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Definability in Substructure Orderings, II: Finite Ordered Sets

Jaroslav Ježek · Ralph McKenzie

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Abstract Let $\mathcal{P}$ be the ordered set of isomorphism types of finite ordered sets (posets), where the ordering is by embeddability. We study first-order definability in this ordered set. We prove among other things that for every finite poset P, the set $\{p, p^\partial\}$ is definable, where p and p^∂ are the isomorphism types of P and its dual poset. We prove that the only non-identity automorphism of $\mathcal{P}$ is the duality map. Then we apply these results to investigate definability in the closely related lattice of universal classes of posets. We prove that this lattice has only one non-identity automorphism, the duality map; that the set of finitely generated and also the set of finitely axiomatizable universal classes are definable subsets of the lattice; and that for each member K of either of these two definable subsets, $\{K, K^\partial\}$ is a definable subset of the lattice. Next, making fuller use of the techniques developed to establish these results, we go on to show that every isomorphism-invariant relation between finite posets that is definable in a certain strongly enriched second-order language L_2 is, after factoring by isomorphism, first-order definable up to duality in the ordered set $\mathcal{P}$. The language L_2 has different types of quantifiable variables that range, respectively, over finite posets, their elements and order-relation, and over arbitrary subsets of posets, functions between two posets, subsets of products of finitely many posets (heteregenous relations), and can make reference to order relations between elements, the application of a function to an element, and the membership of a tuple of elements in a relation.

Keywords Definability · Ordered set · Lattice · Universal class · Category

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J. Ježek (✉)
Department of Mathematics, Charles University, Prague, Czech Republic
e-mail: jezek@karlin.mff.cuni.cz

R. McKenzie
Department of Mathematics, Vanderbilt University, Nashville, TN 37235, USA
e-mail: ralph.mckenzie@vanderbilt.edu

1 Introduction

The set $\mathcal{P}$ of isomorphism types of finite posets, or as we say, finite order types, is denumerable. This set becomes itself a poset under the order induced by the substructure relation—we put $p_0 \leq p_1$, where p_i is the type of the finite poset P_i, iff P_0 is isomorphic to a sub-poset of P_1. In this way we obtain a poset $\langle \mathcal{P}, \leq \rangle$. In this paper, we explore the scope of first-order definitions in the structure $\langle \mathcal{P}, \leq \rangle$. It is an interesting topic because that scope is surprisingly wide: we shall see that in a quite precise sense, first-order definability over this poset is equivalent to second-order definability in the domain of finite posets.

The preceding remarks illustrate, by way of example, what we mean by the phrase "definability in substructure orderings". This paper is the second in a series of four exploring definability in substructure orderings. The paper [3] dealt with finite semilattices; [4] deals with finite distributive lattices; and [5] treats finite lattices. The idea for these explorations arose during our study of some combinatorial properties of these sub-structure orderings (see [1, 2]). We realized also that certain kinds of results on definability in substructure orderings would yield definitive results on definability in the lattice of universal classes of the structures.

The application of definability results for the substructure ordering to obtain definability results for the lattice of universal classes works smoothly for semilattices, for ordered sets and for distributive lattices, but breaks down for lattices because lattices do not form a locally finite class of structures. The results we obtain for the substructure ordering over finite structures are pretty much the same in all four cases, but the proof details are sufficiently different for the different kinds of structures that we did not think it wise to unify all our results in one paper.

By a *universal class* of posets we mean a class K defined by a set of first-order universal sentences, equivalently, a class K closed under forming substructures and ultraproducts. Since every poset is the union of its finite sub-posets, the lattice of universal classes of posets is naturally isomorphic with the lattice of order-ideals of the ordered set $\langle \mathcal{P}, \leq \rangle$, and within this lattice, the principal order-ideals are the same as the strictly join-irreducible elements of the lattice, and they constitute a definable subset of the lattice that is order-isomorphic with $\mathcal{P}$. Thus every subset or relation over the elements of $\mathcal{P}$ that can be shown to be definable in $\langle \mathcal{P}, \leq \rangle$ gives rise to a definable subset or relation in the lattice of universal classes.

A simple but important property of posets is that for every finite collection $\mathcal{F}$ of finite posets, there is a finite poset A such that all members of $\mathcal{F}$ are embeddable into A. From this fact, it is clear that a universal class of posets is finitely generated iff it is contained in a strictly join-irreducible member of the lattice of universal classes. Thus the set of finitely generated universal classes is a definable subset of the lattice. It is easy to show that a universal class K of posets is finitely axiomatizable (in the first-order language of posets) iff up to isomorphism, there are only a finite number of minimal (in the sense of embedding) finite posets lying outside of K. Thus it is easy to write a first-order definition in the language of lattice theory for the class of finitely axiomatizable universal classes: A universal class K is finitely axiomatizable iff there is a strictly join-irreducible universal class O such that for every universal class M, $M \not\leq K \Rightarrow M \cap O \not\leq K$.

We have just proved two of the principal results about universal classes of posets announced in the abstract. The remaining result, that for any universal class K that is either finitely generated or finitely axiomatizable, the set $\{K, K^\partial\}$ is definable in the lattice of universal classes, is not so easy. Our approach is to exhibit two three-element isomorphism types, p_1 and p_1^∂, and show that $\{p_1, p_1^\partial\}$ is definable in $\langle \mathcal{P}, \leq \rangle$, and that when p_1 is taken as a parameter, every member of $\mathcal{P}$ becomes definable. This we accomplish in Section 2 of the paper. We then conclude Section 2 with a derivation of our result that $\{K, K^\partial\}$ is definable in the lattice of universal classes whenever K is a finitely generated, or a finitely axiomatizable, universal class. All these results about universal classes are collected in Theorem 2.35.

In Section 3, building on results obtained in Section 2, we develop a different perspective on first order definability in $\langle \mathcal{P}, \leq \rangle$. In both parts, our principal object of investigation is actually the quasi-ordered set QPOSET whose members are all the posets $\langle A, \leq_A \rangle$ with A a finite subset of the non-negative integers, quasi-ordered by embeddability, so that $\langle A, \leq_A \rangle \leq \langle B, \leq_B \rangle$ means that there is a one-to-one map $f : A \to B$ such that $x \leq_A y \leftrightarrow f(x) \leq_B f(y)$ holds for all $\{x, y\} \subseteq A$. Members of QPOSET will usually be identified notationally with their universes, so that we write $A \in$ QPOSET with a specific choice of a partial order $\leq_A$ on A understood. An exception is that special posets that are to be held fixed throughout our study will be denoted with boldface letters. Here is the first example of this practice: We define $\mathbf{E}_0$ to be the poset with elements 0, 1, 2 and covers $0 \prec 1$ and $0 \prec 2$ (see Fig. 1). We can say more precisely that both in Sections 2 and 3 of the paper, we shall be studying first-order definability in the countable structure QPOSET$'$ = $\langle$QPOSET, $\leq, \mathbf{E}_0\rangle$ with one binary relation and one constant.

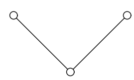
In Section 3, we introduce the category CPOSET whose objects are the members $A \in$ QPOSET with universe identical to $[n] = \{0, 1, \ldots, n-1\}$ for some $n \geq 0$, and whose morphisms are the monotone maps between these posets. The set of morphisms from A to B where A and B are two objects in CPOSET will be denoted CP(A, B).

Here we shall be considering first-order definability in the enriched category CPOSET$'$ obtained by adding to the category structure four fundamental constants. The constants denote two objects, $\mathbf{C}_0 = \langle \{0\}, \leq_0 \rangle$ and $\mathbf{C}_1 = \langle \{0, 1\}, \leq_1 \rangle$ (where this poset has one cover $0 \prec 1$) and the two members of CP$(\mathbf{C}_0, \mathbf{C}_1)$, namely $\mathbf{f}_i : \mathbf{C}_0 \to \mathbf{C}_1$ with $\mathbf{f}_i(0) = i$ (for $i \in \{0, 1\}$).

Our goal in Section 3 will be to prove that the structures QPOSET$'$ and CPOSET$'$ are almost equivalent in terms of the expressibility of first-order language applied to them.

But in fact, we shall show that this equivalence extends to expressibility in a very strong second-order language L_2 applied to the family of structures (posets) which constitutes the set of objects of CPOSET$'$. This language L_2 is an expansion of the first-order language of CPOSET$'$, containing not only variables ranging over objects and morphisms of CPOSET but also quantifiable variables ranging over elements of any object, over arbitrary subsets of objects, over arbitrary functions between two objects, over arbitrary subsets of products of finitely many objects (heteregenous relations), dependent variables giving the universe and the order relation of an

Fig. 1 $\mathbf{E}_0$



object, and the apparatus to denote order relations between elements, application of a function to an element, and membership of a tuple of elements in a relation.

Specifically, we shall prove (Theorem 3.8) that for any positive integer N, any N-ary relation R over QPOSET is first-order definable in QPOSET$'$ iff there is an N-ary relation S over the set of objects of CPOSET such that S is definable in L_2 and we have

$$R = \{(A_0, \ldots, A_{N-1}) \in \text{QPOSET}^N : \text{ there are objects } B_0, \ldots, B_{N-1} \text{ in CPOSET with } B_i \cong A_i \text{ for } i < N \text{ and } (B_0, \ldots, B_{N-1}) \in S\}.$$

The above-described result is surely the central contribution of this paper. Here is a reformulation of it. Let e_0 denote the isomorphism type of the poset $\mathbf{E}_0$. An n-ary relation S over QPOSET (or over the object set of CPOSET) will be called *isomorphism-invariant* iff whenever $A_0 \cong B_0, \ldots, A_{n-1} \cong B_{n-1}$ then $(A_0, \ldots, A_{n-1}) \in S$ iff $(B_0, \ldots, B_{n-1}) \in S$. Then we have: The isomorphism-invariant relations over the objects of CPOSET that are L_2-definable are the same as the isomorphism-invariant relations first-order definable in CPOSET$'$, and the same, after identifying isomorphic posets, as the relations first-order definable in the enriched ordered set $\mathcal{P}' = \langle \mathcal{P}, \leq, e_0 \rangle$.

An easy corollary of Theorem 3.8 is this.

Corollary For every sentence ϕ in the second-order language of posets, there is a formula $\Phi(x)$ in the first-order language of the structure QPOSET$' = \langle$QPOSET$, \leq, \mathbf{E}_0\rangle$ such that a poset A in QPOSET models ϕ if and only if QPOSET$' \models \Phi(A)$.

Specializing the corollary, we find that the set QLATT of members of QPOSET that are lattice-ordered sets, is first-order definable in QPOSET$'$, as is the set QSLATT of meet-semilattice-ordered members of QPOSET and the subset QDLATT of QLATT consisting of the lattice-ordered sets where the lattice is distributive. Moreover, the relation $A \leq_l B$ that holds between A and B in QLATT iff there is a lattice-embedding of A into B is definable, as is the relation $A \leq_{sl} B$ of semilattice embeddability in QSLATT. Each of the quasi-ordered sets $\langle$QSLATT$, \leq_{sl}\rangle$, $\langle$QLATT$, \leq_l\rangle$ and $\langle$QDLATT$, \leq_{dl}\rangle$ is therefore definably present in QPOSET$'$. The authors have studied the first-order definability in these structures in the papers [3, 4] and [5], reaching conclusions parallel to those obtained in this paper.

Birkhoff duality between finite distributive lattices and finite posets yields a second way of definably recovering $\langle$QDLATT$, \leq_{dl}\rangle$ in QPOSET$'$. (This application will be discussed briefly near the end of Section 3.1.)

Finally, we wish to observe that every subset of $\mathcal{P}$ is the set of all isomorphism types of all finite models of some *set* of first-order sentences in the language of posets: Let S be a subset of $\mathcal{P}$, and for every positive integer n, let $A_{n,1}, \ldots, A_{n,p_n}$ be a list of representatives of all the isomorphism types of n-element posets that belong to S. Let ϕ_n be a sentence such that a poset A is a model of ϕ_n iff $|A| = n \Rightarrow A \cong A_i$ for some $1 \leq i \leq p_n$. Clearly, a finite poset A represents an isomorphism type in S iff $A \models \phi_n$ for all $n \geq 1$. Consequently, our results imply that every subset of $\mathcal{P}$ is defined by the simultaneous satisfaction in $\mathcal{P}'$ of some set of formulas $\{\psi_n(x) : n \geq 1\}$ in the first-order language of the structure $\mathcal{P}'$. However, there are subsets of $\mathcal{P}$ that can be defined by a single formula $\psi(x)$ in the first-order language of $\mathcal{P}'$, but cannot be defined as all isomorphism types of finite models of a single sentence in the language of posets. For example, the set of isomorphism types of finite connected posets is such a set. (For the formula $\psi(x)$ see Theorem 2.19 below. A standard model-theoretic argument shows that no single first-order sentence defines the property of connectedness among all finite posets.)

We show in Section 2 (Theorems 2.27 and 2.15) that the relations $\{(A, B, C) : A \cong B + C\}$ (cardinal sum) and $\{(A, B, C) : A \cong B \oplus C\}$ (ordinal sum) are definable in Qposet'. Let us finally remark that it will become obvious in Section 3 that the relation $\{(A, B, C) : A \cong B \times C\}$ is definable in Qposet' (since it is definable in the category Cposet).

Let us remark that the results of this paper imply that the elementary theory of $\langle \mathcal{P}, \leq \rangle$ is undecidable. Indeed, it is well known that where N_0 is the set of nonnegative integers, the structure $\langle N_0, +, \times \rangle$ has undecidable elementary theory; from this it easily follows that also the elementary theory of $\langle N, +, \times \rangle$, where N is the set of positive integers, is undecidable; an obvious mapping is a bijection of N onto the set $\mathcal{A}$ of isomorphism types of finite chains; and we will see that $\mathcal{A}$ is a definable subset and the images of both + and $\times$ are definable operations in $\langle \mathcal{P}, \leq \rangle$.

2 Part I

2.1 Notation and First Results

The elements of Qposet are the finite posets whose elements are non-negative integers. For $A, B \in$ Qposet we put $A \leq B$ iff A is isomorphic with the poset induced by B on a subset of B. We put $A \subseteq B$ iff A is contained in B as a set, and the order in A is the restriction to this set of the order in B—in other words, A is a poset induced by B on a subset of B. Note that A and B are isomorphic, written $A \cong B$, iff $A \leq B$ and $B \leq A$. We denote by $\mathbf{E}_0$ the poset with elements 0, 1, 2 and covers $0 \prec 1$ and $0 \prec 2$, and by $\mathbf{E}_1$ its dual. We set Qposet' equal to the pointed quasi-ordered set $\langle \text{Qposet}, \leq, \mathbf{E}_0 \rangle$.

