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Investigation of the dynamical structures for nonlinear Vakhnenko-Parkes equation using two integration schemes

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Abstract

The dynamic behavior of the Vakhnenko-Parkes equation is examined in this manuscript. This is an important subject because of its implications for comprehending intricate mathematical models describing traveling wave phenomena and solitons. The construction of traveling wave solutions for the Vakhnenko-Parkes equation in closed form is the main issue addressed in the study. The modified auxiliary equation approach and the extended $(\frac{G'}{G^2})$-expansion method are used to address this because they are effective in producing precise solutions of a large class of nonlinear partial differential equations. A visual component to comprehending the behavior of the equation is added by employing $3D$-surface graphs, $2D$-line graphs, and contour plots to explore these solutions graphically. A variety of traveling wave behavior is observed from the obtained solutions. These results imply that the Vakhnenko-Parkes equation and its solutions are complex, offering important insights into the underlying dynamics. The proposed techniques are applied for the first time to study the considered model in this work. A comparison of the obtained results with the previous works is presented to confirm the significance and novelty of the reported results.

Keywords Vakhnenko-Parkes equation · Extended $(\frac{G'}{G^2})$-expansion method · Modified auxiliary equation method · Traveling wave solutions

Mathematics Subject Classification 35C15 · 35Q51 · 37K40 · 37N30

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1 Introduction

Most of the natural occurrences depend on a number of variables that change nonlinearly (Ozisik et al. 2023). To simulate the natural phenomena and dynamic processes, nonlinear partial differential equations (NLPDEs) are extensively used. NLPDEs have a wide range of applications in applied mathematics, plasma physics, bioinformatics, chemistry, fluid dynamics, quantum field theory, artificial intelligence and others. The swift growth of computer sciences and software technologies is directly attributed to the rise of research on NLPDEs. In particular, this upsurge was influenced by the use of computer-based techniques in mathematics and other disciplines.

Evolution equations, which involve derivatives with regard to time, explain how a system changes over time. Partial differential equations (PDEs) that contain nonlinear terms are known as nonlinear PDEs, and they offer a more sophisticated mathematical modeling than linear PDEs (Islam et al. 2022). A number of methods have been developed to find the exact solutions of NLPDEs to explain the underlying physical systems including $\left(\frac{G'}{G^2}\right)$-expansion method and its modified form Behera et al. (2022), Mamun et al. (2021), Duran et al. (2023), and Akram et al. (2024), generalized exponential rational function approach (Duran 2021a), $\left(\frac{G'}{G}, \frac{1}{G}\right)$-expansion method (Duran 2021b; Mamun et al. 2021), exponential function method and its modified form Islam et al. (2018) and Duran et al. (2017), rational $(\frac{1}{\phi'(\xi)})$-expansion approach (Islam et al. 2022), extended tanh-function method (Fan 2000; Islam et al. 2019), extended $(\exp(-\phi(\xi)))$-expansion method (Arshed et al. 2022; Shahen et al. 2021a, b), Sardar sub-equation scheme (Justin et al. 2022), improved tanh method (Yokuş et al. 2022), F-expansion method (Ebaid and Aly 2012), modified auxiliary equation method (Akram et al. 2022, 2024), Hirota's bilinear method (Zuo and Zhang 2019), Kudryashov method (Mirzazadeh et al. 2014), Darboux transformation method (Ma 2019), modified extended tanh-function method (Mamun et al. 2020), improved auxiliary equation approach (Islam et al. 2023a, b), extended Riccati scheme (Islam et al. 2022), rational $\left(\frac{G'}{G}\right)$-expansion (Islam et al. 2023b; Akbar et al. 2023; Islam et al. 2019, 2022), unified method (Foyjonnesa et al. 2022, 2023), $(\frac{G'}{G'+G+A})$-expansion approach (Iqbal et al. 2024), modified extended tanh-function approach (Mamun et al. 2021) and many more.

The surface and interior waves in a rotating ocean can be described by Ostrovsky equation which can be written, as:

$$(u_t + c_0 u_x + p u u_x + q u_{xxx})_x = \gamma u, \tag{1.1}$$

where c_0 is the velocity of dispersion-less linear waves and p is the coefficient of the nonlinear term. Moreover, q and γ are the coefficients of small-scale and large-scale dispersion terms. The small hydrodynamic nonlinearity, $u u_x$ and weak dispersion are combined in this equation.

When $\gamma = 0$, Eq. (1.1) reduces to Korteweg-de Vries (KdV) equation. When $q = 0$, Eq. (1.1) reduces to the form

$$(u_t + c_0 u_x + p u u_x)_x = \gamma u,$$

which is often called the reduced Ostrovsky equation.

It was demonstrated by Wazwaz (2019) that the new integrable equation could be derived from the reduced Ostrovsky equation (Yusufoglu and Bekir 2007)

$$uu_{xxt} - u_x u_{xt} + u^2 u_t = 0. \tag{1.2}$$

The behavior of long surface water waves in a two-dimensional ideal fluid is modeled using the Vakhnenko-Parkes equation. It is a generalization of the well-known KdV equation, and it has been used in research into the stability and solitonic behavior of water waves in a variety of physical systems. The VP equation has numerous applications in various fields of science. It is an essential tool for simulating long surface water waves in oceanography and fluid dynamics, offering insights into wave behavior in many aquatic environments. The equation is used by environmental scientists to comprehend how waves affect coastal ecosystems and regions. The equation generalizes the KdV equation to the domain of solitons and nonlinear waves, allowing for the investigation of solitary wave solutions and their consequences in various physical systems. It is a useful model in mathematical physics to investigate the mathematical characteristics of nonlinear partial differential equations. Studying wave dynamics in geological settings, such as tsunami propagation, is useful to geophysicists. In numerical analysis and simulation, the equation also acts as a benchmark problem, enabling researchers to evaluate the precision and effectiveness of algorithms created to solve nonlinear partial differential equations. In general, the VP equation is essential to the advancement of wave phenomena research in many other fields of science.

There has been a significant amount of work done in the field of mathematics to investigate the various aspects of the solutions to the VP equation, such as their existence and uniqueness and their asymptotic behavior. VP equation has been explored by the modified exponential function method (Yel and Aktürk 2020), the $\exp(-\phi(\xi))$ -expansion method (Roshid et al. 2014), Kudrayashov method (Ibrahim et al. 2019), Hirota's bilinear method (Wazwaz 2019) and generalized Kudryashov method (Kumar and Mann 2022). In 2022, Khater et al. (2022) Khater et al. studied VP equation by Khater II method. In 2022, Kumar and Mann (2022) Kumar et al. studied VP equation by applying three different schemes. In 2014, Roshid et al. (2014) Roshid et al. investigated solitary wave solutions of VP equation via two novel techniques.

In this work, the wave dynamics of the VP equation is explored using two reliable analytical techniques. The proposed techniques, namely, the extended $(\frac{G'}{G^2})$-expansion and the modified auxiliary equation (Akram et al. 2023a, b) methods, are applied to examine the considered VP equation for the first time in this work. The obtained traveling wave solutions not only confirm the previously reported wave behavior for VP equation in the literature but also produce some novel results. A comparison of the presented results with the previous studies available in the literature is carried out to highlight the novel and interesting outcomes of this study.

The extended $(\frac{G'}{G^2})$-expansion and the modified auxiliary equation methods are modern and reliable expansion methods which have been successfully utilized to construct the traveling wave solutions of a large class of NLPDEs arising in mathematical physics in a number of recent studies. The proposed techniques are straight-forward, efficient and usually provide a variety of traveling wave solutions, including solitary waves and solitons. The effectiveness of the proposed methodologies to study the considered VP equation is established through the comparison of the obtained results with the previous literature.

The remaining paper is organized as follows: Sect. 2 presents the conversion of the VP equation into an a reduced equation using traveling wave hypothesis. Section 3 provides a brief overview of the proposed methods. The traveling wave solutions are constructed in Sect. 4. The obtained results are illustrated and discussed in Sect. 5. The conclusion is presented in Sect. 6.

2 Mathematical analysis

The following change of variables is introduced according to the traveling wave hypothesis to obtain the exact solutions of Eq. (1.2)

$$u(x, t) = U(\xi), \; \xi = x - wt. \tag{2.1}$$

Substitution of Eq. (2.1) into Eq. (1.2), reduces it into an equation with ordinary derivatives, as:

$$uu'' - u'u'' + u^2u' = 0. \tag{2.2}$$

Integration of Eq. (2.2) with respect to ξ yields

$$3uu'' - 3(u')^2 + u^3 = 0, \tag{2.3}$$

where the constant of integration is taken as zero.

