



# Entanglement, quantum coherence and quantum Fisher information of two qubit-field systems in the framework of photon-excited coherent states

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## Abstract

Based on coherent state theory, the photon-excited coherent states associated with pseudo-harmonic oscillators (PE-CSPHOs) are considered to investigate the interaction between a system of two qubits (two-level atoms) and a radiation field. By considering the dipole–dipole Hamiltonian and Ising Hamiltonian, we solve the Schrödinger equation to examine the influence of the qubit–qubit interaction (Q–QI) on the dynamics of the Q–Q entanglement and the entanglement between the two qubits and field. Furthermore, we study the dynamics of norm coherence and Fisher information, based on symmetric logarithmic derivative, with respect to the physical parameters of the system and explore the relation among the quantifiers during quantum dynamics.

**Keywords** Excited coherent states · Symmetric logarithmic derivative · Qubit–Qubit interaction · Quantum coherence ·  $I_1$  Norm-coherence

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## 1 Introduction

One of quantum physics' most remarkable aspects is the phenomenon of entanglement. In many applications of quantum technologies, it represents a type of quantum correlation that is crucial to the process (Gühne and Toth 2009), such as in quantum cryptography and computation, quantum teleportation, dense coding, circuit quantum electrodynamics, frequency standard improvement, and quantum metrology. Formally, quantum entanglement is defined in the context of the impossibility of writing the quantum state of a composite system as the product of the states of subsystems. Quantum entanglement has remained obscure despite playing a crucial function and having been quantified (Sperling and Walmsley 2017; Giovannetti et al. 2003). To categorize the quantum entanglement of the quantum states in the quantum regime, various approaches have been taken into consideration (Horodecki et al. 2009; Wootters 2001). Von Neumann entropy is a valuable measure of entanglement in pure states of bipartite quantum systems (Wootters 1998). However, for the situation of mixed states of bipartite quantum systems, entanglement distillation (Bennett et al. 1996), entanglement of formation (Wootters 1998; Kim 2021), and relative entanglement entropy (Vedral et al. 1997) are well established as reliable measures. Furthermore, a measure of quantum entanglement based on a distance obtained from the Fubini-Study metric was proposed (Cocchiarella et al. 2020). Understanding and realizing entangled states has therefore become increasingly crucial. Actually, there are a lot of studies on the entanglement measure of bipartite or multipartite systems, and the phenomenon of entanglement has witnessed tremendous prosperity in recent years (Friis et al. 2019; Beckey et al. 2021; Nezami and Walter 2020).

Recently becoming a basic aspect of quantum physics, quantum coherence is regarded as one of the key resources in quantum information and quantum optics (QIQO) (Walls and Millburn 2010; Levi and Mintert 2014; Bera et al. 2015; Monda et al. 2016; Streltsov et al. 2017; Orszag and Hernandez 2010). The quantum coherence is defined in terms of off-diagonal elements in the quantum states, which is related to the superposition principle of the quantum mechanics theory. Recently, much attention has been attracted to the quantification and characterization of coherence, and various studies have been considered in the literature (Shao et al. 2015; Yuan et al. 2015; Rana et al. 2016; Chitambar and Gour 2016; Winter and Yang 2016; Baumgratz et al. 2014). It is found that quantum coherence is a common and necessary condition for both entanglement and other kinds of quantum correlations. On the other side, the quantum coherence, like the case of quantum correlations, is prone and sensitive to the influence of external noise (Altowyan et al. 2022; Mortezaipour et al. 2018; Algarni et al. 2021a, b; Rahman et al. 2023a), where every realistic quantum system will certainly interact with its environment. The creation, maintenance, and manipulation of quantum systems coherence are difficult. Therefore, it is necessary and crucial to establish and maintain a certain level of coherence in the QIQO fields.