When we say that a subset of Qposet or a relation over Qposet is first-order definable in Qposet', we shall mean definable by a formula in the first-order language with two non-logical symbols, $\leq$ and $\mathbf{E}_0$, and without the equality symbol. As noted above, $\{(A, B) : A \cong B\}$ is definable in Qposet', and it is easily proved (say by induction on the complexity of formulas) that for every formula $\varphi(x_0, \ldots, x_{n-1})$ in this language and for $A_0, B_0, \ldots, A_{n-1}, B_{n-1} \in$ Qposet with $A_i \cong B_i$ for $i < n$ we have $\text{Qposet}' \models \varphi(A_0, \ldots, A_{n-1})$ if and only if $\text{Qposet}' \models \varphi(B_0, \ldots, B_{n-1})$. Thus with our convention about the language (omitting equality) first-order definability in Qposet' is only "up to isomorphism". In particular, $\{\mathbf{E}_0\}$ is not definable, although $\{A : A \cong \mathbf{E}_0\}$ is definable. However, we write that "$\mathbf{E}_0$ is a definable member of Qposet'", meaning that it is definable up to isomorphism; and we shall generally use this language with respect to all definable elements, definable subsets and definable relations over Qposet'.

The relation of isomorphism, definable in Qposet', is an equivalence relation over Qposet that gives rise to the pointed ordered set of isomorphism types, $\mathcal{P}' = \langle \mathcal{P}, \leq, e_0 \rangle$. Via the map sending $A \in$ Qposet to $A/\cong\ \in \mathcal{P}$, definable relations over Qposet' become definable relations over $\mathcal{P}'$, and conversely. Thus working over Qposet' is simply a convenient means to give a more concrete feel to the study of definability over $\mathcal{P}'$.

For every $n \geq 0$ we denote by $\mathbf{C}_n$ the chain of height n,

$$\mathbf{C}_n = \langle \{0, 1, \ldots, n\}, \leq \rangle$$

in which $\leq$ is the usual order. For every $n \geq 0$ we denote by $\mathbf{A}_n$ the $n+1$-element antichain, $\mathbf{A}_n = \langle \{0, 1, \ldots, n\}, \leq \rangle$, in which $\leq$ is the discrete order—$x \leq y$ iff $x = y$ for any elements x, y in $\mathbf{A}_n$. Note that $\mathbf{C}_0 \cong \mathbf{A}_0$.

The height, ht(P), of a finite poset P, is the largest n such that $\mathbf{C}_n \leq P$ (i.e., such that P has an $n+1$-element chain).

The cardinal sum, $A + B$, and ordinal sum, $A \oplus B$, of two posets are defined only up to isomorphism. Thus $C \cong A + B$ if and only if C is the disjoint union of ordered subsets isomorphic respectively to A and to B, such that there are no order relations in C between elements of the two subsets; and $C \cong A \oplus B$ if and only if C is the disjoint union of sub-posets isomorphic respectively to A and to B, such that for every element x of the copy of A in C and for every element y of the copy of B, we have $x < y$ in C.

If $A, B \in$ Qposet we write $A < B$ to indicate that $A \leq B$ and not $B \leq A$, and we say that A is *covered by* B if $A < B$ and there is no $C \in$ Qposet with $A < C < B$. We write $A \prec B$, or Qposet$' \models A \prec B$, to denote that B covers A in Qposet$'$.

The cardinality of A is the number of elements of A, written $|A|$.

For an element e of a poset, $e{\downarrow}$ denotes the principal ideal of the poset generated by e.

Proposition 2.1 *Let a and b be members of* Qposet. *Then $A \prec B$ iff $A \leq B$ and $|B| = |A| + 1$.*

This fact is obvious.

Theorem 2.2 $\{\mathbf{C}_n/\cong : n \geq 0\}$ *and* $\{\mathbf{A}_n/\cong : n \geq 0\}$ *are the only infinite order-ideals in $\mathcal{P}$ that are chains. The set of finite chains is a definable subset of* Qposet$'$ *and each finite chain is a definable member of* Qposet$'$. *The set of finite antichains is a definable subset of* Qposet$'$ *and each finite antichain is a definable member of* Qposet$'$.

Proof If a finite poset $\mathbf{A}$ is neither a chain nor an antichain then $\mathbf{A}_1 \leq \mathbf{A}$ and $\mathbf{C}_1 \leq \mathbf{A}$, so that $(\mathbf{A}/\cong){\downarrow}$ is not a chain in $\mathcal{P}$. In Fig. 2 below, we diagram the lowest four levels of $\mathcal{P}$. The top row consists of the following posets (from left to right): $\mathbf{C}_3$, $\mathbf{C}_0 \oplus \mathbf{E}_0$, $\mathbf{C}_{2,0,-}$ (introduced later), $\mathbf{E}_0 \oplus \mathbf{C}_0$, $\mathbf{C}_0 \oplus \mathbf{A}_2$, $\mathbf{C}_2 + \mathbf{C}_0$, $\mathbf{C}_{2,-,2}$ (introduced later), $\mathbf{E}_1 \oplus \mathbf{C}_0$, $\mathbf{E}_0 + \mathbf{C}_0$, the four-element fence, $\mathbf{A}_1 \oplus \mathbf{A}_1$, $\mathbf{C}_1 + \mathbf{C}_1$, $\mathbf{C}_1 + \mathbf{A}_1$, $\mathbf{E}_1 + \mathbf{C}_0$, $\mathbf{A}_2 \oplus \mathbf{C}_0$, $\mathbf{A}_3$. The next row consists of $\mathbf{C}_2$, $\mathbf{E}_0$, $\mathbf{C}_1 + \mathbf{C}_0$, $\mathbf{E}_1$, $\mathbf{A}_2$; the atoms are $\mathbf{C}_1$ and $\mathbf{A}_1$. We see that $\mathbf{A}_2/\cong$ has six covers in $\mathcal{P}$ while $\mathbf{C}_2/\cong$ has seven covers. Thus $P \in$ Qposet is a chain iff for every $A, B \in$ Qposet with $A \leq P$ and $B \leq P$ we have either $A \leq B$ or $B \leq A$, and there is $Q \in$ Qposet with either $Q \leq P$ or $P \leq Q$ such that $\{R \in$ Qposet $: R \leq Q\}$ has precisely three non-isomorphic members, all of them pairwise comparable, and up to isomorphism Q has precisely seven covers in Qposet. From this, it readily follows that the set of chains is definable, the set of antichains is definable, and each individual chain or antichain is a definable member of Qposet$'$ (meaning, "up to isomorphism", of course). □

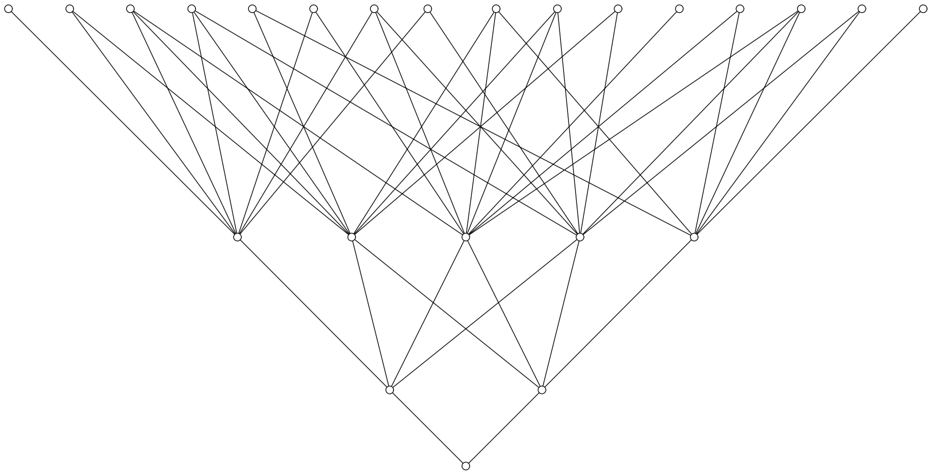


Fig. 2 The lowest four levels of $\mathcal{P}$

Proposition 2.3 *Every finite poset of at most five elements is a definable member of* Qposet$'$.

Proof With a little ingenuity, the reader can extract from Fig. 2 the fact that $\{\mathbf{E}_0, \mathbf{E}_0^\partial\}$ is definable in $\langle$Qposet, $\leq\rangle$, and thus that each poset of at most three elements is definable in Qposet$'$. Then it can be shown that each poset of four or five elements is determined up to isomorphism by the posets that properly embed into it. (The verification that this is so will be left to the reader.) □

Proposition 2.4 *For every positive integer n, the set of finite posets of cardinality n is a definable subset of* Qposet$'$.

This is obvious, from Proposition 2.1.

Remark 2.1 We remarked above that each poset of four or five elements is determined up to isomorphism by the isomorphism types of its proper sub-posets This is not true for smaller posets; as witnessed by $\mathbf{E}_0$ and $\mathbf{E}_1$. It may be that every poset of no fewer than four elements is determined up to isomorphism by the set of isomorphism types of its proper sub-posets. If this were true, it would yield a direct proof of one of the principal results of this paper, that every finite poset is definable in Qposet$'$. (See J. X. Rampon [6].)

2.2 Covers of Chains, and Cutpoints

We have $\mathbf{C}_n \prec \mathbf{C}_{n+1}$ of course. We introduce notation for all the remaining covers of $\mathbf{C}_n$.

For $1 \leq \ell \leq n$, $\mathbf{C}_{n,-,\ell}$ is the poset with elements $a_0, \ldots, a_{n+1}$ and covers $a_i < a_{i+1}$ for $0 \leq i < n$ and $a_{n+1} < a_\ell$.

For $0 \le k < n$, $\mathbf{C}_{n,k,-}$ is the poset with elements $a_0, \ldots, a_{n+1}$ and covers $a_i < a_{i+1}$ for $0 \le i < n$ and $a_k < a_{n+1}$.

For $0 \le k < \ell \le n$ with $k+1 < \ell$, $\mathbf{C}_{n,k,\ell}$ is the poset with elements $a_0, \ldots, a_{n+1}$ and covers $a_i < a_{i+1}$ for $0 \le i < n$ and $a_k < a_{n+1}$ and $a_{n+1} < a_\ell$.

Proposition 2.5 *The covers of* $\mathbf{C}_n$ *are* $\mathbf{C}_{n+1}$*,* $\mathbf{C}_n + \mathbf{C}_0$*, and the posets* $\mathbf{C}_{n,-,\ell}$*,* $\mathbf{C}_{n,k,-}$*, and* $\mathbf{C}_{n,k,\ell}$ *defined above.*

The proof is very easy, using Proposition 2.1.

Proposition 2.6 *For each integer* $n \ge 0$*, every cover of* $\mathbf{C}_n$ *is a definable member of* QPOSET$'$*. Each of the sets* $\{\mathbf{C}_{n,-,1} : n \ge 1\}$*,* $\{\mathbf{C}_{n,-,n} : n \ge 1\}$*,* $\{\mathbf{C}_{n,n-1,-} : n \ge 1\}$*,* $\{\mathbf{C}_{n,0,-} : n \ge 1\}$*,* $\{\mathbf{C}_{n,k,k+2} : n-2 \ge k \ge 0\}$ *is definable in* QPOSET$'$*.*

Proof For $n \ge 1$, we have that $\mathbf{C}_{n,-,1}$ is the only cover of $\mathbf{C}_n$ of height n that does not embed $\mathbf{E}_0$, does embed $\mathbf{E}_1$, and does not embed $\mathbf{C}_{2,-,2}$. (The posets $\mathbf{E}_1$ and $\mathbf{C}_{2,-,2}$ have fewer than five elements and so are definable, by Proposition 2.3.)

It is easy to verify that for $n \ge 1$, $\mathbf{C}_{n,-,n}$ is the only cover of $\mathbf{C}_n$ of height n that does not embed $\mathbf{E}_0$, does embed $\mathbf{E}_1$ and does not embed $\mathbf{E}_1 \oplus \mathbf{C}_1$. (The posets $\mathbf{E}_1$ and $\mathbf{E}_1 \oplus \mathbf{C}_1$ are definable, by Proposition 2.3.)

When $0 \le k$ and $k+2 \le \ell \le n$, we have that $\mathbf{C}_{n,k,\ell}$ is the only cover of $\mathbf{C}_n$ which has height n; embeds $\mathbf{E}_0$ and $\mathbf{E}_1$; embeds $\mathbf{C}_{n-\ell+1,-,1}$ and does not embed $\mathbf{C}_{n-\ell+2,-,1}$; embeds $\mathbf{C}_{k+1,k,-}$ and does not embed $\mathbf{C}_{k+2,k+1,-}$.

For $n \ge 2$ we have that $A \cong \mathbf{C}_{n,k,k+2}$ for some $k \ge 0$ with $k+2 \le n$ iff $A \cong \mathbf{C}_{n,k,\ell}$ for some k, ℓ and A does not embed $\mathbf{N}_5$ (the five-element non-modular lattice).

Finally, we observe that $\mathbf{C}_n + \mathbf{C}_0$ is the only cover of $\mathbf{C}_n$ of height n that embeds neither $\mathbf{E}_0$ nor $\mathbf{E}_1$. □

By a *cutpoint* of a poset A we mean an element $x \in A$ that is comparable to all members of A. Note that if A is a finite poset, say of height n, and A has a cutpoint c of height m, then c is the unique element of A of height m, the co-height of c is $n - m$, and c belongs to every maximal chain in A.

Theorem 2.7 *The relation* $\{(\mathbf{C}_n, \mathbf{C}_k, \mathbf{C}_\ell) : n = k + \ell\}$ *is definable in* QPOSET$'$*.*

Proof For chains $\mathbf{C}_n$, $\mathbf{C}_k$, $\mathbf{C}_\ell$ we have that $n = k + \ell$ iff either $\ell = 0$ and $n = k$, or $\ell = 1$ and $\mathbf{C}_k \prec \mathbf{C}_n$, or $\ell \ge 2$ and $\mathbf{C}_{n+1}$ has a cover A $(= \mathbf{C}_{n+1,k,k+2})$ of height $n+1$ that embeds $\mathbf{C}_{k+1,k,-}$ and does not embed $\mathbf{C}_{k+2,k+1,-}$, and does embed $\mathbf{C}_{\ell,-,1}$ and does not embed $\mathbf{C}_{\ell+1,-,1}$. □

Theorem 2.8 *The relation* $\{(A, \mathbf{C}_m) : A$ *has a cutpoint of height* $m\}$ *is definable in* QPOSET$'$*. The set of topped finite posets* (*those with the largest element*) *and the set of bottomed finite posets* (*those with the least element*) *are definable subsets of* QPOSET$'$*.*

Proof A has a cutpoint of height m iff where $n = \mathrm{ht}(A)$, $\mathbf{C}_m \le \mathbf{C}_n$ and if Q is any cover of $\mathbf{C}_n$ with $Q \le A$, then Q is not isomorphic to $\mathbf{C}_n + \mathbf{C}_0$ or to $\mathbf{C}_{n,k,-}$ for a $k < m$, or to $\mathbf{C}_{n,-,\ell}$ for a $\ell > m$, or to $\mathbf{C}_{n,k,\ell}$ for a k, ℓ satisfying $k < m < \ell$.