3 Description of methods

The mathematical procedure for finding the traveling wave solutions of the VP equation using the two proposed techniques is briefly described as follows:

3.1 Extended $(\frac{G'}{G^2})$-expansion method

According to the extended $(\frac{G'}{G^2})$-expansion approach, the solution of Eq. (2.3) can be written, as:

$$U(\xi) = a_0 + \sum_{k=1}^{N} \left[a_k \left(\frac{G'}{G^2} \right)^k + b_k \left(\frac{G'}{G^2} \right)^{-k} \right], \tag{3.1}$$

where $U = U(\xi)$ satisfies

$$\left(\frac{G'}{G^2} \right)' = \rho + \mu \left(\frac{G'}{G^2} \right)^2, \tag{3.2}$$

for $\mu \neq 0$ and $\rho \neq 1$. The following considerations represent the solution for Eq. (3.2).

If $\mu\rho > 0$, then

$$\frac{G'}{G^2} = \sqrt{\frac{\rho}{\mu}} \left(\frac{E\cos(\sqrt{\mu\rho}\xi) + F\sin(\sqrt{\mu\rho}\xi)}{F\cos(\sqrt{\mu\rho}\xi) - E\sin(\sqrt{\mu\rho}\xi)} \right). \tag{3.3}$$

If $\mu\rho < 0$, then

$$\frac{G'}{G^2} = -\frac{\sqrt{|\mu\rho|}}{\mu} \left(\frac{E\cosh(2\sqrt{|\mu\rho|}\xi) + E\sinh(2\sqrt{|\mu\rho|}\xi) + F}{E\cosh(2\sqrt{|\mu\rho|}\xi) + E\sinh(2\sqrt{|\mu\rho|}\xi) - F} \right). \tag{3.4}$$

If $\mu \neq 0$ and $\rho = 0$, then

$$\frac{G'}{G^2} = -\frac{E}{\mu(E\xi + F)},\tag{3.5}$$

where E and F are constants that can be assigned arbitrary values.

3.2 Modified auxiliary equation method

According to this approach, the solution of Eq. (2.3) can be written, as:

$$U(\xi) = c_0 + \sum_{i=1}^{N} [c_i (k^f)^i + d_i (k^f)^{-i}],\tag{3.6}$$

where c_i's and b_i's are constants. Moreover,

$$f'(\xi) = \frac{\tau + \varepsilon k^{-f} + \sigma k^f}{\ln(k)},\tag{3.7}$$

where ε, τ, σ, and k are arbitrary constants with $k > 0$, $k \neq 1$. The function $k^{f(\xi)}$ has the following values.

If $\tau^2 - 4\varepsilon\sigma < 0$ and $\sigma \neq 0$, then

$$k^{f(\xi)} = \frac{-\tau + \sqrt{4\varepsilon\sigma - \tau^2}\tan\left(\frac{\sqrt{4\varepsilon\sigma - \tau^2}\xi}{2}\right)}{2\sigma} \quad \text{or} \quad k^{f(\xi)} = -\frac{\tau + \sqrt{4\varepsilon\sigma - \tau^2}\cot\left(\frac{\sqrt{4\varepsilon\sigma - \tau^2}\xi}{2}\right)}{2\sigma}.\tag{3.8}$$

If $\tau^2 - 4\varepsilon\sigma > 0$ and $\sigma \neq 0$, then

$$k^{f(\xi)} = -\frac{\tau + \sqrt{\tau^2 - 4\varepsilon\sigma}\tanh\left(\frac{\sqrt{\tau^2 - 4\varepsilon\sigma}\xi}{2}\right)}{2\sigma} \quad \text{or} \quad k^{f(\xi)} = -\frac{\tau + \sqrt{\tau^2 - 4\varepsilon\sigma}\coth\left(\frac{\sqrt{\tau^2 - 4\varepsilon\sigma}\xi}{2}\right)}{2\sigma}.\tag{3.9}$$

If $\tau^2 - 4\varepsilon\sigma = 0$ and $\sigma \neq 0$, then

$$k^{f(\xi)} = -\frac{2 + \tau\xi}{2\sigma\xi}.\tag{3.10}$$

4 Construction of traveling wave solutions

4.1 Results using extended $\left(\frac{G'}{G^2}\right)$-expansion method

The exact traveling wave solutions of Eq. (1.2) are determined as follows:

Implementation of homogeneous balancing principle on the terms uu'' and u^3 of Eq. (2.3) gives $N = 2$ and Eq. (3.1) becomes

$$U(\xi) = a_0 + a_1\left(\frac{G'}{G^2}\right) + b_1\left(\frac{G'}{G^2}\right)^{-1} + a_2\left(\frac{G'}{G^2}\right)^2 + b_2\left(\frac{G'}{G^2}\right)^{-2}.\tag{4.1}$$

A polynomial equation in $\left(\frac{G'}{G^2}\right)$ is determined by using Eq. (4.1) and Eq. (2.3). The following equations are obtained by equating the coefficients of $\left(\frac{G'}{G^2}\right)$ on both sides.

$$
\begin{aligned}
\left(\tfrac{G'}{G^2}\right)^{-1} &: \ -6\mu^2 b_1 b_2 + 6\mu\rho a_0 b_1 + 54\mu\rho a_1 b_2 + 24\rho^2 a_2 b_1 + 3a_0^2 b_1 + 6a_0 a_1 b_2 \\
&\quad + 3a_1 b_1^2 + 6a_2 b_1 b_2 = 0, \\[4pt]
\left(\tfrac{G'}{G^2}\right)^{-2} &: \ -6\mu^2 b_2^2 + 24\mu\rho a_0 b_2 + 12\rho^2 a_1 b_1 + 48\rho^2 a_2 b_2 + 3a_0^2 b_2 + 3a_0 b_1^2 \\
&\quad 6a_1 b_1 b_2 + 3a_2 b_2^2 = 0, \\[4pt]
\left(\tfrac{G'}{G^2}\right)^{-3} &: \ 6\mu\rho b_1 b_2 + 6\rho^2 a_0 b_1 + 30\rho^2 a_1 b_2 + 6a_0 b_1 b_2 + 3a_1 b_2^2 + b_1^3 = 0, \\[4pt]
\left(\tfrac{G'}{G^2}\right)^{-4} &: \ 18\rho^2 a_0 b_2 + 3\rho^2 b_1^2 + 3a_0 b_2^2 + 3b_1^2 b_2 = 0, \\[4pt]
\left(\tfrac{G'}{G^2}\right)^{-5} &: \ 12\rho^2 b_1 b_2 + 3b_1 b_2^2 = 0, \\[4pt]
\left(\tfrac{G'}{G^2}\right)^{-6} &: \ 6\rho^2 b_2^2 + b_2^3 = 0, \\[4pt]
\left(\tfrac{G'}{G^2}\right)^{1} &: \ 24\mu^2 a_1 b_2 + 6\mu\rho a_0 a_1 + 54\mu\rho a_2 b_1 - 6\rho^2 a_1 a_2 + 3a_0^2 a_1 + 6a_0 a_2 b_1 \\
&\quad + 3a_1^2 b_1 + 6a_1 a_2 b_2 = 0, \\[4pt]
\left(\tfrac{G'}{G^2}\right)^{2} &: \ 12\mu^2 a_1 b_1 + 48\mu^2 a_2 b_2 + 24\mu\rho a_0 a_2 - 6\rho^2 a_2^2 + 3a_0^2 a_2 + 3a_0 a_1^2 \\
&\quad + 6a_1 a_2 b_1 + 3a_2^2 b_2 = 0, \\[4pt]
\left(\tfrac{G'}{G^2}\right)^{3} &: \ 6\mu^2 a_0 a_1 + 30\mu^2 a_2 b_1 + 6\mu\rho a_1 a_2 + 6a_0 a_1 a_2 + a_1^3 + 3a_2^2 b_1 = 0, \\[4pt]
\left(\tfrac{G'}{G^2}\right)^{4} &: \ 18\mu^2 a_0 a_2 + 3\mu^2 a_1^2 + 3a_0 a_2^2 + 3a_1^2 a_2 = 0, \\[4pt]
\left(\tfrac{G'}{G^2}\right)^{5} &: \ 12\mu^2 a_1 a_2 + 3a_1 a_2^2 = 0, \\[4pt]
\left(\tfrac{G'}{G^2}\right)^{6} &: \ 6\mu^2 a_2^2 + a_2^3 = 0, \\[4pt]
\left(\tfrac{G'}{G^2}\right)^{0} &: \ 6b_2 \mu^2 a_0 - 3b_1^2 \mu^2 + 24\mu a_1 b_1 \rho + 96\mu\rho a_2 b_2 + 6\rho^2 a_2 a_0 - 3\rho^2 a_1^2 + a_0^3 \\
&\quad + 6a_0 a_1 b_1 + 6a_0 a_2 b_2 + 3a_1^2 b_2 + 3b_1^2 a_2 = 0.
\end{aligned}
\tag{4.2}
$$

By simultaneously resolving the system of algebraic equations, the following values of unknown constants are obtained.

