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Generalized Priestley Quasi-Orders

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Abstract We introduce generalized Priestley quasi-orders and show that subalgebras of bounded distributive meet-semilattices are dually characterized by means of generalized Priestley quasi-orders. This generalizes the well-known characterization of subalgebras of bounded distributive lattices by means of Priestley quasi-orders (Adams, Algebra Univers 3:216–228, 1973; Cignoli et al., Order 8(3):299–315, 1991; Schmid, Order 19(1):11–34, 2002). We also introduce Vietoris families and prove that homomorphic images of bounded distributive meet-semilattices are dually characterized by Vietoris families. We show that this generalizes the well-known characterization (Priestley, Proc Lond Math Soc 24(3):507–530, 1972) of homomorphic images of a bounded distributive lattice by means of closed subsets of its Priestley space. We also show how to modify the notions of generalized Priestley quasi-order and Vietoris family to obtain the dual characterizations of subalgebras and homomorphic images of bounded implicative semilattices, which generalize the well-known dual characterizations of subalgebras and homomorphic images of Heyting algebras (Esakia, Sov Math Dokl 15:147–151, 1974).

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1 Introduction

By the Priestley duality [9, 10], each bounded distributive lattice can be represented as the lattice of clopen upsets of a Priestley space. This provides a generalization of the Stone duality [12] by which each Boolean algebra is represented as the Boolean algebra of clopen subsets of a Stone space. Subalgebras of a given Boolean algebra B can dually be characterized by means of "good" equivalence relations on the Stone space X of B (see, e.g., [7, Sec. 8.2]). On the other hand, equivalence relations on the Priestley space X of a bounded distributive lattice L are no longer sufficient to characterize subalgebras of L. Nevertheless, as follows from [1, 5, 11], subalgebras of L can be characterized by means of "good" quasi-orders on X. The aim of this paper is to solve a similar problem in the more general setting of bounded distributive meet-semilattices. In [2] (see also [4]) we have developed a new "Priestley-like" duality for the category of bounded distributive meet-semilattices and bounded meet-semilattice homomorphisms. In this paper we take advantage of this duality to give the dual characterization of subalgebras of a given bounded distributive meet-semilattice. As a corollary, we obtain the dual characterization of [1, 5, 11] of subalgebras of a bounded distributive lattice.

In [2] (see also [3]) we have also developed a similar duality for bounded implicative semilattices. Based on it, we give the dual characterization of subalgebras of a bounded implicative semilattice. As a particular case, we obtain the dual characterization of [6] of subalgebras of a Heyting algebra, from which the dual characterization of subalgebras of a Boolean algebra follows as a corollary.

In addition, we give the dual characterization of homomorphic images of a bounded distributive meet-semilattice by means of Vietoris families. In the particular case of bounded distributive lattices, this leads to the well-known characterization of homomorphic images of a bounded distributive lattice L by means of closed subsets of the Priestley space of L. We conclude the paper by showing that Vietoris families also provide the dual characterization of homomorphic images of bounded implicative semilattices, and show how in the particular case of Heyting algebras this leads to the dual characterization of [6] of homomorphic images of a Heyting algebra by means of closed upsets of its Esakia space. This immediately leads to the well-known dual characterization of homomorphic images of a Boolean algebra as closed subsets of its Stone space.

Since we rely heavily on the results and techniques developed in [2–4], it might be useful to have [2–4] handy, although we try to give all the needed background from [2–4] in the next section.

2 Duality for Distributive and Implicative Semilattices

In this preliminary section we recall the basics of the duality for distributive and implicative semilattices developed in [2–4].

We recall that a *meet-semilattice* is a commutative idempotent monoid $L = \langle L, \wedge, 1\rangle$. A partial order $\leq$ is defined on L by $a \leq b$ iff $a = a \wedge b$. It is easy to see that $a \wedge b$ is the greatest lower bound of $\{a, b\}$, and that 1 is the largest element of $\langle L, \leq\rangle$. We call L *bounded* if L has a least element, we denote by 0. A bounded meet-semilattice $L = \langle L, \wedge, 0, 1\rangle$ is *distributive* if for each $a, b_1, b_2 \in L$ with $b_1 \wedge b_2 \leq a$, there exist $c_1, c_2 \in L$ such that $b_1 \leq c_1$, $b_2 \leq c_2$, and $a = c_1 \wedge c_2$. Let L and S be bounded meet-semilattices. A map $h : L \to S$ is a *bounded meet-semilattice homomorphism* if for each $a, b \in L$, we have $h(a \wedge b) = h(a) \wedge h(b)$, $h(1) = 1$, and $h(0) = 0$. We denote by BDM the category of bounded distributive meet-semilattices and bounded meet-semilattice homomorphisms. Let also BDL denote the category of bounded distributive lattices and bounded lattice homomorphisms. Since the meet-semilattice reduct of a bounded distributive lattice belongs to BDM, we view BDL as a subcategory of BDM.

A bounded meet-semilattice L is a *bounded implicative semilattice* if for each $a \in L$, the order-preserving map $a \wedge (-) : L \to L$ has a right adjoint, denoted by $a \to (-) : L \to L$. For two bounded implicative semilattices L and S, a map $h : L \to S$ is a *bounded implicative semilattice homomorphism* if h is a bounded meet-semilattice homomorphism and $h(a \to b) = h(a) \to h(b)$ for each $a, b \in L$. We denote by BIM the category of bounded implicative semilattices and bounded implicative semilattice homomorphisms. It is well-known that the meet-semilattice reduct of a bounded implicative semilattice belongs to BDM. Thus, we view BIM as a subcategory of BDM. If a bounded implicative semilattice L is in addition a lattice, then L is a *Heyting algebra*. A *Heyting algebra homomorphism* h from a Heyting algebra A to a Heyting algebra B is a bounded implicative semilattice homomorphism preserving join (that is, $h : A \to B$ is in addition a lattice homomorphism). Let HA denote the category of Heyting algebras and Heyting algebra homomorphisms. Since the implicative semilattice reduct of a Heyting algebra belongs to BIM, we view HA as a subcategory of BIM.

For a partially ordered set $\langle X, \leq\rangle$ and $A \subseteq X$, let $\uparrow A = \{x \in X : \exists a \in A \text{ with } a \leq x\}$ and $\downarrow A = \{x \in X : \exists a \in A \text{ with } x \leq a\}$. If A is the singleton $\{a\}$, then we write $\uparrow a$ and $\downarrow a$ instead of $\uparrow\{a\}$ and $\downarrow\{a\}$, respectively. We call A an *upset* (resp. *downset*) if $A = \uparrow A$ (resp. $A = \downarrow A$). In addition, we denote by A^u the set of upper bounds of A and by A^l the set of lower bounds of A. Thus, A^{ul} denotes the set of lower bounds of the set of upper bounds of A.

A *Priestley space* is a compact ordered topological space $X = \langle X, \tau, \leq\rangle$ satisfying the *Priestley separation axiom*: if $x \not\leq y$, then there is a *clopen* (**clo**sed and **op**en) upset U of X such that $x \in U$ and $y \notin U$. It follows from the Priestley separation axiom that X is Hausdorff and that the clopen sets form a basis for the topology. Thus, each Priestley space is a *Stone space* (that is, it is compact Hausdorff zero-dimensional). For two Priestley spaces X and Y, a map $f : X \to Y$ is a *Priestley morphism* if f is continuous and order-preserving. We denote the category of Priestley spaces and Priestley morphisms by PS.

It is a well-known result of Priestley [9, 10] that BDL is dually equivalent to PS. The functors $(-)_* : \mathsf{BDL} \to \mathsf{PS}$ and $(-)^* : \mathsf{PS} \to \mathsf{BDL}$ establishing the dual equivalence are constructed as follows: If L is a bounded distributive lattice, then $L_* = \langle X, \tau, \leq\rangle$, where X is the set of prime filters of L, $\leq$ is set-theoretic inclusion, and τ is the topology generated by the basis $\{\varphi(a) - \varphi(b) : a, b \in L\}$, where

$$\varphi(a) = \{x \in X : a \in x\}$$

is the Stone map. If $h \in \text{hom}(L, K)$, then $h_* = h^{-1}$. If X is a Priestley space, then X^* is the lattice of clopen upsets of X, and if $f \in \text{hom}(X, Y)$, then $f^* = f^{-1}$.

Esakia's duality for Heyting algebras is a restricted Priestley duality. We recall that an *Esakia space* is a Priestley space $X = \langle X, \tau, \leq \rangle$ in which the downset of each clopen is again clopen. We also recall that an *Esakia morphism* from an Esakia space X to an Esakia space Y is a Priestley morphism f such that for all $x \in X$ and $y \in Y$, from $f(x) \leq y$ it follows that there is $z \in X$ with $x \leq z$ and $f(z) = y$. We denote the category of Esakia spaces and Esakia morphisms by ES. Then it follows from [6] that HA is dually equivalent to ES. In fact, the same functors $(-)_*$ and $(-)^*$, restricted to HA and ES, respectively, establish the desired dual equivalence.

2.1 Duality for Distributive Meet-Semilattices

In [2–4] we generalized the above dualities to the settings of distributive and implicative semilattices. We summarize the main results of [2–4] below. Let $\langle X, \tau, \leq \rangle$ be a Priestley space and X_0 be a dense subset of X. For a clopen subset U of X, let $\max(U)$ denote the set of maximal points of U. We call X_0 *cofinal in* U if $\max(U) \subseteq X_0$. Let U be a clopen upset of X. We call U X_0*-admissible* if X_0 is cofinal in $X - U$. For $x \in X$, let $\mathcal{I}_x$ denote the family of X_0-admissible clopen upsets U of X such that $x \notin U$. A quadruple $X = \langle X, \tau, \leq, X_0 \rangle$ is said to be a *generalized Priestley space* if it satisfies the following five conditions:

(1) $\langle X, \tau, \leq \rangle$ is a Priestley space.
(2) X_0 is a dense subset of X.
(3) For each $x \in X$, there is $y \in X_0$ such that $x \leq y$.
(4) $x \in X_0$ iff $\mathcal{I}_x$ is updirected (that is, $U, V \in \mathcal{I}_x$ imply the existence of $W \in \mathcal{I}_x$ such that $U \cup V \subseteq W$).
(5) $x \leq y$ iff $x \in U$ implies $y \in U$ for each X_0-admissible clopen upset U of X.