Indeed, if c is a cutpoint of height m in A, and $\mathbf{C}_n \prec Q \leq A$, then we have a sub-poset C of A isomorphic to $\mathbf{C}_n$ and a point $q \in A \setminus C$ with $C \cup \{q\}$ (the induced poset) isomorphic to Q. The cutpoint c must be the element of height m in the chain C, and it must be comparable to q. This forces the claimed restrictions on the possibilities for Q. On the other hand, suppose that $0 \leq m \leq n = \mathrm{ht}(A)$ and A has no cutpoint of height m. Choose a sub-poset C of A order-isomorphic to $\mathbf{C}_n$. Let a_m be the element of height m in the induced order on C. Then the height of a_m in A is also m. Since a_m is not a cutpoint of A, there is an element $q \in A$ that is incomparable to a_m. Clearly, $q \notin C$. Where $Q = C \cup \{q\}$, the induced poset on Q is a cover of $\mathbf{C}_n$. The incomparability of q and a_m yields that Q is isomorphic to one of the posets listed in the previous paragraph.

Now, A is topped iff where $n = \mathrm{ht}(A)$, A has a cutpoint of height n. A is bottomed iff A has a cutpoint of height 0. □

2.3 Definability of Some Cardinality Properties

For $n > m \geq 0$ denote by $\mathbf{Y}_{n,m}$ the poset with elements

$$a_0, \ldots, a_{n+1}, a_{n+2}$$

and covers $a_0 < \cdots < a_n$ and $a_m < a_{n+1}$ and $a_m < a_{n+2}$.

Lemma 2.9 *The binary relation*

$$\{(A, B) : A \cong \mathbf{C}_n \ \textit{and}\ B \cong \mathbf{Y}_{n,m} \ \textit{for some}\ n > m \geq 0\}$$

is definable in QPOSET′.

Proof Suppose that $A \cong \mathbf{C}_n$. Then $B \cong \mathbf{Y}_{n,m}$ for some $n > m \geq 0$ iff: A is a $\leq$-maximal subchain of B; there is Q with $A \prec Q \prec B$; $\mathbf{C}_0 \oplus \mathbf{A}_2 \leq B$; $\mathbf{E}_1 \not\leq B$; and $\mathbf{C}_0 \oplus (\mathbf{C}_0 + \mathbf{E}_0) \not\leq B$. □

Theorem 2.10 *The following relation is definable in* QPOSET′*:*

$$\{(A, B) : \ \textit{for some}\ n \geq 0,\ A \cong \mathbf{A}_n \ \textit{and}\ B \cong \mathbf{C}_n\}.$$

Proof This will be a consequence of the following claim.

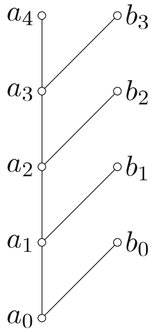
Suppose that $A \cong \mathbf{A}_m$ and $B \cong \mathbf{C}_n$. Then $m \leq n$ iff there is a finite poset P with these properties:

1) $\mathrm{ht}(P) = n$, i.e., $B \leq P$ and if C is a chain and $B < C$ then $C \not\leq P$.
2) P is a rooted tree, i.e., P is bottomed and $\mathbf{E}_1 \not\leq P$.
3) $\mathbf{Y}_{n,m} \not\leq P$ whenever $n > m \geq 0$.
4) $\mathbf{C}_0 \oplus (\mathbf{C}_1 + \mathbf{C}_1) \not\leq P$.
5) $A \leq P$.

We remark that there is a largest poset with properties 1)–4), namely the tree $\mathbf{T}_n$ pictured in Fig. 3 for $n = 4$.

To prove the claim, we observe that clearly $\mathbf{T}_n$ satisfies 1), 2), 3) and 4) and embeds $\mathbf{A}_n$ and does not embed $\mathbf{A}_{n+1}$. Thus it suffices to show that if S satisfies 1), 2), 3) and

Fig. 3 $\mathbf{T}_4$



4) then S is obtained from $\mathbf{T}_n$ by removing some subset (possibly the empty set) of the set of points $\{b_0, \ldots, b_{n-1}\}$.

Thus suppose that S satisfies 1), 2), 3) and 4). Let C be an $n+1$-element chain in S, say $C = \{a_0, \ldots, a_n\}$ with $a_0 < \cdots < a_n$. Now S is a tree with root a_0, by 1) and 2). By 3), for $i < n$, a_i has at most one successor other than a_{i+1}. If $i < n-1$ and a_i has two successors, a_{i+1} and b_i, then by 4) it follows that b_i is a maximal element of S. If $i = n-1$ and a_i has two successors, a_n and b_{n-1}, then since $\mathrm{ht}(S) = n$ it follows again that b_{n-1} is a maximal element. These considerations imply that $\mathbf{S}$ consists of the elements $a_0, \ldots, a_n$ and, for possibly some or all of $i = 0$, $i = 1$, ..., $i = n-1$, a maximal element $b_i \neq a_{i+1}$ that has a_i as its unique subcover. This completes our proof. □

Lemma 2.11 *The relation* $\{(A, E, F) :\ A \cong E \oplus F$ *and* E *is a chain*$\}$ *is definable in* QPOSET$'$.

Proof (A, E, F) belongs to this relation iff there are n, m, ℓ with $\mathrm{ht}(A) = n$, $E \cong \mathbf{C}_m$, $\mathrm{ht}(F) = \ell$ and $n = m + \ell$; and either $F \cong \mathbf{C}_\ell$ and $A \cong \mathbf{C}_n$, or else: there is $0 \le k \le \ell$ such that F has a cutpoint of height i for every $0 \le i < k$ and F has no cutpoint of height k, and A has a cutpoint of height j for every $0 \le j < m + k$ and A has no cutpoint of height $m + k$, and there is a finite un-bottomed poset Q such that $Q \le A$ and $Q \le F$ and whenever R is a finite un-bottomed poset and $R \le A$ or $R \le F$, then $R \le Q$. □

Theorem 2.12 *The following relation is definable in* QPOSET$'$*:*

$$\{(A, B) :\ \text{for some } n \ge 1,\ B \cong \mathbf{C}_n \text{ and } |A| = n\}.$$

Proof Let $n \ge 1$. To prove this theorem, it will suffice to show that for any finite poset A, we have $|A| \ge n$ iff either $\mathbf{A}_1 \not\le A$ (i.e., $\mathbf{A}$ is a chain) and $\mathbf{C}_{n-1} \le A$, or else: $\mathbf{A}_1 \le A$ (i.e., A is not a chain) and for some $m \ge 0$, $\mathrm{ht}(A) = m$ and every finite poset P with the following properties embeds $\mathbf{C}_n$. The properties are:

(1) $\mathbf{C}_0 \oplus A \le P$.
(2) For every chain C and finite poset Q with $\mathbf{C}_m < Q \le A$ and $C \oplus Q \le P$ there is Q' with $\mathbf{C}_m \le Q' \prec Q$ and there is a chain D with $C \prec D$ and $D \oplus Q' \le P$.

To prove that this characterizes the relation $|A| \geq n$, we assume first that A is not a chain and $|A| \geq n$, and say $\mathrm{ht}(A) = m$. We need to show that every $P \in$ QPOSET that satisfies (1) and (2) embeds $\mathbf{C}_n$. So assume that P satisfies (1) and (2). By induction on $k = |A| - |Q|$, using (1) for the base step $k = 0$, and using (2) in the induction step, we can obviously show that for $0 \leq k \leq |A| - m - 1$, we have $\mathbf{C}_k \oplus Q \leq P$ for some $Q \in$ QPOSET satisfying $\mathbf{C}_m \leq Q \leq A$ and $|A| - |Q| = k$. At $k = |A| - m - 1$, it follows that $|Q| = m + 1$ and since $\mathbf{C}_m \leq Q$ then $Q \cong \mathbf{C}_m$: for this k we have $\mathbf{C}_k \oplus \mathbf{C}_m \leq P$. Thus

$$P \geq \mathbf{C}_k \oplus \mathbf{C}_m \cong \mathbf{C}_{m+k+1} \cong \mathbf{C}_{|A|} \geq \mathbf{C}_n\,.$$

Next, we assume that A is not a chain, $|A| < n$, and $\mathrm{ht}(A) = m$. We need to find a poset $P \in$ QPOSET that satisfies (1) and (2) and does not embed $\mathbf{C}_n$. Let $Q_0, \ldots, Q_{p-1}$ be a list that contains exactly one isomorphic copy of each poset $Q \in$ QPOSET such that $\mathbf{C}_m \leq Q \leq A$. Then we put

$$P = \sum_{0 \leq i < p} \mathbf{C}_{k_i} \oplus Q_i$$

where $k_i = |A| - |Q_i|$ for $i < p$. Clearly, $\mathrm{ht}(Q_i) = m$ and $\mathrm{ht}(\mathbf{C}_{k_i} \oplus Q_i) = k_i + m + 1$ so the component $\mathbf{C}_{k_i} \oplus Q_i$ of largest height is the one with the largest value of k_i, that is, when $Q_i \cong \mathbf{C}_m$. For this Q_i we have

$$\mathbf{C}_{k_i} \oplus Q_i \cong \mathbf{C}_{k_i} \oplus \mathbf{C}_m \cong \mathbf{C}_{k_i+m+1} \cong \mathbf{C}_{|A|}\,.$$

It follows that we have $\mathrm{ht}(P) = |A| < n$ and so $\mathbf{C}_n \not\leq P$.

It remains for us to show that P satisfies (1) and (2). The truth of (1) for P is trivial. To prove (2), let $C, Q \in$ QPOSET with $C \cong \mathbf{C}_\ell$ and $\mathbf{C}_m < Q \leq A$ and $C \oplus Q \leq P$. Now $C \oplus Q \cong \mathbf{C}_\ell \oplus Q$ is connected, because it has a bottom element. Thus we have that

$$\mathbf{C}_\ell \oplus Q \leq \mathbf{C}_{k_{i_0}} \oplus Q_{i_0}$$

for some i_0. This implies that

$$\ell + m + 1 = \mathrm{ht}(\mathbf{C}_\ell \oplus Q) \leq \mathrm{ht}(\mathbf{C}_{k_{i_0}} \oplus Q_{i_0}) = k_{i_0} + m + 1\,.$$

Thus $\ell \leq k_{i_0}$. We can assume that $C \oplus Q \subseteq \mathbf{C}_{k_{i_0}} \oplus Q_{i_0}$ (sub-poset).

Since $\mathbf{C}_m < Q \leq A$ then Q is not a chain. Choose a chain $U \subseteq Q$, $U \cong \mathbf{C}_m$, and incomparable elements q_0, q_1 in Q. One of these elements, say q_0, must lie outside U. Both elements q_0, q_1 belong to Q_{i_0}, since elements of $\mathbf{C}_{k_{i_0}}$ are cutpoints in $\mathbf{C}_{m_{i_0}} \oplus Q_{i_0}$. Thus there is i_1 so that $Q_{i_1} \cong Q_{i_0} \setminus \{q_0\}$; and if we put $Q' = Q \setminus \{q_0\}$ then $\mathbf{C}_m \leq Q' \prec Q$. Moreover, $C \oplus Q' \subseteq \mathbf{C}_{k_{i_0}} \oplus Q_{i_0} \setminus \{q_0\} \cong \mathbf{C}_{k_{i_0}} \oplus Q_{i_1}$. Clearly, $\mathbf{C}_{k_{i_1}} \cong \mathbf{C}_0 \oplus \mathbf{C}_{k_{i_0}}$. Thus where $D = \mathbf{C}_0 \oplus C$, we have $C \prec D$, D is a chain, and $D \oplus Q' \leq \mathbf{C}_{k_{i_1}} \oplus Q_{i_1}$, as required. This completes our proof that P satisfies (2). □

2.4 Definability of the Relation $A \cong E \oplus F$

Lemma 2.13 *The relation $\{(A, E, F) :\ A \cong E \oplus F$ and F is a chain$\}$ is definable in* QPOSET$'$.

Proof The proof follows the same pattern as our proof of Lemma 2.11. □

Lemma 2.14 *The relation* $\{(A, E, F) : A \cong E \oplus \mathbf{C}_0 \oplus F\}$ *is definable in* QPOSET$'$.

Proof We have that (A, E, F) lies in this relation if and only if $\mathrm{ht}(E) = m$, say, and $\mathrm{ht}(F) = n$, and $\mathrm{ht}(A) = m + n + 2$ (see Theorem 2.7), and A has a cutpoint of height $m + 1$ (see Theorem 2.8) and for every finite poset R, $\mathbf{C}_{m+1} \oplus R \leq A$ iff $R \leq F$, and $R \oplus \mathbf{C}_{n+1} \leq A$ iff $R \leq E$ (see Lemmas 2.11 and 2.13). □

Theorem 2.15 *The relation*

$$\{(A, E, F) : A \cong E \oplus F\}$$

is definable in QPOSET$'$.

Proof We have that (A, E, F) lies in this relation iff $\mathrm{ht}(E) = m$, say, and $\mathrm{ht}(F) = n$, and $\mathrm{ht}(A) = m + n + 1$, and $\mathbf{C}_m \oplus F \leq A$ and $E \oplus \mathbf{C}_n \leq A$, and either E is topped and there is a unique R with $E \cong R \oplus \mathbf{C}_0$ and for this R we have $A \cong R \oplus \mathbf{C}_0 \oplus F$, or F is bottomed and there is a unique R with $R \cong \mathbf{C}_0 \oplus R$ and for this R we have $A \cong E \oplus \mathbf{C}_0 \oplus R$, or finally: E is not topped and F is not bottomed and there is a finite poset A' such that $A' \cong E \oplus \mathbf{C}_0 \oplus F$ and we have $A \prec A'$ and A has no cutpoint of height $m + 1$ or of depth $n + 1$. □

2.5 Definability of the Relation $A \cong E + F$

Lemma 2.16 *The relation*

$$\{(A, E, F) : E \text{ and } F \text{ are chains and } A \cong E + F\}$$

is definable in QPOSET$'$.