Set 1:

$$
a_0 = -6\mu\rho, \quad a_2 = -6\mu^2, \quad a_1 = 0, \quad b_1 = 0, \quad b_2 = 0.
$$

Set 2:

$$
a_0 = -12\mu\rho, \quad a_1 = 0, \quad a_2 = -6\mu^2, \quad b_1 = 0, \quad b_2 = -6\rho^2.
$$

Using the values given in Sets 1 and 2 accordingly, the following families of solutions are attained.

Family 1: For $\mu\rho > 0$, the trigonometric function solution to the VP Eq. (1.2) is

$$
u_1(x,t) = -6\mu\rho - \frac{6\mu\rho \left(E\cos[\sqrt{\mu\rho}(x-wt)] + F\sin[\sqrt{\mu\rho}(x-wt)]\right)^2}{\left(E\cos[\sqrt{\mu\rho}(x-wt)] - F\sin[\sqrt{\mu\rho}(x-wt)]\right)^2}.
\tag{4.3}
$$

For $\mu\rho < 0$, then the hyperbolic solution to VP Eq. (1.2) can be written, as:

$$u_2(x,t) = -6\mu\rho - \frac{6|\mu\rho|\Big(E\cosh[2\sqrt{|\mu\rho|}(x-wt)] + E\sinh[2\sqrt{|\mu\rho|}(x-wt)] + F\Big)^2}{\Big(E\cosh[2\sqrt{|\mu\rho|}(x-wt)] + E\sinh[2\sqrt{|\mu\rho|}(x-wt)] - F\Big)^2}.$$

(4.4)

When $\mu \neq 0$ and $\rho = 0$, the rational function solution to VP Eq. (1.2) is retrieved, as:

$$u_3(x,t) = -6\mu\rho - \frac{6\mu^2 E^2}{\mu(E\xi + F)^2}.$$

(4.5)

Family 2: For $\mu\rho > 0$, the trigonometric function solution to the VP Eq. (1.2) is

$$u_4(x,t) = -12\mu\rho - \frac{6\mu\rho\big(E\cos[\sqrt{\mu\rho}(x-wt)] + F\sin[\sqrt{\mu\rho}(x-wt)]\big)^2}{\big(E\cos[\sqrt{\mu\rho}(x-wt)] - F\sin[\sqrt{\mu\rho}(x-wt)]\big)^2}$$
$$- \frac{6\mu\rho\big(E\cos[\sqrt{\mu\rho}(x-wt)] - F\sin[\sqrt{\mu\rho}(x-wt)]\big)^2}{\big(E\cos[\sqrt{\mu\rho}(x-wt)] + F\sin[\sqrt{\mu\rho}(x-wt)]\big)^2}.$$

(4.6)

For $\mu\rho < 0$, then the hyperbolic solution to VP Eq. (1.2) can be written, as:

$$u_5(x,t) = -12\mu\rho - \frac{6|\mu\rho|\Big(E\cosh[2\sqrt{|\mu\rho|}(x-wt)] + E\sinh[2\sqrt{|\mu\rho|}(x-wt) + F\Big)^2}{\Big(E\cosh[2\sqrt{|\mu\rho|}(x-wt)] + E\sinh[2\sqrt{|\mu\rho|}(x-wt)] - F\Big)^2}$$
$$- \frac{6\mu^2\rho^2\Big(E\cosh[2\sqrt{|\mu\rho|}(x-wt)] + E\sinh[2\sqrt{|\mu\rho|}(x-wt)] - F\Big)^2}{|\mu\rho|\Big(E\cosh[2\sqrt{|\mu\rho|}(x-wt)] + E\sinh[2\sqrt{|\mu\rho|}(x-wt)] + F\Big)^2}.$$

(4.7)

When $\mu \neq 0$ and $\rho = 0$, the rational function solution to VP Eq. (1.2) is determined, as:

$$u_6(x,t) = -12\mu\rho - \frac{6\mu^2 E^2}{\mu(E(x-wt) + F)^2} - \frac{6\rho^2\mu(E(x-wt) + F)^2}{E^2}.$$

(4.8)

4.2 Results using modified auxiliary equation method

The proposed method is used to attain the exact solutions of Eq. (1.2) as follows:

Implementation of homogeneous balancing principle on the terms uu'' and u^3 of Eq. (2.3) gives $N = 2$ and Eq. (3.6) becomes

$$U(\xi) = c_0 + c_1(k^f) + d_1(k^f)^{-1} + c_2(k^f)^2 + d_2(k^f)^{-2}.$$

(4.9)

A polynomial equation in k^f is obtained by placing Eq. (4.9) into Eq. (2.3). Balancing all the coefficients of k^f to zero gives the following system.

$(k^f)^1$: $6c_0d_1c_2 + 6c_1c_2d_2 - 3c_1^2\tau\varepsilon - 6c_1\varepsilon^2c_2 + 24c_1\sigma^2d_2 + 27d_1\tau^2c_2$
$+ 3c_1\tau^2c_0 + 18c_2\tau\varepsilon c_0 + 24c_1\tau\sigma d_1 + 54d_1\varepsilon\sigma c_2 + 96\tau\sigma c_2d_2$
$+ 6\varepsilon\sigma c_1c_0 + 3c_0^2c_1 + 3c_1^2d_1 = 0,$

$(k^f)^2$: $6c_1d_1c_2 + 12c_1d_1\sigma^2 + 48c_2d_2\sigma^2 + 12c_2c_0\tau^2 - 6c_2^2\varepsilon^2 + 24c_2\varepsilon\sigma c_0$
$- 3c_1c_2\tau\varepsilon + 57d_1c_2\tau\sigma + 9c_1c_0\tau\sigma + 3c_0c_1^2 + 3c_0^2c_2^2 + 3c_2^2d_2 = 0,$

$(k^f)^3$: $6c_0c_1c_2 + 3c_1^2\tau\sigma + 3c_1c_2\tau^2 + 30d_1\sigma^2c_2 - 6c_2^2\varepsilon\tau + 6c_1c_0\sigma^2$
$+ 30c_2c_0\tau\sigma + 6c_1c_2\varepsilon\sigma + c_1^3 + 3c_2^2d_1 = 0,$

$(k^f)^4$: $18c_2c_0\sigma^2 + 3c_1^2\sigma^2 + 15c_1c_2\tau\sigma + 3c_0c_2^2 + 3c_1^2c_2 = 0,$

$(k^f)^5$: $12c_1c_2\sigma^2 + 6c_2^2\tau\sigma + 3c_1c_2^2 = 0,$

$(k^f)^6$: $6c_2^2\sigma^2 + c_2^3 = 0,$

$(k^f)^0$: $12c_1\tau^2d_1 + 24c_1\varepsilon\sigma d_1 + 48c_2\tau^2d_2 + 3c_1c_0\varepsilon\tau + 3d_1\tau\sigma c_0 + 6c_2c_0\varepsilon^2$
$+ 6d_2c_0\sigma^2 + c_0^3 + 51c_1d_2\sigma\tau + 51d_1c_2\varepsilon\tau + 96c_2d_2\varepsilon\sigma - 3c_1^2\varepsilon^2 - 3d_1^2\sigma^2$
$+ 6c_0c_2d_2 + 6c_0c_1d_1 + 3c_1^2d_2 + 3c_2d_1^2 = 0,$

$(k^f)^{-1}$: $6c_0c_1d_2 + 6d_1c_2d_2 + 27c_1d_2\tau^2 - 3d_1^2\tau\sigma + 24d_1c_2\varepsilon^2 - 6d_1d_2\sigma^2$
$+ 3d_1c_0\tau^2 + 6d_1c_0\varepsilon\sigma + 18d_2c_0\tau\sigma + 24c_1d_1\varepsilon\tau + 54c_1d_2\varepsilon\sigma$
$+ 96c_2d_2\varepsilon\tau + 3c_0^2d_1 + 3c_1d_1^2 = 0,$

