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A Note on First-Fit Coloring of Interval Graphs

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Abstract We apply the Column Construction Method (Varadarajan et al., Proceedings of the Fifteenth Annual ACM-SIAM Symposium On Discrete Algorithms, pp. 562–571, 2004) to a minimal clique cover of an interval graph to obtain a new proof that First-Fit is 8-competitive for online coloring interval graphs. This proof also yields a new discovery that in each minimal clique cover of an interval graph G, there is a clique of size $\frac{\omega(G)}{8}$.

Keywords First fit for online graph coloring · Competitive analysis · Column construction method

1 Introduction

Online coloring of interval graphs is motivated by the scenario where resource requests arrive dynamically in an unpredictable order. One heuristic for this purpose is the First-Fit algorithm: allocate the *lowest* color to the current interval that respects the constraints imposed by the intervals that have been colored. Formally, a coloring of a set of intervals by a function h defined on the set of intervals. In other-words, an interval I is said to be assigned a color $h(I)$ which is a positive integer. In the coloring obtained by applying the First-Fit algorithm, if an interval I is assigned a color $h(I)$, then for all colors $1 \leq i < h(I)$, there exists an interval I' such that $I \cap I' \neq \phi$ and $h(I') = i$. The simplicity of First-Fit has made it one of the main choices for the purpose. However, the competitive analysis of First-Fit is far from complete, and this is the central interest of this paper.

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Previous Work In 1974, Woodall proved a $\omega \log \omega$ bound on the number of colors used by First-Fit to color a sequence of intervals. Here ω denotes the size of the maximum clique of the interval graph formed by the given set of intervals. In 1988, Kierstead [4] proved that First-Fit is 40-competitive eliminating the logarithmic factor from Woodall's bound. Around the same time, Chrobak and Ślusarek [1] proved that First-Fit uses at least 4.4ω colors on some instances of interval graphs. Again in 1996, Kierstead and Qin [3] lowered the competitive ratio bound further to 26. In 2004, Varadarajan et al. [5] introduced an interesting technique called *column construction procedure* and analyzed First-Fit. Their analysis first showed that First-Fit is 10-competitive, along with a mention of their belief that their analysis should yield an 8-competitiveness. Followed by this, Trotter et al. (unpublished notes) obtained an upper bound of 8 on the competitive ratio of the First-Fit algorithm. Independently, we also showed a much simpler and mildly tighter analysis to prove the 8-competitiveness of First-Fit. Our simpler analysis was based on a counter-intuitive trick on the construction of [5].

Our Work In this note, we present a clearer proof, involving the minimal clique cover, of the 8-competitiveness of First-Fit. Our counting argument is exactly the same as the column construction procedure in [5]. However, the combinatorial objects on which we apply the counting argument are minimal clique covers. This makes the proofs associated with the counting argument more elegant. Further, as a consequence of this work, we discover, by stumbling on the fact, that in any minimal clique cover of an interval graph, there is a clique of size at least $\frac{\omega(G)}{8}$. To the best of our knowledge such a question regarding the size of cliques in minimal clique covers has never been asked before.

2 Analysis of First-Fit

Let $\{1, \ldots, m\}$ denote a totally ordered set of m-colors, with the total order being $1 < 2 < \ldots < m$. Let L be a vertex coloring of G obtained by using First-Fit. Let $\{K_1, \ldots, K_r\}$ be a minimal clique cover of G where each clique is a maximal clique in G. We also assume that in the linear ordering of maximal cliques of the interval graph G (see [2]), K_i occurs to the left of K_j whenever $i < j$. Given a coloring L of G, we consider the set $\{T_1, \ldots, T_r\}$, where T_i is an infinite sequence which takes values from the set $V(G) \cup \{\texttt{hole}\}$. For $1 \le i \le r$, $j \ge 1$, we define $T_i(j) = v$ if $v \in K_i$ and v is colored j by L. If no vertex in K_i has been assigned the color j, then $T_i(j) = \texttt{hole}$. We refer to the each T_i as a column, and refer to each $T_i(j)$ as a cell. The symbol r is used to denote the number of columns in the following procedure. In the procedure each cell is assigned a label from the set $\{R, \$, F\}$ based on certain rules, and the label assigned to cell $T_i(j)$ is denoted by $label_i(j)$. Finally, each cell has a set of at most two neighboring cells, and this set is denoted by $N_i(j)$ for the cell $T_i(j)$.