Parameter metrology is considered an effective topic in QIQO since experimental uncertainties and errors are inevitable in realistic systems (Abdel-Khalek et al. 2021). The theory of quantum estimation has recently relied heavily on quantum Fisher information (FI) (Helstrom 1976; Giovannetti et al. 2006, 2011; Yao et al. 2014; Abdel-Khalek et al. 2012; Abdel-Khalek 2014). FI is used to understand and display various quantum phenomena such as measurements of gravity accelerations (Lucien 1967), optimal quantum clocks (Bužek et al. 1999), entanglement detection (Li and Luo 2013), etc. Through conditional probabilities (Liu et al. 2017), the classical FI and, consequently,

the Cramér–Rao bound (CRB) of the precision of parameter estimation are reliant on a certain measurement scheme (Liu et al. 2017). Whereas the quantum FI does not depend on the measurement scheme, it relies on the way of parameter accumulation and the input state (Zhang et al. 2013; Wang et al. 2014). Since the estimation error is detected via the CRB, which is inversely proportional to quantum FI, the enhancement and preservation of the amount of quantum FI for a given probe state has a significant issue in the theory of quantum estimation (Braunstein and Caves 1994). Evaluating the quantum FI, obtaining approaches to enhance and preserve it, and finding schemes that allow for higher precision are the main tasks of quantum metrology (Len et al. 2022). Lately, it has been shown that quantum FI is largely related to quantum entanglement (Hyllus et al. 2012; Tóth 2012; Berrada et al. 2012; Berrada 2013). Substantial relations between quantum FI and quantum entropy have been explored (Huber et al. 2017).

Quantum decoherence and entanglement properties of light-matter interaction models have received more attention recently. These models consist of a central system made up of two or three two-level atoms (referred to as two- or three-qubit systems in the quantum information framework) that are resonantly coupled with a cavity field that is prepared in a single number state as well as with each other through dipole–dipole and Ising-like interactions (Torres et al. 2010; Amaro and Pineda 2014). In addition, spin–spin interactions like these have attracted attention in a variety of fields, including optical lattices (Sorensen and Molmer 1999), ion-trapped systems (Porras and Cirac 2004), microcavities (Hartmann et al. 2007), and even the situation of nearly localized and dipolarly coupled two identical molecules (Cusati et al. 2006; Napoli et al. 2006). Based on certain conditions, from an algebraic-structure point of view, intramolecular coupling models to Jaynes-Cummings emerge; furthermore, along this line, some interesting applications on the resonant two-atom JCM have been proposed with the target of implementing novel protocols for unambiguous Bell state discrimination for two qubits (Torres et al. 2016).

A further important concept also widely considered in the QIQO is coherent states. These states were introduced by Schrödinger (Schrödinger 1926) in the context of the quantum harmonic oscillator model. Such states satisfy the minimum uncertainty relation and resemble the classical equations of a harmonic oscillator. Because Glauber identified these harmonic oscillator states as an eigenstate of the boson annihilation operator, they gained significant importance in quantum optics (Glauber 1963). Coherent states can be used to describe a quantum system with the aim of obtaining minimal uncertainty. The essential nonclassical properties of the radiation fields, such as squeezing, photon anti-bunching, and sub-Poissonian photon distribution, have been detected through their description via coherent states (Cusati et al. 2006; Walls 1983; Loudon and Knight 1987; Berrada and Eleuch 2019). The photon-added coherent states for the harmonic oscillator were provided by Agarwal and Tara (Agarwal and Tara 1991, 1992). These states attracted a lot of attention and had a number of practical uses (Dodonov et al. 1998; Sixdernihiers and Penson 2001; Popov 2002; Bellini et al. 2012). There have been several generalizations suggested as a result of their possible uses (Safaeian and Tavassoly 2011; Dehghani et al. 2019). Correspondingly, such coherent states may be useful.