Proof First, a finite poset A is the cardinal sum of two (nonvoid) chains iff A is not a chain, $\mathbf{A}_2 \not\leq A$, $\mathbf{E}_0 \not\leq A$, and $\mathbf{E}_1 \not\leq A$.

Next, a finite poset A satisfies $A \cong \mathbf{C}_m + \mathbf{C}_n$ with $m \leq n$ iff A is the cardinal sum of two nonvoid chains, $\mathrm{ht}(A) = n \geq m$, and $|A| = (m + 1) + (n + 1)$. □

Lemma 2.17 *The relation*

$$\{(A, E, F) : E \text{ is topped, } F \text{ is a chain, and } A \cong E + F\}$$

is definable in QPOSET$'$.

Proof Suppose that E is topped and F is a chain. To begin, assume for the moment that also $\mathrm{ht}(F) > \mathrm{ht}(E)$ and E is not a chain. Then $E \not\leq F \not\leq E$. Under all these assumptions, we claim that $A \cong E + F$ iff $E \leq A$, $F \leq A$, $|A| = |E| + |F|$, whenever $R \leq A$ and $E \leq R$ and $F \leq R$ then $R \cong A$, and finally, if $F \prec Q \leq A$ then $Q \cong \mathbf{C}_0 + F$.

Now drop the assumptions that $\mathrm{ht}(F) > \mathrm{ht}(E)$ and E is not a chain. We claim that $A \cong E + F$ iff: either E is a chain and $A \cong E + F$; or else E is not a chain, and there is a chain $C > F$ such that $\mathrm{ht}(C) > \mathrm{ht}(E)$ and where $A' = E + C$ then $E \leq A \leq A'$ and $|E| + |F| = |A|$. (Use Theorem 2.7 and Theorem 2.12 to see that this characterization is first-order expressible.) □

Lemma 2.18 *The relation*

$$\{(A, E, F):\ E \text{ and } F \text{ are incomparable and topped, and } A \cong E + F\}$$

is definable in QPOSET′.

Proof We have that (A, E, F) belongs to this relation iff $|A| = |E| + |F|$; E and F are topped; $E \not\leq F \not\leq E$; $E \leq A$; $F \leq A$; for all $A' \leq A$, if $E \leq A'$ and $F \leq A'$, then $A' \cong A$; whenever $E \prec Q \leq A$ then $Q \cong E + \mathbf{C}_0$; and whenever $F \prec Q \leq A$ then $Q \cong F + \mathbf{C}_0$. □

Theorem 2.19 *The property of A that it is a connected finite poset is definable in* QPOSET′.

Proof We claim that a finite poset $\mathbf{A}$ is disconnected iff A is neither topped nor bottomed; and either A is the cardinal sum of two nonvoid chains, or else A is not the cardinal sum of any two nonvoid chains, and there are $B < A$ and $C < A$ and (nonvoid) chains D_1, D_2, D_3, D_4 such that where $E' = D_1 \oplus B \oplus D_2$ and $F' = D_3 \oplus C \oplus D_4$ then $E' \not\leq F' \not\leq E'$ and $A \leq E' + F'$.

Proof of the claim: ($\Leftarrow$) If A is connected, then clearly the condition fails.

($\Rightarrow$) Assume that A is disconnected and is not the cardinal sum of two chains. We can write $A \cong B + C$ where B is not a chain. Let h be the maximum of $\mathrm{ht}(B)$, $\mathrm{ht}(C)$. Put $D_1 = \mathbf{C}_{h+1}$, $D_2 = \mathbf{C}_0 = D_3$, and $D_4 = \mathbf{C}_{2h+3}$, and define $E' = D_1 \oplus B \oplus D_2$ and $F' = D_3 \oplus C \oplus D_4$. Now clearly $A \leq E' + F'$. We have that $E' \not\leq F'$ because E' has a non-cutpoint b (belonging to the copy of B in E') of height at least $h + 2$ and F' has no such element. Also, $F' \not\leq E'$ because $\mathrm{ht}(E') \leq 2h + 3$ while $\mathrm{ht}(F') \geq 2h + 4$. □

Lemma 2.20 *The relation*

$$\{(A, E, F):\ E \text{ and } F \text{ are incomparable and connected, and } A \cong E + F\}$$

is definable in QPOSET′.

Proof A triple (A, E, F) belongs to this relation iff E and F are connected and $E \not\leq F \not\leq E$, $E \leq A$, $F \leq A$, $|A| = |E| + |F|$, whenever $R \leq A$ and $E \leq R$ and $F \leq R$ then $R \cong A$, whenever $E \prec Q \leq A$ then $Q \cong E + \mathbf{C}_0$, and whenever $F \prec Q \leq A$ then $Q \cong F + \mathbf{C}_0$. □

Lemma 2.21 *The relation*

$$\{(A, E, F):\ E \text{ is a chain, } F \text{ is connected and } A \cong E + F\}$$

is definable in QPOSET′.

Proof The proof is the same as the proof of Lemma 2.17, using now that the property of being connected is definable. □

By a *maximal connected component* of a finite poset P we shall mean a connected poset Q such that $Q \leq P$ and for every R with $Q < R \leq P$, R is disconnected.

Lemma 2.22 *The relation*

$$\{(A, E) : \ E \text{ is connected and } A \cong E + E\}$$

is definable in QPOSET$'$.

Proof Suppose that E is connected. If E is a chain, then $E + E$ is definable relative to E as in Lemma 2.16. If E is not a chain, then $\mathbf{A}_0 \oplus E$ and $E \oplus \mathbf{A}_0$ are incomparable and connected. Then $B = (\mathbf{A}_0 \oplus E) + (E \oplus \mathbf{A}_0)$ is definable relative to E via Theorem 2.15 and Lemma 2.20. Now $E + E$ is, up to isomorphism, the unique finite poset A such that for some Q, $A \prec Q \prec B$, A is not connected and E is the only maximal component of A. □

Lemma 2.23 *The relation*

$$\{(A, E, F) : \ E \text{ and } F \text{ are connected and } A \cong E + F\}$$

is definable in QPOSET$'$.

Proof Let E, F be connected. If E and F are incomparable, or isomorphic, we can define $E + F$ via the formula of Lemmas 2.20 or 2.22 respectively. Assume otherwise, and say, $E < F$. We claim that $A \cong E + F$ iff the following is true.

A is disconnected, F is a maximal connected component of A, and for every finite poset P that satisfies the conditions below, we have that

$$[\mathbf{C}_0 + (A \oplus \mathbf{A}_0)] \oplus \mathbf{A}_0$$

is a maximal connected component of P.

Let $n = |F| - |E|$. The conditions for P are:

(i) $[\mathbf{C}_n + ((E + E) \oplus \mathbf{A}_0)] \oplus \mathbf{A}_0$ is a maximal connected component of P.

(ii) Every maximal connected component Q of P is of cardinality $3 + |E| + |F|$ and has the form $Q = [C + (S \oplus \mathbf{A}_0)] \oplus \mathbf{A}_0$ where C is a chain and S is disconnected. S has a unique (up to isomorphism) maximal connected component R, and $|C| + |R| = |F| + 1$.

(iii) Let $Q \cong [C + (S \oplus \mathbf{A}_0)] \oplus \mathbf{A}_0$ be a maximal connected component of P with $|C| > 1$. There is a maximal connected component Q' of P such that:

 (1) $Q' \cong [C' + (S' \oplus \mathbf{A}_0)] \oplus \mathbf{A}_0$ where $S \prec S'$ and $C' \prec C$.
 (2) S' is disconnected (of course), and where R and R' are the unique maximal connected components of S and S' respectively, then $R \prec R' \leq F$.

To prove the claim, we first tackle the necessity. Suppose that in fact, $A \cong E + F$. Let P be any member of QPOSET that satisfies (i), (ii) and (iii). Using the conditions recursively, we get a sequence of maximal connected components of P, of the form $Q_0, Q_1, \ldots, Q_n$ where $Q_i \cong [\mathbf{C}_{n-i} + (S_i \oplus \mathbf{A}_0)] \oplus \mathbf{A}_0$; $S_0 = E + E$; S_i is disconnected and $S_i \prec S_{i+1}$ for $0 \leq i < n$; and where R_i is the unique maximal connected component of S_i, we have

$$E \cong R_0 \prec R_1 \prec \cdots \prec R_n .$$

Since $R_n \leq F$ and $|R_n| = |F|$ then $R_n \cong F$.

We claim that $S_i \cong E + R_i$ for all $0 \leq i \leq n$. This is true for $i = 0$. We prove it for $i = 1$ and then inductively for $1 \leq i \leq n$.

For $i = 1$, S_1 has a sub-poset $U \cup V$ where $U \cap V = \emptyset$, each of U and V is isomorphic to E, and there are no order relations between elements of U and elements of V. We have $S_1 \setminus (U \cup V) = \{x_1\}$, say. Since S_1 is disconnected, the element x_1 cannot be related both to some element of U and to some element of V, in S_1. If x_1 were related to no element of $U \cup V$ then $U \cong E$ would be a maximal connected component of S_1, which is false. Thus the connected components of S_1 are U and $V \cup \{x_1\}$, say (or V and $U \cup \{x_1\}$). Clearly, R_1 must be isomorphic to $V \cup \{x_1\}$, and we have $S_1 \cong E + R_1$.

Now suppose that $n > i \geq 1$ and that $S_i \cong E + R_i$. We have that $S_i = U \cup W$ where $U \cong E$ and $W \cong R_i$. Since $S_i \prec S_{i+1}$, we can assume that $S_{i+1} = S_i \cup \{x_{i+1}\}$. Again, we have that the connected components of S_{i+1} must be, either $U \cup \{x_{i+1}\}$ and W, or U and $W \cup \{x_{i+1}\}$. If the first case were to hold, W could not be properly embedded into $U \cup \{x_{i+1}\}$ because $|W| > |U|$, so $R_i \cong W$ would be a maximal connected component of S_{i+1}, giving $R_i \cong R_{i+1}$; but this is false. Thus $R_{i+1} \cong W \cup \{x_{i+1}\}$ and $S_{i+1} \cong E + R_{i+1}$.

This completes our proof that $S_i \cong E + R_i$ for $0 \leq i \leq n$. Since $E + R_n \cong E + F \cong A$, we now have that $[\mathbf{C}_0 + (A \oplus \mathbf{A}_0) \oplus \mathbf{A}_0$ is a maximal connected component of P, as required.

Next, we tackle the proof of sufficiency of our proposed condition to characterize the relation $A \cong E + F$ when E and F are connected and $E < F$. Since E and F are connected, and we are assuming that $E < F$, it is easy to construct a sequence of connected posets $R_i \in$ Qposet $(0 \leq i \leq n)$ such that $E = R_0 \prec R_1 \prec \cdots \prec R_{n-1} \prec R_n = F$. Define P to be the cardinal sum of the posets $[\mathbf{C}_{n-i} + ((E + R_i) \oplus \mathbf{A}_0)] \oplus \mathbf{A}_0$ $(0 \leq i \leq n)$ (the cardinal sum of $n + 1$ connected posets). Since all of the cardinal summands of P are connected, pairwise non-isomorphic, and have the same cardinality, then each connected component of P is a maximal connected component of P. It is obvious that R_i is the unique maximal connected component of $E + R_i$ for each i. In fact, it is obvious that P satisfies (i), (ii) and (iii). Clearly, $[\mathbf{C}_0 + ((E + F) \oplus \mathbf{A}_0)] \oplus \mathbf{A}_0$ is the only maximal connected component of P of the form $[\mathbf{C}_0 + T] \oplus \mathbf{A}_0$ with T connected. Thus if $[\mathbf{C}_0 + (A \oplus \mathbf{A}_0)] \oplus \mathbf{A}_0$ is a maximal connected component of P, then $A \cong E + F$. □

Lemma 2.24 *The relation*

$$\{(A, U, V) : A \cong U + V, \text{ } U \text{ is a maximal connected component of } A, \text{ and } V \text{ is disconnected}\}$$

is definable in Qposet$'$.

Proof (A, U, V) belongs to this relation iff U is a maximal connected component of A; $V \leq A$ and every maximal connected component M of A satisfies $M \cong U$ or $M \leq V$; A and V are disconnected; $A \prec U + (V \oplus \mathbf{A}_0)$; and A is not isomorphic to any $Z + W$ with Z, W connected.

The necessity of these conditions is obvious. For sufficiency, suppose that they are satisfied. We have a poset $W \in$ Qposet, isomorphic with $U + (V \oplus \mathbf{A}_0)$, such that $W = A \cup \{x\}$, $x \notin A$, and A is a sub-poset of W. We can write $W = U' \cup (V' \cup \{a\})$

where $U' \cong U$, $V' \cong V$, $U' \cap V' = \emptyset$, there are no order relations between elements of U' and V', and $a > y$ for all $y \in V'$ while a is incomparable to all elements of U'.

If $x = a$ then $A = U' \cup V' \cong U + V$. It is impossible to have $x \in V'$ because then A is the union of its two connected sub-posets U' and $(V' \cup \{a\}) \setminus \{x\}$, contradicting that A is not isomorphic to the cardinal sum of two connected posets.

It remains to consider the case where $x \in U'$. In this case, $Q = V' \cup \{a\}$ is a connected subset of A, and we must have $Q \leq M$ for some maximal connected component M of A. Since Q is bigger than V, then $Q \leq M \leq U$. But also,

$$A = (U' \setminus \{x\}) \cup Q \cong (U' \setminus \{x\}) + Q$$

in this case, so the connected poset U satisfies $U \leq Q$ as $U \not\leq U' \setminus \{x\}$. Thus $Q \cong U$. Hence $U' \cong U$ has a top element b, and $U' \setminus \{b\} \cong V$. If $b \neq x$ then $U' \setminus \{x\}$ has a top element and is connected. But then, the displayed formula above shows that A is isomorphic to the cardinal sum of two connected posets, a contradiction. We are left with the conclusion that $x = b$. But now $U' \setminus \{x\} \cong V$. Combining this with the fact that $Q \cong U$, the displayed formula now becomes $A \cong V + U$. □

Lemma 2.25 *The relation*

$$\{(A, U, V) : \ A \cong U + V \textit{ and } U \textit{ is a maximal connected component of } A\}$$

is definable in QPOSET$'$.

Proof (A, U, V) belongs to this relation iff either (1) it belongs to the relation of Lemma 2.24; or (2) U, V are connected, $A \cong U + V$ and either $U \cong V$ or $U \not\leq V$. In case (2), Lemma 2.23 shows that the conditions are first-order expressible. □

Lemma 2.26 *The relation*

$$\{(A, E, C) : \ A \cong E + C \textit{ and } C \textit{ is a chain}\}$$

is definable in QPOSET$'$.