$(k^f)^{-2}$: $6c_1d_1d_2 + 12c_1d_1\varepsilon^2 + 48c_2d_2\varepsilon^2 + 12d_2c_0\tau^2 - 6d_2^2\sigma^2 + 9d_1c_0\varepsilon\tau$
$+ 24d_2c_0\varepsilon\sigma + 57c_1d_2\varepsilon\tau - 3d_1d_2\tau\sigma + 3c_0d_1^2 + 3c_0^2d_2 + 3c_2d_2^2 = 0,$

$(k^f)^{-3}$: $6c_0d_1d_2 + 30c_1d_2\varepsilon^2 + 3d_1^2\varepsilon\tau + 3d_1d_2\tau^2 - 6d_2^2\tau\sigma + 6d_1c_0\varepsilon^2$
$+ 30c_0d_2\varepsilon\tau + 6d_1d_2\varepsilon\sigma + d_1^3 + 3c_1d_2^2 = 0,$

$(k^f)^{-4}$: $18d_2c_0\varepsilon^2 + 3d_1^2\varepsilon^2 + 15d_1d_2\varepsilon\tau + 3c_0d_2^2 + 3d_1^2d_2 = 0,$

$(k^f)^{-5}$: $12d_1d_2\varepsilon^2 + 6d_2^2\varepsilon\tau + 3d_1d_2^2 = 0,$

$(k^f)^{-6}$: $6d_2^2\varepsilon^2 + d_2^3 = 0.$

$$(4.10)$$

By simultaneously resolving the system of algebraic equations, the following values of unknown constants are obtained.

Set 1:

$$c_0 = -6\varepsilon\sigma, \quad c_1 = -6\tau\sigma, \quad c_2 = -6\sigma^2, \quad d_1 = 0, \quad d_2 = 0.$$

The following family of solutions of Eq. (1.2) is obtained using the values from Set 1.

Family 1:

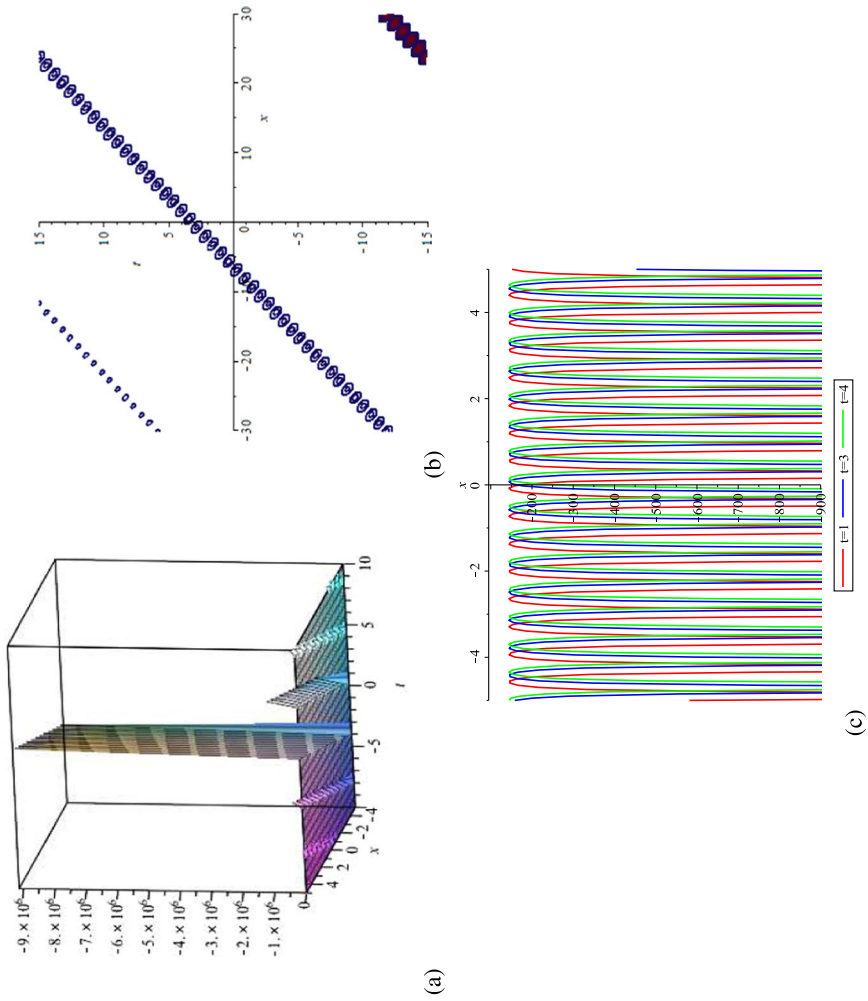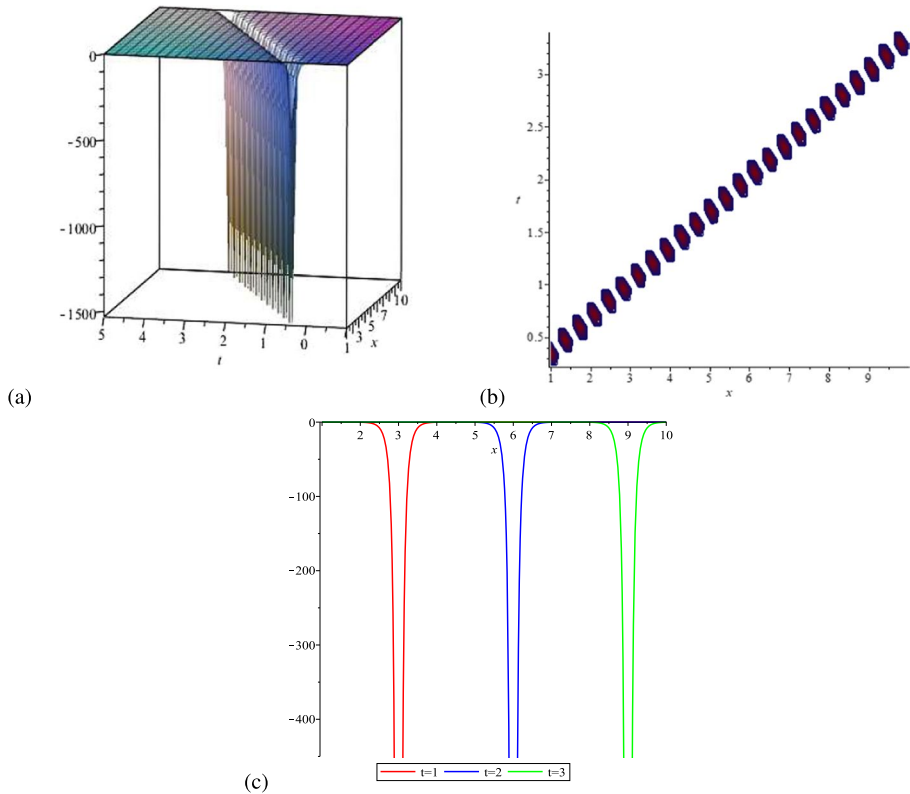When $\tau^2 - 4\varepsilon\sigma < 0, \sigma \neq 0$, then the trigonometric function solution is obtained

(a)

(b)

(c)

Fig. 1 Graphical depiction of $u_1(x, t)$

(a)

(b)

(c)

Fig. 2 Graphical depiction of $u_2(x, t)$

$$u_7(x, t) = -6\varepsilon\sigma - 3\tau\left[-\tau + \sqrt{4\varepsilon\sigma - \tau^2}\tan\left(\frac{1}{2}\sqrt{4\varepsilon\sigma - \tau^2}\xi\right)\right]$$
$$- \frac{3}{2}\left[-\tau + \sqrt{4\varepsilon\sigma - \tau^2}\tan\left(\frac{1}{2}\sqrt{4\varepsilon\sigma - \tau^2}\xi\right)\right]^2. \tag{4.11}$$

When $\tau^2 - 4\varepsilon\sigma > 0, \sigma \neq 0$, then the hyperbolic function solution is obtained

$$u_8(x, t) = -6\varepsilon\sigma + 3\tau\left[\tau + \sqrt{-4\varepsilon\sigma + \tau^2}\tanh\left(\frac{1}{2}\sqrt{-4\varepsilon\sigma + \tau^2}\right)\right]$$
$$- \frac{3}{2}\left[\tau + \sqrt{-4\varepsilon\sigma + \tau^2}\tanh\left(\frac{1}{2}\sqrt{-4\varepsilon\sigma + \tau^2}\xi\right)\right]^2. \tag{4.12}$$

