



Unveiling single soliton solutions for the (3+1)-dimensional negative order KdV–CBS equation in a long wave propagation

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Received: 18 September 2023 / Accepted: 1 January 2024 / Published online: 1 February 2024
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Abstract

In this study, we explore a captivating (3+1)-dimensional negative-order Korteweg–de Vries Calogero–Bogoyavlenskii–Schiff equation, which combines elements of the Korteweg–de Vries equation and the Calogero–Bogoyavlenskii–Schiff equation. Our research investigates how this model characterises long-wave interactions and its relevance in mathematics, physics, and engineering. We employ unified and singular manifold methods to obtain precise travelling wave solutions expressed in various functional forms. By using Maple and Mathematica software to extract valid solutions, including kink-like soliton, singular periodic wave solution, anti-kink solutions, and singular solitons. These methodologies have shown impressive efficiency in solving complex nonlinear equations, offering precise solutions, and streamlining mathematical processes through transformations. This leads to quicker and more accurate outcomes in diverse scientific and engineering applications. Our findings underscore the model’s superiority over existing methods and its importance in comprehending applied mathematical processes, as demonstrated through 3-D and 2-D graphical representations.

Keywords Soliton solutions · (3+1) negative order KdV–CBS equation · Singular manifold method · Unified method

1 Introduction

The integrable model is characterised by non-linear partial differential equations (NLPDEs). Our universe, to a first approximation, is described by linear equations because the response to a small perturbation is generally proportional to the

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perturbation. The theory of three linear partial differential equations, the heat equation, Laplace's equation, and the wave equation, is at the heart of standard courses in mathematical physics. Their exceptional universality explains their fundamental role. However, even if the non-linear adjustments are small, they might have a large impact. It was recognized throughout the previous few decades that "a proper treatment of non-linear effects results in integrable equations with the same high degree of universality as equations of linear mathematical physics." It is more challenging to solve NLPDEs than linear equations, which are crucial in many branches of science. In particular, the study of solutions to linear and NLPDEs plays a magnificent role in solving problems in applied physics. Particularly, explicit NLPDE solutions can be used to describe the wide range of phenomena and their theoretical foundations. They can also serve as inspiration for the explanation and comparison of numerical solutions. It has been noted by several scholars that finding traveling wave solutions to NLPDEs is challenging and that strong and effective strategies are required (Choi et al. 2021). The hierarchy, which is created by each of the non-linear integrable equations, is an indefinite chain of compatible "higher" equations. Furthermore, all known integrable equations (KdV, NLS, Toda chain, KP, and many others) are contemporaries. In general, all of them are contained (as limiting and specific instances, reductions, and equivalent forms derived by changing variables) in a single universal equation (Zabrodin 2018). Many integrable templates have been established in the setting of (3+1)- and (2+1)-dimension equations, while studies on the development of higher-dimensional integrable equations have attracted greater interest recently. The study of integrable models is an exciting area of research since it describes the true aspects and discloses the scientific description of non-linearity in science disciplines. "These non-linear equations include a variety of hidden symmetries that make it possible to build vast classes of precise solutions, such as soliton-type solutions." One essential aspect of totally integrable equations is the idea of many soliton solutions. The fact that soliton theory has empirical applications in scientific domains such as ocean waves, communication technology, non-linear optics, plasma physics, hydrodynamics, quantum mechanics, and many others motivates these investigations (Wazwaz 2021). The soliton theory is greatly influenced by an integrable Korteweg–de Vries (KdV) equation.

All integrable equations have soliton solutions, suggesting a developing non-linear phenomenon in nature. Soliton theory is used in a variety of scientific fields, including telecommunications, transportation phenomena, and many more (Gandarias and Raza 2022). NLPDEs are accustomed to simulating a wide range of chemical, biological, and physical events, and they play an important role in non-linear research (El-Kalaawy and Ibrahim 2012). It gives proper evidence and a greater knowledge of the physical aspects of the problem, leading to further applications (Al-Amr 2014). Many tracks have been built in recent years for physical concerns in order to find accurate solutions for NLPDEs using contemporary computer technology.

There are several effective approaches, such as the Jacobi elliptic function method (Ma et al. 2018), the $(\frac{G'}{G})$ -expansion method (Bilal et al. 2021), the Riccati–Bernoulli Sub-ODE technique (YanG et al. 2015), the extended modified auxiliary equation mapping method (Seadawy et al. 2019), EMRE method (Iqbal et al. 2023), the auxiliary equation methodology (Ma et al. 2009), the modified rational expansion method (Seadawy et al. 2021), the extended direct algebraic method (Faridi et al. 2023a, b), the ϕ^6 -model expansion method (Asjad et al. 2023), the generalized projective Riccati equation method (Vivas-Cortez et al. 2023), the extended sinh-Gordon equation expansion method (Akram et al. 2022) and Riccati mapping method (Nasreen et al. 2019) etc.