Recently, the Tavis-Cummings model (TCM) has been considered a fundamental theoretical model to describe a quantum system that consists of atoms coupled to a radiation field (Guo and Song 2009). Many studies have focused on **the quantum and classical correlations based on** the TCM as an important mathematical model related to theoretical physical systems (López et al. 2007; Tessier et al. 2003; Algarni et al. 2022; Rahman et al. 2023b; Benabdallah et al. 2022; Abouelregal and Marin 2020). In the present manuscript, we develop a model of the interaction between a system of two qubits (two-level atoms)

and a radiation field (RF) in the context of photon-excited coherent states associated with a pseudo-harmonic oscillator (PE-CSPHO). We investigate the impact of the Q–QI on the time evolution of Q–Q entanglement and Q-field entanglement. Furthermore, we analyze the dynamics of the coherence and Fisher information with respect to the physical parameters of the quantum system and explore the corresponding relation among the time evolution of the quantum resources. The results could be important for parameter estimation in QIQO, empirical realization of entangled systems, and coherence control. The manuscript is structured as follows: In Sect. 2, we describe the physical model. In Sect. 3, we present the quantum resources. Section 4 describes the primary outcomes. The final section of the study presents the findings.

## 2 Hamiltonian and quantum dynamics

In the present section, we consider the TCM that describes the Q–QI with a quantized field initially in the PE-CSPHO with the following Hamiltonian:

$$\frac{\hat{H}_{QQ-RF}}{\hbar} = \omega \hat{c}^\dagger \hat{c} + \frac{1}{2} \sum_{j=1}^2 \Omega_j \hat{\sigma}_z^{(j)} + \frac{\hat{H}_I}{\hbar} \tag{1}$$

where  $\omega$  describes the RF frequency and  $\Omega_j$  represents the frequency for the  $j$ th qubit, and the last term is given by

$$\frac{\hat{H}_I}{\hbar} = \sum_{j=1}^2 \mathbf{b}_j \left( \hat{\sigma}_+^{(j)} \hat{c} + \hat{\sigma}_-^{(j)} \hat{c}^\dagger \right), \tag{2}$$

The TQs are described through the operators  $\hat{\sigma}_z, \hat{\sigma}_-$  and  $\hat{\sigma}_+$  which are defined in terms of the  $j$ th qubit basis as  $|+\rangle\langle+| - |-\rangle\langle-|$ ,  $|+\rangle\langle-|$  and  $|-\rangle\langle+|$ , respectively. The state  $|+\rangle_j$  ( $|-\rangle_j$ ) designs the upper (lower) state of the  $j$ th qubit. The radiation field (RF) is described by the annihilation (creation) operator  $\hat{c}$  ( $\hat{c}^\dagger$ ). The coupling coefficients between the  $j$ th qubit and the RF is designed by  $\mathbf{b}_1$  and  $\mathbf{b}_2$  and we assume that  $\mathbf{b}_1 = \mathbf{b}_2 = \mathbf{b}$ . Moreover, we include the effect of Q–QI via the Hamiltonians

$$\frac{\hat{H}_{DD}}{\hbar} = \lambda_D \left( \hat{\sigma}_+^{(1)} \hat{\sigma}_-^{(2)} + \hat{\sigma}_-^{(1)} \hat{\sigma}_+^{(2)} \right) \tag{3}$$

$$\frac{\hat{H}_N}{\hbar} = \lambda_N \left( \hat{\sigma}_z^{(1)} \hat{\sigma}_z^{(2)} \right), \tag{4}$$

where  $\lambda_D$  ( $\lambda_N$ ) represents the dipole–dipole (Ising) coupling parameter.

The total Hamiltonian is then expressed by

$$\hat{H} = \hat{H}_{QQ-RF} + \hat{H}_{DD} + \hat{H}_N. \tag{5}$$

At  $t = 0$ , the solution takes the following form

$$|\Psi(0)\rangle = \frac{1}{\sqrt{2}} \left[ |+-\rangle + |-+\rangle \right] \otimes |\Psi_{RF}(0)\rangle, \tag{6}$$

$|\Psi_{RF}(0)\rangle$  represents the initial state for the RF defined in PE-CSPHO as (Popov et al. 2009)

$$|\Psi_{RF}(0)\rangle = \frac{1}{\sqrt{\sum_{n=0}^{\infty} |U_n|^2}} \sum_{n=0}^{\infty} U_n |n + m\rangle, \tag{7}$$

where

$$U_n = \frac{\alpha^n}{n!} \sqrt{\frac{\Gamma(n + m + 1)\Gamma(n + m + 2k)}{\Gamma(2k)}}, \tag{8}$$

and

$$\Gamma(2k) = \frac{2^{2k-1}\Gamma(k)\Gamma(k + \frac{1}{2})}{\sqrt{\pi}}, \tag{9}$$

with  $k$  designs the Bragmann index and  $m$  is the number of excited photons.