When $\tau^2 - 4\varepsilon\sigma = 0, \sigma \neq 0$, the rational function solution is attained

$$u_9(x, t) = -6\varepsilon\sigma + \frac{3\tau(\tau\xi + 2)}{\xi} - \frac{3}{2}\frac{(\tau\xi + 2)^2}{\xi^2}. \tag{4.13}$$

Set 2:

$$c_0 = -6\varepsilon\sigma, \quad c_1 = 0, \quad c_2 = 0, \quad d_1 = -6\varepsilon\tau, \quad d_2 = -6\varepsilon^2.$$
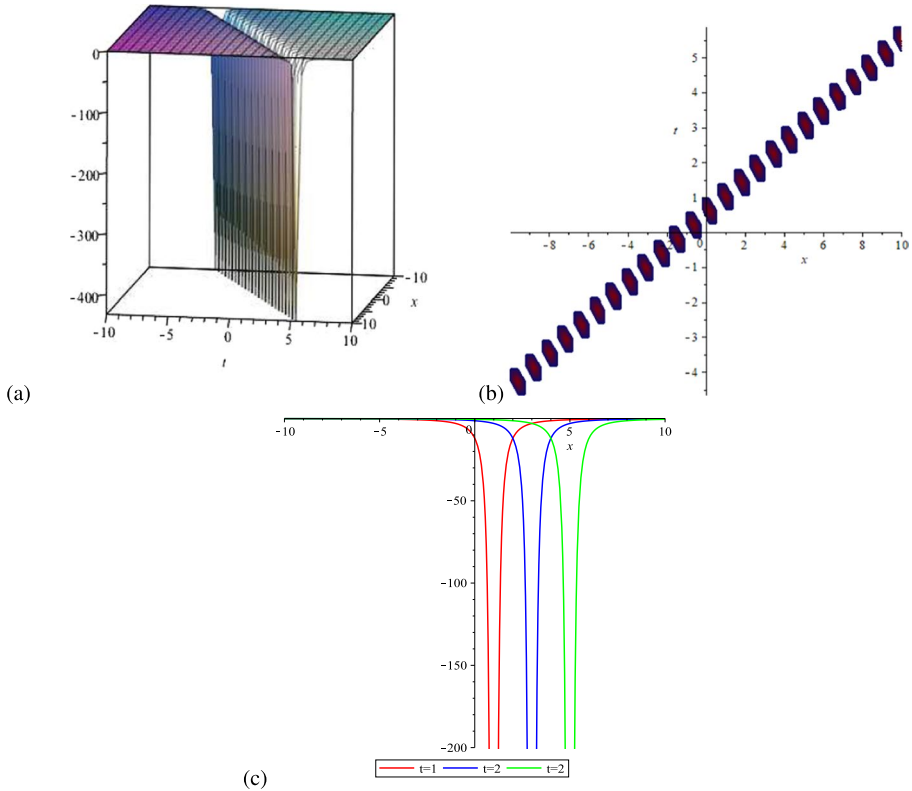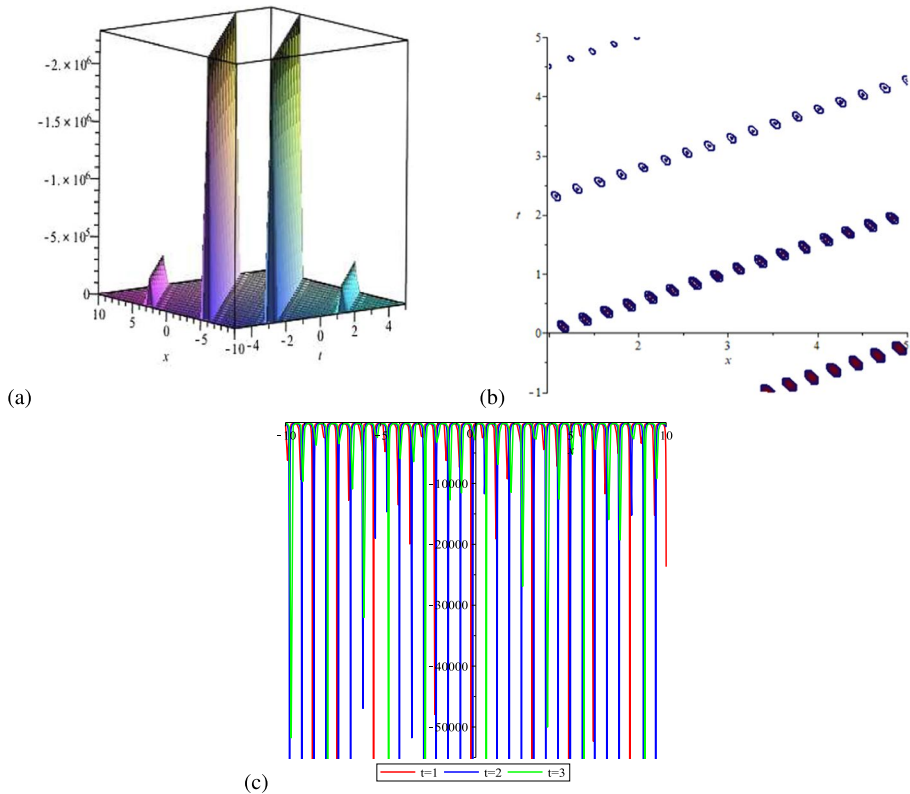
(a)

(b)

(c)

Fig. 3 Graphical depiction of $u_3(x, t)$

The family of solutions of Eq. (1.2) is obtained using the values from Set 2.

Family 2:

When $\tau^2 - 4\varepsilon\sigma < 0, \sigma \neq 0$, then the following trigonometric function solution is obtained

$$u_{10}(x, t) = -6\varepsilon\sigma - \frac{12\varepsilon\tau\sigma}{-\tau + \sqrt{4\varepsilon\sigma - \tau^2}\tan\left(\frac{1}{2}\sqrt{4\varepsilon\sigma - \tau^2}\xi\right)}$$

$$\frac{24\varepsilon^2\sigma^2}{\left[-\tau + \sqrt{4\varepsilon\sigma - \tau^2}\tan\left(\frac{1}{2}\sqrt{4\varepsilon\sigma - \tau^2}\xi\right)\right]^2}.$$

(4.14)

When $\tau^2 - 4\varepsilon\sigma > 0, \sigma \neq 0$, the following hyperbolic function solution is attained

$$u_{11}(x, t) = -6\varepsilon\sigma + \frac{12\varepsilon\tau\sigma}{\tau + \sqrt{-4\varepsilon\sigma + \tau^2}\tanh\left(\frac{1}{2}\sqrt{-4\varepsilon\sigma + \tau^2}\xi\right)}$$

$$- \frac{24\varepsilon^2\sigma^2}{\left[\tau + \sqrt{-4\varepsilon\sigma + \tau^2}\tanh\left(\frac{1}{2}\sqrt{-4\varepsilon\sigma + \tau^2}\xi\right)\right]^2}.$$

(4.15)

(a)

(b)

(c)

t=1 ——— t=2 ——— t=3

Fig. 4 Graphical depiction of $u_4(x, t)$

When $\tau^2 - 4\varepsilon\sigma = 0, \sigma \neq 0$, the rational function solution is yielded

$$u_{12}(x,t) = -6\varepsilon\sigma + \frac{12\varepsilon\tau\sigma\xi}{\tau\xi + 2} - \frac{24\varepsilon^2\sigma^2\xi^2}{(\tau\xi+)^2}. \tag{4.16}$$

5 Results and discussion

In this part of the paper, graphical behavior of the obtained solutions of the VP equation is discussed. In order to illustrate the dynamical behavior of the wave phenomena governed by the VP equation, $3D$ and $2D$ graphical simulations are generated. The parametric values are suitably selected in accordance with the proposed methodologies such that well-defined solution expressions are obtained. The $3D$ surface graph highlights the shape of the traveling wave or soliton whereas the contour graph illustrates the structure of the constructed wave through plots of level curves. The $2D$ line graphs of the solutions are also plotted for increasing values of time to illustrate the progression of the wave along x-axis. In each figure, (a) depicts the $3D$-surface plot, whereas (b) the corresponding $2D$ contour. Part (c) of each figure depicts how the wave travels along x-axis.

