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Analytical treatments of the space–time fractional coupled nonlinear Schrödinger equations

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Abstract
Under investigation in this work is a (2 + 1)-dimensional the space–time fractional coupled nonlinear Schrödinger equations, which describes the amplitudes of circularly-polarized waves in a nonlinear optical fiber. With the aid of conformable fractional derivative and the fractional wave transformation, we derive the analytical soliton solutions in the form of rational soliton, periodic soliton, hyperbolic soliton solutions by four integration method, namely, the extended trial equation method, the $\exp(-\Omega(\eta))$-expansion method and the improved $\tan(\phi(\eta)/2)$-expansion method and semi-inverse variational principle method. Based on the the extended trial equation method, we derive the several types of solutions including singular, kink-singular, bright, solitary wave, compacton and elliptic function solutions. Under certain condition, the 1-soliton, bright, singular solutions are driven by semi-inverse variational principle method. Based on the analytical methods, we find that the solutions give birth to the dark solitons, the bright solitons, combine dark-singular, kink, kink-singular solutions with fractional order for nonlinear fractional partial differential equations arise in nonlinear optics.

Keywords Conformable time-fractional equations · The space–time fractional coupled nonlinear Schrödinger equations · The extended trial equation method · The $\exp(-\Omega(\eta))$-expansion method · The $\tan(\phi(\eta)/2)$-expansion method · The semi-inverse variational principle method

1 Introduction

The common calculus has been studied well and its applications can be encountered in several areas of science and engineering. Relating to fractional Calculus, it is not familiar to several researchers. Indeed, fractional Calculus is a three centuries old mathematical tools. But the searching of the theory of differential Equations of fractional order has just been

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began quite recently (Podlubny 1999; Debnath 2003; Miller and Ross 1993). An expanding of fractional notions in nonlinear optic has also been improved. Fairly, no field of standard analysis has been left unconcerned by fractional Calculus. As we know that many scientific and engineering problems that arise in daily life are modeled through mathematical tools form fractional calculus. Most of the problems that arise are nonlinear in nature and it is not always possible to find analytical solution of such problems. But we use the fractional wave transformation to solve the nonlinear fractional partial differential equations to attain to the exact solution. The analytical methods for finding the exact solutions of nonlinear PDEs have presented, such as, the Exp-function method (Ekici et al. 2017; Manafian and Lakestani 2015; Manafian 2015), the generalized Kudryashov method (Zhou et al. 2016), the extended Jacobi elliptic function expansion method (Ekici et al. 2017; Mirzazadeh et al. 2016), the improve $\tan(\phi/2)$-expansion method (Manafian 2016; Manafian and Lakestani 2016a, b; Manafian 2017; Manafian et al. 2016), the $\exp(-\Omega(\eta))$-expansion method (Abdelrahman et al. 2014), the G'/G-expansion method (Manafian and Lakestani 2017, 2015; Sindi and Manafian 2016), the generalized G'/G-expansion method (Zinati and Manafian 2017), the Bernoulli sub-equation function method (Baskonus 2017; Baskonus and Bulut 2016; Baskonus et al. 2016; Bulut and Baskonus 2016), the sine-Gordon expansion method (Baskonus et al. 2016; Baskonus and Bulut 2016; Baskonus 2016; Yel et al. 2017), the Ricatti equation expansion (Zhou 2016; Inc et al. 2016), the formal linearization method (Mirzazadeh and Eslami 2015), the Lie symmetry (Tchier et al. 2017), Hirota bilinear method (Manafian 2018; Foroutan et al. 2018), and so on.

Consider the space–time fractional (1 + 1)-dimensional coupled nonlinear Schrödinger equation as

$$\begin{aligned} iD_t^{\alpha}u + D_x^{2\beta}u + \delta(|u|^2 + \gamma|v|^2)u &= 0, \quad 0 < \alpha \le 1, \quad 0 < \beta \le 1, \\ iD_t^{\alpha}v + D_x^{2\beta}v + \delta(|v|^2 + \gamma|u|^2)v &= 0, \end{aligned} \tag{1.1}$$

in which $D_t^{\alpha}u = \frac{\partial^{\alpha}u}{\partial t^{\alpha}}$ and $D_x^{2\beta}u = \frac{\partial^{2\beta}u}{\partial x^{2\beta}}$ and δ, γ are nonzero constants, u and v are complex functions of x and t that represent the amplitudes of circularly-polarized waves in a nonlinear optical fiber (Boyd 1992). The nonlinear Schrödinger equation is an example of a universal nonlinear model that describes many physical nonlinear systems. In Esen and coworkers (2018) studied the optical solitons to the space–time fractional (1 + 1)-dimensional coupled nonlinear Schrödinger equation and represented analytical simulation by using the extended sinh-Gordon equation expansion method. Eslami (2016) applied new conformable fractional derivative for converting fractional coupled nonlinear Schrödinger equations using the Kudryashov method. The HSS-like iteration method for the space fractional coupled nonlinear Schrödingere quations was used by Ran and coworkers (2015). Su and Gao (2017) obtained the bilinear forms and dark soliton solutions via the Hirota method for coupled nonlinear Schrödinger equations with time-dependent coefficients. Through the F-expansion method, a set of exact Jacobian solutions of coupled nonlinear Schrödinger equations was been derived by Mvogo and coworkers (2014). Also, Guo and He (2016) obtained the least energy solutions for a weakly coupled fractional Schrödinger system by use of the s-harmonic extension technique.

This paper will adopt four integration schemes that are known as the extended trial equation method, the $\exp(-\Omega(\eta))$-expansion method and the $\tan(\phi(\eta)/2)$-expansion method that will reveal soliton solutions as well as other solutions. The equations for the model studied here to investigate exact solution structures. We note that these equations have not yet been studied using the aforementioned methods. To our knowledge, the extended trial equation method, the $\exp(-\Omega(\eta))$-expansion method and the improved $\tan(\phi(\eta)/2)$-expansion method and the semi-inverse variational principle method for system (1.1) have not been investigated. Under certain condition, the semi-inverse variational principle solutions can be degenerated into the dark, bright, solutions. In Sect. 2, we present the conformable fractional derivative and its properties. In Sect. 3, we will obtain the soliton solutions for system (1.1) via the extended trial equation method. Also, in Sect. 4, the solitary solutions for system (1.1) will be demonstrated. Finally, in Sects. 5 and 6, the ITEM and HSIVPM will be employed to attain to other types of soliton solutions for system (1.1). The conclusions will be outlined in Sect. 7.

2 Conformable fractional derivative and its properties

In this section, we present the outline of the conformable fractional derivative and its properties in which proposed by Khalil et al. (2014). For the further information refere to Refs. Abdeljawad (2015), Eslami (2016).

Definition 1 Suppose $f:(0,\infty)\to R$ be a function. Therefore, the conformable fractional derivative of f of order α is given as

$$T_\alpha(f)(t)=\lim_{\theta\to 0}\frac{f(t+\theta t^{1-\alpha})-f(t)}{\theta}, \tag{2.1}$$

in which $0<\alpha\le 1$.

Theorem 1 *Let $\alpha\in(0,1]$, and f and g be α-differentiable at $t>0$. Then*

1. $T_\alpha(af+bg)=aT_\alpha(f)+bT_\alpha(g),\ \forall a,b\in R.$
2. $T_\alpha(t^\vartheta)=\vartheta t^{\vartheta-\alpha},\ \forall\vartheta\in R.$
3. $T_\alpha(fg)=fT_\alpha(g)+gT_\alpha(f).$
4. $T_\alpha\left(\frac{f}{g}\right)=\frac{gT_\alpha(f)-fT_\alpha(g)}{g^2}.$
3. If *f is differentiable; then $T_\alpha(f)(t)=t^{1-\alpha}\frac{df}{dt}(t)$.*

Theorem 2 *Assume $f,g:(0,\infty)\to R$ be a function such that f is differentiable and also α-differentiable. Then*

$$T_\alpha(fog)(t)=t^{1-\alpha}g'(t)f'(g(t)).$$

We use the following fractional complex wave transformation as:

$$U(x,t)=u(\xi)e^{i\eta},\quad V(x,t)=v(\xi)e^{i\eta},\quad \xi=\mu\left(\frac{x^\beta}{\beta}-c\frac{t^\alpha}{\alpha}\right),\quad \eta=\frac{-kx^\beta}{\beta}+\omega\frac{t^\alpha}{\alpha}+\theta. \tag{2.2}$$

Inserting Eqs. (2.2) into (1.1), gives the following system of nonlinear ordinary differential equation as:

$$\begin{aligned}\mu^2u''+\delta u^3+\delta\gamma uv^2-(\omega+k^2)u&=0,\\ \mu^2v''+\delta v^3+\delta\gamma vu^2-(\omega+k^2)v&=0,\end{aligned} \tag{2.3}$$

by decomposing the real and imaginary parts we get $c = -2k$.

3 Extended trial equation method

For the ETEM consider the following steps as:

Step 1 Suppose the nonlinear space–time fractional partial differential equations as

$$\mathcal{N}\left(u, v, \frac{\partial^{\alpha} u}{\partial t^{\alpha}}, \frac{\partial^{2\beta} u}{\partial t^{2\beta}}, \frac{\partial^{\alpha} v}{\partial t^{\alpha}}, \frac{\partial^{2\beta} v}{\partial t^{2\beta}}, \dots\right) = 0. \tag{3.1}$$

Utilizing the wave transformation

$$U(x,t) = u(\xi)e^{i\eta}, \quad V(x,t) = v(\xi)e^{i\eta}, \quad \xi = \mu\left(\frac{x^{\beta}}{\beta} - c\frac{t^{\alpha}}{\alpha}\right), \quad \eta = \frac{-kx^{\beta}}{\beta} + \omega\frac{t^{\alpha}}{\alpha} + \theta. \tag{3.2}$$

where $c \neq 0$. Putting (3.2) into Eq. (3.1) yields a NODE as,

$$\mathcal{Q}(u, v, u', v', \dots) = 0, \tag{3.3}$$

where prime shows the derivation with respect to ξ.

Step 2 Let the following relation and transformation as follows:

$$u(\eta) = \sum_{i=0}^{\delta} \tau_i \Gamma^i, \tag{3.4}$$

where

$$(\Gamma')^2 = \Omega(\Gamma) = \frac{\Phi(\Gamma)}{\Psi(\Gamma)} = \frac{\xi_\theta \Gamma^\theta + \dots + \xi_1 \Gamma + \xi_0}{\zeta_\epsilon \Gamma^\epsilon + \dots + \zeta_1 \Gamma + \zeta_0}. \tag{3.5}$$

By employing the above relations (3.4) and (3.5), we obtain

$$(u')^2 = \frac{\Phi(\Gamma)}{\Psi(\Gamma)}\left(\sum_{i=0}^{\delta} i\tau_i \Gamma^{i-1}\right)^2, \tag{3.6}$$

$$u'' = \frac{\Phi'(\Gamma)\Psi(\Gamma) - \Phi(\Gamma)\Psi'(\Gamma)}{2\Psi^2(\Gamma)}\left(\sum_{i=0}^{\delta} i\tau_i \Gamma^{i-1}\right) + \frac{\Phi(\Gamma)}{\Psi(\Gamma)}\left(\sum_{i=0}^{\delta} i(i-1)\tau_i \Gamma^{i-2}\right), \tag{3.7}$$

where $\Phi(\Gamma)$ and $\Psi(\Gamma)$ are polynomials of Γ function. Putting the above terms into Eq. (3.3) gains an equation of polynomial $\Lambda(\Gamma)$ of Γ:

$$\Lambda(\Gamma) = \varrho_s \Gamma^s + \dots + \varrho_1 \Gamma + \varrho_0 = 0. \tag{3.8}$$

Based on the balance principle on (3.8), we can find a relation of θ, ϵ and δ.

Step 3 Setting each coefficient of polynomial $\Lambda(\Gamma)$ to zero to derive a system of algebraic equations:

$$\varrho_i = 0, \quad i = 1, 2, \dots, s. \tag{3.9}$$

Solving the system (3.9), we will acquire the values of $\xi_0, \xi_1, \ldots, \xi_\theta$, $\zeta_0, \zeta_1, \ldots, \zeta_\sigma$ and $\tau_0, \tau_1, \ldots, \tau_\delta$.

Step 4 In the continue, we earn the elementary form of the integral by reduction of Eq. (3.5), as follows

$$\pm(\eta-\eta_0)=\int\frac{d\Gamma}{\sqrt{\Omega(\Gamma)}}=\int\sqrt{\frac{\Psi(\Gamma)}{\Phi(\Gamma)}}d\Gamma, \tag{3.10}$$

where η_0 is an free constant.

3.1 Application of ETEM

In this section, the application of the extended trial equation method to the space–time fractional (1 + 1)-dimensional coupled nonlinear Schrödinger equation is presented.