The wavefunction  $|\Psi(t)\rangle$  corresponding to the total Hamiltonian (5) can be provided as

$$|\Psi(t)\rangle = \sum_{n=0}^{\infty} [Y_1(n, t)|++\rangle|n + m\rangle + Y_2(n, t)|+-\rangle|n + m + 1\rangle + Y_3(n, t)|-+\rangle|n + m + 1\rangle + Y_4(n, t)|--\rangle|n + m + 2\rangle]. \tag{10}$$

The whole system has the density matrix  $\rho(t) = |\Psi(t)\rangle\langle\Psi(t)|$ , and as a result, the TQs (RF) density matrix can be determined by performing a trace over the RF (TQs) basis as  $\rho_{QQ[RF]}(t) = \text{Tr}_{RF[QQ]} \{\rho(t)\}$ .

### 3 Quantum quantifiers

The von Neumann entropy in terms of the Qs (or RF) basis is used to quantify the Qs-field entanglement in the present model. It is defined for the density matrix  $\rho_{QQ}$  (or  $\rho_{RF}$ ) as:

$$S_{QQ} = S_{RF} = -\text{Tr}\{\rho_{QQ} \ln(\rho_{QQ})\}. \tag{11}$$

On the other side, the Q-Q entanglement is quantified through the concurrence (Kim 2021) as:

$$C_{QQ} = \max\{0, \xi_1 - \xi_2 - \xi_3 - \xi_4\}, \tag{12}$$

where  $\xi_j$  represent the eigenvalues giving in the decreasing order of  $\rho_{QQ}\tilde{\rho}_{QQ}$  and  $\tilde{\rho}_{QQ}$  is defined by

$$\tilde{\rho}_{QQ} := (\sigma_Y \otimes \sigma_Y)\rho_{QQ}^*(\sigma_Y \otimes \sigma_Y), \tag{13}$$

with  $\rho_{QQ}^*$  is the conjugate of  $\rho_{QQ}$  and  $\sigma_Y$  represents the Y-Pauli operator. The state TQs is said to be in separable state if  $C_{QQ} = 0$  and in a maximally entangled if  $C_{QQ} = 1$ .

The diagonal components of the system density operator affect the dynamical features of coherence in a quantum system. The absolute value of the density matrix's off-diagonal members is used by the quantum coherence. The definition of concurrence based on  $L_1$  norm of coherence is (Baumgratz et al. 2014):

$$C_L = \min_{\delta \in \mathcal{I}} \|\rho - \delta\|_{L_1} \quad (14)$$

where  $\mathcal{I}$  describes the collection of incoherent states where row and column are represented by  $i$  and  $j$  respectively.

For a given process with the parameter  $\beta$ , the quantum FI of the quantum system with density matrix  $\rho$  is defined as (Hyllus et al. 2012; Liu et al. 2013)

$$F = \text{Tr}\{\rho(\beta)\mathbf{R}(\beta)^2\} \quad (15)$$

where the operator  $\mathbf{R}$  defines the symmetric logarithmic derivative verifying

$$2\frac{\partial \mathbf{R}(\beta)}{\partial \beta} = \rho(\beta)\mathbf{R}(\beta) + \mathbf{R}(\beta)\rho(\beta). \quad (16)$$