(a)

(b)

(c)
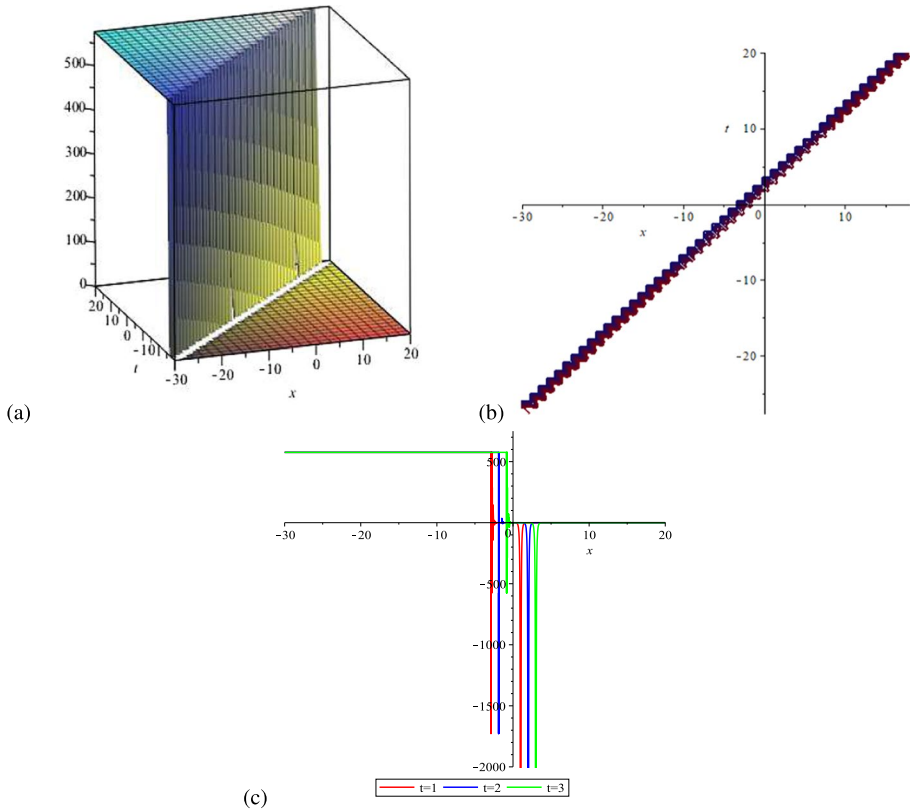
Fig. 5 Graphical depiction of $u_5(x, t)$

The results obtained through the extended $(\frac{G'}{G^2})$-expansion method are graphically expressed in Figs. 1, 2, 3, 4, 5, and 6. Figure 1 shows the behavior of periodic wave solution corresponding to $u_1(x, t)$ for $\mu = 4$, $\rho = 6$, $E = 1$, $F = 1$ and $w = 2$. The corresponding line plot is drawn at $t = 1, 2$ and 3. Figure 2 depicts the dark-singular soliton expressed by $u_2(x, t)$ taking $\mu = 3$, $\rho = -5$, $E = 1$, $F = 1$ and $w = 3$. Figure 3 shows a dark-singular soliton solution $u_3(x, t)$ at $\mu = 2$, $\rho = 0$, $E = 1$, $F = 1$ and $w = 2$. The line graph is plotted at $t = 1, 2$ and 3. Figure 4 shows the behavior of periodic traveling wave given by $u_4(x, t)$ for the parametric values $\mu = 2$, $\rho = 3$, $E = 1$, $F = 1$ and $w = 2$. Figure 5 shows the behavior of kink soliton corresponding to $u_5(x, t)$. The parameters are assigned the values $\mu = 2$, $\rho = -3$, $E = 1$, $F = 1$ and $w = 1$. Figure 6 shows the behavior of dark-singular soliton corresponding to the solution $u_6(x, t)$ at $\mu = 2$, $\rho = -3$, $E = 1$, $F = 1$ and $w = 1$.

The traveling wave solutions obtained through the modifies auxiliary equation method are illustrated in Figs. 7, 8, 9, 10, 11, and 12. Figure 7 shows the periodic wave expressed by $u_7(x, t)$ for $\varepsilon = 1$, $\tau = 1$, $\sigma = 2$ and $w = 1$. Figure 8 depicts a bright soliton for $u_8(x, t)$ taking $\varepsilon = -1$, $\tau = 4$, $\sigma = 3$ and $w = 1$. Figure 9 shows the wave profile of $u_9(x, t)$ which can be identified as dark-singular soliton. The graphs are plotted for parametric values $\varepsilon = 1$, $\tau = 2$, $\sigma = 1$ and $w = 1$. Figure 10 shows the periodic traveling wave behavior of $u_{10}(x, t)$ at $\varepsilon = 1$, $\tau = 2$, $\sigma = 3$ and $w = 2$. Figure 11 shows the construction of dark-singular

(a)

(b)

(c)
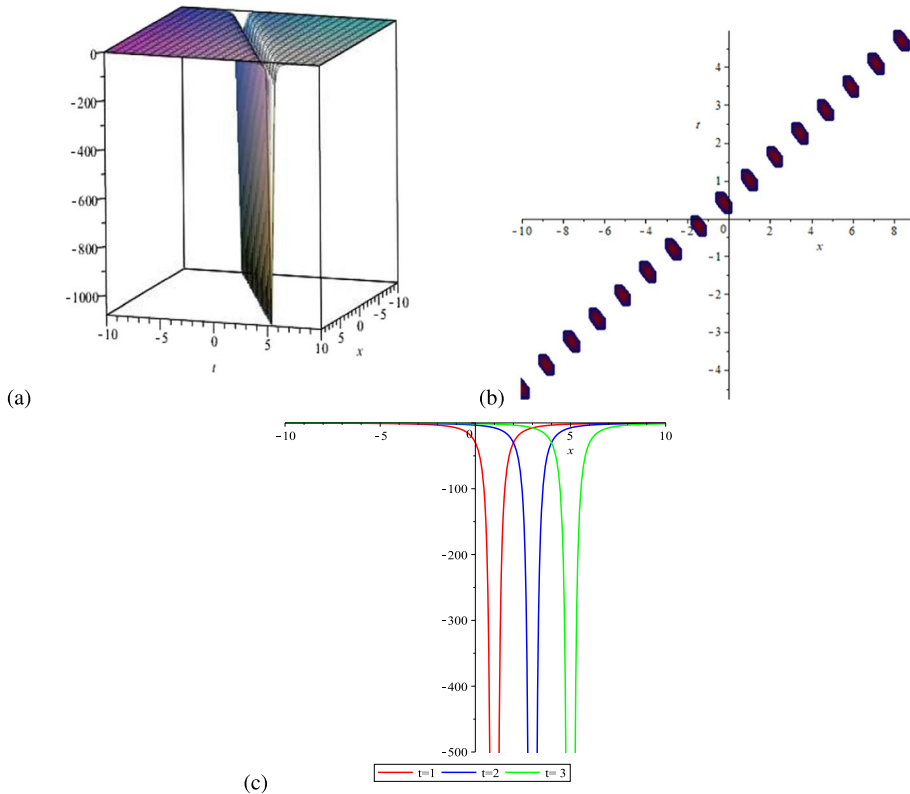
Fig. 6 Graphical depiction of $u_6(x, t)$

soliton with $u_{11}(x, t)$ for $\varepsilon = -1$, $\tau = 4$, $\sigma = 3$ and $w = 5$. Figure 12 also shows a dark-singular soliton $u_{12}(x, t)$ for $\varepsilon = 1$, $\tau = 2$, $\sigma = 1$ and $w = 1$.