The generalized form of the Korteweg–de Vries Calogero–Bogoyavlenskii–Schiff (KdV–CBS) equation with time-dependent variable coefficients of negative order is examined in this article. The beauty of this model is that it deals with the interactions of long wave propagations and has many applications in fluid and quantum mechanics, which is a mixture of two separate equations such as the KdV and CBS equations. Many writers have examined the solutions to this model in the literature using various methodologies. For example, the conservation laws and traveling wave solutions for a negative-order (3+1)-dimensional KdV–CBS equation (Gandarias and Raza 2022), the Exact solutions for (2+1)-dimensional KdV–Calogero–Bogoyavlenskii–Schiff Equation via Symbolic Computation (Li and Chaolu 2020), two new Painlevé integrable KdV–CBS equation and new negative-order KdV–CBS equation (Wazwaz 2021), Nonlocal symmetry, CRE solvability and soliton-cnoidal solutions of the (2+1)-dimensional modified KdV–Calogero–Bogoyavlenskii–Schiff equation (Wang and Wang 2017), Structures of interaction between lump, breather, rogue and periodic wave solutions for new (3+1)-dimensional negative order KdV–CBS model (Raza et al. 2022), Nonlocal symmetry, CRE solvability and soliton-cnoidal solutions of the (2+1)-dimensional modified KdV–Calogero–Bogoyavlenskii–Schiff equation (Wazwaz 2017), Painlevé integrability and analytical solutions of variable coefficients negative order KdV–Calogero–Bogoyavlenskii–Schiff equation using auto-Bäcklund transformation (Singh and Ray 2023). Several solutions have been derived for the Heisenberg ferromagnetic spin chain equation. A multitude of exact traveling wave solutions have been discovered for a modified generalized Vakhnenko equation using the auxiliary equation method. Rich soliton structures have been observed in the Kraenkel–Manna–Merle (KMM) system within ferromagnetic materials. Periodic solutions and solitons have been identified in two complex short pulse (CSP) equations within optical fiber. Unmoving wave solutions and a unique fractal soliton have been found for the (2+1)-dimensional Broer–Kaup equations with variable coefficients. Distinct loop-like solitons have been identified for a generalized Vakhnenko equation stemming from high-frequency wave motion in a relaxing medium and so on.

The literature review indicates that the convergence of long wave propagation via a unified method and singular manifold approach has not received much attention in terms of soliton solutions. Rogue and breather waves are confined waves that occur in an unsteady environment, whereas solitons are stable waves. The rational soliton solution, polynomial solution, and singular manifold approach for long wave propagation are the main innovations of the current work that have been validated. One intention of this article is to show that the negative-order KdV–CBS model yields a wide range of exact solutions for various physical aspects. The singular manifold method and unified method are significant in the realm of mathematical research because they offer powerful and versatile tools for uncovering complex solutions to nonlinear equations. These methods provide a framework to systematically analyse and understand a wide range of phenomena, making them valuable not only for studying specific equations but also for advancing our understanding of various mathematical and physical systems with nonlinear behaviors. Their ability to reveal multiple solution types contributes to the richness of mathematical modelling and problem-solving in diverse fields.

This dissertation aims to find the soliton solutions that the negative-order KdV–CBS equation admits in (3+1) dimensions. First of all, in Sect. 2, we constructed the model and moved to wave reduction. In Sect. 3, a recapitulation of the methods is provided. For the negative-order KdV–CBS equation in (3+1)-dimensions, the line traveling waves $v = V(x + dy + ez - ft)$, where d , f , and e are random values and is a traveling wave

transformation, are studied. Section 4 of this series discusses the use of the singular manifold method and the unified method to derive soliton solutions from the negative-order KdV–CBS equation. These are effective mathematical methods for determining exact solutions to NLPDEs. Three-dimensional soliton solutions are developed. Research findings are illustrated in Sect. 5 of the paper. The Sect. 6 is about results, discussion, and the novelty of this work. In Sect. 7, we addressed the final comments.

2 Governing model

Integrable equations are a popular topic of study because they depict real-world aspects and demonstrate the scientific basis of non-linearity in scientific domains. One essential feature of completely integrable equations is the doctrine of soliton solutions. Recent work on the integrable KdV equation and the integrable CBS equation has resulted in the creation of a (2+1)-dimensional equation of the form. The integrable KdV equation is imperative in solitary wave theory. The non-linear CBS explains the interplay between Riemann waves propagating along the y-axis and long waves propagating along the x-axis. The following is a description of the (2+1)-dimensional KdV–CBS equation:

$$4u_t + \lambda(4uu_y + 2u_x \partial_x^{-1}(u_y) + u_{xy}) + \chi(6uu_x + u_{xx}) = 0, \quad (2.1)$$

The purpose of this study is to modify KdV–CBS Eq. (2.1) to a negative order (3+1)-dimensional KdV–CBS equation. Firstly, we convert Eq. (2.1) into a (3+1)-dimensional KdV–CBS equation given as

$$4u_t + \lambda(4uu_y + 2u_x \partial_x^{-1}(u_y) + u_{xy}) + \varepsilon(4uu_z + 2u_x \partial_x^{-1}(u_z) + u_{xz}) + \chi(6uu_x + u_{xx}) + pu_x + qu_y + ru_z = 0, \quad (2.2)$$

where $\lambda, \varepsilon, \chi, p, q,$ and r are random constants. This equation is formed by combining the z-component of the CBS equation ($4uu_z + 2u_x \partial_x^{-1}(u_z) + u_{xz}$) with three linear components, namely $u_x, u_y,$ and u_z . Equation (2.2) can be simplified to the other nonlinear integrable equations with distinctively important physical characteristics.

- Eq. (2.2) is simplified to a (2+1)-dimensional Calogero–Bogoyavlenskii–Schiff (CBS) equation for $\lambda \neq 0, \varepsilon = \chi = p = q = r = 0$.
- Furthermore, Eq. (2.2) is simplified to a (3+1)-dimensional CBS equation for $\lambda \neq 0, \varepsilon \neq 0, \chi = p = q = r = 0$.
- For $\chi \neq 0, \lambda = \varepsilon = p = q = r = 0$, however, Eq. (2.2) provides the conventional KdV equation.