By balancing u^3, u'' and v^3, v'' in Eq. (2.3), the values of δ, θ, and ϵ, can be determined as:

$$2\delta=\theta-\epsilon-2. \tag{3.11}$$

For different values of δ, θ, and ϵ, we have the following cases:

3.1.1 Case I: $\delta=1, \theta=4$, *and* $\epsilon=0$

Let $\delta=1, \theta=4$, and $\epsilon=0$ for Eqs. (3.4) and (3.5), then we get

$$u(\eta)=\tau_0+\tau_1\Gamma, \tag{3.12}$$

$$v(\eta)=\rho_0+\rho_1\Gamma, \tag{3.13}$$

$$(u'(\eta))^2=\frac{\tau_1^2\left(\xi_4\Gamma^4+\xi_3\Gamma^3+\xi_2\Gamma^2+\xi_1\Gamma+\xi_0\right)}{\zeta_0}, \tag{3.14}$$

$$(v'(\eta))^2=\frac{\rho_1^2\left(\xi_4\Gamma^4+\xi_3\Gamma^3+\xi_2\Gamma^2+\xi_1\Gamma+\xi_0\right)}{\zeta_0}, \tag{3.15}$$

where $\xi_4\neq 0$ and $\zeta_0\neq 0$. With the solve the algebraic equation system (3.9), we can easily get the following sets as:

- *First set*

$$\begin{aligned}&\mu=\mu,\quad \omega=\omega,\quad \rho_0=\pm\tau_0,\quad \rho_1=\pm\tau_1,\\&\tau_0=\tau_0,\quad \tau_1=\tau_1,\quad \xi_0=\xi_0,\quad \xi_1=-\frac{2\zeta_0\tau_0\left(-\omega-k^2+\delta\tau_0^2+\delta\gamma\tau_0^2\right)}{\tau_1\mu^2},\\&k=k,\quad \xi_2=-\frac{\zeta_0\left(-\omega+3\delta\tau_0^2+3\delta\gamma\tau_0^2-k^2\right)}{\mu^2},\quad \xi_3=-\frac{2\delta\zeta_0\tau_0\tau_1(\gamma+1)}{\mu^2},\\&\xi_4=-\frac{\delta\zeta_0\tau_1^2(\gamma+1)}{2\mu^2},\quad \zeta_0=\zeta_0.\end{aligned} \tag{3.16}$$

Putting these results into Eqs. (3.5) and (3.10), then (3.10) can be written as

$$\pm(\eta-\eta_0)=\int\frac{\sqrt{\frac{\zeta_0}{\xi_4}}\,d\Gamma}{\sqrt{\Gamma^4+\frac{\xi_3}{\xi_4}\Gamma^3+\frac{\xi_2}{\xi_4}\Gamma^2+\frac{\xi_1}{\xi_4}\Gamma+\frac{\xi_0}{\xi_4}}}. \tag{3.17}$$

To integrate Eq. (3.17), the following results to the Eq. (1.1) can be found as:

First solution

$$\pm(\eta-\eta_0)=-\frac{\Pi}{\Gamma-\alpha_1},\quad \Pi=\sqrt{-\frac{2\mu^2}{\delta\tau_1^2(\gamma+1)}}, \tag{3.18}$$

$$\begin{aligned}U_1(x,t)=V_1^+(x,t)&=e^{i\left(-\frac{k}{\beta}x^\beta+\frac{\omega}{\alpha}x^\alpha+\theta\right)}\left\{\tau_0+\tau_1\alpha_1-\frac{\Pi\tau_1}{\mu\left(\frac{1}{\beta}x^\beta+\frac{2k}{\alpha}x^\alpha\right)-\eta_0}\right\}.\\ V_1^-(x,t)&=e^{i\left(-\frac{k}{\beta}x^\beta+\frac{\omega}{\alpha}x^\alpha+\theta\right)}\left\{-\tau_0-\tau_1\alpha_1+\frac{\Pi\tau_1}{\mu\left(\frac{1}{\beta}x^\beta+\frac{2k}{\alpha}x^\alpha\right)-\eta_0}\right\}.\end{aligned} \tag{3.19}$$

Second solution

$$\pm(\eta-\eta_0)=\frac{2\Pi}{\alpha_1-\alpha_2}\sqrt{\frac{\Gamma-\alpha_2}{\Gamma-\alpha_1}},\quad \Pi=\sqrt{-\frac{2\mu^2}{\delta\tau_1^2(\gamma+1)}}, \tag{3.20}$$

$$\begin{aligned}U_2(x,t)=&V_2^+(x,t)=e^{i\left(-\frac{k}{\beta}x^\beta+\frac{\omega}{\alpha}x^\alpha+\theta\right)}\left\{\tau_0+\tau_1\alpha_1+\frac{4(\alpha_2-\alpha_1)\tau_1\Pi^2}{4\Pi^2-(\alpha_1-\alpha_2)^2\left[\mu\left(\frac{1}{\beta}x^\beta+\frac{2k}{\alpha}x^\alpha\right)-\eta_0\right]^2}\right\},\\ V_2^-(x,t)=&e^{i\left(-\frac{k}{\beta}x^\beta+\frac{\omega}{\alpha}x^\alpha+\theta\right)}\left\{-\tau_0-\tau_1\alpha_1-\frac{4(\alpha_2-\alpha_1)\tau_1\Pi^2}{4\Pi^2-(\alpha_1-\alpha_2)^2\left[\mu\left(\frac{1}{\beta}x^\beta+\frac{2k}{\alpha}x^\alpha\right)-\eta_0\right]^2}\right\}.\end{aligned} \tag{3.21}$$

Third solution

$$\pm(\eta-\eta_0)=\frac{\Pi}{\alpha_1-\alpha_2}\ln\left|\frac{\Gamma-\alpha_1}{\Gamma-\alpha_2}\right|, \tag{3.22}$$

$$\begin{aligned}U_3(x,t)=&V_3^+(x,t)=e^{i\left(-\frac{k}{\beta}x^\beta+\frac{\omega}{\alpha}x^\alpha+\theta\right)}\left\{\tau_0+\tau_1\alpha_2+\frac{(\alpha_2-\alpha_1)\tau_1}{\exp\left[\frac{\alpha_1-\alpha_2}{\Pi}\left(\mu\left(\frac{1}{\beta}x^\beta+\frac{2k}{\alpha}x^\alpha\right)-\eta_0\right)\right]-1}\right\},\\ V_3^-(x,t)=&e^{i\left(-\frac{k}{\beta}x^\beta+\frac{\omega}{\alpha}x^\alpha+\theta\right)}\left\{-\tau_0-\tau_1\alpha_2-\frac{(\alpha_2-\alpha_1)\tau_1}{\exp\left[\frac{\alpha_1-\alpha_2}{\Pi}\left(\mu\left(\frac{1}{\beta}x^\beta+\frac{2k}{\alpha}x^\alpha\right)-\eta_0\right)\right]-1}\right\}.\end{aligned} \tag{3.23}$$

Fourth solution

$$\pm(\eta-\eta_0)=\frac{\Pi}{\sqrt{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}}\ln\left|\frac{\sqrt{(\alpha_1-\alpha_3)(\Gamma-\alpha_2)}-\sqrt{(\alpha_1-\alpha_2)(\Gamma-\alpha_3)}}{\sqrt{(\alpha_1-\alpha_3)(\Gamma-\alpha_2)}+\sqrt{(\alpha_1-\alpha_2)(\Gamma-\alpha_3)}}\right|, \tag{3.24}$$

$$\begin{aligned}
U_4(x,t)&=V_4^+(x,t)=e^{i\left(-\frac{k}{\beta}x^\beta+\frac{\omega}{\alpha}x^\alpha+\theta\right)}\\
&\times\left\{\tau_0+\tau_1\alpha_1-\frac{2(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\tau_1}{2\alpha_1-\alpha_2-\alpha_3+(\alpha_3-\alpha_2)\cosh\left[\frac{\sqrt{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}}{\Pi}\left(\mu\left(\frac{1}{\beta}x^\beta+\frac{2k}{\alpha}x^\alpha\right)-\eta_0\right)\right]}\right\},\\
V_4^-(x,t)&=e^{i\left(-\frac{k}{\beta}x^\beta+\frac{\omega}{\alpha}x^\alpha+\theta\right)}\\
&\times\left\{-\tau_0-\tau_1\alpha_1+\frac{2(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\tau_1}{2\alpha_1-\alpha_2-\alpha_3+(\alpha_3-\alpha_2)\cosh\left[\frac{\sqrt{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}}{\Pi}\left(\mu\left(\frac{1}{\beta}x^\beta+\frac{2k}{\alpha}x^\alpha\right)-\eta_0\right)\right]}\right\}.
\end{aligned} \tag{3.25}$$

Fifth solution

$$\pm(\eta-\eta_0)=\frac{2\Pi}{\sqrt{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}}F(\varphi,l),\quad \alpha_1>\alpha_2>\alpha_3>\alpha_4, \tag{3.26}$$

where

$$F(\varphi,l)=\int_0^\varphi\frac{d\psi}{\sqrt{1-l^2\sin^2\psi}},\quad \varphi=\arcsin\sqrt{\frac{(\alpha_2-\alpha_4)(\Gamma-\alpha_1)}{(\alpha_1-\alpha_4)(\Gamma-\alpha_2)}},\quad l^2=\frac{(\alpha_2-\alpha_3)(\alpha_1-\alpha_4)}{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}, \tag{3.27}$$

$$\begin{aligned}
U_5(x,t)&=V_5^+(x,t)=e^{i\left(-\frac{k}{\beta}x^\beta+\frac{\omega}{\alpha}x^\alpha+\theta\right)}\\
&\times\left\{\tau_0+\tau_1\alpha_2+\frac{2(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\tau_1}{\alpha_4-\alpha_2+(\alpha_1-\alpha_4)sn^2\left[\frac{\sqrt{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}}{2\Pi}\left(\mu\left(\frac{1}{\beta}x^\beta+\frac{2k}{\alpha}x^\alpha\right)-\eta_0\right)\right]},\frac{(\alpha_2-\alpha_3)(\alpha_1-\alpha_4)}{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}\right\},
\end{aligned} \tag{3.28}$$

$$\begin{aligned}
V_5^-(x,t)&=e^{i\left(-\frac{k}{\beta}x^\beta+\frac{\omega}{\alpha}x^\alpha+\theta\right)}\\
&\times\left\{-\tau_0-\tau_1\alpha_2-\frac{2(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\tau_1}{\alpha_4-\alpha_2+(\alpha_1-\alpha_4)sn^2\left[\frac{\sqrt{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}}{2\Pi}\left(\mu\left(\frac{1}{\beta}x^\beta+\frac{2k}{\alpha}x^\alpha\right)-\eta_0\right)\right]},\frac{(\alpha_2-\alpha_3)(\alpha_1-\alpha_4)}{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}\right\}.
\end{aligned} \tag{3.29}$$

If the modulus $l\to1$, then the dark solutions (3.28) and (3.29), will be as

$$U_{5_1}(x,t) = V^+_{5_1}(x,t) = e^{i\left(-\frac{k}{\beta}x^\beta+\frac{\omega}{\alpha}x^\alpha+\theta\right)} \times\left\{\tau_0+\tau_1\alpha_2+\frac{2(\alpha_1-\alpha_2)(\alpha_1-\alpha_4)\tau_1}{\alpha_4-\alpha_2+(\alpha_1-\alpha_4)\tanh^2\left[\frac{\sqrt{(\alpha_1-\alpha_2)(\alpha_1-\alpha_4)}}{2\Pi}\left(\mu\left(\frac{1}{\beta}x^\beta+\frac{2k}{\alpha}x^\alpha\right)-\eta_0\right)\right]}\right\}, \tag{3.30}$$

$$V^-_{5_1}(x,t) = e^{i\left(-\frac{k}{\beta}x^\beta+\frac{\omega}{\alpha}x^\alpha+\theta\right)} \times\left\{-\tau_0-\tau_1\alpha_2-\frac{2(\alpha_1-\alpha_2)(\alpha_1-\alpha_4)\tau_1}{\alpha_4-\alpha_2+(\alpha_1-\alpha_4)\tanh^2\left[\frac{\sqrt{(\alpha_1-\alpha_2)(\alpha_1-\alpha_4)}}{2\Pi}\left(\mu\left(\frac{1}{\beta}x^\beta+\frac{2k}{\alpha}x^\alpha\right)-\eta_0\right)\right]}\right\}, \tag{3.31}$$

where $\alpha_1=\alpha_2$. Moreover, if $l\to 0$, therefore the compacton solutions (3.28) and (3.29), can be written as

$$U_{5_2}(x,t) = V^+_{5_2}(x,t) = e^{i\left(-\frac{k}{\beta}x^\beta+\frac{\omega}{\alpha}x^\alpha+\theta\right)} \times\left\{\tau_0+\tau_1\alpha_3+\frac{2(\alpha_1-\alpha_3)(\alpha_1-\alpha_3)\tau_1}{\alpha_4-\alpha_3+(\alpha_1-\alpha_4)\sin^2\left[\frac{\sqrt{(\alpha_1-\alpha_3)(\alpha_1-\alpha_3)}}{2\Pi}\left(\mu\left(\frac{1}{\beta}x^\beta+\frac{2k}{\alpha}x^\alpha\right)-\eta_0\right)\right]}\right\}, \tag{3.32}$$

$$V^-_{5_2}(x,t) = e^{i\left(-\frac{k}{\beta}x^\beta+\frac{\omega}{\alpha}x^\alpha+\theta\right)} \times\left\{-\tau_0-\tau_1\alpha_3-\frac{2(\alpha_1-\alpha_3)(\alpha_1-\alpha_3)\tau_1}{\alpha_4-\alpha_3+(\alpha_1-\alpha_4)\sin^2\left[\frac{\sqrt{(\alpha_1-\alpha_2)(\alpha_1-\alpha_4)}}{2\Pi}\left(\mu\left(\frac{1}{\beta}x^\beta+\frac{2k}{\alpha}x^\alpha\right)-\eta_0\right)\right]}\right\}, \tag{3.33}$$

where $\alpha_3=\alpha_4$.

3.1.2 Case II: $\delta=1, \theta=5$, and $\epsilon=1$

Let $\delta=1, \theta=5$, and $\epsilon=1$ for Eqs. (3.4) and (3.5), then we get

$$u(\eta) = \tau_0+\tau_1\Gamma, \tag{3.34}$$

$$v(\eta) = \rho_0+\rho_1\Gamma, \tag{3.35}$$

$$(u'(\eta))^2 = \frac{\tau_1^2(\xi_5\Gamma^5+\xi_4\Gamma^4+\xi_3\Gamma^3+\xi_2\Gamma^2+\xi_1\Gamma+\xi_0)}{\zeta_0+\zeta_1\Gamma}, \tag{3.36}$$

$$(v'(\eta))^2 = \frac{\rho_1^2(\xi_5\Gamma^5+\xi_4\Gamma^4+\xi_3\Gamma^3+\xi_2\Gamma^2+\xi_1\Gamma+\xi_0)}{\zeta_0+\zeta_1\Gamma}, \tag{3.37}$$

where $\xi_5 \neq 0$ and $\zeta_1 \neq 0$. With the solve the algebraic equation system (3.9), we can easily get the following sets

- *First set*

$$
\begin{aligned}
& k = k, \ \ \mu = \mu, \ \ \omega = \frac{3\delta\gamma\rho_0^2\zeta_1 + 3\delta\rho_0^2\zeta_1 - k^2\zeta_1 + \xi_3\mu^2}{\zeta_1}, \ \ \rho_0 = \rho_0, \\
& \rho_1 = \rho_1, \ \ \tau_0 = \pm\rho_0, \ \ \tau_1 = \pm\rho_1, \ \ \xi_0 = 0, \ \ \xi_1 = \xi_1, \\
& \xi_2 = \frac{2\rho_0(2\delta\rho_0^2\zeta_1 + \xi_3\mu^2 + 2\delta\gamma\rho_0^2\zeta_1)}{\rho_1\mu^2}, \ \ \xi_3 = \xi_3, \ \ \xi_4 = -\frac{2\rho_0\delta\rho_1\zeta_1(\gamma+1)}{\mu^2}, \\
& \xi_5 = -\frac{\delta\zeta_1\rho_1^2(\gamma+1)}{2\mu^2}, \ \ \zeta_0 = 0, \ \ \zeta_1 = \zeta_1.
\end{aligned}
\tag{3.38}
$$

Inserting these results into Eqs. (3.5) and (3.10), we have

$$
\pm(\eta - \eta_0) = \int \frac{\sqrt{\frac{\zeta_1}{\xi_5}\Gamma}\, d\Gamma}{\sqrt{\Gamma^5 + \frac{\xi_4}{\xi_5}\Gamma^4 + \frac{\xi_3}{\xi_5}\Gamma^3 + \frac{\xi_2}{\xi_5}\Gamma^2 + \frac{\xi_1}{\xi_5}\Gamma}} = \int \frac{\sqrt{\frac{\zeta_1}{\xi_5}}\, d\Gamma}{\sqrt{\Gamma^4 + \frac{\xi_4}{\xi_5}\Gamma^3 + \frac{\xi_3}{\xi_5}\Gamma^2 + \frac{\xi_2}{\xi_5}\Gamma + \frac{\xi_1}{\xi_5}}}.
\tag{3.39}
$$

According to the before subsection (3.1.1), we can get to similar results.