The graphs show the construction of many dark-singular solitons and bright solitons as well as periodic traveling wave. The bright soliton is characterized by a local increase in the wave amplitude. Bright solitons are significant due to their ability to travel over long distances. Due to their ability to reflect incredibly concentrated and localized events, singular solitons are valuable tools for studying extreme behavior in physical systems and are therefore essential in scientific research. They act as benchmarks in the subject of nonlinear dynamics, helping to comprehend and describe intricate, nonlinear phenomena that defy conventional linear models. Singular solitons are related to shock waves and offer important insights into how shock events interact and propagate across different kinds of materials. Also, visually displaying spatial patterns, supporting data interpretation, enabling clear communication, simplifying difficult material, and boosting analytical depth, contour graphics improve the quality of studies. Their capacity to draw attention to abnormalities and provide backing for predictive modeling adds even more to the study's overall resilience.
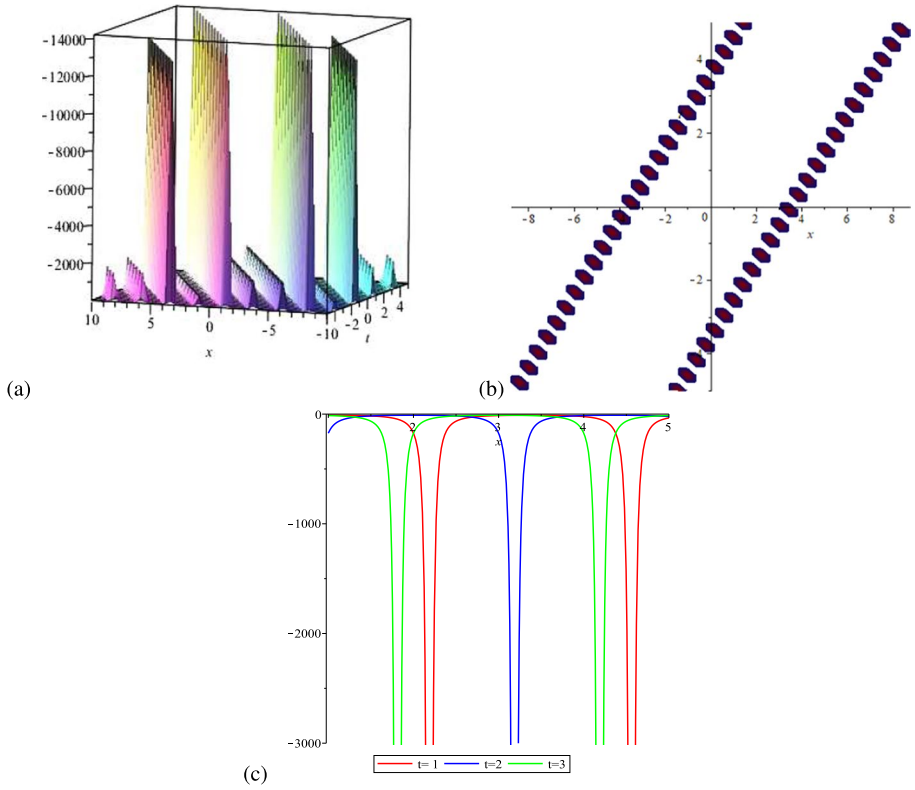
(a)
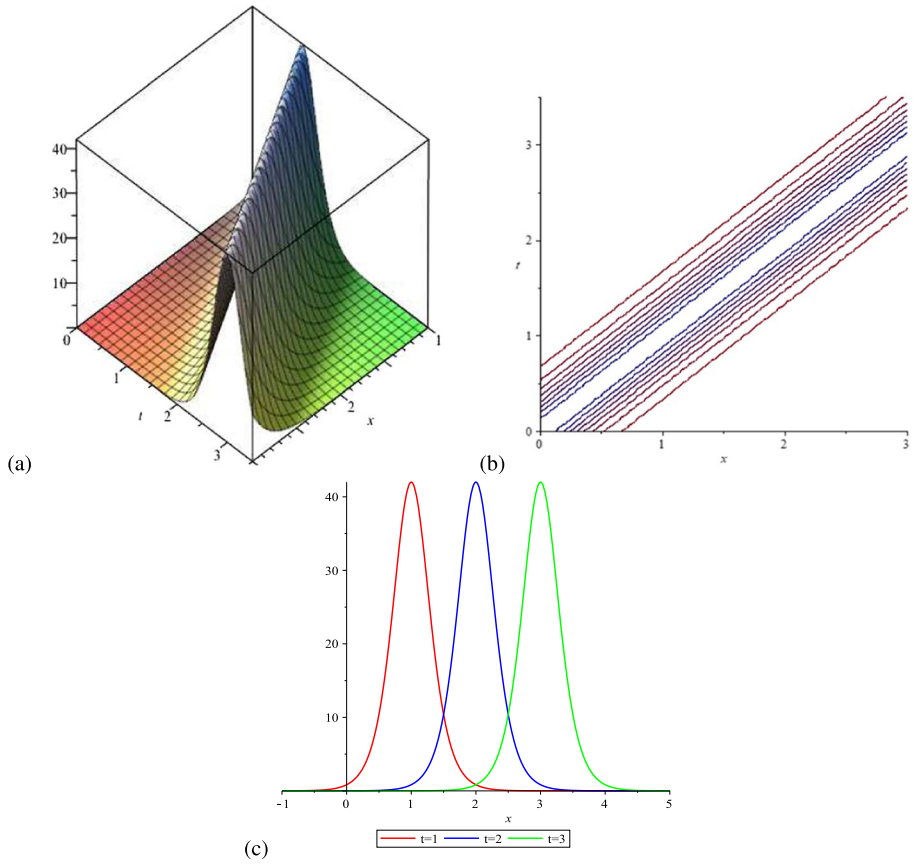
(b)

(c)

Fig. 7 Graphical depiction of $u_7(x, t)$

(a)

(b)

(c)

Fig. 8 Graphical depiction of $u_8(x, t)$

(a)

(b)

(c)
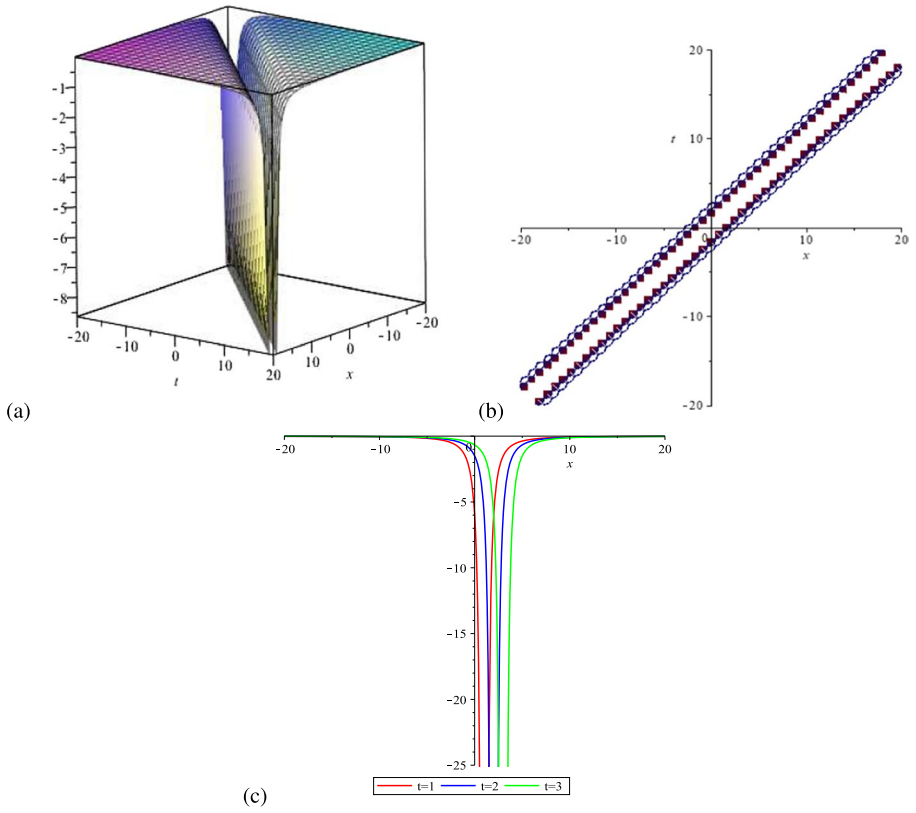
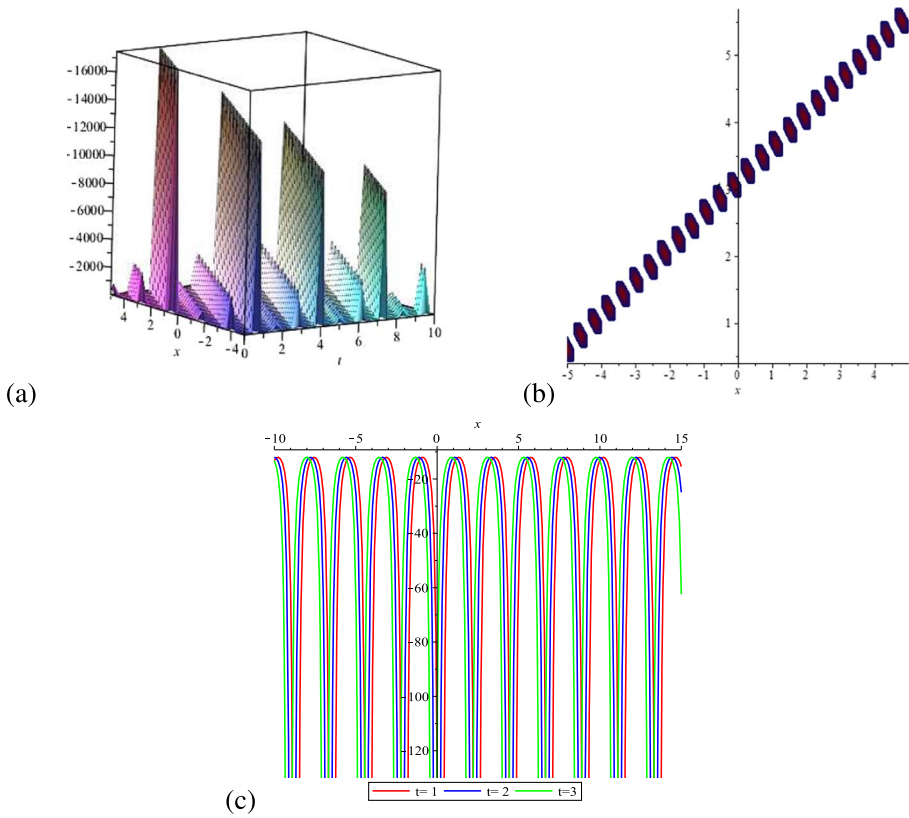**Fig. 9** Graphical depiction of $u_9(x, t)$