For a generalized Priestley space $X = \langle X, \tau, \leq_X, X_0 \rangle$, let X^* denote the set of X_0-admissible clopen upsets of X.

Let X and Y be nonempty sets and $R \subseteq X \times Y$ be a binary relation. For each $x \in X$, let $R[x] = \{y \in Y : xRy\}$; and for each $A \subseteq Y$, let $\Box_R A = \{x \in X : R[x] \subseteq A\}$. Let X and Y be generalized Priestley spaces. We call a binary relation $R \subseteq X \times Y$ a *generalized Priestley morphism* if the following three conditions are satisfied:

(1) If $x \not R y$, then there is a Y_0-admissible clopen upset U of Y such that $R[x] \subseteq U$ and $y \notin U$.
(2) If U is a Y_0-admissible clopen upset of Y, then $\Box_R U$ is a X_0-admissible clopen upset of X.
(3) For each $x \in X$ there is $y \in Y$ such that xRy.

Remark 2.1 It follows from conditions (1) and (2) that $\Box_R(U \cap V) = \Box_R U \cap \Box_R V$ for $U, V \in Y^*$ and that $\Box_R Y = X$; in addition, condition (3) guarantees that $\Box_R \emptyset = \emptyset$. In [2–4] the binary relations R satisfying conditions (1) and (2) were called generalized Priestley morphisms. If in addition R satisfied condition (3), then R was called a *total* generalized Priestley morphism. Since in this paper we are only interested in bounded meet-semilattice homomorphisms, we restrict our attention to the binary relations that satisfy all three conditions (1)–(3) and simply call them generalized Priestley morphisms.

We point out that conditions (1) and (2) imply that for each $B \subseteq Y$ the set $R^{-1}[B] = \{x \in X : \exists y \in B \text{ with } xRy\}$ is a downset of X and that for each $A \subseteq X$ the set $R[A] = \{y \in Y : \exists x \in A \text{ with } xRy\}$ is an upset of Y.

Note that the usual (set-theoretic) composition of two generalized Priestley morphisms may *not* be a generalized Priestley morphism. Let $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ be two generalized Priestley morphisms. We define the composition of R and S as the binary relation $S * R \subseteq X \times Z$ given by

$$x(S * R)z \quad \text{iff} \quad (\forall U \in Z^*)((S \circ R)[x] \subseteq U \Rightarrow z \in U),$$

where $S \circ R$ is the usual set-theoretic composition of R and S. Then $S \circ R \subseteq S * R$, $S * R$ is a generalized Priestley morphism, and if $S \circ R$ is already a generalized Priestley morphism, then $S * R = S \circ R$. With this composition, generalized Priestley spaces and generalized Priestley morphisms form a category we denote by GPS. The categories BDM and GPS turn out to be dually equivalent. The functors $(-)_* : \mathsf{BDM} \to \mathsf{GPS}$ and $(-)^* : \mathsf{GPS} \to \mathsf{BDM}$ that establish their dual equivalence are constructed as follows.

The functor $(-)_*$ Let L be a bounded distributive meet-semilattice. We call a nonempty subset I of L a *Frink ideal* (*F-ideal*) if for each finite subset A of I we have $A^{ul} \subseteq I$. Equivalently, I is an F-ideal iff for each $a_1, \ldots, a_n \in I$ and $c \in L$, whenever $\bigcap_{i=1}^{n} \uparrow a_i \subseteq \uparrow c$, we have $c \in I$. We call an F-ideal I *prime* if $a \wedge b \in I$ implies $a \in I$ or $b \in I$. A subset F of L is said to be an *optimal filter* if $F = L - I$ for some prime F-ideal I of L. It turns out that F-ideals are exactly the traces of prime ideals and optimal filters are exactly the traces of prime filters of the *distributive envelope* $D(L)$ of L, where $D(L)$ is the free distributive lattice generated by the meet-semilattice L (for a proper definition of $D(L)$ see [2, Sec. 4.1] or [4, Sec. 3].) An important property of optimal filters is that they separate filters and F-ideals from each other; that is, if F is a filter and I is an F-ideal such that $F \cap I = \emptyset$, then there exists an optimal filter P such that $F \subseteq P$ and $P \cap I = \emptyset$. We refer to this as the *optimal filter lemma*.

Let F be a proper filter of L. We call F *prime* if for each two filters G and H of L we have $G \cap H \subseteq F$ implies $G \subseteq F$ or $H \subseteq F$. It turns out that each prime filter is optimal, but that the two concepts coincide only when L is a lattice. We also mention that, like in a distributive lattice, there is a 1–1 correspondence between prime filters and prime ideals of L, which is established, as usual, by taking set-theoretic complements. However, the notion of an ideal in L is slightly different from the usual definition of an ideal in a lattice. Namely, a nonempty subset I of L is an *ideal* if I is an updirected downset; that is, I is a downset and $a, b \in I$ implies $\{a, b\}^u \cap I \neq \emptyset$. Equivalently, a nonempty downset I is an ideal iff for each $a, b \in I$ we have $(\uparrow a \cap \uparrow b) \cap I \neq \emptyset$. The notion of a prime ideal is usual: a proper ideal I is *prime* if $a \wedge b \in I$ implies $a \in I$ or $b \in I$. Although there are less prime filters than optimal filters of L, prime filters are still capable of separating filters from ideals; that is, if F is a filter and I is an ideal such that $F \cap I = \emptyset$, then there exists a prime filter P such that $F \subseteq P$ and $P \cap I = \emptyset$. We refer to this as the *prime filter lemma*.

We give a simple example, which is illustrative in separating the concept of ideal from that of F-ideal, and the concept of prime filter from that of optimal filter. Let L be the distributive meet-semilattice shown in Fig. 1. Note that L is in fact an implicative semilattice. Moreover, $I = \{0, a, b\}$ is an F-ideal, but it is not an ideal

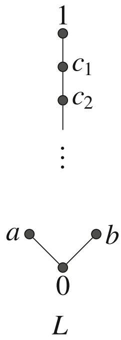

Fig. 1 The implicative semilattice L

of L. Furthermore, $F = \{1, c_1, c_2, \dots\}$ is an optimal filter, which is not a prime filter of L.

Let L_* be the set of optimal filters of L and let L_+ be the set of prime filters of L. For $a \in L$, let $\varphi(a) = \{x \in L_* : a \in x\}$. We set $L_* = \langle L_*, \tau, \leq, L_+\rangle$, where τ is the topology generated by the subbasis $\{\varphi(a) : a \in L\} \cup \{\varphi(b)^c : b \in L\}$ and $\leq$ is set-theoretic inclusion. For $h \in \hom(L, K)$, let $R_h \subseteq K_* \times L_*$ be given by

$$xR_h y \quad \text{iff} \quad h^{-1}(x) \subseteq y$$

for each $x \in K_*$ and $y \in L_*$. We set $f_* = R_h$. Then $(-)_* : \mathsf{BDM} \to \mathsf{GPS}$ is a well-defined functor.

The functor $(-)^*$ For a generalized Priestley space X, we set $X^* = \langle X^*, \cap, X, \emptyset\rangle$. For $R \in \hom(X, Y)$, let $h_R : Y^* \to X^*$ be given by

$$h_R(U) = \Box_R U$$

for each $U \in Y^*$. We set $R^* = h_R$. Then $(-)^* : \mathsf{GPS} \to \mathsf{BDM}$ is a well-defined functor. Moreover, the functors $(-)_* : \mathsf{BDM} \to \mathsf{GPS}$ and $(-)^* : \mathsf{GPS} \to \mathsf{BDM}$ establish the desired dual equivalence of BDM and GPS. More precisely, the natural transformation from the identity functor $\mathrm{id}_{\mathsf{BDM}} : \mathsf{BDM} \to \mathsf{BDM}$ to the functor ${(-)_*}^* : \mathsf{BDM} \to \mathsf{BDM}$ is given by associating with each object L of BDM the morphism $\varphi : L \to {L_*}^*$ of BDM, which is an isomorphism; and the natural transformation from the identity functor $\mathrm{id}_{\mathsf{GPS}} : \mathsf{GPS} \to \mathsf{GPS}$ to the functor ${(-)^*}_* : \mathsf{GPS} \to \mathsf{GPS}$ is given by first defining the order-homeomorphism $\varepsilon : X \to {X^*}_*$ by

$$\varepsilon(x) = \{U \in X^* : x \in U\}$$

for each $x \in X \in \mathsf{GPS}$, and then associating with each object X of GPS the morphism $R_\varepsilon \subseteq X \times {X^*}_*$ of GPS by

$$xR_\varepsilon \nabla \text{ iff } \varepsilon(x) \subseteq \nabla$$

for each $x \in X$ and $\nabla \in {X^*}_*$.

2.2 Duality for Implicative Semilattices

In the case of bounded implicative semilattices, we obtain the following restricted version of the above duality. Let $X = \langle X, \tau, \leq, X_0\rangle$ be a generalized Priestley space.

Then each clopen U in X has the form $\bigcup_{i=1}^{n} \bigcap_{j=1}^{m} (U_i - V_j)$, where $U_i, V_j \in X^*$. We call U an *Esakia clopen* if U has the form $\bigcup_{i=1}^{n} (U_i - V_i)$, where $U_i, V_i \in X^*$, and we call X a *generalized Esakia space* if for each Esakia clopen U, the set $\downarrow U$ is clopen in X. For a generalized Esakia space X, define the binary operation $\to$ on $X^* = \langle X^*, \cap, X, \emptyset \rangle$ by

$$U \to V = \{x \in X : \uparrow x \cap U \subseteq V\}.$$

Then $\langle X^*, \to \rangle$ is a bounded implicative semilattice. Let X and Y be generalized Priestley spaces and $R \subseteq X \times Y$ be a generalized Priestley morphism. We call R a *generalized Esakia morphism* if for each $x \in X$ and $y \in Y_0$, there exists $z \in X_0$ such that $x \leq z$ and $R[z] = \uparrow y$. If $R \subseteq X \times Y$ is a generalized Esakia morphism, then $h_R : Y^* \to X^*$ is a homomorphism of bounded implicative semilattices.