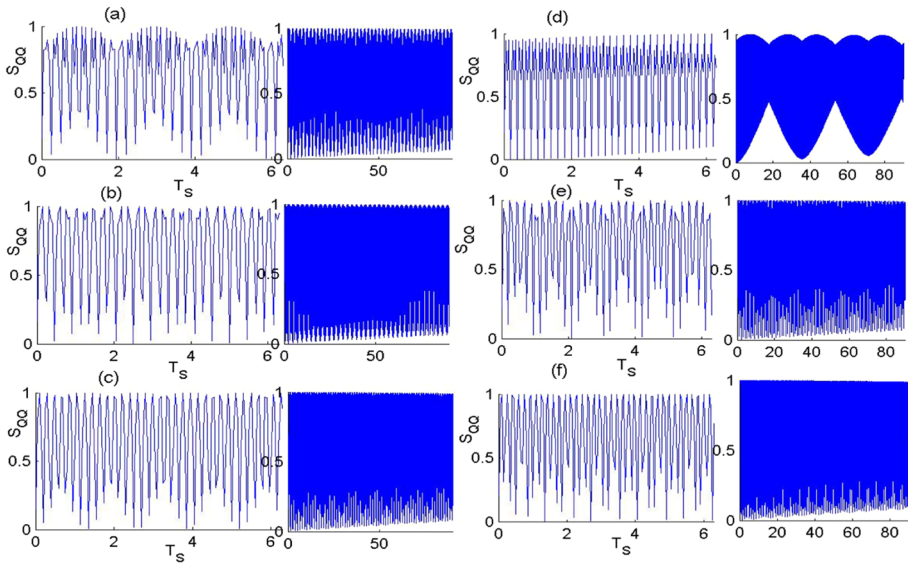
In the next section, based on the density matrix elements in terms of the wave function amplitudes (see "Appendix A"), we are able to display the dynamics of the quantifiers that detect the Qs-field entanglement, Q-Q entanglement, coherence, as well as qubit FI.

## 4 Numerical results and discussion

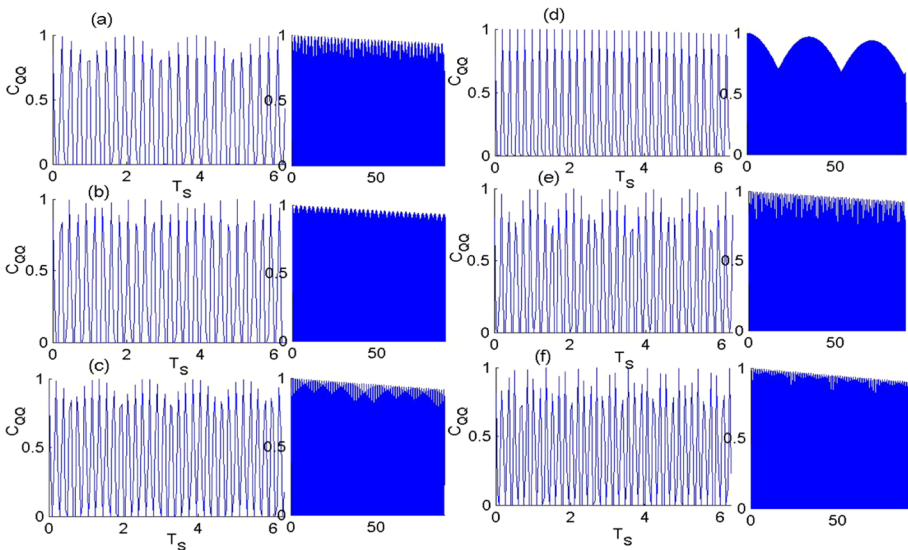
Here, we discuss and analyze the numerical results by depicting the fluctuation throughout time of the quantum resources with respect to the initial parameter values of the quantum system. In Figs. 1, 2, 3 and 4, we show the dynamics of the qubits-field entanglement and Q-Q entanglement, Fisher information as well as quantum coherence under the d-d and Ising interaction effects without and with photons addition for  $k = 3/4$ .

