minor is the edge between c_1 and d_1. However, as $P(d_1, \alpha_4)P(\alpha_4, a_4)P(a_4, \alpha'_4)P(\alpha'_4, c_1)$ is disjoint from the cycle C and $P(\beta'_4, b_4)$, this completes the K_4-minor. Thus we may assume that $P(\beta'_4, b_4)$ intersects only one of $P(c_2, d_3)$ and $P(c_3, d_2)$. Without loss of generality, suppose the intersected path is $P(c_2, d_3)$ and let z be the maximal point of intersection. We note now that there is a cycle formed by

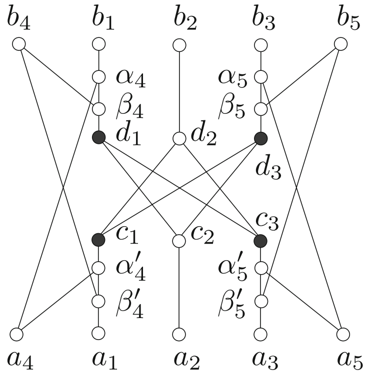
$$P(z, b_4)P(b_4, \beta_4)P(\beta_4, \alpha_4)P(\alpha_4, a_4)P(a_4, \alpha'_4)P(\alpha'_4, c_1)P(c_1, d_3)P(d_3, z).$$

Furthermore, the point d_1 has three distinct paths to this cycle, forming a K_4 minor. Thus the paths $P(a_4, \alpha_4)$, $P(a_4, \alpha'_4)$, $P(b_4, \beta_4)$, and $P(b_4, \beta'_4)$ are all internally disjoint from S as shown in Fig. 4.

We consider the cases where $j \neq 1$ and $j = 1$ separately. (Recall that j is the index such that $\beta_5 \in P(d_j, b_j)$.) For the former, suppose without loss of generality that $j = 3$, as depicted in Fig. 5. In this case, if the following six paths are internally disjoint, they form a K_4-minor with corners c_1, d_1, c_3, and d_3:

- $P(d_1, c_2)P(c_2, d_3)$,
- $P(d_3, \alpha_5)P(\alpha_5, a_5)P(a_5, \alpha'_5)P(\alpha'_5, c_3)$,
- $P(c_3, d_2)P(d_2, c_1)$,
- $P(c_1, \alpha'_4)P(\alpha'_4, a_4)P(a_4, \alpha_4)P(\alpha_4, d_1)$,
- $P(d_1, c_3)$, and
- $P(d_3, c_1)$.

Fig. 5 The case where a_5 and b_5 attach to different paths than a_4 and b_4

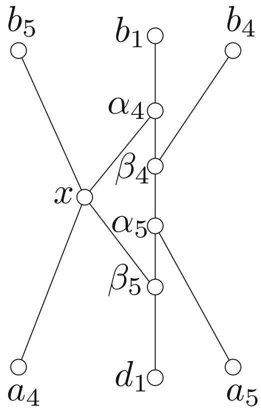


The internal disjointness of each pair of the paths above is clear with the possible exception of the second path and the fourth path. However, if these paths fail to be disjoint, their intersection point has 3 paths to distinct vertices of C, creating a K_4-minor.

The most delicate part of our argument remains in the case where $j = 1$. We consider now the paths that enter $P(d_1, b_1)$. Specifically, we examine the relationships between $P(a_4, \alpha_4)$, $P(a_5, \alpha_5)$, $P(b_4, \beta_4)$, and $P(b_5, \beta_5)$. The paths entering $P(a_1, c_1)$ featuring the α'_i and β'_i will interact identically by duality. It is clear that $P(a_4, \alpha_4)$ and $P(b_4, \beta_4)$ do not intersect, as otherwise $a_4 < b_4$. (A similar argument applies to $P(a_5, \alpha_5)$ and $P(b_5, \beta_5)$.) Suppose then that $P(a_4, \alpha_4)$ and $P(b_5, \beta_5)$ intersect at some point x, while $P(a_5, \alpha_5)$ and $P(b_4, \beta_4)$ do not intersect. Furthermore, if the paths $P(a_4, \alpha_4)$ and $P(b_5, \beta_5)$ intersect more than once, we will assume that x is the minimal such intersection (in terms of the poset).