Let GES denote the category of generalized Esakia spaces and generalized Esakia morphisms. (Again, the composition of two generalized Esakia morphisms is defined as for generalized Priestley morphisms.) Then the restriction of $(-)^*$ to GES is a well-defined functor $(-)^* : \mathsf{GES} \to \mathsf{BIM}$. The converse is also true; that is, the restriction of $(-)_*$ to BIM is a well-defined functor $(-)_* : \mathsf{BIM} \to \mathsf{GES}$. These functors establish the desired dual equivalence of BIM and GES.

2.3 Priestley and Esakia Dualities as Particular Cases

We give a brief account of how Priestley duality between bounded distributive lattices and Priestley spaces and Esakia duality between Heyting algebras and Esakia spaces can be obtained as particular cases of the above dualities.

Let L be a bounded distributive meet-semilattice. If L happens to be a lattice, then the notions of optimal and prime filters of L coincide, so $L_* = L_+$, and so L_* is simply the Priestley space $\langle L_+, \tau, \leq \rangle$. Similarly, if $X = \langle X, \tau, \leq, X_0 \rangle$ is a generalized Priestley space such that $X_0 = X$, then X is simply the Priestley space $\langle X, \tau, \leq \rangle$.

Let X and Y be generalized Priestley spaces and $R \subseteq X \times Y$ be a generalized Priestley morphism. We call R *functional* if for each $x \in X$ there exists $y \in Y$ such that $R[x] = \uparrow y$. If X and Y happen to be Priestley spaces, then functional generalized Priestley morphisms correspond to Priestley morphisms. The correspondence is obtained as follows: If $R \subseteq X \times Y$ is a functional generalized Priestley morphism, then $f_R : X \to Y$ defined by

$$f_R(x) = \text{ the least lement of } R[x]$$

is a Priestley morphism; if $f : X \to Y$ is a Priestley morphism, then $R_f \subseteq X \times Y$ defined by

$$x R_f y \text{ iff } f(x) \leq y$$

is a functional generalized Priestley morphism. Moreover, $f_{R_f} = f$ and $R_{f_R} = R$. Thus, the category PS^{F} of Priestley spaces and functional generalized Priestley morphisms is isomorphic to PS. On the other hand, BDL is dually equivalent to PS^{F}. Priestley duality follows.

Similarly, if L is a bounded implicative semilattice which happens to be a Heyting algebra, then L_* is simply the Esakia space $\langle L_+, \tau, \leq \rangle$; and if $X = \langle X, \tau, \leq, X_0 \rangle$ is a generalized Esakia space in which $X_0 = X$, then X is simply the Esakia space $\langle X, \tau, \leq \rangle$. Moreover, functional generalized Esakia morphisms between Esakia spaces correspond to Esakia morphisms, and so the category ES^F of Esakia spaces and functional generalized Esakia morphisms is isomorphic to ES. Furthermore, HA is dually equivalent to ES^F. Esakia duality follows.

2.4 The Correspondence Between 1–1 and onto Morphisms

As a consequence of the dualities described above, we obtain that 1–1 morphisms in one category correspond to onto morphisms in its dual category and vice versa. Here we give an exact formulation of this for the categories BDM and GPS which contain all the other categories we consider in this paper as subcategories. Let X and Y be generalized Priestley spaces and let $R \subseteq X \times Y$ be a generalized Priestley morphism. We say that R is *onto* if for each $y \in Y$ there exists $x \in X$ such that $R[x] = \uparrow y$. We also say that R is *1–1* if for each $x \in X$ and $U \in X^*$ with $x \notin U$, there exists $V \in Y^*$ such that $R[U] \subseteq V$ and $R[x] \nsubseteq V$. Then we have that $R \subseteq X \times Y$ is 1–1 iff $h_R : Y^* \to X^*$ is onto, and that R is onto iff h_R is 1–1. Consequently, for two bounded distributive meet-semilattices L and K and a homomorphism $h : L \to K$, we have that h is 1–1 iff $R_h \subseteq K_* \times L_*$ is onto, and that h is onto iff R_h is 1–1.

3 Generalized Priestley Quasi-Orders

Let $X = \langle X, \tau, \leq \rangle$ be a Priestley space and let Q be a quasi-order on X extending $\leq$. We call $U \subseteq X$ a *Q-upset* if $x \in U$ and xQy imply $y \in U$. We say that Q is a *Priestley quasi-order* if $x \not\!Q y$ implies there exists a clopen Q-upset U of X such that $x \in U$ and $y \notin U$. In other words, Q is a Priestley quasi-order if xQy iff $x \in U$ implies $y \in U$ for each clopen Q-upset U of X. Let L be a bounded distributive lattice. It is well-known [1, 5, 11] that subalgebras of L dually correspond to Priestley quasi-orders on L_+. In fact, the complete lattice $\langle \mathcal{S}, \subseteq \rangle$ of subalgebras of L is isomorphic to the complete lattice $\langle \mathcal{P}, \supseteq \rangle$ of Priestley quasi-orders on L_+.

We generalize the notion of a Priestley quasi-order to that of a generalized Priestley quasi-order and show that subalgebras of a bounded distributive meet-semilattice L dually correspond to generalized Priestley quasi-orders on L_*. In fact, we introduce a partial order $\leq$ on the set $\mathcal{GP}$ of generalized Priestley quasi-orders on L_* and show that the poset $\langle \mathcal{S}, \subseteq \rangle$ of subalgebras of L is isomorphic to $\langle \mathcal{GP}, \geq \rangle$. We also introduce the notion of a generalized Esakia quasi-order and show that the complete lattice $\langle \mathcal{S}, \subseteq \rangle$ of subalgebras of a bounded implicative semilattice L is isomorphic to the poset $\langle \mathcal{GE}, \geq \rangle$ of generalized Esakia quasi-orders on L_*. In addition, we show how the isomorphism between the lattice of subalgebras of a bounded distributive lattice L and the lattice of Priestley quasi-orders on L_+ and the isomorphism between the lattice of subalgebras of a Heyting algebra A and the lattice of Esakia quasi-orders on A_+ are both easy consequences of our results.

3.1 Subalgebras of Bounded Distributive Meet-Semilattices

By a subalgebra of a bounded distributive meet-semilattice L we mean a bounded distributive meet-semilattice S, which is a $(\wedge, 0, 1)$-subalgebra of L. (Note that not every $(\wedge, 0, 1)$-subalgebra of L is necessarily distributive.)

Lemma 3.1 *Let L be a bounded distributive meet-semilattice and let S be a subalgebra of L.*

(1) *If $x \in S_*$, then there is $y \in L_*$ such that $x = y \cap S$.*
(2) *If $x \in S_+$, then there is $y \in L_+$ such that $x = y \cap S$.*

Proof

(1) Let $x \in S_*$. Consider the filter $G = \uparrow_L x$ of L and the F-ideal J of L generated by $S - x$. We claim that $G \cap J = \emptyset$. If not, then there is $a \in G \cap J$. Therefore, there exist $b \in x$ and $c_1, \ldots, c_n \in S - x$ such that $b \leq_L a$ and $\bigcap_{i=1}^n \uparrow_L c_i \subseteq \uparrow_L a$. Since $\uparrow_L a \subseteq \uparrow_L b$, we have $\bigcap_{i=1}^n \uparrow_L c_i \subseteq \uparrow_L b$. Thus, $\bigcap_{i=1}^n \uparrow_S c_i \subseteq \uparrow_S b$. Since x is an optimal filter of S, we have $S - x$ is an F-ideal of S. Thus, $b \in S - x$, a contradiction. We conclude that $G \cap J = \emptyset$. Then, by the optimal filter lemma, there is $y \in L_*$ such that $G \subseteq y$ and $y \cap J = \emptyset$. Consequently, $y \cap S = x$.
(2) Let $x \in S_+$. Then $S - x$ is a prime ideal of S. We show that $\downarrow_L (S - x)$ is an ideal of L. If $a, b \in \downarrow_L (S - x)$, then there exist $a', b' \in S - x$ such that $a \leq_L a'$ and $b \leq_L b'$. Since $S - x$ is an ideal, $\uparrow_S a' \cap \uparrow_S b' \cap (S - x) \neq \emptyset$. Let $c \in \uparrow_S a' \cap \uparrow_S b' \cap (S - x)$. Then $c \in \uparrow_L a \cap \uparrow_L b \cap \downarrow_L (S - x)$. Thus, $\downarrow_L (S - x)$ is an ideal of L. We claim that $\uparrow_L x \cap \downarrow_L (S - x) = \emptyset$. If $a \in \uparrow_L x \cap \downarrow_L (S - x)$, then there exist $b \in x$ and $c \in S - x$ such that $b \leq_L a \leq_L c$. Thus, $b \leq_S c$, and so $c \in x$, a contradiction. By the prime filter lemma, there is a prime filter y of L such that $\uparrow_L x \subseteq y$ and $y \cap \downarrow_L (S - x) = \emptyset$. Consequently, $x = y \cap S$. □

Definition 3.2 Let L be a bounded distributive meet-semilattice and let S be a subalgebra of L. We set $Y_S = \{x \in L_+ : x \cap S \in S_+\}$.