Proof If E is connected, we can use the formula of Lemma 2.21.

Suppose that E is disconnected. Then (A, E, C) belongs to this relation iff (1) there are U, V such that $E \cong U + V$ and U is a maximal connected component of E of largest cardinality among all maximal connected components of E; (2) C is a chain; (3) there is $B \succ A$ such that $B \cong V + [(U + C) \oplus \mathbf{A}_0]$; and (4) every maximal connected component of A (other than possibly C) has cardinality no greater than $|U|$.

Note that B is first-order definable relative to U, V by Lemma 2.21, Theorem 2.15, and Lemma 2.25.

The necessity of these conditions being obvious, we focus on their sufficiency. Suppose that E is disconnected and the conditions hold. We can assume that $B = \{x\} \cup A$ and A is a sub-poset of B. We can write $B = V' \cup W'$ where there are no order relations linking an element of V' to an element of W', and $V' \cong V$ and $W' \cong (U + C) \oplus \mathbf{A}_0$. Thus W' has a top element. If x is not that top element of W' then either W' or $W' \setminus \{x\}$ is a connected component of A; but this set is bigger than U and bigger than C, contradicting (4). So we must have $A = V' \cup (W' \setminus \{x\})$ where $W' \setminus \{x\} \cong U + C$. This gives $A \cong V + (U + C) \cong E + C$. □

Theorem 2.27 *The relation*

$$\{(A, E, F) : A \cong E + F\}$$

is definable in QPOSET′.

Proof Let C be the chain of cardinality $\max\{|E|, |F|\} + 3 = k + 3$. Define $E' = E + C$ and $F' = F + C$. Then put $B = (E' \oplus \mathbf{A}_0) + (F' \oplus \mathbf{A}_0)$. We know that B is definable relative to E, F. It is the case that $E' + F'$ is up to isomorphism the unique finite poset A' of height $k + 2$ such that for some Q, $A' \prec Q \prec B$, and every connected subset of A' that is not a chain has at most k elements. We have that $E + F$ is, up to isomorphism, the unique poset A of height $k - 1$ such that $A + C + C \cong A'$. The proofs of our assertions are straightforward, and we urge the reader to reconstruct them. □

A poset $Q \in$ QPOSET will be called a *connected component* of $A \in$ QPOSET iff $A \cong Q + R$ for some $R \in$ QPOSET, and Q is connected.

Corollary 2.28 *The relation* $\{(Q, A) :$ *Q is a connected component of A*$\}$ *is definable in* QPOSET′.

Proof This follows from Theorems 2.19 and 2.27. □

2.6 Individual Definability of the Members of QPOSET

We can now prove a key result of this paper:

Theorem 2.29 *Every member of* QPOSET *is a definable member of* QPOSET′.

The proof will be finished at the end of this section. Let us start with some definitions and lemmas.

Definition 2.30 Let $0 \le i < k$ be integers. $\eta_k(i)$ is defined up to isomorphism, as a certain member of QPOSET that encodes the pair (k, i). Namely,

$$\eta_k(i) \cong \mathbf{C}_{i+2} \oplus \mathbf{A}_1 \oplus \mathbf{C}_{k-i}\,.$$

Also, we define

$$\eta_k \cong \sum_{0 \le i < k} \eta_k(i)\,,$$

the cardinal sum of the posets $\eta_k(i)$. The posets $\eta_k(i)$ will be called o-*numbers*. The poset η_k will be called the *k-list of* o-*numbers* (Fig. 4).

Lemma 2.31 *The relation*

$$\{(\mathbf{C}_i, \mathbf{C}_k, \eta_k(i)) : 0 \le i < k\}$$

and the relation

$$\{(\mathbf{C}_k, \eta_k) : 0 < k\}$$

are definable in QPOSET′.

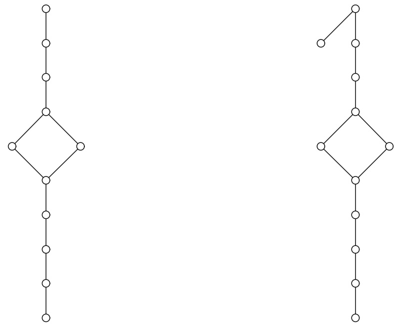


Fig. 4 $\eta_5(2)$ and $\eta'_5(2)$

Proof The definability of the first relation is obvious, from Theorems 2.2 and 2.15. Note that $\mathrm{ht}(\eta_k(i)) = k + 4$. For the second relation, observe that η_k is, up to isomorphism, the $\leq$-least member of QPOSET of height $k + 4$ whose connected components are precisely $\eta_k(0), \ldots, \eta_k(k-1)$. (See Corollary 2.28.) □

Definition 2.32 Let $0 \leq i < k$ be integers. We define $\eta'_k(i) \in$ QPOSET up to isomorphism by the formula

$$\eta'_k(i) \cong \{\mathbf{C}_0 + (\mathbf{C}_{i+2} \oplus \mathbf{A}_1 \oplus \mathbf{C}_{k-i-1})\} \oplus \mathbf{C}_0\,.$$

See Fig. 5.

Definition 2.33 Suppose that $A \in$ QPOSET, $|A| = k$. Let B be any member of QPOSET such that the set of elements of B is $[k] = \{0, 1, \ldots, k-1\}$ and B is isomorphic to A. We define, up to isomorphism, a member of QPOSET that we denote by $P_k(A, B)$.

First, make a poset B^+ isomorphic to $B \oplus \mathbf{A}_2 \oplus \mathbf{C}_0$ by adjoining $k, k+1, k+2, k+3$ to B and defining the order so that $B \subseteq B^+$ as posets, the new elements are above all elements of B, and $k, k+1, k+2$ are incomparable and below $k+3$. Next, find an isomorphic copy of η_k, say $\eta_k \cong N_k \in$ QPOSET with N_k disjoint from $\{0, 1, \ldots, k+3\}$. The set of elements of $P_k(A, B)$ is the disjoint union of N_k and $\{0, 1, \ldots, k+3\}$. For $0 \leq i < k$ let p_i be the top element of the unique copy of $\eta_k(i)$ in N_k. The order on $P_k(A, B)$ is defined so that its covers are those of N_k together with those of B^+ and, for each $0 \leq i < k$ the cover $i \prec p_i$.

Thus $P_k(A, B)$ is the union of its disjoint sub-posets N_k and B^+ and the only order relations in $P_k(A, B)$ besides those in N_k or in B^+ are $x < p_i$ when $x \in B$ and $B \models x \leq i$. Every poset $P_k(A, B)$ will be called an o-*presentation of* A.

For example, see Fig. 5 where $P_k(A, B)$ is pictured for $k = 5$, A is the (isomorphism type of) pentagon, and B is the pentagon labeled as shown in the picture.

Lemma 2.34 *Let* A, B, k *be as above.*

(1) *The isomorphism type of the o-presentation of* A*,* $P_k(A, B)$*, encodes the poset* B *exactly. That is to say, let* B *be a poset with universe* $\{0, \ldots, k-1\}$ *and* B' *be a poset with universe* $\{0, \ldots, \ell-1\}$ *and let* $B \cong A \in$ QPOSET *and* $B' \cong A' \in$ QPOSET*. Then* $P_k(A, B) \cong P_\ell(A', B')$ *iff* $k = \ell$ *and* $B = B'$ (*implying that* $A \cong A'$).

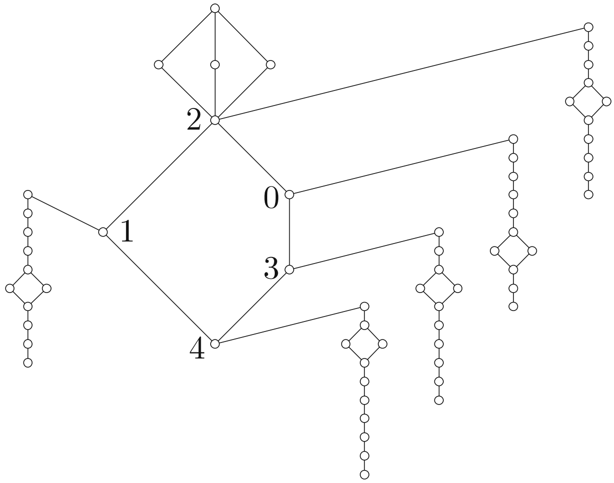


Fig. 5 $P_5(A, B)$ for a pentagon

(2) *Each* o*-presentation* $P_k(A, B)$ *is a definable member of* Qposet' (*only up to isomorphism, of course*).
(3) *The relation* $\{(A, P) :$ *where* $|A| = k$, $P \cong P_k(A, B)$ *for some* $B\}$ *is definable in* Qposet'.

Proof To prove (1), we begin with the assertion that it should be obvious that if $k = \ell$ and $B = B'$ then $P_k(A, B) \cong P_\ell(A', B')$. Conversely, suppose that $P_k(A, B) \cong P_\ell(A', B')$. The height of $P_k(A, B)$ is $k + 4$ and of $P_\ell(A', B')$ is $\ell + 4$, thus $k = \ell$. The posets B and B' thus have the same universe. We need to show that they have the same order. This follows from

Claim: Let $0 \leq i, i' < k$ with $i \neq i'$. Then $B \models i < i'$ iff $\eta'_k(i) + \eta_k(i') \not\leq P_k(A, B)$ (and of course $B' \models i < i'$ iff $\eta'_k(i) + \eta_k(i') \not\leq P_k(A', B')$).

To prove the claim, suppose first that $i \not\leq i'$ in B. We have that the unique copy of $\eta_k(i)$ in $P_k(A, B)$ together with the element i constitutes a sub-poset of $P_k(A, B)$ isomorphic to $\eta'_k(i)$. The unique copy of $\eta_k(i')$ in $P_k(A, B)$ is disjoint from this copy of $\eta'_k(i)$; and the only possible relation involving elements of the two sets is that the top element of the $\eta_k(i')$ might be above i. Since $i \not\leq i'$, then this does not happen. Thus $\eta'_k(i) + \eta_k(i') \leq P_k(A, B)$.

For the converse, assuming that $\eta'_k(i) + \eta_k(i') \leq P_k(A, B)$, then there must be an element $x \in P_k(A, B)$ that is below the top element of the $\eta_k(i)$ and incomparable to all other elements of the $\eta_k(i) + \eta_k(i')$. This element x can only be an element of B, and in fact, where p_i and $p_{i'}$ are the top elements of the $\eta_k(i)$ and $\eta_k(i')$, then we must have $p_i > i \geq x$ and $p_{i'} \not\geq x$ in $P_k(A, B)$. Since $p_{i'} > i'$ and $B \models i \geq x$ then $B \models i' \not\geq i$.

To prove (2), we write first-order properties of the element $P_k(A, B) \in \text{Qposet}'$ that determine it up to isomorphism. In fact, $P_k(A, B)$ is, up to isomorphism, the unique member P of Qposet' satisfying: there is a k-element poset $\overline{B} \in \text{Qposet}$ such that where $\overline{B}^+ \cong \overline{B} \oplus \mathbf{A}_2 \oplus \mathbf{C}_0$ we have

(a) $\text{ht}(P) = k + 4$.
(b) $\eta_k \leq P$, $\overline{B}^+ \leq P$, and $|P| = |\eta_k| + k + 4$.

(c) If $T \in$ Qposet$'$, $T \leq P$, $\eta_k \leq T$ and $\overline{B}^+ \leq T$ then $T \cong P$.
(d) The $\leq$-maximal topped posets embedded in P are, up to isomorphism, $\overline{B}^+$ and, for each $0 \leq i < k$, a poset isomorphic to

$$\{R + (\mathbf{C}_{i+2} \oplus \mathbf{A}_1 \oplus \mathbf{C}_{k-i-1})\} \oplus \mathbf{C}_0$$

for some topped $R \leq \overline{B}$.
($e_{i,i'}$) (Here $0 \leq i, i' < k, i \neq i'$.) $\eta'_k(i) + \eta_k(i') \not\leq P$ iff $B \models i < i'$.

That $P_k(A, B)$ satisfies all of the above properties is easily obtained from the proof of (1) provided that we can show that $P_k(A, B)$ has a unique subset isomorphic to η_k and a unique subset isomorphic to B^+. In order to prove this, recall that $P_k(A, B)$ is the disjoint union of N_k and B^+, and that $N_k \cong \eta_k$. Suppose now that $T \subseteq P_k(A, B)$, $T \cong B^+$. Let $\top$ denote the top element of T. Below $\top$ we have three incomparable elements a_1, a_2, a_3 with a copy of B below all three of the a_i. If $\top \in N_k$ then $\top$ can only be the top element p_i of the copy of a $\eta_k(i)$ inside N_k, and one of a_1, a_2, a_3 must be below i in B (since $\eta_k(i)$ has no three-element antichain). But then the copy of B below a_1, a_2 and a_3 must lie properly below i and actually be a proper subset of B, which is impossible by cardinality considerations. It follows then that $\top \in B^+$. This forces $T \subseteq B^+$ since B^+ is an order-ideal in $P_k(A, B)$. Since $T \cong B^+$, then $T = B^+$. Thus there is only one copy of B^+ in $P_k(A, B)$.

Now let S be a copy of η_k in $P_k(A, B)$. We need to prove that $S = N_k$. Take any $i \in [k]$ and let $S(i)$ be the unique copy of $\eta_k(i)$ inside S, and let $\top_i$ be the top element of $S(i)$. Then the height of $\top_i$ in $P_k(A, B)$ is at least $k + 4$. The only elements of $P_k(A, B)$ having height not less than $k + 3$ in $P_k(A, B)$ lie inside N_k; thus $\top_i \in N_k$. In fact, $\top_i$ must be the top element of the unique copy of $\eta_k(j)$ inside N_k, for a certain $j \in [k]$. Let r be the unique element of height $k + 3$ in $S(i)$. Then likewise, $r \in N_k$, and then r must in fact be the element of height $k + 3$ in the copy of $\eta_k(j)$ in N_k. Now $\top_i$ together with r and the elements below r in $P_k(A, B)$ just constitute this copy of $\eta_k(j)$. Then by cardinality considerations, $S(i)$ is identical with this copy of $\eta_k(j)$. This implies that $j = i$. Our reasoning gives the conclusion that for each $i \in [k]$, the copy of $\eta_k(i)$ in N_k is included in S. By cardinality, we conclude that $S = N_k$, as desired.