We call attention to the fact that the KdV recursion operator can be used to determine both the KdV equation and the CBS equation.

$$\xi = \partial_x^2 + 4v + 2v_x \partial_x^{-1}. \quad (2.3)$$

To put it another way, we arrive at the KdV and CBS formulae.

$$v_t + \xi(v_x) = \partial_x^2(v_x) + 4vv_x + 2v_x \partial_x^{-1}(v_x) = v_t + v_{xxx} + 6vv_x = 0, \quad (2.4)$$

and

$$v_t + \xi(v_y) = \partial_x^2(v_y) + 4vv_y + 2v_x\partial_x^{-1}(v_y) = v_t + v_{xxy} + 2v_x\partial_x^{-1}(v_y) + 4vv_y = 0, \quad (2.5)$$

respectively.

Verosky (1991) extended Olver's work in Olver (1977) by enabling the negative direction to be used to generate a series of equations with rising negative orders. According to Verosky (1991), the series of evolution equations is defined as:

$$u_t + \xi(u_x) = 0, \quad (2.6)$$

and

$$u_t + \xi(u_y) = 0, \quad (2.7)$$

the negative order hierarchy may be utilized to construct the standard KdV and CBS equations.

$$u_t + \xi^{-1}(u_x) = 0, \quad (2.8)$$

and

$$u_t + \xi^{-1}(u_y) = 0, \quad (2.9)$$

or an alternative

$$\xi(u_t) + u_x = 0, \quad (2.10)$$

and

$$\xi(u_t) + u_y = 0, \quad (2.11)$$

the power of ξ moves in the opposite direction for the KdV and CBS equations, respectively. We use the negative order hierarchy to obtain the integrable negative-order KdV (nKdV) equation and the integrable CBS (nCBS) equation such as Eqs. (2.10) and (2.11):

$$u_{xxt} + 4uu_t + 2u_x\partial_x^{-1}(u_t) + u_x = 0, \quad (2.12)$$

and

$$u_{xxt} + 4uu_t + 2u_x\partial_x^{-1}(u_t) + u_y = 0, \quad (2.13)$$

or an alternate

$$u_{xxxt} + 4u_xu_{xt} + 2u_{xx}u_t + u_{xx} = 0, \quad (2.14)$$

and

$$u_{xxxt} + 4u_xu_{xt} + 2u_{xx}u_t + u_{xy} = 0, \quad (2.15)$$

here $u = V_x$ is used. In recent work Olver (1977), we analysed that Eqs.(2.14) and (2.15) are integrable equations that proceed the Painlevé integrable test. In this current study, we pursue the prior exploration of combining the CBS and KdV equations to organize a new combination of the negative-order CBS Eq. (2.15) and the negative-order KdV Eq. (2.14); thus, we establish a new (3+1)-dimensional model.

$$V_{xt} + V_{xxxxy} + 4V_x V_{xy} + 2V_{xx} V_y + \kappa V_{xx} + \tau V_{xy} + \vartheta V_{xz} = 0. \tag{2.16}$$

This is known as the negative-order KdV–CBS equation (nKdV-nCBS). For $\tau = 0$, and $\vartheta = 0$, it is obvious that Eq. (2.16) will be reduced to the negative-order KdV equation (2.14). If $\kappa = 0$ and $\tau = 0$, however, Eq. (2.16) is reduced is to the negative-order CBS equation (2.15). where κ, τ and ϑ are unknown coefficients. The most well-known application of reduction to ODE. A line-travelling wave transformation takes the following form:

$$V(x, y, z, t) = V(\Phi), \quad \Phi = x + dy + ez - ft, \tag{2.17}$$

where d, e and f are unknown constants and Φ is the travelling wave variable. Plugging Eq. (2.17) into Eq. (2.16) gives a non-linear fourth order ODE.

$$dV'''' + (6dV' + \kappa + d\tau + e\vartheta - f)V'' = 0. \tag{2.18}$$

This equation will be investigated in detail in the following section to find new solutions for the governing model. The first goal of this study is to use the singular manifold technique to investigate the soliton solution of (nKdV-nCBS) Eq. (2.16). We intend to use the unified method to derive various soliton solutions for the considered equation.

3 An analysis of the methods

We will examine strategies for this model in this section. First of all, we will apply the singular manifold method. Secondly, we will calculate the polynomial and rational function solutions through the unified method.

3.1 Recapitulation for the singular manifold method

The singular manifold method (SMM) is introduced briefly in this subsection.

Step1. The singular manifold scheme (Estévez and Prada 2005) was used to provide fascinating exact solutions for the given model. The SMM is proposed, which is based on the Painlevé test developed by Weiss (1983) and the solution of Eq. (2.18) is presented as

$$V(\Phi) = \sum_{m=0}^{\infty} v_m \eta(\Phi)^{m-r}, \tag{3.1}$$

where η is an eigen function used to weight the poles and r is the anticipated order of the poles based on dominant behaviour analysis. This series will end at $m = r$ since there is no singularity at $m > r$.

Step2. We may identify a system of algebraic equations by comparing the coefficients of different powers of η . By trying to tackle this system of equations, we get a Schwarzian derivative as

$$\left(\frac{\eta''}{\eta'}\right)' - \frac{1}{2}\left(\frac{\eta''}{\eta'}\right)^2 = [\eta, \Phi]. \tag{3.2}$$

Step3. Solving Eq. (3.2) for $\eta(\Phi)$ and inserted into Eq. (3.1). This results in a solution containing trigonometric and hyperbolic functions.