Lemma 3.3 *Let L be a bounded distributive meet-semilattice, S be a subalgebra of L, and $x \in L_*$. Then $x \cap S = \bigcap\{y \cap S : x \cap S \subseteq y \in Y_S\}$.*

Proof It is clear that $x \cap S \subseteq \bigcap\{y \cap S : x \cap S \subseteq y \in Y_S\}$. Conversely, let $a \notin x \cap S$. Then $a \in S - x$, and so $(x \cap S) \cap \downarrow_S a = \emptyset$. By the prime filter lemma, there is a prime filter z of S such that $x \cap S \subseteq z$ and $a \notin z$. By Lemma 3.1, there is $y \in L_+$ such that $z = y \cap S$. Then $y \in Y_S$ and $a \notin y \cap S$. Thus, $\bigcap\{y \cap S : x \cap S \subseteq y \in Y_S\} \subseteq x \cap S$. □

Let L be a bounded distributive meet-semilattice and let S be a subalgebra of L. Define a binary relation $Q_S \subseteq L_* \times L_*$ by

$$x Q_S y \text{ iff } x \cap S \subseteq y.$$

Lemma 3.4 *The relation Q_S is a quasi-order on L_*. Moreover, for each $x, y \in L_*$, if $x \subseteq y$, then $x Q_S y$.*

Proof Straightforward. □

Let L be a bounded distributive meet-semilattice and S be a subalgebra of L. We characterize the sets $\varphi_L(a)$ with $a \in S$. For $X \subseteq L_*$, let $\downarrow_{Q_S}(X) = y \in L_* : \exists x \in X$ with $y Q_S x$.

Lemma 3.5 *Let L be a bounded distributive meet-semilattice, S be a subalgebra of L, and $a \in S$.*

(1) $\varphi_L(a)$ *is a* Q_S*-upset of* L_*.
(2) $\varphi_L(a) = [\downarrow_{Q_S}(Y_S - \varphi_L(a))]^c$.

Proof

(1) Let $x \in \varphi_L(a)$ and xQ_Sy. Then $x \cap S \subseteq y$, and as $a \in x \cap S$, we have $a \in y$. Thus, $y \in \varphi_L(a)$, and so $\varphi_L(a)$ is a Q_S-upset of L_*.
(2) By (1), $\varphi_L(a)$ is a Q_S-upset of L_*. Therefore, $\varphi_L(a) \cap \downarrow_{Q_S}(Y_S - \varphi_L(a)) = \emptyset$, and so $\varphi_L(a) \subseteq [\downarrow_{Q_S}(Y_S - \varphi_L(a))]^c$. Conversely, suppose that $x \in [\downarrow_{Q_S}(Y_S - \varphi_L(a)]^c$. Then $(\forall y \in Y_S)(xQ_Sy \Rightarrow a \in y)$. If $a \notin x$, then $a \notin x \cap S$. By Lemma 3.3, there exists $y \in Y_S$ such that $x \cap S \subseteq y$ and $a \notin y$. Therefore, xQ_Sy, and so $a \in y$, a contradiction. Thus, $a \in x$, and so $x \in \varphi_L(a)$. It follows that $\varphi_L(a) = [\downarrow_{Q_S}(Y_S - \varphi_L(a))]^c$. □

Lemma 3.6 *Let L be a bounded distributive meet-semilattice, S be a subalgebra of L, and $a \in L$. If $\varphi_L(a) = [\downarrow_{Q_S}(Y_S - \varphi_L(a))]^c$, then $a \in S$.*

Proof Suppose that $a \notin S$. First assume that there exist $a_1, \ldots, a_n \in S$ such that $a = a_1 \vee_L \ldots \vee_L a_n$. Then $\varphi_L(a) = \varphi_L(a_1) \cup \ldots \cup \varphi_L(a_n)$. If $a_1 \vee_S \ldots \vee_S a_n$ exists in S and $a_1 \vee_S \ldots \vee_S a_n = b$, then $\bigcap_{i=1}^n \uparrow_S a_i = \uparrow_S b$ and $a < b$. By the optimal filter lemma, there exists $x \in L_*$ such that $b \in x$ and $a \notin x$. Then $x \notin \varphi_L(a) = [\downarrow_{Q_S}(Y_S - \varphi_L(a))]^c$, and so there exists $y \in Y_S$ such that xQ_Sy and $y \notin \varphi_L(a)$. Since $b \in x \cap S \subseteq y$, we have $\bigcap_{i=1}^n \uparrow_S a_i = \uparrow_S b \subseteq y \cap S \in S_+$. Therefore, $\uparrow_S a_i \subseteq y \cap S$ for some $i \leq n$. Thus, $a_i \in y$, and so $y \in \varphi_L(a_i) \subseteq \varphi_L(a)$, a contradiction. It follows that $a_1 \vee_S \ldots \vee_S a_n$ does not exist in S. Let $F = \bigcap_{i=1}^n \uparrow_S a_i$ and I be the F-ideal of S generated by $\{a_1, \ldots, a_n\}$. If there is $c \in F \cap I$, then $\bigcap_{i=1}^n \uparrow_S a_i = \uparrow_S c$. This implies that $c = a_1 \vee_S \ldots \vee_S a_n$, a contradiction. Thus, $F \cap I = \emptyset$, and by the optimal filter lemma, there exists $x \in S_*$ such that $F \subseteq x$ and $x \cap I = \emptyset$. By Lemma 3.1, there exists $y \in L_*$ such that $x = y \cap S$. If $y \in \varphi_L(a)$, then $y \in \varphi_L(a_1) \cup \ldots \cup \varphi_L(a_n)$. So $a_i \in y$ for some $i \leq n$, which is a contradiction as $a_i \in I$ and $(y \cap S) \cap I = x \cap I = \emptyset$. Thus, $y \notin \varphi_L(a) = [\downarrow_{Q_S}(Y_S - \varphi_L(a))]^c$, and so there exists $z \in Y_S - \varphi_L(a)$ such that yQ_Sz. Therefore, $y \cap S \subseteq z$, so $\bigcap_{i=1}^n \uparrow_S a_i \subseteq y \cap S \subseteq z$, and so there is $i \leq n$ such that $\uparrow_S a_i \subseteq z$. Thus, $a_i \in z$, and so $z \in \varphi_L(a_i) \subseteq \varphi_L(a)$, a contradiction. It follows that our assumption that there exist $a_1, \ldots, a_n \in S$ such that $a = a_1 \vee_L \ldots \vee_L a_n$ is false.

Consider the filter $\uparrow_L a$ and the F-ideal J of L generated by $\downarrow_L a \cap S$. If there is $b \in \uparrow_L a \cap J$, then $a \leq_L b$ and there exist $c_1, \ldots, c_n \in \downarrow_L a \cap S$ such that $\bigcap_{i=1}^n \uparrow_L c_i \subseteq \uparrow_L b$. So $a \in \bigcap_{i=1}^n \uparrow_L c_i \subseteq \uparrow_L b \subseteq \uparrow_L a$, and so $\bigcap_{i=1}^n \uparrow_L c_i = \uparrow_L a$. Therefore, $a = c_1 \vee_L \cdots \vee_L c_n$, a contradiction. Thus, $\uparrow_L a \cap J = \emptyset$, and by the optimal filter lemma, there exists $x \in L_*$ such that $\uparrow_L a \subseteq x$ and $x \cap J = \emptyset$. Now consider the filter G of L generated by $x \cap S$ and the ideal $\downarrow_L a$. If there is $b \in G \cap \downarrow_L a$, then $c \leq_L b \leq_L a$ for some $c \in x \cap S$. Therefore, $c \leq_L a$, so $c \in \downarrow_L a \cap S$, which is not possible because $c \in x$ and $x \cap (\downarrow_L a \cap S) = \emptyset$. Therefore, $G \cap \downarrow_L a = \emptyset$, and by the prime filter lemma, there is $y \in L_+$ such that $G \subseteq y$ and $y \cap \downarrow_L a = \emptyset$. Thus, $x \cap S \subseteq y$ and $a \notin y$. So xQ_Sy and $y \in Y_S - \varphi_L(a)$, and so $x \notin [\downarrow_{Q_S}(Y_S - \varphi_L(a))]^c = \varphi_L(a)$, a contradiction. Consequently, our assumption that $a \notin S$ is false, and so $a \in S$. □

Let L be a bounded distributive meet-semilattice and let S be a subalgebra of L. We set

$$Y_S{}^* = \{\varphi_L(a) : \varphi_L(a) = [\downarrow_{Q_S}(Y_S - \varphi_L(a))]^c\}.$$

The next lemma is an immediate consequence of Lemmas 3.5 and 3.6.

Lemma 3.7 *Let L be a bounded distributive meet-semilattice and let S be a subalgebra of L. Then $Y_S{}^* = \{\varphi_L(a) : a \in S\}$.*

Lemma 3.8 *Let L be a bounded distributive meet-semilattice and let S be a subalgebra of L. For each $x \in L_+$, we have $x \in Y_S$ iff for each $a_1, \ldots, a_n \in S$ with $x \notin \varphi_L(a_1) \cup \ldots \cup \varphi_L(a_n)$, there is $a \in S$ such that $x \notin \varphi_L(a)$ and $\varphi_L(a_1) \cup \ldots \cup \varphi_L(a_n) \subseteq \varphi_L(a)$.*