In Fig. 1, we show the influence of the Q-QI on the dynamical behavior of von Neumann entropy in the absence and presence of the photon-added number. In general, the function has a dynamical behavior with rapid oscillations and takes values between 0 and 1, depending on the quantum system parameters. In the absence of Q-QI and photon excitation, we can observe that the interaction of the quantized field with the two atoms gives rise to an oscillating entanglement with a gradual augmentation in the minimum value of oscillations as time goes on and preserves the entanglement for longer times. As can be seen when the Q-QI is taken into consideration, this leads to enhanced oscillations of the function and enhances the entanglement of the qubits-field state during the evolution. However, the presence of photon excitation affects the behavior of entanglement in the absence of Q-QI, and entanglement preservation is more effective in this situation. On the other hand, the dynamical behavior of the concurrence is less affected by the addition of photons in the presence of Q-Q interactions.

Figure 2 exhibits the entanglement's temporal development of the Q-Q state without and with the effect of Q-QI and photons excitation. The plots show an oscillating behavior for the Q-Q entanglement with a large frequency during the evolution. We can find that the Q-Q entanglement is affected in a similar way as the Qs-field entanglement with respect to the system parameters  $\lambda_D, \lambda_N$  and  $m$ . Interestingly, the dynamics of the concurrence reveals the occurrences of the sudden death and sudden birth phenomena on entanglement in the current

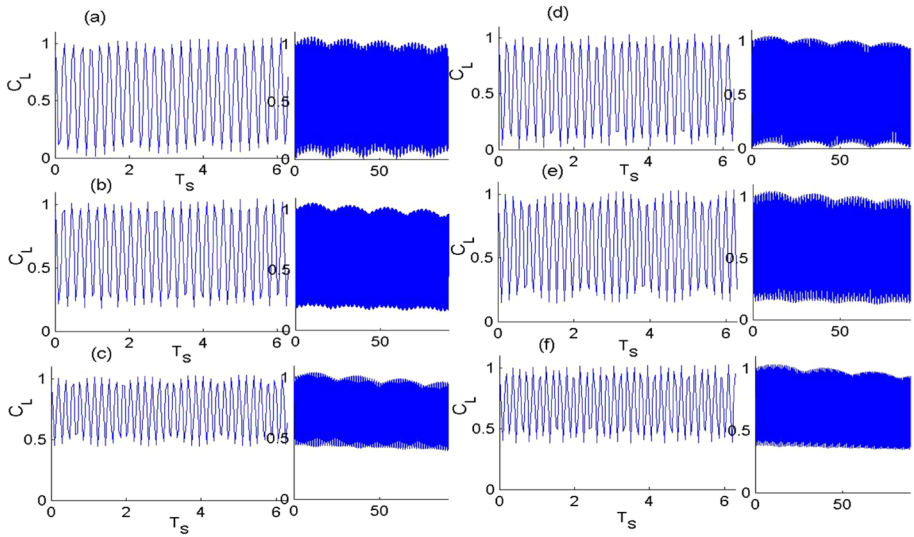


**Fig. 1** The von entropy  $S_{QQ}$  versus the scaled time  $T_S$  for the state of TQs initially in a maximally entangled state and the RF in PE-CSPHO for  $\beta = \sqrt{20}$ ,  $k = 3/4$ . **a–c** Are for  $m = 0$  and **d–f** are for  $m = 20$ . The d–d parameter  $\lambda_D$  and Ising parameter  $\lambda_N$  are taken as follow:  $(\lambda_D, \lambda_N) = (0, 0)$  for **(a, d)**,  $(\lambda_D, \lambda_N) = (10, 0)$  for **(b, e)** and  $(\lambda_D, \lambda_N) = (0, 10)$  for **(c, f)**