Now consider rerouting the path $P(d_1, b_1)$ through x. The new path will be the concatenation of $P(d_1, \beta_5)$, $P(\beta_5, x)$, $P(x, \alpha_4)$, and $P(\alpha_4, b_1)$. We then choose the new vertices $\hat{\alpha}_4$, $\hat{\alpha}_5$, $\hat{\beta}_4$, $\hat{\beta}_5$ appropriately, recalling that they are chosen to be maximal or minimal amongst possible options. The new paths $P(a_4, \hat{\alpha}_4)$ and $P(b_5, \hat{\beta}_5)$ are internally disjoint by construction. Suppose now that α_5 is a element of the path $P(\beta_5, \alpha_4)$ and consider the cycle

Fig. 6 Rerouting $P(d_1, b_1)$ via x



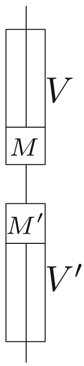
formed by $P(\beta_5, x)$, $P(x, \alpha_4)$, and $P(\alpha_5, \beta_5)$. (See Fig. 6.) Observe that there are three disjoint paths—namely, $P(x, a_4)$, $P(\beta_5, d_1)$, and $P(\alpha_5, a_5)$—emanating from the cycle. Since a_4, a_5, and d_1 all connect to the path $P(a_1, c_1)$, these three vertices are all in the same connected component after deleting the cycle. Therefore, we have found a K_4-minor. In a similar manner, we may assume that β_4 is not on the path $P(\beta_5, \alpha_4)$. Thus we have that $\hat{\alpha}_5 = \alpha_5$ and $\hat{\beta}_4 = \beta_4$, and furthermore by our assumptions, the paths $P(a_5, \hat{\alpha}_5)$ and $P(\hat{\beta}_4, b_4)$ do not intersect.

Now consider the case where, in addition, $P(a_5, \alpha_5)$ and $P(b_4, \beta_4)$ intersect at some point y, again choosing y as the minimal intersection point. Since $\beta_5 < x < \alpha_4$, $\beta_4 < y < \alpha_5$, $\beta_4 < \alpha_4$, and $\beta_5 < \alpha_5$, we have that $\{\beta_4, \beta_5\} < \{\alpha_4, \alpha_5\}$. Since a_i is incomparable to b_i in the poset, we must have that x and y are incomparable as well. This implies that any intersection between $P(x, \beta_5)$ and $P(y, \beta_4)$ occurs at a point less than both x and y on these paths. Similarly, any intersection between $P(x, \alpha_4)$ and $P(y, \alpha_5)$ must be greater than both x and y. It is then easy to see that there is a $(K_4 - e)$-minor with corners x, y, $\{\beta_4, \beta_5\}$, and $\{\alpha_4, \alpha_5\}$, possibly adding intersection points between $P(x, \beta_5)$ and $P(y, \beta_4)$ to $\{\beta_4, \beta_5\}$ and intersection points between $P(x, \alpha_4)$ and $P(y, \alpha_5)$ to $\{\alpha_4, \alpha_5\}$. The missing connection to complete a K_4-minor is between x and y. However, notice that x and y are connected by a path through a_4, a_5, and $P(a_1, c_1)$, giving the needed path to complete the minor. □

We are now able to make a fairly strong assumption about the pairwise intersections of $P(a_4, \alpha_4)$, $P(a_5, \alpha_5)$, $P(b_4, \beta_4)$, and $P(b_5, \beta_5)$. Of the six possible crossings, the only two that can occur are $P(a_4, \alpha_4)$ with $P(a_5, \alpha_5)$ and $P(b_4, \beta_4)$ with $P(b_5, \beta_5)$. Furthermore, these intersections imply that $\alpha_4 = \alpha_5$ or $\beta_4 = \beta_5$, respectively, by the maximality of the β_i and minimality of the α_j.

Having established these intersection limitations (and the corresponding ones for the α_i' and β_i'), we consider the graph formed by contracting each of $P(a_i, \alpha_i)$, $P(a_i, \alpha_i')$, $P(b_i, \beta_i)$, and $P(b_i, \beta_i')$ for $i = 4, 5$ to a single edge. In fact, we go further and contract (arbitrarily) all the edges we can while ensuring that the α_i, β_i, α_i', β_i', a_i, and b_i are not identified for $i = 4, 5$. Since it is possible for some of these vertices to have been equal at the outset, we are then left with a graph with at most 12 vertices. (We refer to the vertices as having labels to allow that, for example, α_4 and α_5 may refer to the same vertex.) The resulting graph is built up from a path in which the vertices with labels $V = \{\alpha_4, \alpha_5, \beta_4, \beta_5\}$ appear consecutively, as do the vertices with labels $V' = \{\alpha_4', \alpha_5', \beta_4', \beta_5'\}$. In addition to