- *Second set*

$$
\begin{aligned}
& k = k, \ \ \mu = \mu, \ \ \omega = \frac{3\delta\gamma\rho_0^2\zeta_1 + 3\delta\rho_0^2\zeta_1 - k^2\zeta_1 + 2\delta\rho_0\rho_1\zeta_0 + 2\delta\gamma\rho_0\rho_1\zeta_0 + \xi_3\mu^2}{\zeta_1}, \ \ \rho_0 = \rho_0, \ \ \rho_1 = \rho_1, \\
& \tau_0 = \pm\rho_0, \ \ \tau_1 = \pm\rho_1, \ \ \xi_0 = \xi_0, \ \ \xi_1 = \frac{4\delta\rho_0^3\zeta_0^2\zeta_1 + 4\rho_0^2\zeta_0^3\delta\gamma\rho_1 + 4\rho_0^2\zeta_0^3\delta\rho_1 + 4\rho_0^3\zeta_0^2\zeta_1\delta\gamma + 2\rho_0\zeta_0^2\mu^2\xi_3 + \mu^2\zeta_1^2\xi_0\rho_1}{\zeta_1\rho_1\mu^2\zeta_0}, \\
& \xi_2 = \frac{4\delta\rho_0^3\zeta_1^2\gamma + 4\delta\rho_0^2\rho_1\zeta_0\zeta_1 + 2\delta\rho_0\rho_1^2\zeta_0^2 + \mu^2\rho_1\xi_3\zeta_0 + 4\delta\gamma\rho_0^2\zeta_0\rho_1\zeta_1 + 2\rho_0\zeta_1\mu^2\xi_3 + 2\delta\gamma\rho_0\rho_1^2\zeta_0^2 + 4\delta\rho_0^3\zeta_1^2}{\zeta_1\rho_1\mu^2}, \\
& \xi_3 = \xi_3, \ \ \xi_4 = -\frac{\rho_1\delta(\gamma\rho_1\zeta_0 + 4\gamma\rho_0\zeta_1 + \rho_1\zeta_0 + 4\rho_0\zeta_1)}{2\mu^2}, \ \ \xi_5 = -\frac{\delta\zeta_1\rho_1^2(\gamma+1)}{2\mu^2}, \ \ \zeta_0 = \zeta_0, \ \ \zeta_1 = \zeta_1.
\end{aligned}
\tag{3.40}
$$

Substituting these results into Eqs. (3.5) and (3.10), then (3.10) can be written as

$$
\pm(\eta - \eta_0) = \int \frac{\sqrt{\frac{\zeta_0}{\xi_5} + \frac{\zeta_1}{\xi_5}\Gamma}\, d\Gamma}{\sqrt{\Gamma^5 + \frac{\xi_4}{\xi_5}\Gamma^4 + \frac{\xi_3}{\xi_5}\Gamma^3 + \frac{\xi_2}{\xi_5}\Gamma^2 + \frac{\xi_1}{\xi_5}\Gamma + \frac{\xi_0}{\xi_5}}}.
\tag{3.41}
$$

To integrate Eq. (3.41), we must discuss the following families:

Family 1 Suppose $F(\Gamma) = \Gamma^5 + \frac{\xi_4}{\xi_5}\Gamma^4 + \frac{\xi_3}{\xi_5}\Gamma^3 + \frac{\xi_2}{\xi_5}\Gamma^2 + \frac{\xi_1}{\xi_5}\Gamma + \frac{\xi_0}{\xi_5}$ can be written in the following form:

$$F(\Gamma) = (\Gamma - \alpha_1)^5, \tag{3.42}$$

where α_1 is a free constant. Then, we have

$$\pm(\eta - \eta_0) = \int \frac{\sqrt{\frac{\zeta_0}{\xi_5} + \frac{\zeta_1}{\xi_5}\Gamma}\, d\Gamma}{(\Gamma - \alpha_1)^2\sqrt{\Gamma - \alpha_1}} = -\frac{2}{3}\frac{\left(\frac{\zeta_0}{\xi_5} + \frac{\zeta_1}{\xi_5}\Gamma\right)^{\frac{3}{2}}}{\left(\frac{\zeta_0}{\xi_5} + \frac{\zeta_1}{\xi_5}\alpha_1\right)(\Gamma - \alpha_1)^{\frac{3}{2}}}, \tag{3.43}$$

or

$$\Gamma = \frac{\frac{\zeta_0}{\xi_5} + \alpha_1\left[-\frac{3}{2}\left(\frac{\zeta_0}{\xi_5} + \frac{\zeta_1}{\xi_5}\alpha_1\right)(\eta - \eta_0)\right]^{\frac{3}{2}}}{\left[-\frac{3}{2}\left(\frac{\zeta_0}{\xi_5} + \frac{\zeta_1}{\xi_5}\alpha_1\right)(\eta - \eta_0)\right]^{\frac{3}{2}} - \frac{\zeta_1}{\xi_5}}. \tag{3.44}$$

Substituting (3.44) into (3.34) and (3.35), the exact solutions, can be found, respectively, as:

$$\begin{aligned} U_6(x,t) =& V_6^+(x,t) = e^{i\left(-\frac{k}{\beta}x^\beta + \frac{3\delta\gamma\rho_0^2\zeta_1 + 3\delta\rho_0^2\zeta_1 - k^2\zeta_1 + 2\delta\rho_0\rho_1\zeta_0 + 2\delta\gamma\rho_0\rho_1\zeta_0 + \xi_3\mu^2}{\alpha\zeta_1}x^\alpha + \theta\right)} \\ &\times\left\{\tau_0 + \tau_1\frac{-\frac{2\zeta_0\mu^2}{\delta\zeta_1\rho_1^2(\gamma+1)} + \alpha_1\left[\frac{3\mu^2}{\delta\zeta_1\rho_1^2(\gamma+1)}\left(\zeta_0 + \zeta_1\alpha_1\right)\left(\mu\left(\frac{1}{\beta}x^\beta + \frac{2k}{\alpha}x^\alpha\right) - \eta_0\right)\right]^{\frac{3}{2}}}{\left[\frac{3\mu^2}{\delta\zeta_1\rho_1^2(\gamma+1)}\left(\zeta_0 + \zeta_1\alpha_1\right)\left(\mu\left(\frac{1}{\beta}x^\beta + \frac{2k}{\alpha}x^\alpha\right) - \eta_0\right)\right]^{\frac{3}{2}} + \frac{2\mu^2}{\delta\rho_1^2(\gamma+1)}}\right\}, \end{aligned} \tag{3.45}$$

$$\begin{aligned} V_6^-(x,t) =& e^{i\left(-\frac{k}{\beta}x^\beta + \frac{3\delta\gamma\rho_0^2\zeta_1 + 3\delta\rho_0^2\zeta_1 - k^2\zeta_1 + 2\delta\rho_0\rho_1\zeta_0 + 2\delta\gamma\rho_0\rho_1\zeta_0 + \xi_3\mu^2}{\alpha\zeta_1}x^\alpha + \theta\right)} \\ &\times\left\{-\tau_0 - \tau_1\frac{-\frac{2\zeta_0\mu^2}{\delta\zeta_1\rho_1^2(\gamma+1)} + \alpha_1\left[\frac{3\mu^2}{\delta\zeta_1\rho_1^2(\gamma+1)}\left(\zeta_0 + \zeta_1\alpha_1\right)\left(\mu\left(\frac{1}{\beta}x^\beta + \frac{2k}{\alpha}x^\alpha\right) - \eta_0\right)\right]^{\frac{3}{2}}}{\left[\frac{3\mu^2}{\delta\zeta_1\rho_1^2(\gamma+1)}\left(\zeta_0 + \zeta_1\alpha_1\right)\left(\mu\left(\frac{1}{\beta}x^\beta + \frac{2k}{\alpha}x^\alpha\right) - \eta_0\right)\right]^{\frac{3}{2}} + \frac{2\mu^2}{\delta\rho_1^2(\gamma+1)}}\right\}. \end{aligned} \tag{3.46}$$

Family 2 Suppose $F(\Gamma) = \Gamma^5 + \frac{\xi_4}{\xi_5}\Gamma^4 + \frac{\xi_3}{\xi_5}\Gamma^3 + \frac{\xi_2}{\xi_5}\Gamma^2 + \frac{\xi_1}{\xi_5}\Gamma + \frac{\xi_0}{\xi_5}$ can be written in the following form:

$$F(\Gamma) = (\Gamma - \alpha_1)^4(\Gamma - \alpha_2), \tag{3.47}$$

where α_1 and α_2 are free constants. Then, we have

$$\pm(\eta - \eta_0) = \int \frac{\sqrt{\frac{\zeta_0}{\xi_5} + \frac{\zeta_1}{\xi_5}\Gamma}\, d\Gamma}{(\Gamma - \alpha_1)^2\sqrt{\Gamma - \alpha_2}} = -\frac{1}{2}\frac{(\Gamma - \alpha_1)(\Gamma - \alpha_2)\sqrt{\frac{\zeta_0}{\xi_5} + \frac{\zeta_1}{\xi_5}\Gamma}\,[2\Pi_1\Pi_2 + \Pi_3\ln(Y)]}{(\alpha_1 - \alpha_2)\Pi_1\Pi_2\sqrt{(\Gamma - \alpha_1)^4(\Gamma - \alpha_2)}}, \tag{3.48}$$

where

$$\Pi_1 = \sqrt{(\alpha_1 - \alpha_2)\left(\frac{\zeta_0}{\xi_5} + \frac{\zeta_1}{\xi_5}\alpha_1\right)}, \quad \Pi_2 = \sqrt{\frac{\zeta_1}{\xi_5}\Gamma^2 + \left(\frac{\zeta_0}{\xi_5} - \frac{\zeta_1}{\xi_5}\alpha_2\right)\Gamma - \frac{\zeta_0}{\xi_5}\alpha_2},$$
$$\Pi_3 = (\alpha_1 - \Gamma)\left(\frac{\zeta_0}{\xi_5} - \frac{\zeta_1}{\xi_5}\alpha_2\right), \tag{3.49}$$

$$\Pi_4 = \Gamma\left(\frac{\zeta_0}{\xi_5} - \frac{\zeta_1}{\xi_5}\alpha_2 + 2\frac{\zeta_1}{\xi_5}\alpha_1\right) + \frac{\zeta_0}{\xi_5}\alpha_1 - 2\frac{\zeta_0}{\xi_5}\alpha_2 - \frac{\zeta_1}{\xi_5}\alpha_1\alpha_2, \quad Y = \frac{\Pi_4 + 2\Pi_1\Pi_2}{\Gamma - \alpha_1} \tag{3.50}$$

Family 3 Suppose $F(\Gamma) = \Gamma^5 + \frac{\xi_4}{\xi_5}\Gamma^4 + \frac{\xi_3}{\xi_5}\Gamma^3 + \frac{\xi_2}{\xi_5}\Gamma^2 + \frac{\xi_1}{\xi_5}\Gamma + \frac{\xi_0}{\xi_5}$ can be written in the following form:

$$F(\Gamma) = (\Gamma - \alpha_1)^3(\Gamma - \alpha_2)^2, \tag{3.51}$$

where α_1 and α_2 are free constants. Then, we have

$$\pm(\eta - \eta_0) = \int \frac{\sqrt{\frac{\zeta_0}{\xi_5} + \frac{\zeta_1}{\xi_5}\Gamma}\, d\Gamma}{\sqrt{(\Gamma - \alpha_1)^3(\Gamma - \alpha_2)^2}} = -\frac{\sqrt{\frac{\zeta_0}{\xi_5} + \frac{\zeta_1}{\xi_5}\Gamma}\,[2(\alpha_1 - \alpha_2)\Pi_2 + (\Gamma - \alpha_1)\Pi_1 \ln(Y)]}{(\alpha_1 - \alpha_2)^2\Pi_2\sqrt{\Gamma - \alpha_1}}, \tag{3.52}$$

where

$$\Pi_1 = \sqrt{-(\alpha_1 - \alpha_2)\left(\frac{\zeta_0}{\xi_5} + \frac{\zeta_1}{\xi_5}\alpha_2\right)}, \quad \Pi_2 = \sqrt{\frac{\zeta_1}{\xi_5}\Gamma^2 + \left(\frac{\zeta_0}{\xi_5} - \frac{\zeta_1}{\xi_5}\alpha_1\right)\Gamma - \frac{\zeta_0}{\xi_5}\alpha_1}, \tag{3.53}$$

$$\Pi_3 = \Gamma\left(\frac{\zeta_0}{\xi_5} - \frac{\zeta_1}{\xi_5}\alpha_1 + 2\frac{\zeta_1}{\xi_5}\alpha_2\right) + \frac{\zeta_0}{\xi_5}\alpha_2 - 2\frac{\zeta_0}{\xi_5}\alpha_1 - \frac{\zeta_1}{\xi_5}\alpha_1\alpha_2, \quad Y = \frac{\Pi_3 + 2\Pi_1\Pi_2}{\Gamma - \alpha_2} \tag{3.54}$$