3.2 Recapitulation for the unified method

The obtained solutions of Eq. (2.18) using the unified method (UM) are classified as rational function solutions and polynomial function solutions. This subsection includes a brief overview of UM (Raza and Rafiq 2021).

Step1. The Polynomial Function Solution

The UM recommends that to achieve the polynomial function solution of Eq. (2.18),

$$\begin{aligned} V(\Phi) &= \sum_{k=0}^N a_k \eta^k(\Phi), \\ (\eta'(\Phi))^\Omega &= \sum_{k=0}^{\Omega N} b_k \eta^k(\Phi), \quad \Omega = 1, 2. \end{aligned} \quad (3.3)$$

In this context, the unknown constants to be found are a_k and b_k and the solution generated by Eq. (3.3) validates Eq. (2.18). It is crucial to note that the numeric values of N and Ω in Eq. (2.18) must be found by the homogeneous balancing principle (Gawad et al. 2013). Similarly, the indeterminate coefficients in Eq. (3.3) must be specifically identified using the criterion of consistency constraint. In UM, the results of Eq. (3.3) for elementary or elliptic solutions for $\Omega = 1$ or $\Omega = 2$, respectively.

Step2. The Rational Function Solution

The key idea behind this strategy is to claim that Eq. (2.18) has a rational solution as,

$$\begin{aligned} V(\Phi) &= \frac{\sum_{k=0}^n h_k \eta^k(\Phi)}{\sum_{k=0}^m I_k \eta^k(\Phi)}, \quad n \geq m, \\ (\eta'(\Phi))^\omega &= \sum_{k=0}^{\omega\alpha} j_k \eta^k(\Phi). \quad \omega = 1, 2. \end{aligned} \quad (3.4)$$

In this context, the unknown constants to be found are h_k , I_k and j_k and the solution given by Eq. (3.4) validates the Eq. (2.18). It is crucial to note that the numeric values of α and ω in Eq. (2.18) must be found by homogeneous balancing between linear and non linear terms (Gawad et al. 2013). Similarly, the indeterminate coefficients in Eq. (3.4) must be specifically identified using the criterion of consistency constraint. The UM approach solves Eq. (3.4) for elementary or elliptic solutions for $\omega = 1$ or $\omega = 2$, respectively. The following procedures must be followed to obtain results for Eq. (2.18) across the UM in theoretical frameworks such as polynomial function or rational function solutions:

- System of algebraic equations using any symbolic computer software are solved.
- The auxiliary equations are computed.
- Furthermore, the precise solution of the nonlinear equation from Eq. (3.3) (or Eq. (3.4)) is obtained.

4 Construction the wave solutions of nKdV-nCBS equation

Many strategies for solving NLPDEs have been developed during the last several decades. These strategies are distinguished by their validity in tackling broad classes of NLPDE. In this section, we extracted the soliton solutions for Eq. (2.16) by using the singular manifold method and the unified method.

4.1 Extraction of solution using the singular manifold method

The singular manifold method (Estévez and Prada 2005) is used to determine the soliton solution of the (3+1)-dimensional nKdV-nCBS model. The solution of Eq. (2.18) is represented in series form as

$$V(\Phi) = \sum_{m=0}^{\infty} v_m \eta(\Phi)^{m-r}, \tag{4.1}$$

where r predicts the order of the poles based on dominating behaviour analysis and η is an eigen function that calculates the pole weights. This series will be terminated at the conclusion, i.e., $m = r$. There is no singularity at $m > r$ and the dominating behaviour results between V'''' and $V'V''$ in Eq. (2.18) gives $r = 1$. Therefore, Eq. (4.1) may be expressed as

$$V(\Phi) = \frac{v_0(\Phi)}{\eta} + v_1(\Phi). \tag{4.2}$$

This is the shortest Bäcklund sequence. Swapping Eq. (4.2) in Eq. (2.18) and associating the different powers of η equal zero yields the following set of equations:

$$\eta^{-5} : 2\eta' - v_0 = 0, \tag{4.3}$$

$$\eta^{-4} : 4v_0'(v_0 - \eta') + \eta'' v_0 \left(\frac{v_0}{\eta'} - 6\right) = 0, \tag{4.4}$$

$$\begin{aligned} \eta^{-3} : & 8dv_0\eta'\eta''' + 6dv_0(\eta'')^2 + 24dv_0'\eta''(\eta' - \frac{1}{4}v_0) + 12\eta' \left(dv_0''(\eta' - \frac{1}{2}v_0) + \eta'v_0(dv_1' \right. \\ & \left. + \frac{1}{6}(-f + \kappa + d\tau + e\vartheta)) - d(v_0')^2 \right) = 0, \end{aligned} \tag{4.5}$$

$$\begin{aligned} \eta^{-2} : & -dv_0\eta'''' - 4dv_0'''\eta' - 4dv_0'\eta'''' + (-6dv_0'' - 6v_0(dv_1' + \frac{1}{6}(-f + \kappa + d\tau + e\vartheta)))\eta'' \\ & + 6dv_0'v_0'' - 12\eta' \left(\frac{1}{2}dv_1''v_0 + v_0'(dv_1' + \frac{1}{6}(-f + \kappa + d\tau + e\vartheta)) \right) = 0, \end{aligned} \tag{4.6}$$

$$\eta^{-1} : dv_0'''' + (6dv_1' + d\tau + e\vartheta - f + \kappa)v_0'' + 6dv_0'v_1'' = 0, \tag{4.7}$$

$$\eta^0 : dv_1''' + v_1''(6dv_1' + d\tau + e\vartheta - f + \kappa) = 0. \quad (4.8)$$