Column Construction Procedure In each iteration, this procedure constructs a subset of a set that was constructed in the previous iteration. In the procedure λ is a positive non-zero number, whose value will be chosen based on the analysis of the procedure. The initial subset is $C_1 = \{j | 1 \le j \le r\}$. For each $j \in C_1$, if T_{1j} is not $\texttt{hole}$, then $label_j(1) = R$, otherwise $label_j(1) = \$$. Let $N_1(1) = \{2\}$, $N_r(1) = \{r - 1\}$,

$N_j(1) = \{j-1, j+1\}$ for each r such that $2 \le j \le r-1$. For each $i \ge 2$ such that C_{i-1} is non-empty, C_i is constructed from C_{i-1} by applying the following set of rules in the order in which they are presented. If C_{i-1} is empty, the procedure stops and these rules are not applied.

- For each $j \in C_{i-1}$, if $T_j(i)$ is not `hole`, then add j to C_i, and assign R to $label_j(i)$.
- For each $j \in C_{i-1} \setminus C_i$, let $D_j(i) = \{k | k \in N_j(i-1), k \in C_i\}$. If $D_j(i)$ is non-empty, then add j to C_i, and assign \$ to $label_j(i)$.
- For each $j \in C_{i-1} \setminus C_i$, and for each $l \in N_j(i-1)$, let $h = min\{b | l \in N_j(b)\}$. If $\rho_j(i-1, h) > \frac{(i-h)}{\lambda}$, then add j to C_i, and assign F to $label_j(i)$.
- *Neighbors of cells in C_i:* For each $j \in C_i$, let $n_{left} \in C_i$ be a number such that $n_{left} < j$ and for each k such that $n_{left} < k < j$, $k \notin C_i$. Similarly, let $n_{right} \in C_i$ be a number such that $j < n_{right}$ and for each k such that $j < k < n_{right}$, $k \notin C_i$. The neighbor set of cell $T_j(i)$ is defined as follows: $N_j(i) = \{n_{left}\}$, if n_{right} is undefined. $N_j(i) = \{n_{right}\}$, if n_{left} is undefined. Otherwise, $N_j(i) = \{n_{left}, n_{right}\}$. It is useful to note the fact that we are dealing with interval graphs is used here, in this definition of the neighbors of a cell.

2.1 Bounding the Height of the Columns

Here we prove our claims regarding the labeling obtained from the column construction procedure presented.

Lemma 1 *For each $i \ge 1$, the set of maximal cliques $\{K_l | l \in C_i\}$ is a clique cover of $\{v | L(v) \ge i\}$.*

Proof The proof is by induction on i. The base case, for $i = 1$, is very straightforward, as C_1 is the indices of cliques in the minimal clique cover of G. Let us assume that the claim holds for $j = i-1$. We now prove this for i. Let v be a vertex such that $L(v) \ge i$. By the induction hypothesis v is covered by a clique K_r, $r \in C_{i-1}$. Let us consider $Cov_v(i-1) = \{r | r \in C_{i-1}$ and $v \in K_r\}$. Let $B_v(i) = \{u | L(u) = i, \{u, v\} \in E(G)\}$. By the induction hypothesis, for each $u \in B_v(i)$, there exists an $r' \in C_{i-1}$ such that $u \in K_{r'}$. By the algorithm, each such $r' \in C_i$, and let $Cov_{n(v)}(i-1)$ denote the set of such r'. If there exists an $r' \in Cov_{n(v)}(i-1)$ such that $v \in K_{r'}$, then the lemma is proved for this case. Otherwise, due to the linear order of the elements of C_{i-1} it follows that one neighbor of an element of $Cov_v(i-1)$, say r'', is an element of $Cov_{n(v)}(i-1)$. By the algorithm, r'' is added to C_i with the symbol \$. In this case, v is an element of $K_{r''}$ where $r'' \in C_i$. Hence the lemma. □

Let m' be the smallest value such that $C_{m'+1} =$ `empty`. Let j be a value such that $j \in C_0$ and $j \in C_{m'}$. In the following theorem, all the arguments are with respect to this j that we consider.