Family 4 Suppose $F(\Gamma) = \Gamma^5 + \frac{\xi_4}{\xi_5}\Gamma^4 + \frac{\xi_3}{\xi_5}\Gamma^3 + \frac{\xi_2}{\xi_5}\Gamma^2 + \frac{\xi_1}{\xi_5}\Gamma + \frac{\xi_0}{\xi_5}$ can be written in the following form:

$$F(\Gamma) = (\Gamma - \alpha_1)^2(\Gamma - \alpha_2)^2(\Gamma - \alpha_3), \tag{3.55}$$

where α_1, α_2 and α_3 are free constants. Then, we have

$$\begin{aligned}\pm(\eta - \eta_0) &= \int \frac{\sqrt{\frac{\zeta_0}{\xi_5} + \frac{\zeta_1}{\xi_5}\Gamma}\, d\Gamma}{\sqrt{(\Gamma - \alpha_1)^3(\Gamma - \alpha_2)^2}} \\ &= -\frac{\sqrt{\Gamma - \alpha_3}\sqrt{\frac{\zeta_0}{\xi_5} + \frac{\zeta_1}{\xi_5}\Gamma}\,[(\alpha_2 - \alpha_3)\Pi_1 \ln(Y_1) - (\alpha_1 - \alpha_3)\Pi_2 \ln(Y_2)]}{\Pi_3\Pi_6},\end{aligned} \tag{3.56}$$

where

$$\Pi_1 = \sqrt{(\alpha_1 - \alpha_3)\left(\frac{\zeta_0}{\xi_5} + \frac{\zeta_1}{\xi_5}\alpha_1\right)}, \quad \Pi_2 = \sqrt{(\alpha_2 - \alpha_3)\left(\frac{\zeta_0}{\xi_5} + \frac{\zeta_1}{\xi_5}\alpha_2\right)}, \tag{3.57}$$

$$\begin{aligned}\Pi_3 =&\sqrt{\frac{\zeta_1}{\xi_5}\Gamma^2+\left(\frac{\zeta_0}{\xi_5}-\frac{\zeta_1}{\xi_5}\alpha_3\right)\Gamma-\frac{\zeta_0}{\xi_5}\alpha_3},\\ \Pi_4 =&\Gamma\left(\frac{\zeta_0}{\xi_5}-\frac{\zeta_1}{\xi_5}\alpha_3+2\frac{\zeta_1}{\xi_5}\alpha_1\right)+\frac{\zeta_0}{\xi_5}\alpha_1-2\frac{\zeta_0}{\xi_5}\alpha_3-\frac{\zeta_1}{\xi_5}\alpha_1\alpha_3,\end{aligned} \tag{3.58}$$

$$\Pi_5 =\Gamma\left(\frac{\zeta_0}{\xi_5}-\frac{\zeta_1}{\xi_5}\alpha_3+2\frac{\zeta_1}{\xi_5}\alpha_2\right)+\frac{\zeta_0}{\xi_5}\alpha_2-2\frac{\zeta_0}{\xi_5}\alpha_3-\frac{\zeta_1}{\xi_5}\alpha_2\alpha_3, \tag{3.59}$$

$$\Pi_6 =(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)(\alpha_2-\alpha_3),\quad Y_1=\frac{\Pi_4+2\Pi_1\Pi_3}{\Gamma-\alpha_1},\quad Y_2=\frac{\Pi_5+2\Pi_2\Pi_3}{\Gamma-\alpha_2}. \tag{3.60}$$

3.1.3 Case III: $\delta = 2, \theta = 6,$ *and* $\epsilon = 0$

Let $\delta = 2, \theta = 6$, and $\epsilon = 0$ for Eqs. (3.4) and (3.5), then we get

$$u(\eta) =\tau_0+\tau_1\Gamma+\tau_2\Gamma^2, \tag{3.61}$$

$$v(\eta) =\rho_0+\rho_1\Gamma+\rho_2\Gamma^2, \tag{3.62}$$

$$(u'(\eta))^2 =\frac{(\tau_1+2\tau_2\Gamma)^2(\xi_6\Gamma^6+\xi_5\Gamma^5+\xi_4\Gamma^4+\xi_3\Gamma^3+\xi_2\Gamma^2+\xi_1\Gamma+\xi_0)}{\zeta_0}, \tag{3.63}$$

$$(v'(\eta))^2 =\frac{(\rho_1+2\rho_2\Gamma)^2(\xi_6\Gamma^6+\xi_5\Gamma^5+\xi_4\Gamma^4+\xi_3\Gamma^3+\xi_2\Gamma^2+\xi_1\Gamma+\xi_0)}{\zeta_0}, \tag{3.64}$$

where $\xi_6 \neq 0$ and $\zeta_0 \neq 0$. With the solve the algebraic equation system (3.9), we can easily get the following sets

- *First set*

$$\begin{aligned}
&k=k,\ \ \mu=\mu,\ \ \omega=\omega,\ \ \rho_0=\tau_0,\ \ \rho_1=\tau_1,\ \ \rho_2=\rho_2,\ \ \tau_0=\tau_0,\ \ \tau_1=\tau_1,\ \ \tau_2=\rho_2,\\
&\xi_0=-\frac{\zeta_0(256\tau_0\rho_2^3(\tau_0^2\delta\gamma+\delta\tau_0^2-\omega-k^2)-32\tau_1^2\rho_2^2(-\omega+3\tau_0^2\delta\gamma-k^2+3\delta\tau_0^2)+16\tau_1^4\delta\tau_0\rho_2(1+\gamma)-\tau_1^6\delta(1+\gamma))}{512\mu^2\rho_2^4},\\
&\xi_1=-\frac{\zeta_0\tau_1(96\delta\gamma\rho_2^2\tau_0^2-32k^2\rho_2^2+96\delta\tau_0^2\rho_2^2-32\omega\rho_2^2+\tau_1^4\delta\gamma+\tau_1^4\delta-16\tau_1^2\delta\gamma\tau_0\rho_2-16\delta\tau_0\tau_1^2\rho_2)}{128\rho_2^3\mu^2},\\
&\xi_2=-\frac{32\rho_2^2(-\omega+3\tau_0^2\delta\gamma-k^2+3\delta\tau_0^2)+48\delta\tau_0\tau_1^2\rho_2(1+\gamma)-3\tau_1^4\delta(1+\gamma)}{128\rho_2^2\mu^2},\\
&\xi_3=-\frac{\delta\zeta_0\tau_1(\gamma\tau_1^2+\tau_1^2+16\gamma\tau_0\rho_2+16\tau_0\rho_2)}{16\rho_2\mu^2},\ \ \xi_4=-\frac{\delta\zeta_0(11\gamma\tau_1^2+11\tau_1^2+16\gamma\tau_0\rho_2+16\tau_0\rho_2)}{32\mu^2},\\
&\xi_5=-\frac{3\delta\zeta_0\tau_1\rho_2(\gamma+1)}{8\mu^2},\ \ \xi_6=-\frac{\delta\zeta_0\rho_2^2(\gamma+1)}{8\mu^2},\ \ \zeta_0=\zeta_0.
\end{aligned} \tag{3.65}$$

Inserting these results into Eqs. (3.5) and (3.10), then (3.10) can be written as

$$\pm(\eta-\eta_0)=\int\frac{\sqrt{\frac{\zeta_0}{\xi_6}}\,d\Gamma}{\sqrt{\Gamma^6+\frac{\xi_5}{\xi_6}\Gamma^5+\frac{\xi_4}{\xi_6}\Gamma^4+\frac{\xi_3}{\xi_6}\Gamma^3+\frac{\xi_2}{\xi_6}\Gamma^2+\frac{\xi_1}{\xi_6}\Gamma+\frac{\xi_0}{\xi_6}}}. \tag{3.66}$$

Integrating (3.66), we obtain the solutions to the Eq. (1.1) the following families:

Family 1 Suppose $F(\Gamma)=\Gamma^6+\frac{\xi_5}{\xi_6}\Gamma^5+\frac{\xi_4}{\xi_6}\Gamma^4+\frac{\xi_3}{\xi_6}\Gamma^3+\frac{\xi_2}{\xi_6}\Gamma^2+\frac{\xi_1}{\xi_6}\Gamma+\frac{\xi_0}{\xi_6}$ can be written in the following form:

$$F(\Gamma)=(\Gamma-\alpha_1)^6, \tag{3.67}$$

where α_1 is a free constant. Then, we have

$$\pm(\eta-\eta_0)=\int\frac{\Pi}{F(\Gamma)}d\Gamma\Longrightarrow\Gamma=\frac{2(\eta-\eta_0)\alpha_1\pm\sqrt{-2\Pi(\eta-\eta_0)}}{2(\eta-\eta_0)},\quad \Pi=\sqrt{-\frac{8\mu^2}{\delta\rho_2^2(\gamma+1)}}, \tag{3.68}$$

$$\begin{aligned}U_7(x,t)=V_7^+(x,t)=e^{i\left(-\frac{k}{\beta}x^\beta+\frac{\omega}{\alpha}x^\alpha+\theta\right)}&\left\{\tau_0+\tau_1\frac{2(\eta-\eta_0)\alpha_1\pm\sqrt{-2\Pi\left(\mu\left(\frac{1}{\beta}x^\beta+\frac{2k}{\alpha}x^\alpha\right)-\eta_0\right)}}{2\left(\mu\left(\frac{1}{\beta}x^\beta+\frac{2k}{\alpha}x^\alpha\right)-\eta_0\right)}\right.\\&\left.+\tau_2\left(\frac{2(\eta-\eta_0)\alpha_1\pm\sqrt{-2\Pi\left(\mu\left(\frac{1}{\beta}x^\beta+\frac{2k}{\alpha}x^\alpha\right)-\eta_0\right)}}{2\left(\mu\left(\frac{1}{\beta}x^\beta+\frac{2k}{\alpha}x^\alpha\right)-\eta_0\right)}\right)^2\right\},\\V_7^-(x,t)=e^{i\left(-\frac{k}{\beta}x^\beta+\frac{\omega}{\alpha}x^\alpha+\theta\right)}&\left\{-\tau_0-\tau_1\frac{2(\eta-\eta_0)\alpha_1\pm\sqrt{-2\Pi\left(\mu\left(\frac{1}{\beta}x^\beta+\frac{2k}{\alpha}x^\alpha\right)-\eta_0\right)}}{2\left(\mu\left(\frac{1}{\beta}x^\beta+\frac{2k}{\alpha}x^\alpha\right)-\eta_0\right)}\right.\\&\left.-\tau_2\left(\frac{2(\eta-\eta_0)\alpha_1\pm\sqrt{-2\Pi\left(\mu\left(\frac{1}{\beta}x^\beta+\frac{2k}{\alpha}x^\alpha\right)-\eta_0\right)}}{2\left(\mu\left(\frac{1}{\beta}x^\beta+\frac{2k}{\alpha}x^\alpha\right)-\eta_0\right)}\right)^2\right\}.\end{aligned} \tag{3.69}$$

Family 2 Suppose $F(\Gamma)=\Gamma^6+\frac{\xi_5}{\xi_6}\Gamma^5+\frac{\xi_4}{\xi_6}\Gamma^4+\frac{\xi_3}{\xi_6}\Gamma^3+\frac{\xi_2}{\xi_6}\Gamma^2+\frac{\xi_1}{\xi_6}\Gamma+\frac{\xi_0}{\xi_6}$ can be written in the following form:

$$F(\Gamma)=(\Gamma-\alpha_1)^5(\Gamma-\alpha_2), \tag{3.70}$$

where α_1 and α_2 are free constants. Then, we have

$$\pm(\eta-\eta_0)=\int\frac{\Pi}{F(\Gamma)}d\Gamma\Longrightarrow\Gamma=\frac{2(\eta-\eta_0)\alpha_1\pm\sqrt{-2\Pi(\eta-\eta_0)}}{2(\eta-\eta_0)},\quad \Pi=\sqrt{-\frac{8\mu^2}{\delta\rho_2^2(\gamma+1)}}, \tag{3.71}$$

where

$$M=9(\eta-\eta_0)^2(\alpha_1-\alpha_2)^4-16,\ \ S=\frac{(\alpha_1-\alpha_2)^5}{\sqrt{M}},\ \ Z=2(\alpha_1-\alpha_2)^3+6(\eta-\eta_0)S, \tag{3.72}$$

$$\Gamma=\alpha_1+\frac{\sqrt[3]{ZM^2}}{M}-\frac{4(\alpha_1-\alpha_2)^2}{\sqrt[3]{ZM^2}}, \tag{3.73}$$

$$\begin{aligned}U_8(x,t)&=V_8^+(x,t)=e^{i\left(-\frac{k}{\beta}x^\beta+\frac{\omega}{\alpha}x^\alpha+\theta\right)}\\&\times\left\{\tau_0+\tau_1\left[\alpha_1+\frac{\sqrt[3]{ZM^2}}{M}-\frac{4(\alpha_1-\alpha_2)^2}{\sqrt[3]{ZM^2}}\right]+\tau_2\left[\alpha_1+\frac{\sqrt[3]{ZM^2}}{M}-\frac{4(\alpha_1-\alpha_2)^2}{\sqrt[3]{ZM^2}}\right]^2\right\},\\V_8^-(x,t)&=e^{i\left(-\frac{k}{\beta}x^\beta+\frac{\omega}{\alpha}x^\alpha+\theta\right)}\\&\times\left\{-\tau_0-\tau_1\left[\alpha_1+\frac{\sqrt[3]{ZM^2}}{M}-\frac{4(\alpha_1-\alpha_2)^2}{\sqrt[3]{ZM^2}}\right]-\tau_2\left[\alpha_1+\frac{\sqrt[3]{ZM^2}}{M}-\frac{4(\alpha_1-\alpha_2)^2}{\sqrt[3]{ZM^2}}\right]^2\right\}.\end{aligned} \tag{3.74}$$