(a)

(b)

(c)

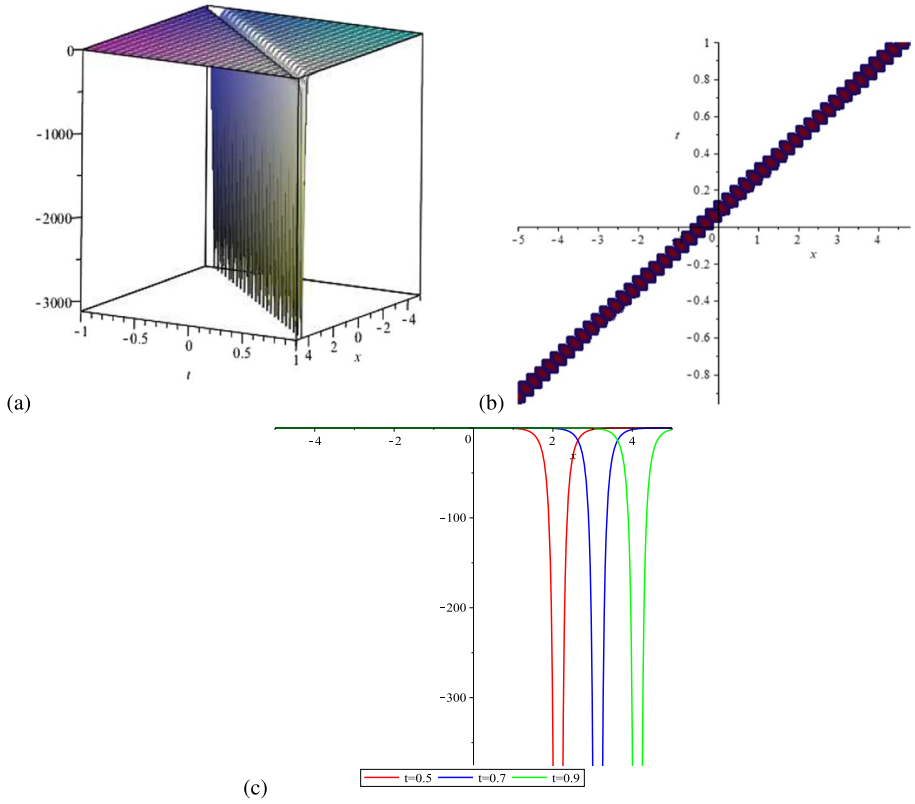Fig. 10 Graphical depiction of $u_{10}(x, t)$

(a)

(b)

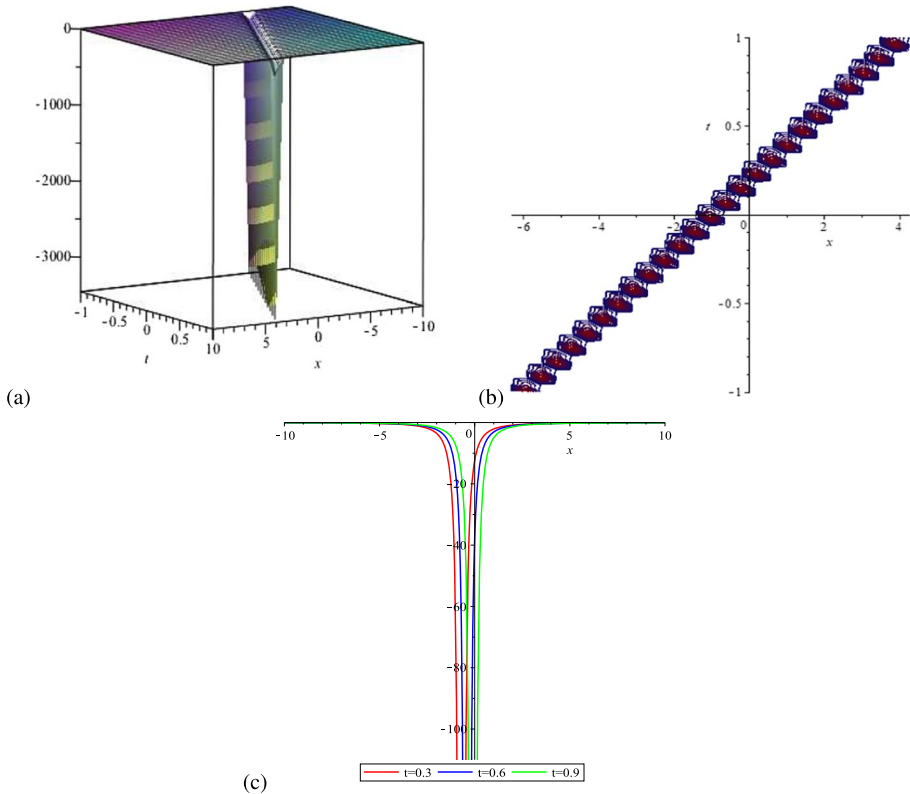(c)

Fig. 11 Graphical depiction of $u_{11}(x, t)$

Fig. 12 Graphical depiction of $u_{12}(x, t)$

The comparison of the reported results with the previous studies in the literature show that the presented results not only confirm some of the previously reported wave behavior for the VP equation but also provide more detailed insight into the traveling waves described by the afore-mentioned equation. The authors of Yel and Aktürk (2020); Khater et al. (2022) reported bright soliton solution and periodic wave solution but failed to discuss dark-singular solitons. A few traveling wave solutions of VP equation were reported in Roshid et al. (2014) including bright soliton and periodic wave but no dark-singular soliton was reported. Only the bright soliton was constructed in Ibrahim et al. (2019). The VP equation was also examined in Kumar and Mann (2022) but no bright soliton was reported. These comparisons and observations confirm the novelty and significance of the results presented in the current manuscript.

6 Conclusion

In this paper, we studied the dynamics of VP equation using the modified auxiliary equation and extended $\frac{G'}{G^2}$-expansion techniques. Using these techniques, we were able to find traveling wave solutions of the considered equation in the form of

rational, hyperbolic, and trigonometric functions. We observed dynamical features corresponding to the suggested solutions, such as bright and dark singular solitons and periodic solitary waves, by performing numerical simulations with properly selected parameters. Our investigation revealed that the suggested techniques were simple, dependable, and effective. This method's adaptability suggested that it may be used in the future to solve other nonlinear partial differential equations analytically. Through numerical simulations, this study offered a thorough investigation of the solitary wave dynamics for the VP equation, providing analytical answers and insightful information. The importance and novelty of the obtained results was established by comparing the obtained results with the previous studies. Moreover, the potential physical applications were also described. In future, the VP equation will be studied using fractional order derivative to gain further interesting and useful results.

Data availability and materials There is no data set need to be accessed.

Declarations

Conflict of interest The authors declare that they have no Conflict of interest to report regarding the present study.

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