Addressing Eq. (4.3) to Eq. (4.8) using maple generates

$$v_0 = 2\eta', v_1 = \int \left(-\frac{2}{3} \frac{\eta'''}{\eta'} + \frac{1}{2} \left(\frac{\eta''}{\eta'} \right)^2 - \frac{1}{6} \frac{d\tau + e\vartheta - f + \kappa}{d} \right) d\Phi + K_1. \quad (4.9)$$

Substituting Eq. (4.9) into Eq. (4.2), we get

$$V(\Phi) = \frac{2\eta'}{\eta} + v_1. \quad (4.10)$$

This is the Bäcklund transformation for the Eq. (2.16). The following result is obtained by plugging Eq. (4.9) into Eq. (4.7)

$$\left(\frac{\eta''}{\eta'} \right)' - \frac{1}{2} \left(\frac{\eta''}{\eta'} \right)^2 = C_1. \quad (4.11)$$

That is the Schwarzian derivative of the eigen function η . Upon integrating three times, we obtain

$$\eta(\Phi) = \frac{1}{\sqrt{2}} \left(-\sqrt{2} + 4 \tan\left(\frac{1}{\sqrt{2}}(\Phi + 1)\right) \right), \quad (4.12)$$

We get the feasible solutions of the Eq. (2.16) presented below by back substituting in Eq. (4.12) and noticing the relationship between $V(\Phi)$ and $v(x, y, z, t)$ indicated in Eq. (2.17).

$$V_1(\Phi) = \frac{2\eta'}{\eta} + v_1, \quad (4.13)$$

where $\Phi = x + dy + ez - ft$. The above obtained solution is the combined soliton-like solution.

4.2 Extraction of solutions using the unified method

The unified method provides the solution of Eq. (2.16) in two ways: polynomial function solutions and rational function solutions (Raza and Rafiq 2021).

4.2.1 The polynomial form of solutions

To obtain the polynomial form of solutions of the nKdV-nCBS, we consider

$$V(\Phi) = \sum_{k=0}^N a_k \eta^k(\Phi), \quad (4.14)$$

$$(\eta'(\Phi))^\Omega = \sum_{k=0}^{\Omega N} b_k \eta^k(\Phi), \quad \Omega = 1, 2.$$

Where a_k and b_k are independent variables. In Eq. (2.18), we used homogeneous balancing between V'''' and $V''V'$. We get $N = (k - 1)$; $k = 2$.

When $k = 2$ and $\Omega = 1$ or $\Omega = 2$, we restrict our focus to enhance these solutions. As a consequence, we suppose that the solution in the form of a polynomial of Eq. (2.18) has the following:

$$\begin{aligned}
 V(\Phi) &= a_0 + a_1\eta(\Phi), \\
 (\eta'(\Phi))^\Omega &= \sum_{k=0}^{\Omega N} b_k \eta^k(\Phi), \quad \Omega = 1, 2.
 \end{aligned}
 \tag{4.15}$$

(a): The Solitary Wave Solution:

These solutions could be derived by replacing $\Omega = 1$ in Eq. (4.15),

$$\begin{aligned}
 V(\Phi) &= a_0 + a_1\eta(\Phi), \\
 \eta'(\Phi) &= b_0 + b_1\eta(\Phi) + b_2\eta^2(\Phi).
 \end{aligned}
 \tag{4.16}$$

Plugging Eq. (4.16) into Eq. (2.18) and equating the coefficients of $\eta(\Phi)$ equals zero provides a system of equations. Solving these algebraic systems of equations with Mathematica or Maple software yields the solution.

$$\begin{aligned}
 b_0 &= 0, \quad a_1 = -2b_2, \\
 b_1 &= \frac{\sqrt{f - \kappa - d\tau - e\vartheta}}{\sqrt{d}}.
 \end{aligned}
 \tag{4.17}$$

Going to plug the solution of an auxiliary equation $\eta'(\Phi) = b_0 + b_1\eta(\Phi) + b_2\eta^2(\Phi)$ and Eq. (4.17) into Eq. (2.18), we get the Kink-like soliton solution of the Eq. (2.16) as

$$V_2(\Phi) = a_0 + \sqrt{-f + \kappa - \tau + e\vartheta} \left(1 + \tanh\left(\frac{1}{\sqrt{2}} \sqrt{-f + \kappa - \tau + e\vartheta}(\Phi + c(1))\right) \right),
 \tag{4.18}$$

where, $\Phi = x + dy + ez - ft$. The kink-like soliton solution provided by $V_2(x, y, z, t)$ in compliance with the specifications: $a_0 = 1 = \kappa = \tau = e, \vartheta = 3, c(1) = 0, d = f = -1$. Now, we get

$$V_2(x, y, z, t) = 1 + \sqrt{2} \left(1 + \tanh\left(\frac{(x - y + z + t)}{\sqrt{2}}\right) \right).
 \tag{4.19}$$

(b): The Soliton Solution:

In this context by putting $\Omega = 2$ into Eq. (4.15). We get

$$\begin{aligned}
 V(\Phi) &= a_0 + a_1\eta(\Phi), \\
 \xi'(\chi) &= \xi(\chi) \sqrt{q_0 + q_1\xi(\chi) + q_2\xi^2(\chi)}.
 \end{aligned}
 \tag{4.20}$$