Theorem 2 *If the First-Fit strategy for online coloring interval graphs uses m colors, then there exists a clique of size at least $\frac{m}{8}$. Therefore, First-Fit is 8-competitive.*

Proof To prove the theorem, we show that the number of R in T_j is at least $\frac{m'}{8}$. We first place an upper bound on the number of \$, denoted by δ_j, in the multi-set

$\{label_j(i)|1 \le i \le m'\}$. This bound is placed by considering the contribution of the left and right neighbors of T_j in C_i, $1 \le i \le m'$. First we consider the left neighbors. Let $h_0 = 0, h_1, \ldots, h_k$ be k positive integers, and $l_k < l_{k-1} < \ldots < l_1 < j$ be the elements of C_1, such that the following three properties hold:

- For each $1 \le i \le k$, and for each y such that $\left(\sum_{c=1}^{i-1} h_c\right) + 1 \le y \le \left(\sum_{c=1}^{i} h_c\right)$, $l_i \in N_j(y)$.
- For each s, $1 \le s \le k$, and for each x such that $\left(\sum_{c=1}^{s-1} h_c\right) + 1 \le x \le \left(\sum_{c=1}^{s} h_c\right)$, $j \in N_{l_s}(x)$.
- For each $1 \le s \le k$, $l_s \notin C_{(\sum_{c=1}^{s} h_c)+1}$.

From the Column Construction Procedure, these three properties ensure that for each $1 \le s \le k$, the number of R symbols in the multi-set $\{label_{l_s}(i)|(\sum_{c=1}^{s-1} h_c) + 1 \le i \le (\sum_{c=1}^{s} h_c)\}$ is at most $\frac{h_s}{\lambda}$. Since $\sum_{i=1}^{k} h_i \le m'$, it follows that the number of \$ symbols in $\{label_j(i)|1 \le i \le m'\}$ due to the R symbols in the columns $T_{l_1}, \ldots, T_{l_k}$ is at most $\frac{m'}{\lambda}$. The contribution due to the right neighbors of T_j is analyzed symmetrically, using the same argument, and the value of the upper bound is $\frac{m'}{\lambda}$. Therefore, the number of \$ symbols in T_j is at most $\frac{2m'}{\lambda}$.

We now place an upper bound on the number of F symbols in T_j. To do this, we consider a set of intervals J satisfying the following properties

- Any two intervals in J are disjoint, and the set $\{x|label_j(x) = F\}$ is a subset of $U = \cup_{I \in J} I$. For each interval $I \in J$, $label_j(max(I)) = F$, where $max(I)$ is the largest element in I. The interval I is $[min(I), max(I)]$, where $min(I)$ is the smallest integer such that $\bigcap_{s=min(I)}^{max(I)-1} N_j(s)$ is non-empty, and the number of R symbols in the multi-set $\{label_j(s)|min(I) \le s \le max(I) - 1\}$ is more than $\frac{(max(I)-min(I))}{\lambda}$.

It is easy to see that J is well defined and exists. The intervals of J can be constructed iteratively using the set $Fset = \{x|label_j(x) = F\}$. In the first iteration we construct an interval I for which $max(I)$ is the largest element in the set. $min(I)$ is chosen exactly as mentioned in the description of J. $min(I)$ is a well defined parameter since $label_j(max(I)) = F$. The elements of I are then removed from *Fset*, to yield the *Fset* for the next iteration, which finds another element of J. The construction procedure terminates when *Fset* becomes the empty set.

Let h denote the cardinality of the multi-set $\{x|label_j(x) \ne \$ \text{ and } x \in U\}$. Clearly, by the column construction procedure, the number of R in the multi-set $\{label_j(x)| x \in U\}$ is more than $\frac{h}{\lambda}$. Further, it is clear that $h \le m' - \delta_j$. Therefore, it follows that the number of R symbols in the multi-set $\{label_j(i)|1 \le i \le m'\}$ is at least $m' - \delta_j - h + \frac{h}{\lambda} = m' - \delta_j - h\left(1 - \frac{1}{\lambda}\right) \ge \frac{m'}{\lambda}\left(1 - \frac{2}{\lambda}\right)$. This is maximized for $\lambda = 4$. Further, the number of R symbols in the multi-set $\{label_j(i)|1 \le i \le m'\}$ is a lower bound on the size of the clique K_j. Therefore, the size of the maximum clique $\omega(G) \ge \frac{m'}{8}$. In other words, First-Fit is 8-competitive. □

Corollary 3 *In any minimal clique cover of an interval graph, there is a clique of size at least* $\frac{\omega(G)}{8}$.

Proof We consider the sequence of intervals ordered in increasing order of their left end points, and then apply First-Fit coloring to this sequence. We know that for this order, the number of colors used is $\omega(G)$. Now we apply the Column Construction Procedure using this coloring, and a minimal clique cover as the columns. By the above two theorems we conclude that there is a clique of size at least $\frac{\omega(G)}{8}$ in this minimal clique cover. Hence the corollary. □

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