This completes our proof that $P_k(A, B)$ satisfies the properties. Now assume that P satisfies these properties. The first three, (a), (b), (c), imply that P is the disjoint union of a subset N'_k isomorphic to η_k and a subset isomorphic to $\overline{B}^+$ and that P contains just one subset isomorphic to η_k and just one subset isomorphic to $\overline{B}^+$. To simplify notation, we can assume that the copy of $\overline{B}^+$ in P is

$$\overline{B}^+ = \overline{B} \cup \{a_1, a_2, a_3\} \cup \{t\}$$

where a_1, a_2, a_3 are incomparable, above all elements of $\overline{B}$, and below t.

Now if t were above some element of N'_k then $t\!\downarrow$ in P would be a proper extension of $\overline{B}^+$, and so by (d), $\overline{B} \cup \{a_1, a_2, a_3\}$ would be order-embeddable into

$$R + (\mathbf{C}_{i+2} \oplus \mathbf{A}_1 \oplus \mathbf{C}_{k-i-1})$$

for some $i \in [k]$ and R a topped subset of $\overline{B}$. This is clearly impossible, since it would force $\overline{B}$ to be properly embeddable into itself. Hence the only comparabilities in P

between an element of N'_k and an element of $\overline{B}^+$ must be of the form $x > y$ with $x \in N'_k$ and $y \in \overline{B}^+$.

For $i \in [k]$ let p_i and r_i be the elements of height $k+4$ and $k+3$ in the copy of $\eta_k(i)$ inside N'_k. We claim that the only elements of N'_k that can possibly be above an element of $\overline{B}^+$ are the p_i. If this fails to be the case, then for some i we have $r_i > y$ with $y \in \overline{B}^+$. By (d), every $\leq$-maximal topped subset of P has height at most $k+4$. (Note that $\mathrm{ht}(B) < k$.) Thus $p_i{\downarrow}$ in P is clearly a $\leq$-maximal topped subset of P and by (d), $r_i{\downarrow} \leq B$ or

$$r_i{\downarrow} \cong \mathbf{C}_{i+2} \oplus \mathbf{A}_1 \oplus \mathbf{C}_{k-i-1} .$$

Since the height of r_i in P is at least $k+3$ and the height of the order-ideal $\overline{B}$ is less than k, then the second alternative must prevail. But this implies that $r_i{\downarrow}$ in P is contained in N'_k and so we have a contradiction (to the assumption that $r_i > y \in \overline{B}^+$).

Thus the only relations between N'_k and $\overline{B}^+$ are of the form $p_i > y$, $y \in \overline{B}^+$. Actually, (d), with height considerations, now implies that for each $i \in [k]$, $p_i{\downarrow} \cap \overline{B}^+$ is nonvoid and this set has a largest element, y_i, and this element y_i belongs to $\overline{B}$.

From what we have shown up to here, it is easily established that P has a unique copy of $\eta_k(i)$ for each $i \in [k]$. Then the conditions ($\mathrm{e}_{i,i'}$) yield that the map $i \mapsto y_i$ is one-to-one. Since $|\overline{B}| = k$, then this map is also onto $\overline{B}$. Finally, conditions (e) show that $B \models i < i'$ iff $\overline{B} \models y_i < y_{i'}$, so we have $\overline{B} \cong B$. This ends our proof of (2).

To prove (3), suppose that $A \in$ QPOSET and $|A| = k$, and that $P \in$ QPOSET. Then we claim that $P \cong P_k(A, B)$ for some B if and only if the following hold:

- (α) $\mathrm{ht}(P) = k+4$.
- (β) $\eta_k \leq P$, $A^+ \leq P$, and $|P| = |\eta_k| + k + 4$.
- (γ) If $T \in$ QPOSET$'$, $T \leq P$, $\eta_k \leq T$ and $A^+ \leq T$ then $T \cong P$.
- (δ) The $\leq$-maximal topped posets embedded in P are, up to isomorphism, A^+ and, for each $0 \leq i < k$, a poset isomorphic to $\{R + (\mathbf{C}_{i+2} \oplus \mathbf{A}_1 \oplus \mathbf{C}_{k-i-1})\} \oplus \mathbf{C}_0$ for some topped $R \leq A$.
- ($\varepsilon_{i,i'}$) (Here $0 \leq i, i' < k$, $i \neq i'$, and otherwise i, i' are arbitrary.) Either $\eta'_k(i) + \eta_k(i') \leq P$ or $\eta'_k(i') + \eta_k(i) \leq P$.

The proof of this claim parallels our proof of (2), and is left for the reader to supply. □

Let us finish the proof of Theorem 2.29. Let $A \in$ QPOSET. Say $|A| = k$. Choose $B \cong A$ with the universe of B identical to $\{0, 1, \ldots, k-1\}$. By Lemma 2.34(2), $P_k(A, B)$ is a definable member of QPOSET$'$. Now A is, up to isomorphism, the unique $R \in$ QPOSET such that $R \oplus \mathbf{A}_2 \oplus \mathbf{C}_1$ is a $\leq$-maximal topped sub-poset of $P_k(A, B)$.

2.7 Universal Classes of Posets

For a class K of posets, denote by K^∂ the class of the posets dual to the posets of K. The mapping $K \mapsto K^\partial$ is clearly an automorphism of the lattice of universal classes of posets.

Theorem 2.35 *The lattice of universal classes of posets has only two automorphisms: the identity and the map $K \mapsto K^\partial$. The set of all finitely axiomatizable and also the set* *of all finitely generated universal classes of posets are definable subsets of this lattice, and each member of either of these two definable subsets is an element definable up to the two automorphisms of this lattice.*

Proof As we mentioned in the introduction, the lattice of universal classes of posets is isomorphic to the lattice of order-ideals of the poset $\langle \mathcal{P}, \leq \rangle$, and also isomorphic to the lattice $\mathbf{L}$ of order-ideals of the quasi-ordered set $\langle \text{QPOSET}, \leq \rangle$. The members of $\mathbf{L}$ are the subsets $K \subseteq \text{QPOSET}$ such that $A \leq B \in K$ implies $A \in K$. Under the isomorphism between these lattices, the finitely generated order-ideals are carried onto the finitely generated universal classes, and the set-complements of the finitely generated order-filters are carried onto the finitely axiomatizable universal classes. We proved in the introduction that the set of finitely generated universal classes is a definable subset of the lattice, and the set of finitely axiomatizable universal classes is definable.

Thus let I be an order-ideal in QPOSET that is either finitely generated or the complement of a finitely generated order-filter. We need to show that $\{I, I^{\partial}\}$ is first-order definable in the lattice $\mathbf{L}$. There are finitely many finite posets $A_1, \ldots, A_n$ so that either we have

$$I = \{B \in \text{QPOSET} : B \leq A_i \text{ for some } 1 \leq i \leq n\}$$

or we have

$$I = \{B \in \text{QPOSET} : \text{ for all } i \text{ with } 1 \leq i \leq n \;\; B \not\geq A_i\}\,.$$

For $A \in \text{QPOSET}$ put $A{\downarrow} = \{B \in \text{QPOSET} : B \leq A\}$. The set of strictly join-irreducible members of $\mathbf{L}$, definable in $\mathbf{L}$, is precisely the set of order-ideals of QPOSET of the form $A{\downarrow}$ (for $A \in \text{QPOSET}$). Thus Theorem 2.29 implies that each of $A_1{\downarrow}, \ldots, A_n{\downarrow}$ is a definable member of the pointed lattice $(\mathbf{L}, \mathbf{E}_0{\downarrow})$. Thus for $1 \leq i \leq n$ there is a first-order lattice-theoretic formula $\varphi_i(x, y)$ so that $A_i{\downarrow}$ is the unique member x of $\mathbf{L}$ such that $\mathbf{L} \models \varphi_i(x, \mathbf{E}_0{\downarrow})$. Also, there is a formula $\varepsilon(x)$ so that $\mathbf{E}_0{\downarrow}, \mathbf{E}_0^{\partial}{\downarrow}$ are the only elements of $\mathbf{L}$ that satisfy $\varepsilon(x)$. (Because the set $\{\mathbf{E}_0, \mathbf{E}_0^{\partial}\}$ is definable in QPOSET'; see the proof of Proposition 2.3.)

Define $\Phi(x)$ to be the formula

$$(\exists y)(\exists x_1, \ldots, x_n)\left[\varepsilon(y) \wedge \bigwedge_{1 \leq i \leq n} \varphi_i(x_i, y) \wedge x = x_1 + \cdots + x_n\right];$$

and $\Psi(x)$ to be the formula

$$(\exists y)(\exists x_1, \ldots, x_n)[\varepsilon(y) \wedge \bigwedge_{1 \leq i \leq n} \varphi_i(x_i, y)$$
$$\wedge\, (\forall z)\left[z \leq x \leftrightarrow \bigwedge_{1 \leq i \leq n} x_i \not\leq z\right].$$

In the first formula, $+$ is the symbol for the lattice join operation in $\mathbf{L}$.

We claim that for $x \in \mathbf{L}$, $\mathbf{L} \models \Phi(x)$ iff $x = I$ or $x = I^{\partial}$ where I is the order-ideal generated by $A_1, \ldots, A_n$; and $\mathbf{L} \models \Psi(x)$ iff $x = J$ or $x = J^{\partial}$ where J is the largest order-ideal containing none of $A_1, \ldots, A_n$.

We shall prove just the claim for $\Psi(x)$ and J. Suppose first that $U \in \mathbf{L}$ and $\mathbf{L} \models \Psi(U)$. Let Y and $X_1, \ldots, X_n$ be the elements of $\mathbf{L}$ that witness the satisfaction of

$\Psi(U)$. Then $\mathbf{L} \models \varepsilon(Y)$ and $\mathbf{L} \models \varphi_i(X_i, Y)$ for $i = 1, \ldots, n$. It follows that $Y = \mathbf{E}_0{\downarrow}$ or $Y = \mathbf{E}_0^\partial{\downarrow}$. If $Y = \mathbf{E}_0{\downarrow}$ then it follows that $X_i = A_i{\downarrow}$ for $i = 1, \ldots, n$. In this case, the fact that $\mathbf{L} \models \Psi(U)$ tells us that U is the largest member of $\mathbf{L}$ that fails to intersect $\{A_1, \ldots, A_n\}$, i.e., $U = J$. In the case that $Y = \mathbf{E}_0^\partial$, consider U^∂ $(= \{A^\partial : A \in U\})$. Since ∂ is an automorphism of $\langle$Qposet, $\leq\rangle$, it induces an automorphism of $\mathbf{L}$. It follows that $\mathbf{L} \models \Psi(U^\partial)$ with witnesses $Y^\partial = \mathbf{E}_0{\downarrow}$ and X_i^∂. This puts us in the first case, and we can conclude that $U^\partial = J$. So it follows that $U = J^\partial$ in this case. Since it is more or less obvious that $\mathbf{L} \models \Psi(J)$ and $\mathbf{L} \models \Psi(J^\partial)$, we regard the proof of Theorem 2.35, as regards definability, to be finished.

It remains to show that $U \mapsto U^\partial$ is the only non-identity automorphism of $\mathbf{L}$. Here is a proof. Let σ be any automorphism of $\mathbf{L}$. Since $\{\mathbf{E}_0{\downarrow}, \mathbf{E}_0^\partial{\downarrow}\}$ is a definable subset of $\mathbf{L}$ then $\sigma(\mathbf{E}_0{\downarrow})$ belongs to this set. Thus if $\sigma(\mathbf{E}_0{\downarrow}) \neq \mathbf{E}_0{\downarrow}$ then $\tau(\mathbf{E}_0{\downarrow}) = \mathbf{E}_0{\downarrow}$ where τ is the automorphism $U \mapsto \sigma(U)^\partial$. We now show that any automorphism which fixes the element $\mathbf{E}_0{\downarrow}$ must be the identity. It will follow that σ is the identity, or σ followed by the map 'dual' is the identity; so that σ is the identity or the map $U \mapsto U^\partial$.

So finally, suppose that σ is an automorphism of $\mathbf{L}$ and that $\sigma(\mathbf{E}_0{\downarrow}) = \mathbf{E}_0{\downarrow}$. For every $A \in$ Qposet there is, as we noted above, a lattice-theoretic formula $\varphi(x, y)$ such that $A{\downarrow}$ is the unique element $U \in \mathbf{L}$ for which $\mathbf{L} \models \varphi(A{\downarrow}, \mathbf{E}_0{\downarrow})$. Since $\mathbf{L} \models \varphi(A{\downarrow}, \mathbf{E}_0{\downarrow})$ then $\mathbf{L} \models \varphi(\sigma(A{\downarrow}), \sigma(\mathbf{E}_0{\downarrow}))$; but since σ fixes $\mathbf{E}_0{\downarrow}$ then $\mathbf{L} \models \varphi(\sigma(A{\downarrow}), \mathbf{E}_0{\downarrow})$, and $\sigma(A{\downarrow}) = A{\downarrow}$ is forced. Thus the fixed points of σ include all the $A{\downarrow}$ and, consequently, every point of $\mathbf{L}$ is fixed by σ, as every member of $\mathbf{L}$ is the join in $\mathbf{L}$ of some subset of the family of members of the form $A{\downarrow}$. □

3 Part II

3.1 Introduction to Definability in Cposet and Cposet′

The category Cposet has for its set Obj of objects the members of Qposet of the form $A = \langle [n], \leq_A \rangle$ where $[n] = \{0, \ldots, n-1\}$, $n > 0$. For every $A, B \in$ Obj the set CP(A, B) of morphisms in Cposet is the set of triples $f = (A, \alpha, B)$ where α is a monotone map from A to B, i.e., a map from the universe of A to the universe of B such that whenever $x \leq y$ in A then $\alpha(x) \leq \alpha(y)$ in B. The identity morphism in CP(A, A) is denoted as 1_A. Thus $1_A = (A, \mathrm{id}_A, A)$ where id_A is the identity function on A. Composition of morphisms in Cposet is, for every triple of objects A, B, C a mapping CP$(A, B) \times$ CP$(B, C) \to$ CP(A, C). If $f = (A, \alpha, B) \in$ CP(A, B) and $g = (B, \beta, C) \in$ CP(B, C), the composition $f \circ g$ (written also as fg) is

$$f \circ g = (A, \beta \circ \alpha, C)$$

where for $x \in A$, $\{\beta \circ \alpha\}(x) = \beta(\alpha(x)) \in C$. When $f \in$ CP(A, B), the *domain of* f is A and the *co-domain of* f *is* B. Note that since a morphism f is actually of the form $f = (A, \alpha, B)$, the domain and the co-domain of f are unique. That is to say, for objects $A, B, C, D \in$ Obj, we have CP$(A, B) \cap$ CP$(C, D) = \emptyset$ unless $A = C$ and $B = D$.