Proof Suppose that $x \in Y_S$. Then $x \in L_+$ and $x \cap S \in S_+$. Let $a_1, \ldots, a_n \in S$ with $x \notin \varphi_L(a_1) \cup \ldots \cup \varphi_L(a_n)$. If $\bigcap_{i=1}^n \uparrow_S a_i \subseteq x \cap S$, then as $x \cap S \in S_+$, there is $i \leq n$ such that $\uparrow_S a_i \subseteq x \cap S$, so $x \in \varphi_L(a_i) \subseteq \varphi_L(a_1) \cup \ldots \cup \varphi_L(a_n)$, a contradiction. Therefore, $\bigcap_{i=1}^n \uparrow_S a_i \not\subseteq x \cap S$. Thus, there is $a \in S$ such that $a \in \bigcap_{i=1}^n \uparrow_S a_i$ and $a \notin x$. This implies that $\varphi_L(a_1) \cup \ldots \cup \varphi_L(a_n) \subseteq \varphi_L(a)$ and $x \notin \varphi_L(a)$. Conversely, suppose that $x \in L_+$ and for each $a_1, \ldots, a_n \in S$ with $x \notin \varphi_L(a_1) \cup \ldots \cup \varphi_L(a_n)$, there is $a \in S$ such that $x \notin \varphi_L(a)$ and $\varphi_L(a_1) \cup \ldots \cup \varphi_L(a_n) \subseteq \varphi_L(a)$. We show that $x \cap S$ is a prime filter of S. If not, then there exist filters F_1 and F_2 of S such that $F_1 \cap F_2 \subseteq x \cap S$, $F_1 \not\subseteq x \cap S$, and $F_2 \not\subseteq x \cap S$. Let $a_1 \in F_1 - x$ and $a_2 \in F_2 - x$. Then $x \notin \varphi_L(a_1) \cup \varphi_L(a_2)$. By the assumption, there exists $a \in S$ such that $x \notin \varphi_L(a)$ and $\varphi_L(a_1) \cup \varphi_L(a_2) \subseteq \varphi_L(a)$. Therefore, $a \notin x$ and $a_1, a_2 \leq a$. Thus, $a \in \uparrow_S a_1 \cap \uparrow_S a_2 \subseteq F_1 \cap F_2 \subseteq x$, a contradiction. It follows that $x \cap S$ is a prime filter of S, and so $x \in Y_S$. □

The properties of Q_S and Y_S suggest the following definition. Let $X = \langle X, \tau, \leq, X_0 \rangle$ be a generalized Priestley space, Q be a quasi-order on X extending $\leq$, and $Y \subseteq X_0$. We set

$$Y^* = \{U \in X^* : U = [\downarrow_Q(Y - U)]^c\}.$$

Definition 3.9 Let X be a generalized Priestley space. We call a pair $\langle Q, Y\rangle$ a *generalized Priestly quasi-order* if $\langle Q, Y\rangle$ satisfies the following conditions:

(1) Q is a quasi-order on X extending $\leq$.
(2) $Y \subseteq X_0$.
(3) For each $x \in X$ there is $y \in Y$ such that xQy.
(4) If $x \in X_0$, then $x \in Y$ iff for each $U_1, \ldots, U_n \in Y^*$ with $x \notin U_1 \cup \ldots \cup U_n$, there is $V \in Y^*$ such that $x \notin V$ and $U_1 \cup \ldots \cup U_n \subseteq V$.
(5) xQy iff $(\forall U \in Y^*)(x \in U \Rightarrow y \in U)$.

Theorem 3.10 *Let L be a bounded distributive meet-semilattice and let S be a subalgebra of L. Then $\langle Q_S, Y_S\rangle$ is a generalized Priestley quasi-order on L_*.*

Proof By Lemma 3.4, $\langle Q_S, Y_S\rangle$ satisfies condition (1) of Definition 3.9. The definition of Y_S implies that $\langle Q_S, Y_S\rangle$ satisfies condition (2). We show that $\langle Q_S, Y_S\rangle$ satisfies condition (3). For each $x \in L_*$ by Lemma 3.3, there exists $y \in Y_S$ such that $x \cap S \subseteq y$, so xQ_Sy, and so condition (3) is satisfied. That $\langle Q_S, Y_S\rangle$ satisfies condition

(4) follows from Lemmas 3.7 and 3.8. Finally, it follows from Lemmas 3.5 and 3.6 and the definition of Q_S that $\langle Q_S, Y_S\rangle$ satisfies condition (5). Thus, $\langle Q_S, Y_S\rangle$ is a generalized Priestley quasi-order on L_*. □

Lemma 3.11 *Let X be a generalized Priestley space and let (Q, Y) be a generalized Priestley quasi-order on X. Then:*

(1) *$\langle Y^*, \cap, X, \emptyset\rangle$ is a bounded distributive meet-semilattice.*
(2) *Y^* is a subalgebra of X^*.*

Proof

(1) Since $[\downarrow_Q(Y-X)]^c = (\downarrow_Q\emptyset)^c = \emptyset^c = X$, we have $X \in Y^*$. Also, $[\downarrow_Q(Y-\emptyset)]^c = (\downarrow_Q Y)^c = X^c = \emptyset$, and so $\emptyset \in Y^*$. Next we show that Y^* is closed under $\cap$. Let $U, V \in Y^*$. Then $U, V \in X^*$, so $U \cap V \in X^*$. Moreover, $U\cap V = [\downarrow_Q(Y-U)]^c \cap [\downarrow_Q(Y-U)]^c = [\downarrow_Q(Y-U) \cup \downarrow_Q(Y-V)]^c = [\downarrow_Q((Y-U)\cup(Y-V))]^c = [\downarrow_Q(Y-(U\cap V))]^c$. Thus, $U\cap V \in Y^*$. Finally, we show that Y^* is distributive. Suppose that $U, V, W \in Y^*$ and $U \cap V \subseteq W$. Let $x \notin W = [\downarrow_Q(Y-W]^c$. Then there exists $y \in Y - W$ such that xQy. We have $y \in U$ or $y \notin U$. If $y \in U$, then $y \notin V$, so $y \notin W \cup V$. Therefore, by condition (4) of Definition 3.9, there exists $Z_x \in Y^*$ such that $y \notin Z_x$ and $W \cup V \subseteq Z_x$. If $y \notin U$, then $y \notin W \cup U$, so there exists $Z_x \in Y^*$ such that $y \notin Z_x$ and $W \cup U \subseteq Z_x$. In both cases we have $x \notin Z_x$ because Z_x is a Q-upset of X and $xQy \notin Z_x$. Then $W^c = \bigcup\{Z_x^c : x \notin W\}$. Since X is compact, there is a finite $A \subseteq (W\cup V)^c$ and a finite $B \subseteq (W \cup U)^c$ such that $W^c = \bigcup\{Z_x^c : x\in A\} \cup \bigcup\{Z_x^c : x \in B\}$. Let $U' = \bigcap\{Z_x : x \in A\}$ and $V' = \bigcap\{Z_x : x \in B\}$. Then $U', V' \in Y^*$, $U \subseteq U'$, $V \subseteq V'$, and $W = U' \cap V'$. Thus, $\langle Y^*, \cap, X, \emptyset\rangle$ is a bounded distributive meet-semilattice.
(2) follows from (1). □

Let X be a generalized Priestley space and let $\mathcal{GP}$ denote the set of generalized Priestley quasi-orders on X. For $\langle Q, Y\rangle, \langle R, Z\rangle \in \mathcal{GP}$, we set $\langle Q, Y\rangle \leq \langle R, Z\rangle$ if $Y \subseteq Z$ and xQy implies xRy for all $x, y \in Y$. Clearly $\leq$ is a partial order on $\mathcal{GP}$.

Theorem 3.12 *For a bounded distributive meet-semilattice L, the poset $\langle \mathcal{S}, \subseteq\rangle$ of subalgebras of L is isomorphic to $\langle \mathcal{GP}, \geq\rangle$.*

Proof Suppose that S is a subalgebra of L. By Theorem 3.10, $\langle Q_S, Y_S\rangle$ is a generalized Priestley quasi-order on L_*. Conversely, if $\langle Q, Y\rangle$ is a generalized Priestley quasi-order on L_*, then Lemma 3.11 implies that Y^* is a subalgebra of ${L_*}^*$, thus is isomorphic to a subalgebra of L. We show that this correspondence is 1–1. If S is a subalgebra of L, then Lemma 3.7 implies that S is isomorphic to ${Y_S}^*$. If $\langle Q, Y\rangle$ is a generalized Priestley quasi-order on L_*, then we show that ε is an isomorphism between $\langle Q, Y\rangle$ and $\langle Q_{Y^*}, Y^*\rangle$. By the definition of Q_{Y^*} and condition (5) of Definition 3.9, $\varepsilon(x) Q_{Y^*} \varepsilon(y)$ iff $\varepsilon(x) \cap Y^* \subseteq \varepsilon(y)$ iff $(\forall U \in Y^*)(U \in \varepsilon(x) \Rightarrow U \in \varepsilon(y))$ iff $(\forall U \in Y^*)(x \in U \Rightarrow y \in U)$ iff xQy. Moreover, $\varepsilon(x) \in Y_{Y^*}$ iff $\varepsilon(x) \in ({L_*}^*)_+$ and $\varepsilon(x) \cap Y^* \in (Y^*)_+$. Now, if $x \in Y$, then $x \in L_+$, so $\varepsilon(x) \in ({L_*}^*)_+$. Also, by condition (4) of Definition 3.9, $Y^* - (\varepsilon(x) \cap Y^*)$ is a prime ideal of Y^*, and so $\varepsilon(x) \cap Y^* \in (Y^*)_+$. Therefore, $\varepsilon(x) \in Y_{Y^*}$. Conversely, suppose that $\varepsilon(x) \in$

Y_{Y^*}. Then $\varepsilon(x) \in ({L_*}^*)_+$ and $\varepsilon(x) \cap Y^* \in (Y^*)_+$. So $x \in L_+$. To see that $x \in Y$, let $a_1, \ldots, a_n \in L$ be such that $\varphi_L(a_1), \ldots, \varphi_L(a_n) \in Y^*$ and $x \notin \varphi_L(a_1) \cup \ldots \cup \varphi_L(a_n)$. Then $\varphi_L(a_1), \ldots, \varphi_L(a_n) \notin \varepsilon(x)$ and so $\varphi_L(a_1), \ldots, \varphi_L(a_n) \notin \varepsilon(x) \cap Y^*$. Since $Y^* - (\varepsilon(x) \cap Y^*)$ is a prime ideal of Y^*, there is $b \in L$ such that $\varphi_L(b) \in Y^* - (\varepsilon(x) \cap Y^*)$ and $\varphi_L(a_1) \cup \ldots \cup \varphi_L(a_n) \subseteq \varphi_L(b)$. Therefore, $x \notin \varphi_L(b)$, and so condition (4) of Definition 3.9 implies that $x \in Y$.