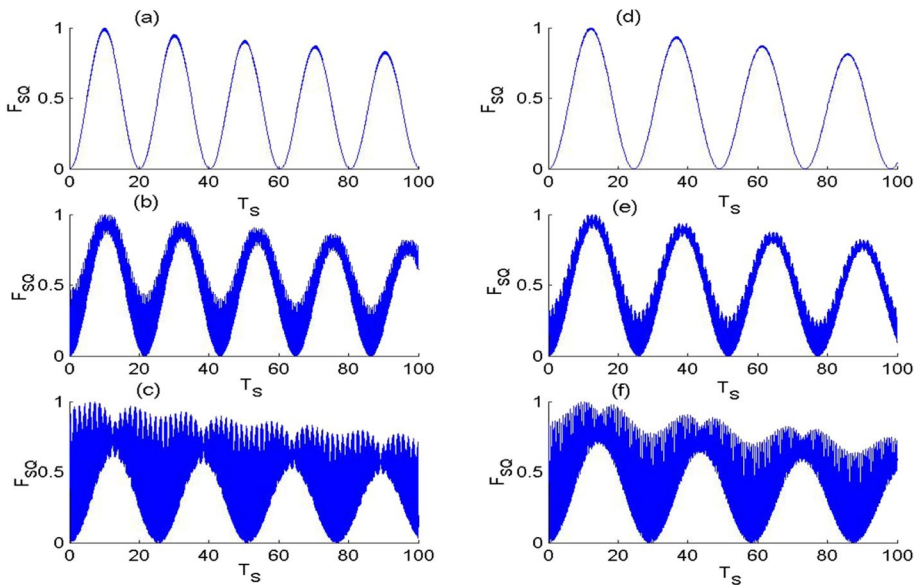


**Fig. 2** The concurrence  $C_{QQ}$  against the scaled time  $T_S$  for the state of TQs initially in a maximally entangled state and the RF in PE-CSPHO for  $\beta = \sqrt{20}$ ,  $k = 3/4$ . **a–c** Are for  $m = 0$  and **d–f** are for  $m = 20$ . The d–d parameter  $\lambda_D$  and Ising parameter  $\lambda_N$  are taken as follow:  $(\lambda_D, \lambda_N) = (0, 0)$  for **(a, d)**,  $(\lambda_D, \lambda_N) = (10, 0)$  for **(b, e)** and  $(\lambda_D, \lambda_N) = (0, 10)$  for **(c, f)**





**Fig. 3** The quantum coherence  $C_L$  versus the scaled time  $T_S$  for the state of TQs initially in a maximally entangled state and the RF in PE-CSPHO for  $\beta = \sqrt{20}$ ,  $k = 3/4$ . **a–c** Are for  $m = 0$  and **d–f** are for  $m = 20$ . The d–d parameter  $\lambda_D$  and Ising parameter  $\lambda_N$  are taken as follow:  $(\lambda_D, \lambda_N) = (0, 0)$  for **(a, d)**,  $(\lambda_D, \lambda_N) = (10, 0)$  for **(b, e)** and  $(\lambda_D, \lambda_N) = (0, 10)$  for **(c, f)**



**Fig. 4** The atomic FI,  $F_{SQ}$ , versus the scaled time  $T_S$  for the state of TQs initially in a maximally entangled state and the RF in PE-CSPHO for  $\beta = \sqrt{20}$ ,  $k = 3/4$ . **a–c** Are for  $m = 0$  and Figs. **d–f** are for  $m = 20$ . The d–d parameter  $\lambda_D$  and Ising parameter  $\lambda_N$  are taken as follow:  $(\lambda_D, \lambda_N) = (0, 0)$  for **(a, d)**,  $(\lambda_D, \lambda_N) = (10, 0)$  for **(b, e)** and  $(\lambda_D, \lambda_N) = (0, 10)$  for **(c, f)**



model. Moreover, the maximum values of Q–Q entanglement decrease with time. These results confirm that the qubits–field entanglement and Q–Q entanglement can be controlled and generated during the time evolution.

In Fig. 3, we illustrate the effect of the Q–QI on the dynamical behavior of qubit coherence with and without the photon-added number. We find that the measure of coherence exhibits a quasi-periodic behavior with fast oscillations. The presence of Q–QI leads to an increase in the minimum value of atom coherence during the dynamics. Whereas the photons excitation affects the behavior of coherence in the presence of d–d interaction and has a very slight effect in the other cases. Referring to the response of entanglement dynamics discussed above, coherence and entanglement are in general quantum resources of different natures.