Fig. 7 Relation of V, M, V', and M' in the graph after contractions



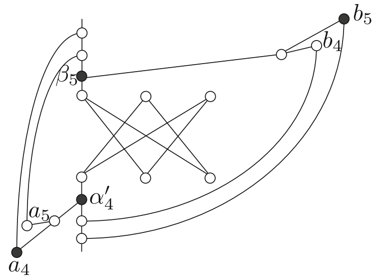


Fig. 8 There must also be a path from b_5 to a_4

this primary path, the graph resulting from the contraction contains a collection of 4 paths of length 2 (via the a_i and b_i) connecting vertices in V and V'.

We now show that in all but one case (to be described later), this graph has a K_4-minor. Let M' be the labels of the maximum vertices (with respect to the poset) of those with labels in V' and similarly define M as the labels of the minimal vertices of those with labels in V. Since M and M' each correspond to a single vertex, they cannot contain two labels with the same subscript and must respect the ordering on elements with the same subscript. Thus, $M \in \{\{\beta_4\}, \{\beta_5\}, \{\beta_4, \beta_5\}\}$ and $M' \in \left\{\left\{\alpha'_4\right\}, \left\{\alpha'_5\right\}, \left\{\alpha'_4, \alpha'_5\right\}\right\}$. We now construct the K_4-minor using M, $V - M$, M', and $V' - M'$ as the corners. Figure 7 makes clear that (with appropriate contractions), $V - M$, M, M', $V' - M'$ is a path oflength 3.

To construct the K_4-minor, it suffices to show that there are connections between (1) M and $V' - M'$, (2) M' and $V - M$, and (3) $V - M$ and $V' - M'$. Since α_i and α'_i are connected via a_i for $i = 4, 5$, and β_i and β'_i are connected via b_i for $i = 4, 5$, the first two cases are immediately resolved because of the possible contents of M and M'. Furthermore, for the third pair, the only situation where we do not immediately see a connection between $V - M$ and $V' - M'$ is when $M = \{\beta_4, \beta_5\}$ and $M' = \{\alpha'_4, \alpha'_5\}$.

In this case, the poset must contain the paths depicted in Fig. 8. However, in this case there is a path between a_4 and b_5 since $a_4 < b_5$ in the poset. This path is not depicted in Fig. 8. If this path is disjoint from C it is straightforward to verify that there is a K_4-minor with corners a_4, b_5, α'_4, and β_5. Otherwise, as a_i is incomparable to b_i for $i = 4, 5$, the path from a_4 to b_5 can only intersect C in the paths $P(c_2, d_3)$ or $P(c_3, d_2)$. Without loss of generality, suppose the path from a_4 to b_5 intersects C at $P(c_2, d_3)$. Then there is a K_4-minor with corners a_4, b_5, β_5, and $P(c_2, d_3)$.

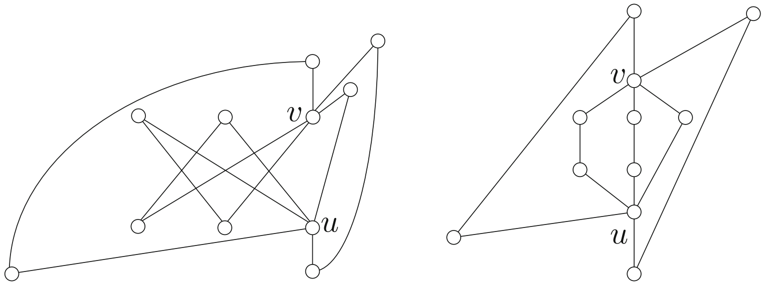


Fig. 9 A poset containing $S_5 - x$ (*left*) with cover graph G (*right*) redrawn to help show G does not contain a K_4-minor, and therefore $\mathrm{tw}(G) = 2$

We conclude this section by observing that for any $x \in S_5$, there is a poset containing $S_5 - x$ and having a cover graph of treewidth 2. We show an example in Fig. 9, with the poset on the left and a redrawing of the cover graph on the right. Notice that $S_5 - x$ is the subposet formed by the elements other than u and v. (Since the graph is clearly K_4-minor-free, it has treewidth at most 2.) This implies that Theorem 5.2 is best possible.

6 Update on Question 5.1

While this paper was under review, Joret, Micek, Trotter, Wang, and Wiechert announced that they have resolved Question 5.1 in the affirmative [7] with a bound on the dimension of 1276.

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