Family 3 Suppose $F(\Gamma)=\Gamma^6+\frac{\xi_5}{\xi_6}\Gamma^5+\frac{\xi_4}{\xi_6}\Gamma^4+\frac{\xi_3}{\xi_6}\Gamma^3+\frac{\xi_2}{\xi_6}\Gamma^2+\frac{\xi_1}{\xi_6}\Gamma+\frac{\xi_0}{\xi_6}$ can be written in the following form:

$$F(\Gamma)=(\Gamma-\alpha_1)^3(\Gamma-\alpha_2)^3, \tag{3.75}$$

where α_1 and α_2 are free constants. Then, we have

$$\pm(\eta-\eta_0)=\int\frac{\Pi}{F(\Gamma)}d\Gamma\Longrightarrow\Gamma=\frac{1}{2}(\alpha_1+\alpha_2)-\frac{(\alpha_1-\alpha_2)^3(\eta-\eta_0)}{2\sqrt{(\eta-\eta_0)^2(\alpha_1-\alpha_2)^4-16}}, \tag{3.76}$$

$$\begin{aligned}U_9(x,t)=&V_9^+(x,t)=e^{i\left(-\frac{k}{\beta}x^\beta+\frac{\omega}{\alpha}x^\alpha+\theta\right)}\left\{\tau_0+\tau_1\left[\frac{1}{2}(\alpha_1+\alpha_2)-\frac{(\alpha_1-\alpha_2)^3\left(\mu\left(\frac{1}{\beta}x^\beta+\frac{2k}{\alpha}x^\alpha\right)-\eta_0\right)}{2\sqrt{\left(\mu\left(\frac{1}{\beta}x^\beta+\frac{2k}{\alpha}x^\alpha\right)-\eta_0\right)^2(\alpha_1-\alpha_2)^4-16}}\right]\right.\\&\left.+\tau_2\left[\frac{1}{2}(\alpha_1+\alpha_2)-\frac{(\alpha_1-\alpha_2)^3(\eta-\eta_0)}{2\sqrt{\left(\mu\left(\frac{1}{\beta}x^\beta+\frac{2k}{\alpha}x^\alpha\right)-\eta_0\right)^2(\alpha_1-\alpha_2)^4-16}}\right]^2\right\},\end{aligned} \tag{3.77}$$

$$V_9^-(x,t) = e^{i\left(-\frac{k}{\beta}x^\beta+\frac{\omega}{\alpha}x^\alpha+\theta\right)}\left\{-\tau_0-\tau_1\left[\frac{1}{2}(\alpha_1+\alpha_2)-\frac{(\alpha_1-\alpha_2)^3\left(\mu\left(\frac{1}{\beta}x^\beta+\frac{2k}{\alpha}x^\alpha\right)-\eta_0\right)}{2\sqrt{\left(\mu\left(\frac{1}{\beta}x^\beta+\frac{2k}{\alpha}x^\alpha\right)-\eta_0\right)^2(\alpha_1-\alpha_2)^4-16}}\right]\right.$$
$$\left.-\tau_2\left[\frac{1}{2}(\alpha_1+\alpha_2)-\frac{(\alpha_1-\alpha_2)^3(\eta-\eta_0)}{2\sqrt{\left(\mu\left(\frac{1}{\beta}x^\beta+\frac{2k}{\alpha}x^\alpha\right)-\eta_0\right)^2(\alpha_1-\alpha_2)^4-16}}\right]^2\right\}. \tag{3.78}$$

4 The exp(− Ω(η))-Expansion Method

In this section, we describe the exp(− Ω(η))-expansion method (Khan and Akbar 2014; Rayhanul Islam et al. 2015) with the following steps as:

Step 1 Suppose the nonlinear space–time fractional partial differential equations as

$$\mathcal{N}\left(u, v, \frac{\partial^\alpha u}{\partial t^\alpha}, \frac{\partial^{2\beta} u}{\partial t^{2\beta}}, \frac{\partial^\alpha v}{\partial t^\alpha}, \frac{\partial^{2\beta} v}{\partial t^{2\beta}}, \dots\right) = 0. \tag{4.1}$$

Utilizing the wave transformation

$$U(x,t) = u(\xi)e^{i\eta}, \quad V(x,t) = v(\xi)e^{i\eta}, \quad \xi = \mu\left(\frac{x^\beta}{\beta} - c\frac{t^\alpha}{\alpha}\right), \;\; \eta = \frac{-kx^\beta}{\beta} + \omega\frac{t^\alpha}{\alpha} + \theta, \tag{4.2}$$

where $k, \omega \neq 0$. Putting (4.2) into Eq. (4.1) gains a NLODE as,

$$\mathcal{Q}(u, v, u', v', \dots) = 0, \tag{4.3}$$

where prime shows the derivation with respect to ξ.

Step 2 We suppose the solution be based on the exp(− Ω(η)) as:

$$u(\xi) = \sum_{j=0}^{M} a_j \exp(-j\Omega(\xi)) + \sum_{j=1}^{M} b_j \exp(j\Omega(\xi)), \tag{4.4}$$

where $0 \le a_j \le M$ and $1 \le b_j \le M$, are constants such that $a_M, b_M \neq 0$, and, $\Omega = \Omega(\xi)$ satisfies the following ordinary differential;

$$\Omega' = \theta \exp(\Omega(\xi)) + \exp(-\,\Omega(\xi)) + \lambda. \tag{4.5}$$

The closed form solutions (Hafez et al. 2014, 2015) can be determined from Eq. (4.5):

Solution-1 If $\theta \neq 0$ and $\lambda^2 - 4\theta > 0$, then we have

$$\Omega(\xi) = \ln\left(-\frac{\sqrt{\lambda^2-4\theta}}{2\theta}\tanh\left(\frac{\sqrt{\lambda^2-4\theta}}{2}(\xi+E)\right) - \frac{\lambda}{2\theta}\right), \tag{4.6}$$

where E is integral constant.

Solution-2 If $\theta \neq 0$ and $\lambda^2 - 4\theta < 0$, then we have

$$\Omega(\xi) = \ln\left(\frac{\sqrt{-\lambda^2+4\theta}}{2\mu}\tan\left(\frac{\sqrt{-\lambda^2+4\theta}}{2}(\xi+E)\right)-\frac{\lambda}{2\theta}\right). \tag{4.7}$$

Solution-3 If $\theta = 0$, $\lambda \neq 0$, and $\lambda^2 - 4\theta > 0$, then we get

$$\Omega(\xi) = -\ln\left(\frac{\lambda}{\exp(\lambda(\xi+E))-1}\right). \tag{4.8}$$

Solution-4 If $\theta \neq 0$, $\lambda \neq 0$, and $\lambda^2 - 4\theta = 0$, then we get

$$\Omega(\xi) = \ln\left(-\frac{2\lambda(\xi+E)+4}{\lambda^2(\eta+E)}\right). \tag{4.9}$$

Solution-5 If $\theta = 0$, $\lambda = 0$, and $\lambda^2 - 4\theta = 0$, then we get

$$\Omega(\xi) = \ln(\xi+E), \tag{4.10}$$

where $a_j(0 \leq j \leq M), b_j(1 \leq j \leq M)$, λ and θ are constants to be determined. The value M can be identified by taking the balance principle which is based on the relationship between the highest order derivatives and the highest degree of the nonlinear terms occurring in Eq. (4.3).

Step 3 Substituting (4.4) into Eq. (4.3) with the value of M obtained in Step 2. collecting the coefficients of $F(\eta)$, then setting each coefficient to zero, we can get a set of over-determined equations for $a_0, a_1, b_1, a_2, b_2, \ldots, a_M, b_M$ λ, and θ with the aid of symbolic computation Maple. Solving the algebraic equations including coefficients of $a_0, a_1, b_1, a_2, b_2, \ldots, a_M, b_M$ λ, and θ into (4.4) we get to exact solution of considered problem.

4.1 Application of EEM

With the help of the EEM and by balancing u^3, u'' and v^3, v'' in Eq. (2.3) we can acquire the balance number $M = 1$, therefore the exact solutions become,

$$u(\eta) = a_0 + a_1\exp(-\Omega(\eta)) + b_1\exp(\Omega(\eta)), \tag{4.11}$$

$$v(\eta) = c_0 + c_1\exp(-\Omega(\eta)) + d_1\exp(\Omega(\eta)). \tag{4.12}$$

Inserting (4.11) and (4.12) into Eq. (2.3) and comparing the terms, we will reach the following results as:

Case 1

$$k = k,\ \ \lambda = 0,\ \ \mu = \mu,\ \ \omega = 2\mu^2\theta - k^2,\ \ \theta = \theta,\ \ a_0 = 0,\ \ a_1 = 0,\ \ b_1 = \theta\mu\sqrt{-\frac{2}{\delta(\gamma+1)}},$$
$$c_0 = 0,\ \ c_1 = 0,\ \ d_1 = \theta\mu\sqrt{-\frac{2}{\delta(\gamma+1)}}. \tag{4.13}$$

By employing (4.6) and (4.13), the exact dark solutions become,

$$U_1(x,t) = -\mu\sqrt{\frac{2\theta}{\delta(\gamma+1)}}e^{i\left(-\frac{k}{\beta}x^\beta+\frac{2\mu^2\theta-k^2}{\alpha}x^\alpha+\theta\right)}\tanh\left[\sqrt{-\theta}\left(\mu\left(\frac{1}{\beta}x^\beta+\frac{2k}{\alpha}x^\alpha\right)+E\right)\right], \tag{4.14}$$

$$V_1(x,t) = -\mu\sqrt{\frac{2\theta}{\delta(\gamma+1)}}e^{i\left(-\frac{k}{\beta}x^{\beta}+\frac{2\mu^2\theta-k^2}{\alpha}x^{\alpha}+\theta\right)}\tanh\left[\sqrt{-\theta}\left(\mu\left(\frac{1}{\beta}x^{\beta}+\frac{2k}{\alpha}x^{\alpha}\right)+E\right)\right], \tag{4.15}$$

when $\theta < 0$. By employing (4.7) and (4.13), the exact periodic solutions get,

$$U_2(x,t) = -\mu\sqrt{\frac{-2\theta}{\delta(\gamma+1)}}e^{i\left(-\frac{k}{\beta}x^{\beta}+\frac{2\mu^2\theta-k^2}{\alpha}x^{\alpha}+\theta\right)}\tan\left[\sqrt{\theta}\left(\mu\left(\frac{1}{\beta}x^{\beta}+\frac{2k}{\alpha}x^{\alpha}\right)+E\right)\right], \tag{4.16}$$

$$V_2(x,t) = -\mu\sqrt{\frac{-2\theta}{\delta(\gamma+1)}}e^{i\left(-\frac{k}{\beta}x^{\beta}+\frac{2\mu^2\theta-k^2}{\alpha}x^{\alpha}+\theta\right)}\tan\left[\sqrt{\theta}\left(\mu\left(\frac{1}{\beta}x^{\beta}+\frac{2k}{\alpha}x^{\alpha}\right)+E\right)\right], \tag{4.17}$$

when $\theta > 0$.

Case 2

$$k=k,\ \ \lambda=0,\ \ \mu=\mu,\ \ \omega=2\mu^2\theta-k^2,\ \ \theta=\theta,\ \ a_0=0,\ \ a_1=\mu\sqrt{-\frac{2}{\delta(\gamma+1)}},\ \ b_1=0,$$
$$c_0=0,\ \ c_1=\mu\sqrt{-\frac{2}{\delta(\gamma+1)}},\ \ d_1=0. \tag{4.18}$$

By employing (4.6) and (4.18), the exact singular solutions become,

$$U_3(x,t) = -\mu\sqrt{\frac{2\theta}{\delta(\gamma+1)}}e^{i\left(-\frac{k}{\beta}x^{\beta}+\frac{2\mu^2\theta-k^2}{\alpha}x^{\alpha}+\theta\right)}\coth\left[\sqrt{-\theta}\left(\mu\left(\frac{1}{\beta}x^{\beta}+\frac{2k}{\alpha}x^{\alpha}\right)+E\right)\right], \tag{4.19}$$

$$V_3(x,t) = -\mu\sqrt{\frac{2\theta}{\delta(\gamma+1)}}e^{i\left(-\frac{k}{\beta}x^{\beta}+\frac{2\mu^2\theta-k^2}{\alpha}x^{\alpha}+\theta\right)}\coth\left[\sqrt{-\theta}\left(\mu\left(\frac{1}{\beta}x^{\beta}+\frac{2k}{\alpha}x^{\alpha}\right)+E\right)\right], \tag{4.20}$$

when $\theta < 0$. By employing (4.7) and (4.18), the exact solutions get,

$$U_4(x,t) = -\mu\sqrt{\frac{-2\theta}{\delta(\gamma+1)}}e^{i\left(-\frac{k}{\beta}x^{\beta}+\frac{2\mu^2\theta-k^2}{\alpha}x^{\alpha}+\theta\right)}\cot\left[\sqrt{\theta}\left(\mu\left(\frac{1}{\beta}x^{\beta}+\frac{2k}{\alpha}x^{\alpha}\right)+E\right)\right], \tag{4.21}$$

$$V_4(x,t) = -\mu\sqrt{\frac{-2\theta}{\delta(\gamma+1)}}e^{i\left(-\frac{k}{\beta}x^{\beta}+\frac{2\mu^2\theta-k^2}{\alpha}x^{\alpha}+\theta\right)}\cot\left[\sqrt{\theta}\left(\mu\left(\frac{1}{\beta}x^{\beta}+\frac{2k}{\alpha}x^{\alpha}\right)+E\right)\right], \tag{4.22}$$

when $\theta > 0$.

Case 3

$$\begin{aligned} &k=k,\ \ \lambda=0,\ \ \mu=\mu,\ \ \omega=-6\mu^2\theta-k^2,\ \ \theta=\theta,\ \ a_0=0,\\ &a_1=\mu\sqrt{-\frac{2}{\delta(\gamma+1)}},\ \ b_1=\mu\theta\sqrt{-\frac{2}{\delta(\gamma+1)}},\\ &c_0=0,\ \ c_1=\mu\sqrt{-\frac{2}{\delta(\gamma+1)}},\ \ d_1=\mu\theta\sqrt{-\frac{2}{\delta(\gamma+1)}}. \end{aligned} \tag{4.23}$$

By employing (4.6) and (4.23), the exact dark-singular solutions catch,

$$U_5(x,t) = -\mu\sqrt{\frac{2\theta}{\delta(\gamma+1)}}e^{i\left(-\frac{k}{\beta}x^\beta+\frac{2\mu^2\theta-k^2}{\alpha}x^\alpha+\theta\right)}\left\{\tanh\left[\sqrt{-\theta}\left(\mu\left(\frac{1}{\beta}x^\beta+\frac{2k}{\alpha}x^\alpha\right)+E\right)\right] + \coth\left[\sqrt{-\theta}\left(\mu\left(\frac{1}{\beta}x^\beta+\frac{2k}{\alpha}x^\alpha\right)+E\right)\right]\right\}, \tag{4.24}$$

$$V_5(x,t) = -\mu\sqrt{\frac{2\theta}{\delta(\gamma+1)}}e^{i\left(-\frac{k}{\beta}x^\beta+\frac{2\mu^2\theta-k^2}{\alpha}x^\alpha+\theta\right)}\left\{\tanh\left[\sqrt{-\theta}\left(\mu\left(\frac{1}{\beta}x^\beta+\frac{2k}{\alpha}x^\alpha\right)+E\right)\right] + \coth\left[\sqrt{-\theta}\left(\mu\left(\frac{1}{\beta}x^\beta+\frac{2k}{\alpha}x^\alpha\right)+E\right)\right]\right\}, \tag{4.25}$$

when $\theta < 0$. By employing (4.7) and (4.23), the exact periodic solutions get,

$$U_6(x,t) = -\mu\sqrt{\frac{-2\theta}{\delta(\gamma+1)}}e^{i\left(-\frac{k}{\beta}x^\beta+\frac{2\mu^2\theta-k^2}{\alpha}x^\alpha+\theta\right)}\left\{\tan\left[\sqrt{\theta}\left(\mu\left(\frac{1}{\beta}x^\beta+\frac{2k}{\alpha}x^\alpha\right)+E\right)\right] + \cot\left[\sqrt{\theta}\left(\mu\left(\frac{1}{\beta}x^\beta+\frac{2k}{\alpha}x^\alpha\right)+E\right)\right]\right\}, \tag{4.26}$$

$$V_6(x,t) = -\mu\sqrt{\frac{-2\theta}{\delta(\gamma+1)}}e^{i\left(-\frac{k}{\beta}x^\beta+\frac{2\mu^2\theta-k^2}{\alpha}x^\alpha+\theta\right)}\left\{\tan\left[\sqrt{\theta}\left(\mu\left(\frac{1}{\beta}x^\beta+\frac{2k}{\alpha}x^\alpha\right)+E\right)\right] + \cot\left[\sqrt{\theta}\left(\mu\left(\frac{1}{\beta}x^\beta+\frac{2k}{\alpha}x^\alpha\right)+E\right)\right]\right\}, \tag{4.27}$$

when $\theta > 0$.