It happens to be true that a morphism $f \in$ CP(A, B) is one-to-one on elements iff whenever $g, h \in$ CP(U, A) for some object U then $gf = hf \leftrightarrow g = h$. Also, f is onto the set of elements of B iff whenever $g, h \in$ CP(B, V) for some object V then $fg = fh \leftrightarrow g = h$. Thus the properties of a morphism that it is injective, or surjective, are (first-order) definable in Cposet. We have that $f \in \mathrm{CP}(A, B)$ is an isomorphism iff there is $g \in \mathrm{CP}(B, A)$ with $fg = 1_A$ and $gf = 1_B$.

A morphism $f = (A, \alpha, B)$ (or the monotone map α) is called an *embedding* iff for all $x, y \in A$ it is the case that $x \leq y$ in A iff $\alpha(x) \leq \alpha(y)$ in B. The property of being an embedding is definable in Cposet as well, but this requires a little care.

To see it, note that $\mathbf{C}_0$ is the unique terminal object in Cposet; i.e., for every object A there is a unique morphism $A \to \mathbf{C}_0$. Thus $\mathbf{C}_0$ is definable. There are two objects C with the property that $|\mathrm{CP}(C, C)| = 3$, namely $\langle [2], \leq \rangle$, with $\leq$ the usual order, and its dual, $\langle [2], \geq \rangle$. These two objects are isomorphic, and in that sense, either one deserves to be labeled as $\mathbf{C}_1$. Now one can verify that a morphism $f \in \mathrm{CP}(A, B)$ is an embedding iff whenever $C \in \mathrm{Obj}$ and $|\mathrm{CP}(C, C)| = 3$ and $\mathrm{CP}(\mathbf{C}_0, C) = \{\varepsilon_0, \varepsilon_1\}$, and $u, v \in \mathrm{CP}(\mathbf{C}_0, A)$ and there is $q \in \mathrm{CP}(C, B)$ with $\varepsilon_0 q = uf$ and $\varepsilon_1 q = vf$, then there is $p \in \mathrm{CP}(C, A)$ with $\varepsilon_0 p = u$ and $\varepsilon_1 p = v$.

Thus not only the properties of a morphism that it be injective, or surjective, or an isomorphism, but also the property that it be an embedding, are all first-order definable in Cposet. It follows that the quasi-order relation $\leq$ of Qposet, restricted to Cposet, is definable in Cposet. Since every member of Qposet is isomorphic to a member of Cposet, then every subset or relation first-order definable in Qposet is first-order definable in Cposet (or rather its restriction to Cposet is so).

Our goal in the remainder of this paper is to obtain a converse to the result of the last paragraph. Namely, we shall show that every isomorphism-invariant relation on objects in Cposet that is definable in the first-order language of Cposet$'$ (or even definable in the second-order language L_2 described in the introduction) is actually first-order definable in the much more modest structure Qposet$'$. We hope that the following observations will render the more technical work in the next section more readable.

In Qposet, we have only the posets as objects, and the relation of embeddability between objects, to work with. The internal structure of an object (the elements, and the order relation) are officially unavailable. In Cposet, we have only the objects and the morphisms and their compositions. The internal structure of the objects is officially unavailable in Cposet. Nevertheless, we have a way of reading the elements of an object in Cposet: Clearly, $[n]$, the set of elements of $A = \langle [n], \leq_A \rangle$ is naturally bijective with $\mathrm{CP}(\mathbf{C}_0, A)$. In Cposet$'$, we can name $\mathbf{C}_1 = \langle \{0, 1\}, \leq \rangle$ and also name the maps $\mathbf{f}_0 = \{(0, 0)\}$ and $\mathbf{f}_1 = \{(0, 1)\}$, and with this help we can also read the order $\leq_A$ in the object A. In fact, where $f, g \in \mathrm{CP}(\mathbf{C}_0, A)$ and say $f = (\mathbf{C}_0, \alpha, A)$ and $g = (\mathbf{C}_0, \beta, A)$ and $\alpha(0) = x$ and $\beta(0) = y$ then $x \leq_A y$ iff there is $h \in \mathrm{CP}(\mathbf{C}_1, A)$ such that $\mathbf{f}_0 h = f$ and $\mathbf{f}_1 h = g$. In fact,

$$A = \langle [n], \leq_A \rangle \cong \langle \mathrm{CP}(\mathbf{C}_0, A), \leq_d \rangle = \widetilde{A}$$

where the order $\leq_d$ on $\mathrm{CP}(\mathbf{C}_0, A)$ is defined by the formula expressed in the last sentence. The isomorphism is via the map $i \mapsto (\mathbf{C}_0, \{(0, i)\}, A)$ for $0 \leq i < n$. Here, both the set of elements and the order of the second poset $\widetilde{A}$ have first-order definitions in the language of Cposet$'$. This means that first-order language applied to the structure Cposet$'$ is equivalent in expressive power to a certain second-order language L' applied to another structure that exists inside Cposet$'$. This second-order language L' has variables ranging over the collection $\{\widetilde{A} : A \in \mathrm{Obj}\}$, has for each $A \in \mathrm{Obj}$ variables ranging over the elements of $\widetilde{A}$, and has for every $A, B \in \mathrm{Obj}$ variables ranging over the set of monotone maps from $\widetilde{A}$ to $\widetilde{B}$. All these variables can be quantified. In this language L$'$ we can express equality of elements, of structures, of monotone maps, the application of a map to an element, order-inclusions between elements.

To illustrate the power of these ideas, note that the property that the range of a monotone map $f : \widetilde{A} \to \widetilde{B}$ is a convex subset of $\widetilde{B}$ can be easily expressed by a formula in L'. This formula can be converted to a formula $\phi(x, Y, Z)$ in the first-order language of Cposet$'$ so that Cposet$' \models \phi(f, A, B)$ iff $A, B \in$ Obj, $f \in$ CP(A, B) and the range of the underlying function of the morphism f is a convex subset of the poset B. In this way, the relation between $A, B \in$ Obj that holds iff A is a surjective monotone image of a convex subset of B, is first-order definable in Cposet$'$. Via the results proved in the next section, this relation is definable in Qposet$'$.

According to Birkhoff duality, there is an order $\ll$ on $\mathcal{P}$ under which it becomes isomorphic to the set of isomorphism types of finite distributive lattices ordered by embeddability. The observation in the previous paragraph establishes that this order is first-order definable in $\mathcal{P}'$.

We can go further. The language L' can be enriched to full second-order language L_2 without changing the situation. To show that L_2-expressibility is no stronger than first-order expressibility over Cposet$'$ requires only one additional simple observation. Let $A_1, \ldots, A_n$ be any objects of Cposet and R be any nonvoid subset of the Cartesian product $A_1 \times \cdots \times A_n$. Setting $k = |R|$, there is a bijective map $\beta : [k] \to R$. Via projections, this gives maps $\beta_i : [k] \to U(A_i)$ (where $U(A_i)$ is the set of elements of A_i) such that $(x_1, \ldots, x_n) \in R$ (where $x_i \in U(A_i)$) iff for some $y \in [k]$, $\beta_i(y) = x_i$ for $i \in \{1, \ldots, n\}$. Now where $A = \mathbf{A}_k = \langle [k], \leq \rangle$ is the k-element antichain, we have that A is an object of Cposet and the maps β_i are actually monotone, $A \to A_i$. Thus we have morphisms $p_i = (A, \beta_i, A_i)$, $i \in \{1, \ldots, n\}$. In particular, choosing $n = 2$ for illustration, we find that arbitrary (non-void) relations $\widetilde{R} \subseteq U(\widetilde{A_1}) \times U(\widetilde{A_2})$ can be parameterized by triples (B, p_1, p_2) where B ranges over all objects of Cposet$'$ while p_i ranges over CP(B, A_i). Here with the proper choice of (B, p_1, p_2) we have

$$\widetilde{R} = \{(q_1, q_2) \in \mathrm{CP}(\mathbf{C}_0, A_1) \times \mathrm{CP}(\mathbf{C}_0, A_2) : \text{ for some } q \in \mathrm{CP}(\mathbf{C}_0, B)$$
$$q_i = qp_i \text{ for } i = 1, 2\} .$$

3.2 Interpreting Cposet$'$ in Qposet$'$

We wish to build a copy of the structure Cposet$'$ inside Qposet$'$, in such a way that the fundamental relations of Cposet$'$—"$A \in$ Obj"; "$f \in$ CP(A, B)"; "$g \in$ CP(A, B) and $f \in$ CP(B, C) and $h = gf \in$ CP(A, C)"—are translated to relations in Qposet$'$ that are first-order definable in that structure. The relation that links a member A of Qposet$'$ to a member P of Qposet$'$ that plays the role (in the copy) of some $B \in$ Cposet that is isomorphic to A, should be first-order definable in Qposet$'$ as well. In this way, we shall be enabled to construct a translation (or mapping) sending any first-order formula $\Phi(X_1, \ldots, X_M)$ over Cposet$'$ whose free variables $X_1, \ldots, X_n$ range over Obj to a first-order formula $\widehat{\Phi}(x_1, \ldots, x_n)$ over Qposet$'$ so that Qposet$' \models \widehat{\Phi}(A_1, \ldots, A_n)$ (for elements $A_i \in$ Qposet) iff for some $B_i \cong A_i$, $B_i \in$ Obj we have Cposet$' \models \Phi(B_1, \ldots, B_n)$. From the observations with which we concluded Section 3.1, it will follow also that such a translation can be extended to all formulas $\Phi(X_1, \ldots, X_n)$ of L_2.

Most of the technical work involved in building this copy of CPOSET′ inside QPOSET′ has already been accomplished in Section 2. Given $A \in$ QPOSET, $k = |A|$, and $B \in$ Obj with $A \cong B$, we have the poset $P_k(A, B) \in$ QPOSET (Definition 2.33). In a sense, this poset has both an existence in the quasi-ordered set QPOSET, and a parallel existence in the category CPOSET: A is encoded in $P_k(A, B)$ in terms definable in QPOSET′, as the up-to-isomorphism unique $Q \in$ QPOSET such that Q^+ is isomorphic to a ≤-maximal topped subset of $P_k(A, B)$. A presentation of $B \in$ Obj is encoded in $P_k(A, B)$ also, by a first-order formula over QPOSET′.

Much as the elements of B are encoded in CPOSET′ by the members of CP($\mathbf{C}_0$, B), and the relation over CP($\mathbf{C}_0$, B) encoding the order relation in B is defined by a first-order formula over CPOSET′, we have that the elements of B are encoded in QPOSET′ by the posets $\eta_k(i)$ $(0 \leq i < k)$, taken up to isomorphism, and the relation between the $\eta_k(i)$ that corresponds to the order in B (again taken up to isomorphism between posets) is first-order definable in QPOSET′. This is the content of Definition 2.33 and Lemma 2.34.

Thus in our model of CPOSET′ built inside QPOSET′, the role of members of Obj will be played by the posets $P \cong P_k(B, B)$ (corresponding to $B \in$ Obj with $k = |B|$). We have seen in Lemma 2.34(3) that the set of all such P is definable in QPOSET′ (and this will be critical in ensuring that our translation of formulas works as advertised). The role of equality in CPOSET′ (between objects, or between morphisms) will be played by the relation of isomorphism in QPOSET′.

It is now time to reveal how we propose to encode the morphisms of CPOSET′ by members of QPOSET′.

Definition 3.1 Suppose that $0 \leq i < k$ are integers. Recall the definition of $\eta_k(i)$ and η_k in Definition 2.30. We now put

$$\lambda_k(i) \cong \mathbf{C}_0 \oplus \mathbf{A}_2 \oplus \eta_k(i)\,;$$

$$\lambda_k \cong \sum_{0 \leq i < k} \lambda_k(i)\,.$$

For example, λ_3 can be easily recognized in the right-hand part of Fig. 6 (without the bottom element).

Observe that all of the posets $\lambda_k(i)$ and λ_k have height $k + 6$.

Lemma 3.2 *The relation*

$$\{(\mathbf{C}_i, \mathbf{C}_k, \lambda_k(i)) : \ k > 0 \textit{ and } 0 \leq i < k\}$$

and the relation

$$\{(\mathbf{C}_k, \lambda_k) : k > 0\}$$

are definable in QPOSET′.

The proof is similar to that of Lemma 2.31

Definition 3.3 Suppose that m and n are positive integers and α is a function $[m] \to [n]$. We define $F(m, \alpha, n)$, up to isomorphism, as a member of QPOSET. The poset $F(m, \alpha, n)$ will be called the *f-presentation of* α.

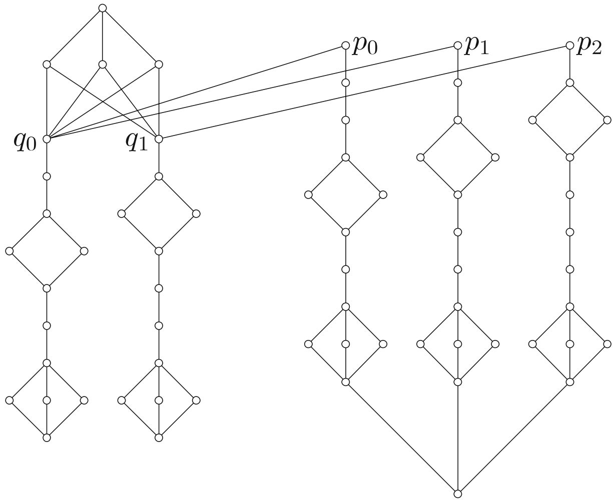


Fig. 6 $F(3, \alpha, 2)$ with $\alpha(0) = \alpha(1) = 0$ and $\alpha(2) = 1$

The universe of this poset is the disjoint union of subsets isomorphic to $\mathbf{C}_0 \oplus \lambda_m$ and to $\lambda_n \oplus \mathbf{A}_2 \oplus \mathbf{C}_0$. The order in $F(m, \alpha, n)$ is defined so that the covers are those in the copy of $\mathbf{C}_0 \oplus \lambda_m$, together with those in the copy of $\lambda_n \oplus \mathbf{A}_2 \oplus \mathbf{C}_0$, and where p_i is the maximal element in the copy of $\mathbf{C}_0 \oplus \lambda_m$ which is the top element of a copy of $\mathbf{C}_0 \oplus \lambda_m(i)$ in $\mathbf{C}_0 \oplus \lambda_m$ (for $0 \leq i < m$), and q_j is the unique element x in the copy of $\lambda_n \oplus \mathbf{A}_2 \oplus \mathbf{C}_0$ such that $x\!\downarrow$ is isomorphic to $\lambda_n(j)$ (for $0 \leq j < n$), an additional cover $q_{\alpha(i)} < p_i$ for each $0 \leq i < m$. (See Fig. 6.)