Let $S, T \in \mathcal{S}$ with $S \subseteq T$. Then $x \in Y_T$ implies $x \in L_+$ and $x \cap T \in T_+$. Since S is a subalgebra of T, from $x \cap T \in T_+$ it follows that $x \cap S \in S_+$. Therefore, $Y_T \subseteq Y_S$. Moreover, if xQ_Ty, then $x \cap T \subseteq y$, so $x \cap S \subseteq y$, and so xQ_Sy. Thus, $\langle Q_T, Y_T\rangle \leq \langle Q_S, Y_S\rangle$. Conversely, suppose that $S \not\subseteq T$. Then there exists $a \in S - T$. Since $a \notin T$, by Lemma 3.5, there exist $x \in \varphi_L(a)$ and $y \in Y_T - \varphi_L(a)$ with xQ_Ty. As $a \in S$, we have $\varphi_L(a) = [{\downarrow}_{Q_S}(Y_S - \varphi_L(a)]^c$. Therefore, $y \notin Y_S$ or $x\not{Q}_Sy$. Thus, $Y_T \not\subseteq Y_S$ or xQ_Ty does not imply xQ_Sy, and so $\langle Q_T, Y_T\rangle \not\leq \langle Q_S, Y_S\rangle$. Consequently, for $S, T \in \mathcal{S}$ we have $S \subseteq T$ iff $\langle Q_T, Y_T\rangle \leq \langle Q_S, Y_S\rangle$, which together with the 1–1 correspondence between subalgebras of L and generalized Priestley quasi-orders on L_* gives us that $\langle \mathcal{S}, \subseteq\rangle$ is isomorphic to $\langle \mathcal{GP}, \geq\rangle$. □

3.2 Subalgebras of Bounded Distributive Lattices

Now we show how Theorem 3.12 implies easily the well-known correspondence between subalgebras of a bounded distributive lattice L and Priestley quasi-orders on L_+.

Let L be a bounded distributive lattice and S be a subalgebra of L. Consider $\langle Y_S, Q_S\rangle$. It follows from conditions (1) and (5) of Definition 3.9 that Q_S is a Priestley quasi-order on $L_+ = L_*$. Moreover, since $L_+ = L_*$ and ${Y_S}^*$ is closed under $\cup$, conditions (2) and (4) of Definition 3.9 imply that $Y_S = L_+$. Thus, the generalized Priestley quasi-order $\langle Y_S, Q_S\rangle$ simply becomes the Priestley quasi-order Q_S. Moreover, given two generalized Priestley quasi-orders $\langle Q, Y\rangle$ and $\langle R, Z\rangle$ on L_+, we obviously have that $\langle Q, Y\rangle \leq \langle R, Z\rangle$ iff $Q \subseteq R$. Consequently, the poset $\langle \mathcal{GP}, \leq\rangle$ of generalized Priestley quasi-orders is equal to the poset $\langle \mathcal{P}, \subseteq\rangle$ of Priestley quasi-orders on L_+, and so Theorem 3.12 implies the following well-known dual characterization of subalgebras of L.

Corollary 3.13 [1, 5, 11] *For a bounded distributive lattice L, the complete lattice $\langle \mathcal{S}, \subseteq\rangle$ of subalgebras of L is isomorphic to the poset $\langle \mathcal{P}, \supseteq\rangle$ of Priestley quasi-orders on L_+.*

3.3 Subalgebras of Bounded Implicative Semilattices

By a subalgebra of a bounded implicative semilattice L we mean a $(\wedge, \to, 0)$-subalgebra of L. We turn to the dual characterization of subalgebras of bounded implicative semilattices. Let X be an Esakia space and let Q be a Priestley quasi-order on X. Define $\sim_Q$ on X by

$$x \sim_Q y \text{ iff } xQy \text{ and } yQx.$$

Clearly $\sim_Q$ is an equivalence relation on X. We call Q an *Esakia quasi-order* on X if $(\forall x, y \in X)(xQy \Rightarrow (\exists z \in X)(x \leq z \;\&\; z \sim_Q y))$.

Definition 3.14 Let X be a generalized Esakia space and let $\langle Q, Y\rangle$ be a generalized Priestley quasi-order on X. We call $\langle Q, Y\rangle$ a *generalized Esakia quasi-order* on X if $(\forall x \in X)(\forall y \in Y)(xQy \Rightarrow (\exists z \in Y)(x \leq z \;\&\; z \sim_Q y))$.

For a generalized Esakia space X, let $\mathcal{GE}$ denote the set of generalized Esakia quasi-orders on X, and let $\leq$ be the restriction of $\leq$ from $\mathcal{GP}$ to $\mathcal{GE}$.

Lemma 3.15 *Let L be a bounded implicative semilattice and let S be a subalgebra of L. Then $\langle Q_S, Y_S\rangle$ is a generalized Esakia quasi-order on L_*.*

Proof Since S is a subalgebra of L, it follows from Theorem 3.10 that $\langle Q_S, Y_S\rangle$ is a generalized Priestley quasi-order on L_*. Let $x \in L_*$, $y \in Y_S$, and xQ_Sy. Then $x \cap S \subseteq y$. We have $S - y$ is an ideal of S. Therefore, $\downarrow_L(S - y)$ is an ideal of L. Let F be the filter of L generated by $x \cup (y \cap S)$. We show that $F \cap \downarrow_L(S - y) = \emptyset$. If there exists $a \in F \cap \downarrow_L(S - y)$, then there exist $b \in x$, $c \in y \cap S$, and $d \in S - y$ such that $b \wedge c \leq a \leq d$. Therefore, $b \wedge c \leq d$, and so $b \leq c \to d$. Thus, $c \to d \in x \cap S$, and so $c \to d \in y$. As $c \in y$, it follows that $d \in y \cap S$, a contradiction. Consequently, $F \cap \downarrow_L(S - y) = \emptyset$. By the prime filter lemma, there exists $z \in L_+$ such that $F \subseteq z$ and $z \cap \downarrow_L(S - y) = \emptyset$. Therefore, $z \cap S = y \cap S \in S_+$. Thus, $z \in Y_S$, $x \subseteq z$, and $z \sim_{Q_S} y$. Consequently, $\langle Q_S, Y_S\rangle$ is a generalized Esakia quasi-order on L_*. □

Lemma 3.16 *Let X be a generalized Esakia space and let $\langle Q, Y\rangle$ be a generalized Esakia quasi-order on X. Then Y^* is a subalgebra of X^*.*

Proof By Lemma 3.11, it is sufficient to show that Y^* is closed under $\to$. Let $U, V \in Y^*$. Then $U = [\downarrow_Q(Y - U)]^c$ and $V = [\downarrow_Q(Y - V)]^c$. We show that $U \to V = [\downarrow_Q(Y - (U \to V))]^c$. Let $x \in U \to V$, $y \in Y$, and xQy. We claim that $\uparrow y \cap U \subseteq V$. If not, then there exists $z \in \uparrow y \cap U$ such that $z \notin V$. Then $y \leq z \in U$ and $z \in \downarrow_Q(Y - V)$. Therefore, there exists $z' \in Y$ such that zQz' and $z' \notin V$. Thus, $xQy \leq zQz'$, and so xQz'. Since $\langle Q, Y\rangle$ is a generalized Esakia quasi-order on X, there exists $z'' \in Y$ such that $x \leq z''$ and $z'' \sim_Q z'$. So $zQz'Qz''$, and so zQz''. Since $z \in U$ and U is a Q-upset of X, we have $z'' \in U$. Therefore, $z'' \in \uparrow x \cap U$. As $x \in U \to V$, we have $\uparrow x \cap U \subseteq V$. Thus, $z'' \in V$, and since V is a Q-upset of X and $z''Qz'$, we have $z' \in V$, a contradiction. Consequently, $\uparrow y \cap U \subseteq V$, so $y \in U \to V$, and so $x \in [\downarrow_Q(Y - (U \to V))]^c$. Thus, $U \to V \subseteq [\downarrow_Q(Y - (U \to V))]^c$. Conversely, suppose that $x \notin U \to V$. Then there exists $z \in X$ such that $z \in \uparrow x \cap U$ and $z \notin V$. Therefore, $x \leq z \in U$ and $z \in \downarrow_Q(Y - V)$. Thus, there exists $y \in Y$ such that zQy and $y \notin V$. Since $\langle Q, Y\rangle$ is a generalized Esakia quasi-order on X, there is $z' \in Y$ such that $z \leq z'$ and $z' \sim_Q y$. Thus, $z' \in U$ and as U is a Q-upset of X and $z'Qy$, we also have $y \in U$. Therefore, $y \notin U \to V$, and since xQy, we obtain $x \in \downarrow_Q(Y - (U \to V))$. Consequently, $x \notin [\downarrow_Q(Y - (U \to V))]^c$, so $[\downarrow_Q(Y - (U \to V))]^c \subseteq U \to V$, and so $U \to V = [\downarrow_Q(Y - (U \to V))]^c$. It follows that $U \to V \in Y^*$. □

Theorem 3.17 *For a bounded implicative semilattice L, the complete lattice $\langle \mathcal{S}, \subseteq\rangle$ of subalgebras of L is isomorphic to the poset $\langle \mathcal{GE}, \geq\rangle$ of generalized Esakia quasi-orders on L_*.*

Proof Apply Theorem 3.12 and Lemmas 3.15 and 3.16. □

3.4 Subalgebras of Heyting Algebras

Now we show how Theorem 3.17 implies the dual characterization of subalgebras of a Heyting algebra A by means of Esakia quasi-orders on $A_+ = A_*$. Let A be a Heyting algebra and let S be a subalgebra of A. Then S is a bounded sublattice of A, so Q_S is a Priestley quasi-order on A_+. Moreover, since S is a $(\wedge, \to, 0)$-subalgebra of A, we have that Q_S is in fact an Esakia quasi-order on A_+. Consequently, the poset $\langle \mathcal{GE}, \leq \rangle$ of generalized Esakia quasi-orders is equal to the poset $\langle \mathcal{E}, \subseteq \rangle$ of Esakia quasi-orders on A_+. Thus, Theorem 3.17 implies the following dual characterization of subalgebras of A.