Substituting Eq. (4.20) with Eq. (2.18), the following values of constants have been calculated as

$$\begin{aligned}
 a_0 = a_0, a_1 = a_1, b_0 &= \frac{1}{2} \frac{-2d\tau - 2e\vartheta + 2f - 2\kappa}{d}, \\
 b_1 &= \sqrt{\frac{d\tau - e\vartheta - f + \kappa}{d}}, b_2 = \frac{1}{4} a_1^2.
 \end{aligned}
 \tag{4.21}$$

Going to plug the solution of an auxiliary equation $\eta'(\Phi) = \eta(\Phi) \sqrt{b_0 + b_1\eta(\Phi) + b_2\eta^2(\Phi)}$ and Eq. (4.21) into Eq. (4.20), we get the singular soliton solution of the Eq. (2.16) under some specifications such as $a_0 = a_1 = d = \kappa = y = z = 1, e = \tau = -1, \vartheta = 3$ and $f = 2$;

$$V_3(x, y, z, t) = \frac{2(\sqrt{5} - 10)e^{\sqrt{5}} - e^{(-t + \frac{1}{2}x)2\sqrt{5}}}{2e^{\sqrt{5}} - e^{(-t + \frac{1}{2}x)2\sqrt{5}}}.
 \tag{4.22}$$

(c): The Elliptic Wave Solution:

The complex elliptic wave solution is found in this subsection. In this context, $\Omega = 2$ is used in the auxiliary equation given by Eq. (4.15).

$$\begin{aligned}
 V(\Phi) &= a_0 + a_1\eta(\Phi), \\
 \eta'(\Phi) &= \sqrt{b_0 + b_2\eta^2(\Phi) + b_4\eta^4(\Phi)}.
 \end{aligned}
 \tag{4.23}$$

This is obtained by substituting Eq. (4.23) with Eq. (2.18) in the same way as we did in the preceding case,

$$\begin{aligned}
 \kappa &= -d\tau + 2db_2 - e\vartheta + f, a_0 = a_0, \\
 b_0 &= \frac{1}{4} \frac{b_2^2}{b_4}, a_1 = 2\sqrt{b_4},
 \end{aligned}
 \tag{4.24}$$

where b_k and $k = 0, 2, 4$. are distinct constants. Various solutions for different values of b_k are achieved in Jacobi elliptic functions. The scheme of classification in Zhang (2009), viz.

$$b_0 = \frac{1}{4} \frac{b_2^2}{b_4}, b_2 < 0, b_4 > 0.
 \tag{4.25}$$

We take $b_2 = -1$ and $b_4 = 4$ then $b_0 = \frac{1}{16}$. Where $\eta(\Phi) = \sqrt{-\frac{1}{2} \frac{b_2}{b_4}} \tanh(\sqrt{-\frac{1}{2} b_2} \Phi)$ and the kink-like soliton solution to the Eq. (2.16) will lead to the formation:

$$V_4(x) = \frac{1}{\sqrt{2}} \tanh\left(\frac{1}{\sqrt{2}} \sqrt{x + dy + ez - ft}\right) 2 + a_0.
 \tag{4.26}$$

4.2.2 The rational function solutions

In order to acquire the rational solutions for the proposed model Eq. (2.16), consider

$$V(\Phi) = \frac{\sum_{k=0}^n h_k \eta^k(\Phi)}{\sum_{k=0}^m I_k \eta^k(\Phi)}, \quad n \geq m$$

$$(\eta'(\Phi))^\omega = \sum_{k=0}^{\omega\alpha} j_k \eta^k(\Phi), \quad \omega = 1, 2,$$
(4.27)

where h_k, I_k and j_k are specific constants. By investigating the homogeneous balancing of V''' and $V''V'$ in Eq. (2.18). We get $n - m = (k - 1); k = 1, 2, 3, \dots$

We concentrate our efforts to obtain these solutions for $k = 1$, so $(n = m)$ and $\omega = 2$. Here, two aspects arise as a consequence of Eq. (4.27).

(a): The Periodic Solution: We consider that,

$$V(\Phi) = \frac{h_0 + h_1 \eta(\Phi)}{I_0 + I_1 \eta(\Phi)},$$

$$\eta'(\Phi) = \sqrt{j_0^2 - j_2^2 \eta^2(\Phi)}.$$
(4.28)

Similarly, by inserting Eq. (4.28) into Eq. (2.18), we acquire a set of algebraic equations based on the polynomial coefficients $\eta(\Phi)$. By solving the algebraic system of equations, we obtain

$$I_1 = \frac{h_1 I_0}{h_0}.$$
(4.29)

By resolving the auxiliary equation, $\eta'(\Phi) = \sqrt{j_0^2 - j_2^2 \eta^2(\Phi)}$ and inserting Eq. (4.29) into Eq. (4.28), we have the singular periodic wave solution of the Eq. (2.18), as

$$V_5(\Phi) = \frac{h_0 j_2 + h_1 j_0 \sin(j_2(\Phi + c(1)))}{I_0 j_2 + I_1 j_0 \sin(j_2(\Phi + c(1)))}.$$
(4.30)

Where, $\Phi = x + dy + ez - ft$.