Case 4

$$\begin{aligned} &k = k,\ \ \lambda = 2\sqrt{\theta},\ \ \mu = \mu,\ \ \omega = -k^2,\ \ \theta = \theta,\\ &a_0 = -\frac{2\mu\sqrt{\theta}}{\sqrt{-2(\gamma+1)\delta}},\ \ a_1 = 0,\ \ b_1 = \mu\theta\sqrt{-\frac{2}{\delta(\gamma+1)}},\\ &c_0 = \mu\sqrt{-\frac{2\theta}{\delta(\gamma+1)}},\ \ c_1 = 0,\ \ d_1 = \mu\theta\sqrt{-\frac{2}{\delta(\gamma+1)}}. \end{aligned} \tag{4.28}$$

By employing (4.9) and (4.28), the exact singular solutions overtake,

$$U_7(x,t) = e^{i\left(-\frac{k}{\beta}x^\beta+\frac{2\mu^2\theta-k^2}{\alpha}x^\alpha+\theta\right)} \times\left\{-\frac{2\mu\sqrt{\theta}}{\sqrt{-2(\gamma+1)\delta}} - \mu\theta\sqrt{-\frac{2}{\delta(\gamma+1)}}\left(\frac{4\theta\left(\mu\left(\frac{1}{\beta}x^\beta+\frac{2k}{\alpha}x^\alpha\right)+E\right)}{4\sqrt{\theta}\left(\mu\left(\frac{1}{\beta}x^\beta+\frac{2k}{\alpha}x^\alpha\right)+E\right)+4}\right)\right\}, \tag{4.29}$$

$$V_7(x,t)=e^{i\left(-\frac{k}{\beta}x^\beta+\frac{2\mu^2\theta-k^2}{\alpha}x^\alpha+\theta\right)}\times\left\{-\frac{2\mu\sqrt{\theta}}{\sqrt{-2(\gamma+1)\delta}}-\mu\theta\sqrt{-\frac{2}{\delta(\gamma+1)}}\left(\frac{4\theta\left(\mu\left(\frac{1}{\beta}x^\beta+\frac{2k}{\alpha}x^\alpha\right)+E\right)}{4\sqrt{\theta}\left(\mu\left(\frac{1}{\beta}x^\beta+\frac{2k}{\alpha}x^\alpha\right)+E\right)+4}\right)\right\}, \tag{4.30}$$

when $\theta \neq 0$.

Case 5

$$k=k,\;\; \lambda=\frac{c_0\sqrt{-2(\gamma+1)\delta}}{\mu},\;\; \mu=\mu,\;\; \omega=c_0^2(\gamma+1)\delta+2\mu^2\theta-k^2,\;\; \theta=\theta,\;\; a_0=c_0,\;\; a_1=0,$$
$$b_1=\mu\theta\sqrt{-\frac{2}{\delta(\gamma+1)}},\;\; c_0=c_0,\;\; c_1=0,\;\; d_1=\mu\theta\sqrt{-\frac{2}{\delta(\gamma+1)}}. \tag{4.31}$$

By employing (4.6) and (4.31), the exact dark solutions become,

$$U_8(x,t)=V_8(x,t)=e^{i\left(-\frac{k}{\beta}x^\beta+\frac{c_0^2(\gamma+1)\delta+2\mu^2\theta-k^2}{\alpha}x^\alpha+\theta\right)}\left\{c_0+\mu\theta\sqrt{-\frac{2}{\delta(\gamma+1)}}\times\left[-\frac{\sqrt{-2(c_0^2\delta(\gamma+1)+2\mu^2\theta)}}{2\mu\theta}\tanh\left(\frac{\sqrt{-2(c_0^2\delta(\gamma+1)+2\mu^2\theta)}}{2\mu}\left(\mu\left(\frac{1}{\beta}x^\beta+\frac{2k}{\alpha}x^\alpha\right)+E\right)\right)-\frac{c_0\sqrt{-2(\gamma+1)\delta}}{2\mu\theta}\right]\right\}, \tag{4.32}$$

when $2c_0^2\delta(\gamma+1)+4\mu^2\theta<0$. By employing (4.7) and (4.31), the exact periodic solutions catch,

$$U_9(x,t)=V_9(x,t)=e^{i\left(-\frac{k}{\beta}x^\beta+\frac{c_0^2(\gamma+1)\delta+2\mu^2\theta-k^2}{\alpha}x^\alpha+\theta\right)}\left\{c_0+\mu\theta\sqrt{-\frac{2}{\delta(\gamma+1)}}\times\left[-\frac{\sqrt{2(c_0^2\delta(\gamma+1)+2\mu^2\theta)}}{2\mu\theta}\tan\left(\frac{\sqrt{2(c_0^2\delta(\gamma+1)+2\mu^2\theta)}}{2\mu}\left(\mu\left(\frac{1}{\beta}x^\beta+\frac{2k}{\alpha}x^\alpha\right)+E\right)\right)-\frac{c_0\sqrt{-2(\gamma+1)\delta}}{2\mu\theta}\right]\right\}, \tag{4.33}$$

when $2c_0^2\delta(\gamma+1)+4\mu^2\theta>0$.

Case 6

$$
\begin{aligned}
& k=k,\quad \lambda=\frac{2}{3}\sqrt{\frac{3\theta(\gamma+3)}{(\gamma-1)}},\quad \mu=\mu,\quad \omega=\frac{3k^2(\gamma-1)-4\mu^2\theta(\gamma-3)}{3(1-\gamma)},\\
& \theta=\theta,\quad a_0=\frac{1}{3}\frac{\mu\sqrt{6\theta(\gamma+3)}}{\sqrt{\delta(1-\gamma^2)}},\quad a_1=0,\\
& b_1=\mu\theta\sqrt{-\frac{2}{\delta(\gamma+1)}},\quad c_0=\mu\sqrt{\frac{2\theta(\gamma+3)}{3\delta(1-\gamma^2)}},\quad c_1=0,\\
& d_1=\mu\theta\sqrt{-\frac{2}{\delta(\gamma+1)}}.
\end{aligned}
\tag{4.34}
$$

By employing (4.6) and (4.34), the exact dark solutions get,

$$
\begin{aligned}
U_{10}(x,t)=&V_{10}(x,t)=e^{i\left(-\frac{k}{\beta}x^\beta+\frac{3k^2(\gamma-1)-4\mu^2\theta(\gamma-3)}{3\alpha(1-\gamma)}x^\alpha+\theta\right)}\left\{\frac{1}{3}\frac{\mu\sqrt{6\theta(\gamma+3)}}{\sqrt{\delta(1-\gamma^2)}}+\mu\theta\sqrt{-\frac{2}{\delta(\gamma+1)}}\right.\\
&\left.\times\left[-\frac{\sqrt{-8\theta(\gamma-3)}}{2\theta\sqrt{3(\gamma-1)}}\tanh\left(\frac{\sqrt{-8\theta(\gamma-3)}}{2\sqrt{3(\gamma-1)}}\left(\mu\left(\frac{1}{\beta}x^\beta+\frac{2k}{\alpha}x^\alpha\right)+E\right)\right)-\frac{2}{3}\sqrt{\frac{3\theta(\gamma+3)}{2\theta(\gamma-1)}}\right]\right\},
\end{aligned}
\tag{4.35}
$$

when $\frac{\theta(\gamma-3)}{\gamma-1}<0$. By employing (4.7) and (4.34), the exact periodic solutions become,

$$
\begin{aligned}
U_{11}(x,t)=&V_{11}(x,t)=e^{i\left(-\frac{k}{\beta}x^\beta+\frac{3k^2(\gamma-1)-4\mu^2\theta(\gamma-3)}{3\alpha(1-\gamma)}x^\alpha+\theta\right)}\left\{\frac{1}{3}\frac{\mu\sqrt{6\theta(\gamma+3)}}{\sqrt{\delta(1-\gamma^2)}}+\mu\theta\sqrt{-\frac{2}{\delta(\gamma+1)}}\right.\\
&\left.\times\left[\frac{\sqrt{8\theta(\gamma-3)}}{2\theta\sqrt{3(\gamma-1)}}\tan\left(\frac{\sqrt{8\theta(\gamma-3)}}{2\sqrt{3(\gamma-1)}}\left(\mu\left(\frac{1}{\beta}x^\beta+\frac{2k}{\alpha}x^\alpha\right)+E\right)\right)-\frac{2}{3}\sqrt{\frac{3\theta(\gamma+3)}{2\theta(\gamma-1)}}\right]\right\},
\end{aligned}
\tag{4.36}
$$

when $\frac{\theta(\gamma-3)}{\gamma-1}>0$.

Case 7

$$
\begin{aligned}
& k=k,\quad \lambda=\frac{c_0\sqrt{-2(\gamma+1)\delta}}{\mu},\quad \mu=\mu,\quad \omega=c_0^2(\gamma+1)\delta+2\mu^2\theta-k^2,\\
& \theta=\theta,\quad a_0=c_0,\quad a_1=\mu\sqrt{-\frac{2}{\delta(\gamma+1)}},\\
& b_1=0,\quad c_0=c_0,\quad c_1=\mu\sqrt{-\frac{2}{\delta(\gamma+1)}},\quad d_1=0.
\end{aligned}
\tag{4.37}
$$

By employing (4.6) and (4.31), the exact solutions catch,

$$U_{12}(x,t) = V_{12}(x,t) = e^{i\left(-\frac{k}{\beta}x^{\beta} + \frac{c_0^2(\gamma+1)\delta+2\mu^2\theta-k^2}{\alpha}x^{\alpha}+\theta\right)}\Bigg\{c_0 + \mu\sqrt{-\frac{2}{\delta(\gamma+1)}} \times\left[-\frac{\sqrt{-2(c_0^2\delta(\gamma+1)+2\mu^2\theta)}}{2\mu\theta}\tanh\left(\frac{\sqrt{-2(c_0^2\delta(\gamma+1)+2\mu^2\theta)}}{2\mu}\left(\mu\left(\frac{1}{\beta}x^{\beta}+\frac{2k}{\alpha}x^{\alpha}\right)+E\right)\right)-\frac{c_0\sqrt{-2(\gamma+1)\delta}}{2\mu\theta}\right]^{-1}\Bigg\}, \tag{4.38}$$

when $2c_0^2\delta(\gamma+1)+4\mu^2\theta<0$. By employing (4.7) and (4.31), the exact solutions overtake,

$$U_{13}(x,t) = V_{13}(x,t) = e^{i\left(-\frac{k}{\beta}x^{\beta} + \frac{c_0^2(\gamma+1)\delta+2\mu^2\theta-k^2}{\alpha}x^{\alpha}+\theta\right)}\Bigg\{c_0 + \mu\sqrt{-\frac{2}{\delta(\gamma+1)}} \times\left[-\frac{\sqrt{2(c_0^2\delta(\gamma+1)+2\mu^2\theta)}}{2\mu\theta}\tan\left(\frac{\sqrt{2(c_0^2\delta(\gamma+1)+2\mu^2\theta)}}{2\mu}\left(\mu\left(\frac{1}{\beta}x^{\beta}+\frac{2k}{\alpha}x^{\alpha}\right)+E\right)\right)-\frac{c_0\sqrt{-2(\gamma+1)\delta}}{2\mu\theta}\right]^{-1}\Bigg\}, \tag{4.39}$$

when $2c_0^2\delta(\gamma+1)+4\mu^2\theta>0$.

5 Description of ITEM

In this section, we describe the ITEM (Manafian 2016; Manafian and Lakestani 2016a, b) with the following steps as:

Step 1 Suppose the nonlinear space–time fractional partial differential equations as

$$\mathcal{N}\left(u, v, \frac{\partial^{\alpha}u}{\partial t^{\alpha}}, \frac{\partial^{2\beta}u}{\partial t^{2\beta}}, \frac{\partial^{\alpha}v}{\partial t^{\alpha}}, \frac{\partial^{2\beta}v}{\partial t^{2\beta}}, \dots\right) = 0. \tag{5.1}$$

Utilizing the wave transformation

$$U(x,t) = u(\xi)e^{i\eta}, \quad V(x,t) = v(\xi)e^{i\eta}, \quad \xi = \mu\left(\frac{x^{\beta}}{\beta} - c\frac{t^{\alpha}}{\alpha}\right), \ \ \eta = \frac{-kx^{\beta}}{\beta} + \omega\frac{t^{\alpha}}{\alpha} + \theta, \tag{5.2}$$

where $k, \omega \neq 0$. Putting (5.2) into Eq. (5.1) gains a NLODE as,

$$Q(u, v, u', v', \dots) = 0, \tag{5.3}$$

where prime shows the derivation with respect to ξ.