Corollary 3.18 *For a Heyting algebra A, the complete lattice $\langle \mathcal{S}, \subseteq \rangle$ of subalgebras of A is isomorphic to the poset $\langle \mathcal{E}, \supseteq \rangle$ of Esakia quasi-orders on A_+.*

Let X be an Esakia space and let $\sim$ be an equivalence relation on X. For $x \in X$ let $[x] = \{y \in X : x \sim y\}$, and for $Y \subseteq X$ let $[Y] = \bigcup\{[y] : y \in Y\}$. We call $Y \subseteq X$ *saturated* if $Y = [Y]$. We say that $\sim$ is an *Esakia equivalence relation* if $\sim$ satisfies the following two conditions:

(1) $x \not\sim y$ implies there exists a saturated clopen U of X such that $x \in U$ and $y \notin U$.
(2) $(\forall x, y, z \in X)((x \sim y \;\&\; y \leq z) \Rightarrow (\exists z \in X)(x \leq z \;\&\; z \sim y))$.

We point out that if E satisfies only condition (1), then E is an equivalence relation which is a Priestley quasi-order. We call such equivalence relations *Priestley equivalence relations*. Thus, an Esakia equivalence relation is a Priestley equivalence relation satisfying condition (2).

There is a 1–1 correspondence between Esakia quasi-orders and Esakia equivalence relations on an Esakia space X. Indeed, it is easy to verify that if Q is an Esakia quasi-order on X, then $\sim_Q$ is an Esakia equivalence relation on X. Conversely, if $\sim$ is an Esakia equivalence relation on X, then $Q_\sim = (\leq \circ \sim)$ is an Esakia quasi-order on X. Moreover, $Q_{\sim_Q} = Q$ and $\sim_{Q_\sim} = \sim$. Thus, Corollary 3.18 implies the following well-known characterization of subalgebras of A.

Corollary 3.19 [6] *For a Heyting algebra A, the complete lattice of subalgebras of A (ordered by $\subseteq$) is isomorphic to the poset of Esakia equivalence relations on A_+ (ordered by $\supseteq$).*

If A is a Boolean algebra, then the Stone space of X is the space of ultrafilters of X. Therefore, $\leq$ becomes simply $=$. Thus, Esakia equivalence relations become simply Priestley equivalence relations, and so we obtain the following well-known characterization of subalgebras of a Boolean algebra:

Corollary 3.20 (see, e.g., [7, Sec. 8.2]) *For a Boolean algebra A, the complete lattice of subalgebras of A (ordered by $\subseteq$) is isomorphic to the poset of Priestley equivalence relations on A_+ (ordered by $\supseteq$).*

4 Vietoris Families

In this section we give the dual descriptions of homomorphic images of bounded distributive meet-semilattices and bounded implicative semilattices. We also show how our dual descriptions lead to the well-known dual descriptions of homomorphic images of bounded distributive lattices, Heyting algebras, and Boolean algebras.

4.1 Bounded Distributive Meet-Semilattices

Let L be a bounded distributive meet-semilattice. Since homomorphic images of L are onto homomorphisms of L, dually they correspond to 1–1 generalized Priestley morphisms R from some generalized Priestley space to L_*. We give a description of homomorphic images of L purely in terms of L_* without referring to any generalized Priestley space other than L_*.

We start by considering a generalized Priestley space $X = \langle X, \tau, \leq, X_0\rangle$ and a 1–1 generalized Priestley morphism $R \subseteq X \times L_*$. Set

$$\mathfrak{F}_R = \{R[x] : x \in X\} \text{ and } (\mathfrak{F}_R)_0 = \{R[x] : x \in X_0\}.$$

Then $\mathfrak{F}_R$ and $(\mathfrak{F}_R)_0$ are families of nonempty closed upsets of L_* such that $(\mathfrak{F}_R)_0 \subseteq \mathfrak{F}_R$. Since R is 1–1, we have $x \leq y$ iff $R[y] \subseteq R[x]$. Define the *Vietoris topology* (or the *hit-or-miss topology*) τ_V on $\mathfrak{F}_R$ as follows: For $a \in L$, set

$$\begin{aligned} H_a &= \{R[x] : R[x] \cap \varphi(a)^c \neq \emptyset\} \text{ and} \\ M_a &= \{R[x] : R[x] \cap \varphi(a)^c = \emptyset\} = \{R[x] : R[x] \subseteq \varphi(a)\}. \end{aligned}$$

Clearly M_a and H_a are set-theoretic complements of each other. We let

$$\mathfrak{B}_V = \{M_a : a \in L\} \cup \{H_a : a \in L\}$$

be a subbasis for τ_V.

Lemma 4.1 *Let L be a bounded distributive meet-semilattice, X be a generalized Priestley space, and $R \subseteq X \times L_*$ be a 1–1 generalized Priestley morphism. Then for each $F \in \mathfrak{F}_R$ we have $F = \bigcap\{\varphi(a) : F \subseteq \varphi(a)\}$.*

Proof Let $F \in \mathfrak{F}_R$. Then there exists $x \in X$ such that $F = R[x]$. Clearly $R[x] \subseteq \bigcap\{\varphi(a) : R[x] \subseteq \varphi(a)\}$. Suppose that $y \notin R[x]$. Then $x \not R y$, and as R is a generalized Priestley morphism, there exists $a \in L$ such that $y \notin \varphi(a)$ and $R[x] \subseteq \varphi(a)$. Thus, $R[x] = \bigcap\{\varphi(a) : R[x] \subseteq \varphi(a)\}$. □

Lemma 4.2 *Let L be a bounded distributive meet-semilattice, X be a generalized Priestley space, and $R \subseteq X \times L_*$ be a 1–1 generalized Priestley morphism. Then $\langle \mathfrak{F}_R, \tau_V, \supseteq\rangle$ is a Priestley space which is order-homeomorphic to $\langle X, \tau, \leq\rangle$.*

Proof Since R is 1–1, we have $x \leq y$ iff $R[y] \subseteq R[x]$, so $\langle X, \leq\rangle$ is order-isomorphic to $\langle \mathfrak{F}_R, \supseteq\rangle$. As τ_V is a Vietoris topology and L_* is compact, we obtain that $\langle \mathfrak{F}_R, \tau_V\rangle$ is compact as well. To see that $\langle \mathfrak{F}_R, \tau_V, \supseteq\rangle$ satisfies the Priestley separation axiom, let $F \not\supseteq G$ with $F, G \in \mathfrak{F}_R$. By Lemma 4.1, there exists $a \in L$ such that $F \subseteq \varphi(a)$ and $G \not\subseteq \varphi(a)$. Thus, $F \in M_a$ and $G \notin M_a$, and as M_a is a clopen upset of $\langle \mathfrak{F}_R, \tau_V, \supseteq\rangle$, we obtain that $\langle \mathfrak{F}_R, \tau_V, \supseteq\rangle$ is a Priestley space. We show that $\langle \mathfrak{F}_R, \tau_V\rangle$ is homeomorphic to $\langle X, \tau\rangle$. Define $f : X \to \mathfrak{F}_R$ by $f(x) = R[x]$. Then f is a bijection. Moreover, for $a \in L$ and $x \in X$ we have $x \in f^{-1}(M_a)$ iff $f(x) \in M_a$ iff $R[x] \subseteq \varphi(a)$ iff $x \in \Box_R(\varphi(a))$. Consequently, $f^{-1}(M_a) = \Box_R(\varphi(a))$ and $f^{-1}(H_a) = \Box_R(\varphi(a))^c$, and so f is continuous. Finally, since f is a continuous map between compact Hausdorff spaces, f is a homeomorphism. □

Lemma 4.3 *Let L be a bounded distributive meet-semilattice, X be a generalized Priestley space, and $R \subseteq X \times L_*$ be a 1–1 generalized Priestley morphism. Then $\langle \mathfrak{F}_R, \tau_V, \supseteq, (\mathfrak{F}_R)_0\rangle$ is a generalized Priestley space.*