As a final part of the numerical analysis, we apply the definition given in Eq. (15) to plot the time variation of the quantum FI according to the parameter values of the quantum system. As can be seen from the first column of Fig. 4, the FI has quasi-periodic behavior with a gradual reduction in its maximum value during the dynamics. Interestingly, the presence of Q–QI induces oscillations in the dynamics of atomic FI prolonging its preservation during evolution. This may be seen as the transfer of information between the atoms resulting from the Q–QI and qubits–field interaction. On the other hand, the graphs in column II of Fig. 4 demonstrate that the quantity of qubit FI increases when photons are excited during the time evolution, where the quantum FI is protected compared to the case where photons are not excited, and therefore improves the accuracy of parameter.

## 5 Conclusions

A system of two qubits and a radiation field interacting in the context of photon-excited coherent states associated with pseudo-harmonic oscillators (PE-CSPHOs) has been explored in the current study. We have examined the influence of the Q–QI on the dynamical behavior of Qs–field entanglement and Q–Q entanglement. Furthermore, we have analyzed the dynamics of the coherence and Fisher information with respect to the physical parameter of the system and explored the corresponding relation among the quantum resources during the evolution. Interestingly, we have shown that the quantum resources exhibit an oscillating behavior with amplitude and frequency that depends on the values of the interaction parameters and photons excitation. Our results support the suggested quantum systems' capacity to maintain entanglement, coherence, and accuracy of parameter estimation throughout dynamics in the presence of dipole and Ising interactions and provide new useful insights. Also, these results may be significant for the empirical realization of the preparation of entangled states and control of coherence, as well as parameter estimation in QIQO.

## Appendix A

Acting the Hamiltonian operator  $\hat{H}$  Eq. (5) on the wave function  $|\psi(\mathbf{t})\rangle$  and applying the Schrödinger equation,

$$i\hbar \frac{\partial |\psi(\mathbf{t})\rangle}{\partial t} = \hat{H} |\psi(\mathbf{t})\rangle, \quad (17)$$

hence this allows us to determine  $\mathbf{Y}_j(\mathbf{n}, \mathbf{t})$ , for  $\hbar = 1$ , that are verifying the system of coupled ode

$$\frac{dY_1}{dt} = -i\lambda_N Y_1 - ib\sqrt{n+m+1}(Y_2 + Y_3) \quad (18)$$

$$\frac{dY_2}{dt} = -i(\lambda_N Y_2 + \lambda_D Y_3) - ib\left[\sqrt{n+m+1}Y_1 + \sqrt{n+m+2}Y_4\right] \quad (19)$$

$$\frac{dY_3}{dt} = -i(\lambda_N Y_3 + \lambda_D Y_2) - ib\left[\sqrt{n+m+1}Y_1 + \sqrt{n+m+2}Y_4\right] \quad (20)$$

$$\frac{dY_4}{dt} = -i\lambda_N Y_4 - ib\sqrt{n+m+2}(Y_2 + Y_3). \quad (21)$$

The above coupled system (18)–(21) is solved numerically under the initial conditions

$$Y_1(0) = Y_4(0) = 0 \quad (22)$$

$$Y_2(0) = Y_3(0) = \frac{U_n}{\sqrt{2 \sum_{n=0}^{\infty} |U_n|^2}} \quad (23)$$

where  $U_n$  is given in Eq. (8).

**Author contributions** SA-K: Conceptualization, methodology, writing—original draft, writing—reviewing and editing. MA: Visualization, supervision, project administration, reviewing and editing. KB: Writing—original draft, writing—reviewing and editing. MM: Validation, investigation, reviewing. All authors have read and agreed to the published version of the manuscript.

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## Declarations

**Competing interests** The authors declare no conflict of interest.

**Ethical approval** Not applicable.

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