Step 2 Suppose the closed form solution of Eq. (5.3) can be determined as follows:

$$u(\xi) = \sum_{k=0}^{M} a_k \tan^k\left(\frac{\Phi(\xi)}{2}\right) + \sum_{k=1}^{M} b_k \cot^k\left(\frac{\Phi(\xi)}{2}\right), \tag{5.4}$$

where $a_k (0 \le k \le M), b_k (1 \le k \le M)$ are constants to be determined, such that $a_M \neq 0, b_M \neq 0$, and $\phi = \phi(\xi)$ satisfies the following NLODE:

$$\phi'(\eta) = a\sin(\phi(\eta)) + b\cos(\phi(\eta)) + c. \tag{5.5}$$

Step 3 To determine M. This, usually, can be accomplished by balancing the linear term(s) of highest order with the highest-order nonlinear term(s) in Eq. (5.3). Moreover, precisely, we define the degree of $u(\eta)$ as $D(u(\eta)) = M$, which gives rise to degree of another expression as follows:

$$D\left(\frac{d^q u}{d\eta^q}\right) = M + q, \quad D\left(u^p \left(\frac{d^q u}{d\eta^q}\right)^s\right) = Mp + s(M + q). \tag{5.6}$$

Step 4 Inserting (5.4) into Eq. (5.3) with the value of M obtained in Step 3. collecting the coefficients of $\tan(\phi/2)^k$, $\cot(\phi/2)^k (k = 0, 1, 2, \dots, M)$, then setting each coefficient to zero, we can get a set of over-determined equations for $a_0, a_k, b_k (k = 1, 2, \dots,)$ a, b, and c with the aid of symbolic computation Maple. Solving the algebraic equations, then substituting $a_0, a_1, \dots, a_M, b_1, \dots, b_M, k, \omega$ in (5.4).

Consider the following special solutions of Eq. (5.5):

Family 1 When $\Delta = a^2 + b^2 - c^2 < 0$ and $b - c \neq 0$, then $\phi(\xi) = 2\arctan\left[\frac{a}{b-c} - \frac{\sqrt{-\Delta}}{b-c}\tan\left(\frac{\sqrt{-\Delta}}{2}\widetilde{\xi}\right)\right]$.

Family 2 When $\Delta = a^2 + b^2 - c^2 > 0$ and $b - c \neq 0$, then $\phi(\xi) = 2\arctan\left[\frac{a}{b-c} + \frac{\sqrt{\Delta}}{b-c}\tanh\left(\frac{\sqrt{\Delta}}{2}\widetilde{\xi}\right)\right]$.

Family 3 When $a^2 + b^2 - c^2 > 0$, $b \neq 0$ and $c = 0$, then $\phi(\xi) = 2\arctan\left[\frac{a}{b} + \frac{\sqrt{b^2+a^2}}{b}\tanh\left(\frac{\sqrt{b^2+a^2}}{2}\widetilde{\xi}\right)\right]$.

Family 4 When $a^2 + b^2 - c^2 < 0$, $c \neq 0$ and $b = 0$, then $\phi(\xi) = 2\arctan\left[-\frac{a}{c} + \frac{\sqrt{c^2-a^2}}{c}\tan\left(\frac{\sqrt{c^2-a^2}}{2}\widetilde{\xi}\right)\right]$.

Family 5 When $a^2 + b^2 - c^2 > 0$, $b - c \neq 0$ and $a = 0$, then $\phi(\xi) = 2\arctan\left[\sqrt{\frac{b+c}{b-c}}\tanh\left(\frac{\sqrt{b^2-c^2}}{2}\widetilde{\xi}\right)\right]$.

Family 6 When $a = 0$ and $c = 0$, then $\phi(\xi) = \arctan\left[\frac{e^{2b\widetilde{\xi}}-1}{e^{2b\widetilde{\xi}}+1}, \frac{2e^{b\widetilde{\xi}}}{e^{2b\widetilde{\xi}}+1}\right]$.

Family 7 When $b = 0$ and $c = 0$, then $\phi(\xi) = \arctan\left[\frac{2e^{a\widetilde{\xi}}}{e^{2a\widetilde{\xi}}+1}, \frac{e^{2a\widetilde{\xi}}-1}{e^{2a\widetilde{\xi}}+1}\right]$.

Family 8 When $a^2 + b^2 = c^2$, then $\phi(\xi) = -2\arctan\left[\frac{(b+c)(a\widetilde{\xi}+2)}{a^2\widetilde{\xi}}\right]$.

Family 9 When $c = a$, then $\phi(\xi) = -2\arctan\left[\frac{(a+b)e^{b\widetilde{\xi}}-1}{(a-b)e^{b\widetilde{\xi}}-1}\right]$.

Family 10 When $c = -a$, then $\phi(\xi) = 2\arctan\left[\frac{e^{b\widetilde{\xi}}+b-a}{e^{b\widetilde{\xi}}-b-a}\right]$.

Family 11 When $b = -c$, then $\phi(\xi) = -2\arctan\left[\frac{ae^{a\widetilde{\xi}}}{ce^{a\widetilde{\xi}}-1}\right]$.

Family 12 When $b = 0$ and $a = c$, then $\phi(\xi) = -2\arctan\left[\frac{c\widetilde{\xi}+2}{c\widetilde{\xi}}\right]$.

Family 13 When $a = 0$ and $b = c$, then $\phi(\xi) = 2\arctan\left[c\widetilde{\xi}\right]$.

Family 14 When $a = 0$ and $b = -c$, then $\phi(\xi) = -2\arctan\left[\frac{1}{c\widetilde{\xi}}\right]$,

Family 15 When $a = 0$ and $b = 0$, then $\phi(\xi) = c\widetilde{\xi} + C$.

Family 16 When $b = c$ then $\phi(\xi) = 2\arctan\left[\frac{e^{a\widetilde{\xi}}-c}{a}\right]$, where $\widetilde{\xi} = \xi + C$.

5.1 Application of ITEM

By employing the ITEM for Eq. (2.3) and by balancing u^3, u'' and v^3, v'' in Eq. (2.3), we obtain the balance number $M = 1$. Therefore the closed form of solution will be as

$$u(\eta) = \sum_{j=0}^{1} a_j \tan^j(\phi/2) + \sum_{j=1}^{1} b_j \cot^j(\phi/2), \tag{5.7}$$

$$v(\eta) = \sum_{j=0}^{1} c_j \tan^j(\phi/2) + \sum_{j=1}^{1} d_j \cot^j(\phi/2). \tag{5.8}$$

Inserting (5.8) into Eq. (2.3) and comparing the terms, we will reach the following results as the below cases:

Case 1

$$\begin{aligned} &a = 0,\ \ b = b,\ \ c = c,\ \ k = k,\ \ \mu = \frac{d_1\sqrt{-2\delta(\gamma+1)}}{b+c},\ \ \omega = \frac{d_1^2\delta(b-c)(\gamma+1) - k^2(b+c)}{b+c},\\ &a_0 = 0,\ \ a_1 = 0,\ \ b_1 = \pm d_1,\ \ c_0 = 0,\ \ c_1 = 0,\ \ d_1 = d_1. \end{aligned} \tag{5.9}$$

By employing Family 5, (5.5) and (5.9), the exact singular solutions become,

$$\begin{aligned} U_1(x,t) =V_1^+(x,t) = d_1\sqrt{\frac{b-c}{b+c}} e^{i\left(-\frac{k}{\beta}x^\beta + \frac{d_1^2\delta(b-c)(\gamma+1)-k^2(b+c)}{\alpha(b+c)}x^\alpha + \theta\right)} \coth\left(\frac{\sqrt{b^2-c^2}}{2}\left(\frac{d_1\sqrt{-2\delta(\gamma+1)}}{b+c}\left(\frac{1}{\beta}x^\beta + \frac{2k}{\alpha}x^\alpha\right)+C\right)\right),\\ V_1^-(x,t) = -d_1\sqrt{\frac{b-c}{b+c}} e^{i\left(-\frac{k}{\beta}x^\beta + \frac{d_1^2\delta(b-c)(\gamma+1)-k^2(b+c)}{\alpha(b+c)}x^\alpha + \theta\right)} \coth\left(\frac{\sqrt{b^2-c^2}}{2}\left(\frac{d_1\sqrt{-2\delta(\gamma+1)}}{b+c}\left(\frac{1}{\beta}x^\beta + \frac{2k}{\alpha}x^\alpha\right)+C\right)\right). \end{aligned} \tag{5.10}$$

By employing Family 13, (5.5) and (5.9), the exact singular solutions get,

$$\begin{aligned} U_2(x,t) =V_2^+(x,t) = d_1 e^{i\left(-\frac{k}{\beta}x^\beta + \frac{-k^2}{\alpha}x^\alpha + \theta\right)} \frac{c}{\frac{d_1\sqrt{-2\delta(\gamma+1)}}{2c}\left(\frac{1}{\beta}x^\beta + \frac{2k}{\alpha}x^\alpha\right) + C},\\ V_2^-(x,t) = -d_1 e^{i\left(-\frac{k}{\beta}x^\beta + \frac{-k^2}{\alpha}x^\alpha + \theta\right)} \frac{c}{\frac{d_1\sqrt{-2\delta(\gamma+1)}}{2c}\left(\frac{1}{\beta}x^\beta + \frac{2k}{\alpha}x^\alpha\right) + C}. \end{aligned} \tag{5.11}$$

By employing Family 16, (5.5) and (5.9), the exact exponential solutions catch,

$$\begin{aligned} U_3(x,t) =V_3^+(x,t) = d_1 e^{i\left(-\frac{k}{\beta}x^\beta + \frac{-k^2}{\alpha}x^\alpha + \theta\right)} \left[\frac{e^{a\frac{d_1\sqrt{-2\delta(\gamma+1)}}{2c}\left(\frac{1}{\beta}x^\beta + \frac{2k}{\alpha}x^\alpha\right)+C} - c}{a}\right],\\ V_3^-(x,t) = -d_1 e^{i\left(-\frac{k}{\beta}x^\beta + \frac{-k^2}{\alpha}x^\alpha + \theta\right)} \left[\frac{e^{a\frac{d_1\sqrt{-2\delta(\gamma+1)}}{2c}\left(\frac{1}{\beta}x^\beta + \frac{2k}{\alpha}x^\alpha\right)+C} - c}{a}\right]. \end{aligned} \tag{5.12}$$

Case 2

$$\begin{aligned}&a=\frac{c_0\sqrt{-2\delta(\gamma+1)}}{\mu},\ \ b=-\frac{2d_1\delta(\gamma+1)+\mu c\sqrt{-2\delta(\gamma+1)}}{\mu\sqrt{-2\delta(\gamma+1)}},\\&c=c,\ \ k=k,\ \ \mu=\mu,\\&\omega=\delta(c_0^2+d_1^2)(\gamma+1)+\mu d_1c\sqrt{-2\delta(\gamma+1)}-k^2,\ \ a_0=c_0,\\&a_1=0,\ \ b_1=d_1,\ \ c_0=c_0,\ \ c_1=0,\ \ d_1=d_1.\end{aligned}\tag{5.13}$$

By employing Family 1 ($\Delta=-\frac{2}{\mu^2}(\delta(c_0^2+d_1^2)(\gamma+1)c\mu d_1+\sqrt{-2\delta(\gamma+1)})$), (5.5) and (5.19), the exact solutions overtake,

$$\begin{aligned}U_4(x,t)=&V_4^+(x,t)=d_1e^{i\left(-\frac{k}{\beta}x^\beta+\frac{\delta(c_0^2+d_1^2)(\gamma+1)+\mu d_1c\sqrt{-2\delta(\gamma+1)}-k^2}{\alpha}x^\alpha+\theta\right)}\\&\times\left[\frac{a}{b-c}-\frac{\sqrt{-\Delta}}{b-c}\tan\left(\frac{\sqrt{-\Delta}}{2}\left(\mu\left(\frac{1}{\beta}x^\beta+\frac{2k}{\alpha}x^\alpha\right)+C\right)\right)\right]^{-1},\\V_4^-(x,t)=&-d_1e^{i\left(-\frac{k}{\beta}x^\beta+\frac{\delta(c_0^2+d_1^2)(\gamma+1)+\mu d_1c\sqrt{-2\delta(\gamma+1)}-k^2}{\alpha}x^\alpha+\theta\right)}\\&\times\left[\frac{a}{b-c}-\frac{\sqrt{-\Delta}}{b-c}\tan\left(\frac{\sqrt{-\Delta}}{2}\left(\mu\left(\frac{1}{\beta}x^\beta+\frac{2k}{\alpha}x^\alpha\right)+C\right)\right)\right]^{-1},\end{aligned}\tag{5.14}$$

in which $\delta(\gamma+1)<0$ and $c\mu d_1<0$. By employing Family 2 ($\Delta=-\frac{2}{\mu^2}(\delta(c_0^2+d_1^2)(\gamma+1)c\mu d_1+\sqrt{-2\delta(\gamma+1)})$), (5.5) and (5.19), the exact periodic solutions catch,

$$\begin{aligned}U_5(x,t)=&V_5^+(x,t)=d_1e^{i\left(-\frac{k}{\beta}x^\beta+\frac{\delta(c_0^2+d_1^2)(\gamma+1)+\mu d_1c\sqrt{-2\delta(\gamma+1)}-k^2}{\alpha}x^\alpha+\theta\right)}\\&\times\left[\frac{a}{b-c}+\frac{\sqrt{\Delta}}{b-c}\tanh\left(\frac{\sqrt{\Delta}}{2}\left(\mu\left(\frac{1}{\beta}x^\beta+\frac{2k}{\alpha}x^\alpha\right)+C\right)\right)\right]^{-1},\\V_5^-(x,t)=&-d_1e^{i\left(-\frac{k}{\beta}x^\beta+\frac{\delta(c_0^2+d_1^2)(\gamma+1)+\mu d_1c\sqrt{-2\delta(\gamma+1)}-k^2}{\alpha}x^\alpha+\theta\right)}\\&\times\left[\frac{a}{b-c}+\frac{\sqrt{\Delta}}{b-c}\tanh\left(\frac{\sqrt{\Delta}}{2}\left(\mu\left(\frac{1}{\beta}x^\beta+\frac{2k}{\alpha}x^\alpha\right)+C\right)\right)\right]^{-1},\end{aligned}\tag{5.15}$$

in which $\delta(\gamma+1)<0$ and $\sqrt{-2\delta(\gamma+1)}<\delta(c_0^2+d_1^2)(\gamma+1)c\mu d_1$.