Proof It follows from Lemma 4.2 that $\langle \mathfrak{F}_R, \tau_V, \supseteq\rangle$ is a Priestley space which is order-homeomorphic to $\langle X, \tau, \leq\rangle$. This implies that $(\mathfrak{F}_R)_0 = f(X_0)$ is dense in $\mathfrak{F}$. Moreover, for $F \in \mathfrak{F}_R$, we have $F = R[x]$ for some $x \in X$. Since X is a generalized Priestley space, there exists $y \in X_0$ such that $x \leq y$. Therefore, $R[y] \subseteq R[x]$, and so there is $G \in (\mathfrak{F}_R)_0$ such that $F \supseteq G$. For $F, G \in \mathfrak{F}_R$, it follows from Lemma 4.1 that $F \supseteq G$ iff $(\forall a \in L)(F \in M_a \Rightarrow G \in M_a)$. Thus, conditions (1), (2), (3), and (5) of the definition of a generalized Priestley space are satisfied. To see that condition (4) is satisfied as well, let $F \in (\mathfrak{F}_R)_0$. Then $F = R[x]$ for some $x \in X_0$. Let $M_a, M_b \in \mathcal{I}_F$. Then $F \not\subseteq \varphi(a), \varphi(b)$, so $R[x] \not\subseteq \varphi(a), \varphi(b)$, and so $x \notin \Box_R\varphi(a), \Box_R\varphi(b)$. Therefore, $\Box_R\varphi(a), \Box_R\varphi(b) \in \mathcal{I}_x$ and as $x \in X_0$, there exists $U \in X^*$ such that $x \notin U$ and $\Box_R\varphi(a), \Box_R\varphi(b) \subseteq U$. Since R is 1–1, $x \notin U$ implies there exists $c \in L$ such that $R[U] \subseteq \varphi(c)$ and $R[x] \not\subseteq \varphi(c)$. Thus, $F = R[x] \notin M_c$, and so $M_c \in \mathcal{I}_F$. Let $G \in M_a$ and let $G = R[y]$ for some $y \in X$. Then $R[y] \subseteq \varphi(a)$, so $y \in \Box_R\varphi(a) \subseteq U$, and so $G = R[y] \subseteq R[U] \subseteq \varphi(c)$. Therefore, $G \in M_c$, and so $M_a \subseteq M_c$. Similarly, $M_b \subseteq M_c$. Thus, $\mathcal{I}_F$ is updirected. Now let $\mathcal{I}_F$ be updirected and let $F = R[x]$. We show that $x \in X_0$. Let $U, V \in \mathcal{I}_x$. Then $x \notin U, V$. Since R is 1–1, there exist $a, b \in L$ such that $R[U] \subseteq \varphi(a)$, $R[x] \not\subseteq \varphi(a)$, $R[V] \subseteq \varphi(b)$, and $R[x] \not\subseteq \varphi(b)$. Therefore, $F \notin M_a, M_b$, and so $M_a, M_b \in \mathcal{I}_F$. As $\mathcal{I}_F$ is updirected, there exists $c \in L$ such that $M_c \in \mathcal{I}_F$ and $M_a, M_b \subseteq M_c$. Thus, $x \notin \Box_R\varphi(c)$ and $\Box_R\varphi(a), \Box_R\varphi(b) \subseteq \Box_R\varphi(c)$, and so $\Box_R\varphi(c) \in \mathcal{I}_x$ and $U, V \subseteq \Box_R\varphi(c)$. Consequently, $\mathcal{I}_x$ is updirected, so $x \in X_0$, and so $F \in (\mathfrak{F}_R)_0$. It follows that condition (4) of the definition of generalized Priestley space is also satisfied, and so $\langle \mathfrak{F}_R, \tau_V, \supseteq, (\mathfrak{F}_R)_0\rangle$ is a generalized Priestley space. □

Definition 4.4 Let L be a bounded distributive meet-semilattice. We call a pair $(\mathfrak{F}, \mathfrak{F}_0)$ of families of nonempty closed upsets of L_* a *Vietoris family* if the following conditions are satisfied:

(1) $\mathfrak{F}_0 \subseteq \mathfrak{F}$.
(2) $F = \bigcap\{\varphi(a) : F \subseteq \varphi(a)\}$ for each $F \in \mathfrak{F}$.
(3) $\langle \mathfrak{F}, \tau_V, \supseteq, \mathfrak{F}_0\rangle$ is a generalized Priestley space.

Let L be a bounded distributive meet-semilattice. For a Vietoris family $(\mathfrak{F}, \mathfrak{F}_0)$ we define $R_{\mathfrak{F}} \subseteq \mathfrak{F} \times L_*$ by

$$F R_{\mathfrak{F}} x \text{ iff } x \in F.$$

Lemma 4.5 *Let L be a bounded distributive meet-semilattice and let $(\mathfrak{F}, \mathfrak{F}_0)$ be a Vietoris family. Then $R_{\mathfrak{F}} \subseteq \mathfrak{F} \times L_*$ is a 1–1 generalized Priestley morphism.*

Proof First we show that $R_{\mathfrak{F}} \subseteq \mathfrak{F} \times L_*$ is a generalized Priestley morphism. Suppose that $F \in \mathfrak{F}$, $x \in L_*$, and $F \not\!R_{\mathfrak{F}} x$. Then $x \notin F$, and as $F = \bigcap\{\varphi(a) : F \subseteq \varphi(a)\}$, there exists $a \in L$ such that $x \notin \varphi(a)$ and $F \subseteq \varphi(a)$. Thus, there is $a \in L$ such that $x \notin \varphi(a)$ and $R_{\mathfrak{F}}[F] \subseteq \varphi(a)$, and so condition (1) of the definition of a generalized Priestley morphism is satisfied. Now let $a \in L$ and $F \in \mathfrak{F}$. We have $F \in \Box_{R_{\mathfrak{F}}}(\varphi(a))$ iff $R_{\mathfrak{F}}[F] \subseteq \varphi(a)$ iff $\{x \in L_* : F R_{\mathfrak{F}} x\} \subseteq \varphi(a)$ iff $\{x \in L_* : x \in F\} \subseteq \varphi(a)$ iff $F \subseteq \varphi(a)$ iff $F \in M_a$. Thus, $\Box_{R_{\mathfrak{F}}}(\varphi(a)) = M_a$, and so condition (2) of the definition of a generalized Priestley morphism is satisfied. To see that condition (3) is also satisfied, let $F \in \mathfrak{F}$. Since $F \neq \emptyset$, there exists $x \in F$. This, by the definition of $R_{\mathfrak{F}}$, gives us $F R_{\mathfrak{F}} x$. Therefore, for each $F \in \mathfrak{F}$ there exists $x \in L_*$ such that $F R_{\mathfrak{F}} x$. It follows that $R_{\mathfrak{F}}$ is a generalized Priestley morphism. We show that $R_{\mathfrak{F}}$ is 1–1. Let $F \notin M_a$. Then $R_{\mathfrak{F}}[M_a] \subseteq \varphi(a)$ and $R_{\mathfrak{F}}[F] = F \not\subseteq \varphi(a)$. Thus, $R_{\mathfrak{F}}$ is 1–1. □

Lemma 4.6 *Let L be a bounded distributive meet-semilattice.*

(1) *If $R \subseteq X \times L_*$ is a 1–1 generalized Priestley morphism, then for each $x \in X$ and $y \in L_*$ we have xRy iff $R[x] R_{\mathfrak{F}_R} y$.*
(2) *If $(\mathfrak{F}, \mathfrak{F}_0)$ is a Vietoris family, then $\mathfrak{F} = \mathfrak{F}_{R_{\mathfrak{F}}}$.*

Proof

(1) For $x \in X$ and $y \in L_*$, we have $R[x] R_{\mathfrak{F}_R} y$ iff $y \in R[x]$ iff xRy.
(2) We have $F \in \mathfrak{F}_{R_{\mathfrak{F}}}$ iff $(\exists G \in \mathfrak{F})(F = R_{\mathfrak{F}}[G] = G)$ iff $F \in \mathfrak{F}$. Thus, $\mathfrak{F}_{R_{\mathfrak{F}}} = \mathfrak{F}$. □

Since homomorphic images of a bounded distributive meet-semilattice L are dually characterized by 1–1 generalized Priestley morphisms of L_*, by putting Lemmas 4.1, 4.2, 4.3, 4.5, and 4.6 together, we obtain:

Theorem 4.7 *Homomorphic images of a bounded distributive meet-semilattice L are dually characterized by Vietoris families on L_*.*

4.2 Bounded Distributive Lattices

Now let L be a bounded distributive lattice. We show how our characterization of homomorphic images of L simplifies considerably and becomes the usual characterization in case we are interested in onto bounded lattice homomorphisms of L.

Lemma 4.8 *Let X be a Priestley space. Then 1–1 functional generalized Priestley morphisms $R \subseteq Y \times X$, where Y is a Priestley space, correspond to closed subsets of X.*

Proof Let $R \subseteq Y \times X$ be a 1–1 functional generalized Priestley morphism. By [2, Lem. 11.19.1], $f_R : X \to Y$ given by $f_R(x) =$ the least lement of $R[x]$ is an embedding. Thus, $f_R(Y)$ is a closed subset of X which is order-homeomorphic to Y. Conversely, if Y is a closed subset of X, then the identity map $f : Y \to X$ is an embedding. By [2, Cor. 11.20.1], $R_f \subseteq Y \times X$ given by $xR_f y$ iff $f(x) \leq y$ is a 1–1 functional generalized Priestley morphism. It is obvious that this correspondence is a bijection. □

Thus, we arrive at the following well-known characterization of homomorphic images of bounded distributive lattices.

Corollary 4.9 [10] *Let L be a bounded distributive lattice. Homomorphic images of L dually correspond to closed subsets of L_+.*

Proof Homomorphic images of L dually correspond to 1–1 functional generalized Priestley morphisms $R \subseteq X \times L_+$, where X is a Priestley space. These, by Lemma 4.8, correspond to closed subsets of L_+. □

4.3 Bounded Implicative Semilattices, Heyting Algebras, and Boolean Algebras

Let L be a bounded implicative semilattice. As an immediate consequence of the bounded distributive meet-semilattice case, we obtain that homomorphic images of L are dually characterized by those Vietoris families $(\mathfrak{F}, \mathfrak{F}_0)$ on L_* for which $(\mathfrak{F}, \tau_V, \supseteq, \mathfrak{F}_0)$ is a generalized Esakia space. Another dual description of homomorphic images of L can be obtained through filters of L. It is well-known [8, Thm. 3.2] that homomorphic images of L are characterized by filters of L. In [2, Thm. 11.11] (see also [4, Thm. 8.5]) we characterized filters of L dually as such closed upsets C of L_* for which $L_* - C = \downarrow(L_+ - C)$. This leads to the following alterative dual description of homomorphic images of L.

Theorem 4.10 *Homomorphic images of a bounded implicative semilattice L are dually characterized by closed upsets C of L_* such that $L_* - C = \downarrow(L_+ - C)$.*

In the case of Esakia spaces, $L_* = L_+$, so the condition of Theorem 4.10 on the closed upset C is redundant, and so we obtain the following well-known characterization of homomorphic images of Heyting algebras.

Corollary 4.11 [6] *Homomorphic images of a Heyting algebra A are dually characterized by closed upsets of A_+.*

If A is a Boolean algebra, then upsets of the Stone space A_+ of A are simply subsets of A_+. Thus, Corollary 4.11 implies the following well-known characterization of homomorphic images of Boolean algebras.

Corollary 4.12 (see, e.g., [7, Sec. 8.1]) *Homomorphic images of a Boolean algebra A are dually characterized by closed subsets of A_+.*

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