(b): The Soliton Solution:

Here, we consider that

$$V(\Phi) = \frac{h_0 + h_1 \eta(\Phi)}{I_0 + I_1 \eta(\Phi)},$$

$$\eta'(\Phi) = \sqrt{j_0 + j_1 \eta(\Phi) + j_2 \eta^2(\Phi)}.$$
(4.31)

We create a system of equations using the polynomial coefficients of $\eta(\Phi)$ by entering Eq. (4.31) into Eq. (2.18). By calculating this system of equations, we obtain

$$h_0 = \frac{i\sqrt{d}j_1 I_1}{\sqrt{-f + \kappa + d\tau + e\vartheta}}, \quad j_0 = -\frac{dj_1}{4(-f + \kappa + d\tau + e\vartheta)}, \quad I_0 = 0,$$

$$j_2 = -\frac{-f + \kappa + d\tau + e\vartheta}{d}.$$
(4.32)

By solving the auxiliary equation $\eta'(\Phi) = \sqrt{j_0 + j_1 \eta(\Phi) + j_2 \eta^2(\Phi)}$ and plugging Eq. (4.32) into Eq. (4.31), we have the anti-kink soliton solution of the Eq. (2.16), as

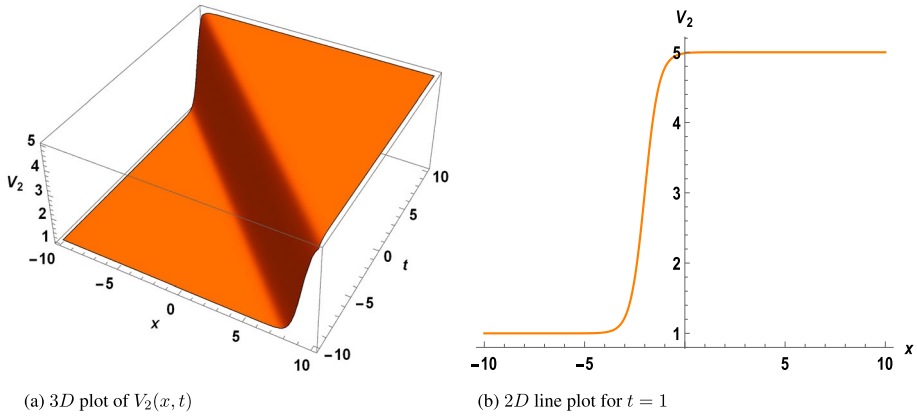


Fig. 1 For the kink-like soliton solution Eq. (4.19), there are three-dimensional and two-dimensional conspiracies for $\lambda = -1, a_0 = 1 = \kappa = \tau = e, \vartheta = 3, y = z = 0, c(1) = 1, d = f = -1$

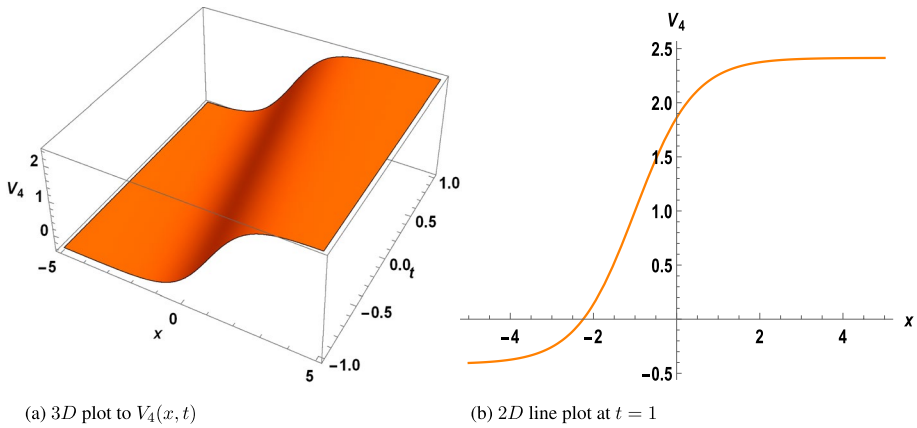


Fig. 2 For the kink-like soliton solution Eq. (4.26), there are three-dimensional and two-dimensional conspiracies for $d = 1, e = -1, f = 2, a_0 = 1, y = z = 0, b_2 = -1, b_4 = 4$

$$V_6(\Phi) = \frac{h_1}{I_1} + \frac{4j_1 \sqrt{f - \kappa - d\tau - e\vartheta}}{\sqrt{d(-2j_1 + e\sqrt{\frac{-f + \kappa + d\tau + e\vartheta}{d}}(\Phi + c(1)))}}, \tag{4.33}$$

where, $\Phi = x + dy + ez - ft$.

5 Pictorial representation of research results

This section presents the acquired results as well as some graphical representations of valuable techniques for interpreting the found solutions for the (3+1) nKdV-nCBS problem. As a result, 3D and 2D graphs of certain obtained solutions for relevant parameters. The 3D charts of the kink-like soliton solution $V_2(x, t)$ for $\lambda = -1$ are depicted at $a_0 = 1 = \kappa = \tau = e, \vartheta = 3, y = z = 0, d = f = -1,$ and $c(1) = 0$