Definition 3.4 Suppose that m and n are positive integers, $0 \leq i < m$ and $0 \leq j < n$. We define a poset $\lambda_{m,n}(i, j)$ up to isomorphism by the formula

$$\lambda_{m,n}(i, j) \cong [\{\mathbf{C}_1 \oplus \mathbf{A}_2 \oplus \mathbf{C}_{i+2} \oplus \mathbf{A}_1 \oplus \mathbf{C}_{m-i-1}\} + \lambda_n(j)] \oplus \mathbf{C}_0 \,.$$

Lemma 3.5

(1) *$F(m, \alpha, n) \cong F(m', \alpha', n')$ iff $m = m'$, $n = n'$ and $\alpha = \alpha'$.*

(2) *The relation*

$$\{(\mathbf{C}_m, \mathbf{C}_n, \mathbf{C}_i, \mathbf{C}_j, L) :\ 0 \leq i < m,\ 0 \leq j < n \text{ and } L \cong \lambda_{m,n}(i, j)\}$$

is definable in QPOSET′.

(3) *The relation*

$$\{(\mathbf{C}_m, \mathbf{C}_n, F) :\ m > 0, n > 0 \text{ and } F \cong F(m, \alpha, n) \text{ for some } \alpha : [m] \to [n]\}$$

is definable in QPOSET′.

Proof The proof of (2) is straightforward.

For the proof of (1) and (3), it is necessary to show that $F = F(m, \alpha, n)$ contains a unique copy of $\mathbf{C}_0 \oplus \lambda_m$ and a unique copy of $\lambda_n \oplus \mathbf{A}_2 \oplus \mathbf{C}_0$. This task is straightforward, if tedious, and is left to the reader. $F(m, \alpha, n)$, then, can be characterized up to isomorphism as the member $Q \in$ QPOSET such that $\mathbf{C}_0 \oplus \lambda_m \leq Q$, $\lambda_n \oplus \mathbf{A}_2 \oplus \mathbf{C}_0 \leq Q$,

$$|Q| = |\lambda_m| + |\lambda_n| + 5\,,$$

every $P \leq Q$ such that $\mathbf{C}_0 \oplus \lambda_m \leq P$ and $\lambda_n \oplus \mathbf{A}_2 \oplus \mathbf{C}_0 \leq P$ satisfies $P \cong Q$, and the $\leq$-maximal topped posets $R \leq Q$ are $\lambda_n \oplus \mathbf{A}_2 \oplus \mathbf{C}_0$ and for every $0 \leq i < m$, the poset $\lambda_{m,n}(i, \alpha(i))$.

This characterization easily yields both (1) and (3). □

Now suppose that $B_1, B_2 \in$ Obj and

$$f = (B_1, \alpha, B_2) \in \mathrm{CP}(B_1, B_2)\,.$$

Say $B_i = \langle [m_i], \leq_i \rangle$, $i \in \{1, 2\}$ so that $\alpha : [m_1] \to [m_2]$. We are encoding B_i as (any member of QPOSET isomorphic to) $P_i = P_{m_i}(B_i, B_i)$. We encode f as (any triple coordinatewise isomorphic to) $M(f) = (P_1, F(m_1, \alpha, m_2), P_2)$.

Proposition 3.6 *Let $B_1, B_2 \in$ Obj and $U, V, W \in$ QPOSET.*

(1) *If $(U, V, W) \cong M(f)$, $f = (B_1, \alpha, B_2) \in \mathrm{CP}(B_1, B_2)$, then f (and α) are uniquely determined and for all $i \in [m_1]$ and $j \in [m_2]$, we have that $\alpha(i) = j$ is equivalent to $\lambda_{m_1,m_2}(i, j) \leq V$.*

(2) *$(U, V, W) \cong M(f)$ for some $f = (B_1, \alpha, B_2) \in \mathrm{CP}(B_1, B_2)$ iff: where $m_i = |B_i|$, we have $U \cong P_{m_1}(B_1, B_1)$, $W \cong P_{m_2}(B_2, B_2)$, and $V \cong F(m_1, \alpha, m_2)$ for some $\alpha : [m_1] \to [m_2]$; and whenever we have $0 \leq i, i' < m_1$ and $0 \leq j, j' < m_2$, $j \neq j'$, and $\lambda_{m_1,m_2}(i, j) \leq V$ and $\lambda_{m_1,m_2}(i', j') \leq V$, then $\eta'_{m_2}(j) + \eta_{m_2}(j') \leq W$ implies $\eta'_{m_1}(i) + \eta_{m_1}(i') \leq U$.*

The proof is straightforward.

Proposition 3.7 *Let $B_1, B_2, B_3 \in$ Obj, $f \in \mathrm{CP}(B_1, B_2)$, $g \in \mathrm{CP}(B_2, B_3)$ and, say $|B_i| = m_i$ and $f = (B_1, \alpha, B_2)$ and $g = (B_2, \beta, B_3)$. Let $M(f) \cong (P_1, F, P_2)$ and $M(g) \cong (P_2, G, P_3)$. Then $M(fg) \cong (P_1, H, P_3)$, where H is, up to isomorphism, the unique member of* QPOSET *of the form $F(m_1, \gamma, m_3)$ that satisfies: for all $i \in [m_1]$, $j \in [m_2]$, $k \in [m_3]$ we have that $\lambda_{m_1,m_2}(i, j) \leq F$ and $\lambda_{m_2,m_3}(j, k) \leq G$ imply that $\lambda_{m_1,m_3}(i, k) \leq H$.*

The proof is straightforward.

Theorem 3.8 *Let N be a positive integer and R be an isomorphism-invariant N-ary relation over* QPOSET*. Then R is first-order definable over* QPOSET$'$ *iff the restriction of R to* Obj *is first-order definable over the category* CPOSET$'$ *(or equivalently, is L_2-definable over* CPOSET$'$*).*

Proof Since the property that a morphism is an embedding is definable in CPOSET$'$, the non-obvious direction in this theorem is the passage from CPOSET$'$ definability to QPOSET$'$ definability.

So let $R \subseteq \text{QPOSET}^N$ be isomorphism-invariant and let $S = R \cap \text{Obj}^N$, and assume that

$$S = \{(B_0, \ldots, B_{N-1}) \in \text{Obj}^N : \text{CPOSET}' \models \Phi(B_0, \ldots, B_{N-1}\},$$

where $\Phi(X_0, \ldots, X_{N-1})$ is a formula of the first-order language of CPOSET' whose free variables are the object variables $X_0, \ldots, X_{N-1}$. We need to build a formula $\tilde{\Phi}(x_0, \ldots, x_{N-1})$ in the first-order language of QPOSET' so that for any $A_0, \ldots, A_{N-1} \in$ QPOSET and where $A_i \cong B_i \in \text{Obj}$ and $k_i = |A_i|$ for $0 \leq i < N$ we have

$$\begin{gathered} \text{CPOSET}' \models \Phi(B_0, \ldots, B_{N-1}) \\ \text{iff} \\ \text{QPOSET}' \models \tilde{\Phi}(P_{k_0}(A_0, B_0), \ldots, P_{k_{N-1}}(A_{N-1}, B_{N-1})). \end{gathered}$$

We can then take $\Psi(x_0, \ldots, x_{N-1})$ to be:

$$\begin{gathered} \text{there exist } u_i \ (0 \leq i < N) \text{ so that } \tilde{\Phi}(u_0, \ldots, u_{N-1}) \text{ and} \\ \text{“}u_i \cong P_{k_i}(x_i, y_i) \text{ for some } y_i \text{ where } k_i = |x_i|, \text{ for } 0 \leq i < N\text{”} \end{gathered}$$

and it will follow that

$$R = \{(A_0, \ldots, A_{N-1}) \in \text{QPOSET}^N : \text{QPOSET}' \models \Psi(A_0, \ldots, A_{N-1})\}.$$

To construct $\tilde{\Phi}$, we extend the list of free variables in Φ to a list of all the object variables that have an occurrence, free or bound, in Φ; say this list is $X_0, \ldots, X_{M-1}$ ($M \geq N$). We make a list $f_0, \ldots, f_{K-1}$ of all the morphism variables that occur in Φ. We introduce variables $x_0, \ldots, x_{M-1}$ and $y_0, \ldots, y_{K-1}$ from the first-order language of QPOSET' to correspond to the X_i and f_j. Now by induction on length of a formula, we define a mapping that sends all the sub-formulas ϕ of Φ to corresponding formulas $\tilde{\phi}$ in the first-order language of QPOSET'.

(1) If ϕ is $X_i = X_j$ then $\tilde{\phi}$ is $x_i \leq x_j \wedge x_j \leq x_i$.
(2) If ϕ is $f_s = f_t$ then $\tilde{\phi}$ is $y_s \leq y_t \wedge y_t \leq y_s$.
(3) If ϕ is $f_s \in \text{CP}(X_i, X_j)$ then $\tilde{\phi}$ is

$$\begin{gathered} (\exists u_i, u_j)(\text{“there are } v_i, v_j \text{ so that where } k_i = |u_i|, k_j = |u_j| \text{ we have} \\ x_i = P_{k_i}(u_i, v_i) \text{ and } x_j = P_{k_j}(u_j, v_j) \text{ and} \\ (x_i, y_s, x_j) = M(f) \text{ for some } f \in \text{CP}(v_i, v_j)\text{”}) \end{gathered}$$

(4) If ϕ is

$$\begin{gathered} f_{r_0} \in \text{CP}(X_{s_0}, X_{s_1}) \wedge f_{r_1} \in \text{CP}(X_{s_1}, X_{s_2}) \wedge \\ \wedge f_{r_2} = f_{r_0} \circ f_{r_1} \end{gathered}$$

then $\tilde{\phi}$ is

$$\begin{gathered} (\exists u_{s_0}, u_{s_1}, u_{s_2})(\text{“there are } v_{s_0}, v_{s_1}, v_{s_2} \text{ so that} \\ \text{where } k_i = |u_{s_i}| \text{ we have } x_{s_i} = P_{k_i}(u_{s_i}, v_{s_i}) \text{ for } i \in \{0, 1, 2\} \\ \text{and } (x_{s_0}, y_{r_0}, x_{s_1}) = M(f) \text{ for some } f \in \text{CP}(v_{s_0}, v_{s_1}) \\ \text{and } (x_{s_1}, y_{r_1}, x_{s_2}) = M(g) \text{ for some } g \in \text{CP}(v_{s_1}, v_{s_2}) \\ \text{and } (x_{s_0}, y_{r_2}, x_{s_2}) = M(fg)\text{”}). \end{gathered}$$

(5) If ϕ is $\neg\psi$, or $\psi \wedge \chi$ then $\tilde{\phi}$ is $\neg\tilde{\psi}$, or $\tilde{\psi} \wedge \tilde{\chi}$.

(6) If ϕ is $(\exists X_i)\psi$ then $\tilde{\phi}$ is

$$(\exists x_i)([(\exists u_i)(\text{“there is } v_i \text{ so that where } k_i = |u_i|, x_i = P_{k_i}(u_i, v_i)\text{”})] \wedge \tilde{\psi}) .$$

(7) If ϕ is $(\forall X_i)\psi$ then $\tilde{\phi}$ is

$$(\forall x_i)([(\exists u_i)(\text{“there is } v_i \text{ so that where } k_i = |u_i|, x_i = P_{k_i}(u_i, v_i)\text{”})] \to \tilde{\psi}) .$$

(8) If ϕ is $(\exists f_s \in \mathrm{CP}(X_i, X_j))\psi$ then $\tilde{\phi}$ is

$$\begin{gathered}(\exists y_s)[(\exists u_i, u_j)(\text{“there are } v_i, v_j \text{ so that}\\ \text{where } k_i = |u_i|, k_j = |u_j| \text{ we have } x_i = P_{k_i}(u_i, v_i)\\ \text{and } x_j = P_{k_j}(u_j, v_j) \text{ and } (x_i, y_s, x_j) = M(f)\\ \text{for some } f \in \mathrm{CP}(v_i, v_j)\text{”})\wedge\tilde{\psi}].\end{gathered}$$

(9) If ϕ is $(\forall f_s \in \mathrm{CP}(X_i, X_j))\psi$ then $\tilde{\phi}$ is

$$\begin{gathered}(\forall y_s)[(\exists u_i, u_j)(\text{“there are } v_i, v_j \text{ so that}\\ \text{where } k_i = |u_i| \text{ and } k_j = |u_j| \text{ we have } x_i = P_{k_i}(u_i, v_i)\\ \text{and } x_j = P_{k_j}(u_j, v_j) \text{ and } (x_i, y_s, x_j) = M(f)\\ \text{for some } f \in \mathrm{CP}(v_i, v_j)\text{”})\to \tilde{\psi}].\end{gathered}$$

One can prove by induction on the length of ϕ that for all sub-formulas $\phi(\bar{X}, \bar{f})$ of Φ, and for all $B_i \in \mathrm{Obj}$, $0 \le i < M$ and $f_j = (U_j, \alpha_j, V_j) \in \mathrm{CP}(U_j, V_j)$, $0 \le j < K$, and where $|B_i| = b_i$, $|U_j| = u_j$ and $|V_j| = v_j$ we have

$$\begin{gathered}\textsc{Cposet}' \models \phi(B_0, \ldots, B_{M-1}; f_0, \ldots, f_{K-1}) \text{ iff } \textsc{Qposet}' \models\\ \tilde{\phi}(P_{\alpha_0}(B_0, B_0), \ldots, P_{\alpha_{M-1}}(B_{M-1}, B_{M-1}); F(u_0, \alpha_0, v_0), \ldots,\\ F(u_{K-1}, \alpha_{K-1}, v_{K-1})).\end{gathered}$$

Taking $\phi = \Phi$ we then have the desired result. □

Remark 3.1 We have organized and written the material of Section 3 in a way that we hope makes it readable for most algebraists and order-theorists. It is quite possible that there is a more elegant way to express the essential fact of Theorem 3.8 within set theory. Specifically, we believe that if one deals directly with the quasi-ordered set $\langle \mathcal{HF}, \le \rangle$ whose members are the posets $\langle A, \le \rangle$ such that A belongs to the set HF of all hereditarily finite sets, quasi-ordered by embeddability $\le$, then it should be possible to prove that every isomorphism-invariant finitary relation over $\mathcal{HF}$ that is first-order definable in the model $\langle HF, \varepsilon \rangle$ (where ε is the membership relation in the domain HF), is also first-order definable in the quasi-ordered set $\langle \mathcal{HF}, \le \rangle$. ($HF$ is the smallest set containing the empty set and closed under the binary operation $x \cup \{y\}$; with the restricted membership relation it is a model of finite set theory.)

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