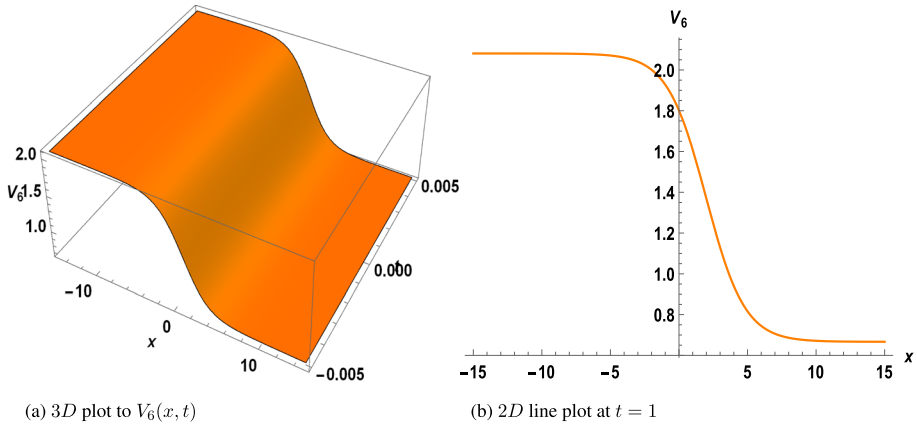


Fig. 3 Outlook of anti-kink soliton solution Eq. (4.33) for $j_1 = 1 = h_1 = f = \theta = \kappa = e$, $d = 2$, $-1 = I_1 = \tau$, $c_1 = 0$, $y = 0$ and $z = 0$

in Fig. 1a. Also, the 2D line plot of $V_2(x)$ is illustrated for $t = 1$ in Fig. 1b. Similarly, Fig. 2a presents the 3D chart of kink-like soliton solution $V_4(x, t)$ arguments for $d = 1$, $e = -1$, $f = 2$, $a_0 = 1$, $y = z = 0$, $b_2 = -1$, and $b_4 = 4$. Furthermore, in Fig. 2b the 2D line plot of $V_4(x)$ is illustrated for $t = 1$. Now in Fig. 3a depicts the 3D graph of the anti-kink soliton solution provided by $V_6(x, t)$ in compliance with the specifications $j_1 = 1 = h_1 = \theta = f = \kappa = e$, $\tau = -1 = I_1$, $d = 2$, $c_1 = 0$, $y = 0$, and $z = 0$. Furthermore, the 2D line plot of $V_6(x)$ is illustrated for $t = 1$ in Fig. 3b.

6 Results and discussion

In this section, we compare this article with Ali et al. (2022); Cui et al. (2023); Wazwaz (2021); Das et al. (2023). These research articles explore intriguing aspects of nonlinear wave equations, with a focus on obtaining unique solutions.

- We possess an individual soliton resolution $V_3(x, y, z, t)$ in the aforementioned equation Eq. (4.22) that has already been discovered in the realm of literature (Ali et al. 2022; Wazwaz 2021; Das et al. 2023). e.g; Eq. (12) in Ali et al. (2022), Eqs.(28)-(39) in Wazwaz (2021) and Eq. (46) in Das et al. (2023).
- We possess exclusive and unparalleled solutions, denoted as $V_1(x, y, z, t)$ and $V_5(x, y, z, t)$, that have yet to be discovered within the realm of literature.
- Similarly, there are other solutions such as $V_2(x, y, z, t)$, $V_4(x, y, z, t)$, and $V_6(x, y, z, t)$ that bear resemblance to the findings mentioned in [35]. e.g; Eq. (79) and Eq. (82).

In contrast, this article, titled "Unveiling Unique Solitons for the (3+1)-dimensional negative order KdV–CBS equation in a long wave propagation," offers a broader scope by presenting not only kink-like solutions but also anti-kink solutions, singular solutions, and singular periodic wave solutions. These additional solution types are obtained through the utilization of the singular manifold method and unified method, indicating a more comprehensive investigation into the behavior of the equation. These results have the potential to advance our understanding of nonlinear wave dynamics and offer practical applications across a wide

range of scientific and engineering disciplines. They contribute to the exploration of complex wave behaviors and their utility in solving real-world problems.

6.1 Novelty of this study

The novelty of the study of symbolic computation lies in its innovative approach to handling highly complex and non-linear equations symbolically. This research likely introduces novel symbolic solutions and employs advanced symbolic algorithms, pushing the boundaries of what can be achieved in symbolic mathematics. Additionally, the study may offer a fresh perspective on the symbolic representation of singular solitons, singular periodic wave solutions, kink-like solitons, and anti-kink soliton solutions, potentially opening new avenues for understanding and manipulating these solutions in the realm of computer algebra systems.

7 Concluding remarks

In this study, we investigated the (3+1) nKdV-nCBS equation and obtained novel solitary wave solutions using the singular manifold approach and the unified method. These solutions differ from previous approaches and provide a precise representation of the model's dynamic behavior, enhancing our understanding of the nKdV-nCBS equation and nonlinear wave phenomena by offering crucial insights into wave characteristics. The explicit determination of numerous soliton solutions with Maple and Mathematica tools surpassed prior methods. The 3D plots of numerical solutions offer a tangible perspective on nonlinear events, facilitating a deeper comprehension of the model's dynamics. These results hold the potential to advance our understanding of nonlinear wave dynamics and find practical applications across a wide range of scientific and engineering disciplines. They contribute to the exploration of complex wave behaviours and their utility in addressing real-world problems. In summary, this work establishes a robust foundation for further exploration, encompassing both theoretical and practical applications of the derived soliton solutions and the methods employed.

Funding There is no funding source.

Data availability The corresponding author can provide the data supporting the findings of this study upon a reasonable request.

Declarations

Conflict of interest The authors have not disclosed any conflict of interest.

Ethical approval and consent to participate The authors declare that there is no conflict with publication ethics.

Consent to publication The authors declare that there is no conflict with publication of this paper.

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