Case 3

$$\begin{aligned}&a=0,\ \ b=b,\ \ c=c,\ \ k=k,\ \ \mu=\frac{c_1\sqrt{-2\delta(\gamma+1)}}{b-c},\ \ \omega=\frac{c_1^2\delta(b+c)(\gamma+1)-k^2(b-c)}{b-c},\\&a_0=0,\ \ a_1=\pm c_1,\ \ b_1=0,\ \ c_0=0,\ \ c_1=c_1,\ \ d_1=0.\end{aligned}\tag{5.16}$$

By employing Family 5, (5.5) and (5.9), the exact dark solutions get,

$$
\begin{aligned}
U_6(x,t) =& V_6^+(x,t) = c_1\sqrt{\frac{b+c}{b-c}}e^{i\left(-\frac{k}{\beta}x^\beta+\frac{c_1^2\delta(b+c)(\gamma+1)-k^2(b-c)}{\alpha(b-c)}x^\alpha+\theta\right)}\\
&\times\tanh\left(\frac{\sqrt{b^2-c^2}}{2}\left(\frac{c_1\sqrt{-2\delta(\gamma+1)}}{b-c}\left(\frac{1}{\beta}x^\beta+\frac{2k}{\alpha}x^\alpha\right)+C\right)\right),\\
V_6^-(x,t) =& -c_1\sqrt{\frac{b+c}{b-c}}e^{i\left(-\frac{k}{\beta}x^\beta+\frac{d_1^2\delta(b-c)(\gamma+1)-k^2(b+c)}{\alpha(b+c)}x^\alpha+\theta\right)}\\
&\times\coth\left(\frac{\sqrt{b^2-c^2}}{2}\left(\frac{c_1\sqrt{-2\delta(\gamma+1)}}{b-c}\left(\frac{1}{\beta}x^\beta+\frac{2k}{\alpha}x^\alpha\right)+C\right)\right).
\end{aligned}
\tag{5.17}
$$

By employing Family 11, (5.5) and (5.9), the exact solutions catch,

$$
\begin{aligned}
U_7(x,t) =& V_7^+(x,t) = -c_1e^{i\left(-\frac{k}{\beta}x^\beta+\frac{-k^2}{\alpha}x^\alpha+\theta\right)}\left[\frac{ae^{a\left(\frac{c_1\sqrt{-2\delta(\gamma+1)}}{-2c}\left(\frac{1}{\beta}x^\beta+\frac{2k}{\alpha}x^\alpha\right)+C\right)}}{ce^{a\left(\frac{c_1\sqrt{-2\delta(\gamma+1)}}{-2c}\left(\frac{1}{\beta}x^\beta+\frac{2k}{\alpha}x^\alpha\right)+C\right)}-1}\right],\\
V_7^-(x,t) =& c_1e^{i\left(-\frac{k}{\beta}x^\beta+\frac{-k^2}{\alpha}x^\alpha+\theta\right)}\left[\frac{ae^{a\left(\frac{c_1\sqrt{-2\delta(\gamma+1)}}{-2c}\left(\frac{1}{\beta}x^\beta+\frac{2k}{\alpha}x^\alpha\right)+C\right)}}{ce^{a\left(\frac{c_1\sqrt{-2\delta(\gamma+1)}}{-2c}\left(\frac{1}{\beta}x^\beta+\frac{2k}{\alpha}x^\alpha\right)+C\right)}-1}\right].
\end{aligned}
\tag{5.18}
$$

Case 4

$$
\begin{aligned}
& a=\frac{c_0\sqrt{-2\delta(\gamma+1)}}{\mu},\quad c=-\frac{2c_1\delta(\gamma+1)+\mu b\sqrt{-2\delta(\gamma+1)}}{\mu\sqrt{-2\delta(\gamma+1)}},\\
& b=b,\quad k=k,\quad \mu=\mu,\\
& \omega=\delta(c_0^2-c_1^2)(\gamma+1)+\mu bc_1\sqrt{-2\delta(\gamma+1)}-k^2,\quad a_0=c_0,\\
& a_1=0,\quad b_1=d_1,\quad c_0=c_0,\quad c_1=0,\quad d_1=d_1.
\end{aligned}
\tag{5.19}
$$

By employing Family 1 ($\Delta=-\frac{2}{\mu^2}(\delta(c_0^2-c_1^2)(\gamma+1)c\mu c_1+\sqrt{-2\delta(\gamma+1)})$, (5.5) and (5.19), the exact kink-singular solutions overtake,

$$
\begin{aligned}
U_8(x,t) =& V_8^+(x,t) = c_1e^{i\left(-\frac{k}{\beta}x^\beta+\frac{\delta(c_0^2-c_1^2)(\gamma+1)+\mu bc_1\sqrt{-2\delta(\gamma+1)}-k^2}{\alpha}x^\alpha+\theta\right)}\\
&\times\left[\frac{a}{b-c}-\frac{\sqrt{-\Delta}}{b-c}\tan\left(\frac{\sqrt{-\Delta}}{2}\left(\mu\left(\frac{1}{\beta}x^\beta+\frac{2k}{\alpha}x^\alpha\right)+C\right)\right)\right],\\
V_8^-(x,t) =& -c_1e^{i\left(-\frac{k}{\beta}x^\beta+\frac{\delta(c_0^2-c_1^2)(\gamma+1)+\mu bc_1\sqrt{-2\delta(\gamma+1)}-k^2}{\alpha}x^\alpha+\theta\right)}\\
&\times\left[\frac{a}{b-c}-\frac{\sqrt{-\Delta}}{b-c}\tan\left(\frac{\sqrt{-\Delta}}{2}\left(\mu\left(\frac{1}{\beta}x^\beta+\frac{2k}{\alpha}x^\alpha\right)+C\right)\right)\right],
\end{aligned}
\tag{5.20}
$$

in which $\delta(\gamma+1)<0$ and $c\mu c_1(c_0^2-c_1^2)<0$. By employing Family 2 $(\Delta=-\frac{2}{\mu^2}(\delta(c_0^2-c_1^2)(\gamma+1)c\mu c_1+\sqrt{-2\delta(\gamma+1)})$, (5.5) and (5.19), the exact solutions get,

$$\begin{aligned} U_9(x,t)=&V_9^+(x,t)=d_1 e^{i\left(-\frac{k}{\beta}x^\beta+\frac{\delta(c_0^2-c_1^2)(\gamma+1)+\mu bc_1\sqrt{-2\delta(\gamma+1)}-k^2}{\alpha}x^\alpha+\theta\right)}\\ &\times\left[\frac{a}{b-c}+\frac{\sqrt{\Delta}}{b-c}\tanh\left(\frac{\sqrt{\Delta}}{2}\left(\mu\left(\frac{1}{\beta}x^\beta+\frac{2k}{\alpha}x^\alpha\right)+C\right)\right)\right],\\ V_9^-(x,t)=&-d_1 e^{i\left(-\frac{k}{\beta}x^\beta+\frac{\delta(c_0^2-c_1^2)(\gamma+1)+\mu bc_1\sqrt{-2\delta(\gamma+1)}-k^2}{\alpha}x^\alpha+\theta\right)}\\ &\times\left[\frac{a}{b-c}+\frac{\sqrt{\Delta}}{b-c}\tanh\left(\frac{\sqrt{\Delta}}{2}\left(\mu\left(\frac{1}{\beta}x^\beta+\frac{2k}{\alpha}x^\alpha\right)+C\right)\right)\right], \end{aligned} \tag{5.21}$$

in which $\delta(\gamma+1)<0$ and $\sqrt{-2\delta(\gamma+1)}<\delta(c_0^2-c_1^2)(\gamma+1)c\mu c_1$.

Case 5

$$\begin{aligned} &a=0,\ \ b=b,\ \ c=c,\ \ k=k,\ \ \mu=\frac{d_1\sqrt{-2\delta(\gamma+1)}}{b+c},\ \ \omega=-\frac{2d_1^2\delta(b-c)(\gamma+1)+k^2(b+c)}{b+c},\\ &a_0=0,\ \ a_1=\pm\frac{(b-c)d_1}{b+c},\ \ b_1=d_1,\ \ c_0=0,\ \ c_1=\pm\frac{(b-c)d_1}{b+c},\ \ d_1=d_1. \end{aligned} \tag{5.22}$$

By employing Family 5, (5.5) and (5.9), the exact dark-singular solutions become,

$$\begin{aligned} U_{10}(x,t)=&V_{10}^+(x,t)=e^{i\left(-\frac{k}{\beta}x^\beta-\frac{2d_1^2\delta(b-c)(\gamma+1)+k^2(b+c)}{(b+c)\alpha}x^\alpha+\theta\right)}\\ &\times\left\{\frac{(b-c)d_1}{b+c}\sqrt{\frac{b+c}{b-c}}\tanh\left(\frac{\sqrt{b^2-c^2}}{2}\left(\frac{d_1\sqrt{-2\delta(\gamma+1)}}{b+c}\left(\frac{1}{\beta}x^\beta+\frac{2k}{\alpha}x^\alpha\right)+C\right)\right)+\right.\\ &\left.d_1\sqrt{\frac{b-c}{b+c}}\coth\left(\frac{\sqrt{b^2-c^2}}{2}\left(\frac{d_1\sqrt{-2\delta(\gamma+1)}}{b+c}\left(\frac{1}{\beta}x^\beta+\frac{2k}{\alpha}x^\alpha\right)+C\right)\right)\right\}, \end{aligned} \tag{5.23}$$

$$\begin{aligned} V_{10}^-(x,t)=&e^{i\left(-\frac{k}{\beta}x^\beta-\frac{2d_1^2\delta(b-c)(\gamma+1)+k^2(b+c)}{(b+c)\alpha}x^\alpha+\theta\right)}\\ &\times\left\{-\frac{(b-c)d_1}{b+c}\sqrt{\frac{b+c}{b-c}}\tanh\left(\frac{\sqrt{b^2-c^2}}{2}\left(\frac{d_1\sqrt{-2\delta(\gamma+1)}}{b+c}\left(\frac{1}{\beta}x^\beta+\frac{2k}{\alpha}x^\alpha\right)+C\right)\right)+\right.\\ &\left.d_1\sqrt{\frac{b-c}{b+c}}\coth\left(\frac{\sqrt{b^2-c^2}}{2}\left(\frac{d_1\sqrt{-2\delta(\gamma+1)}}{b+c}\left(\frac{1}{\beta}x^\beta+\frac{2k}{\alpha}x^\alpha\right)+C\right)\right)\right\}. \end{aligned} \tag{5.24}$$

6 The SIVPM

The main steps of the SIVPM are as follows:

Step 1 Suppose the nonlinear space–time fractional partial differential equations as

$$\mathcal{N}\left(u, v, \frac{\partial^\alpha u}{\partial t^\alpha}, \frac{\partial^{2\beta} u}{\partial t^{2\beta}}, \frac{\partial^\alpha v}{\partial t^\alpha}, \frac{\partial^{2\beta} v}{\partial t^{2\beta}}, \dots\right) = 0. \tag{6.1}$$

Utilizing the wave transformation

$$U(x,t) = u(\xi)e^{i\eta}, \quad V(x,t) = v(\xi)e^{i\eta}, \quad \xi = \mu\left(\frac{x^\beta}{\beta} - c\frac{t^\alpha}{\alpha}\right), \quad \eta = \frac{-kx^\beta}{\beta} + \omega\frac{t^\alpha}{\alpha} + \theta, \tag{6.2}$$

where $k, \omega \neq 0$. Putting (6.2) into Eq. (6.1) acquires a NLODE as,

$$Q(u, v, u', v', \dots) = 0, \tag{6.3}$$

where prime shows the derivation with respect to ξ.

Step 2 Based on the He's semi-inverse method, then the trial-functional can be written as,

$$J(U) = \int L d\xi, \tag{6.4}$$

where L is an unknown function of U and its derivatives.

Step 3 Utilizing the Ritz method, the various forms of solitary wave solutions, can be acquired such as

$$u(\xi) = F sech(G\xi), \tag{6.5}$$

$$u(\xi) = F sech^2(G\xi), \tag{6.6}$$

$$u(\xi) = F\sqrt{sech(G\xi)}, \tag{6.7}$$

$$u(\xi) = \frac{F}{D + \cosh(G\xi)}, \tag{6.8}$$

where F and G are constants to be further determined. Upon putting (6.5)–(6.8) into (6.4) and making J stationary with respect to F and G results in

$$\frac{\partial J}{\partial F} = 0, \tag{6.9}$$

$$\frac{\partial J}{\partial G} = 0. \tag{6.10}$$

Solving Eqs. (6.9) and (6.10), we obtain F and G. Therefore, the solitary wave solutions (6.5), (6.6) and (6.8) are well determined.

6.1 Application of HSIVP

By employing He's semi-inverse principle (He 2006; Kohl et al. 2009; Zhang 2007), one can catch the variational formulation as follows

$$J_1 = \int_0^\infty \left[\frac{1}{2}\mu^2\left(\frac{du(\xi)}{d\xi}\right)^2 + \frac{\delta}{4}u(\xi)^4 + \frac{1}{2}\delta\gamma u(\xi)^2 v(\xi)^2 - \frac{1}{2}(\omega + k^2)u(\xi)^2\right] d\xi, \tag{6.11}$$

$$J_2=\int_0^{\infty}\left[\frac{1}{2}\mu^2\left(\frac{dv(\xi)}{d\xi}\right)^2+\frac{\delta}{4}v(\xi)^4+\frac{1}{2}\delta\gamma u(\xi)^2v(\xi)^2-\frac{1}{2}(\omega+k^2)v(\xi)^2\right]d\xi. \tag{6.12}$$

Utilizing a Ritz-like method, a solitary wave solution read as

6.1.1 Case I

$$u(\xi)=Fsech(G\xi),\quad v(\xi)=Fsech(G\xi), \tag{6.13}$$

where F and G are unknown constants to be further determined. Upon inserting (6.13) into (6.11) or (6.12) and carrying out the integration gives

$$J_1=\frac{1}{6}\mu^2F^2G+\frac{1}{6}\frac{\delta F^4}{G}+\frac{1}{3}\frac{\delta\gamma F^4}{G}-\frac{1}{12}\frac{F^2(\omega+k^2)}{G}. \tag{6.14}$$

Making J_1 stationary with A and B supplies

$$\frac{\partial J_1(F,G)}{\partial F}=\frac{1}{3}\mu^2FG+\frac{2}{3}\frac{\delta F^3}{G}+\frac{4}{3}\frac{\delta\gamma F^3}{G}-\frac{1}{6}\frac{F(\omega+k^2)}{G}=0, \tag{6.15}$$

$$\frac{\partial J_1(F,G)}{\partial G}=\frac{1}{6}\mu^2F^2-\frac{1}{6}\frac{\delta F^4}{G^2}-\frac{1}{3}\frac{\delta\gamma F^4}{G^2}+\frac{1}{12}\frac{F^2(\omega+k^2)}{G^2}=0. \tag{6.16}$$

Solving Eqs. (6.15) and (6.16), one can obtain

$$F=\sqrt{\frac{\omega+k^2}{3\delta(2\gamma+1)}},\quad G=\frac{1}{\mu}\sqrt{-\frac{\omega+k^2}{6}}. \tag{6.17}$$

The domain of definition of above relations is:

$$\omega+k^2<0,\quad \delta(2\gamma+1)<0. \tag{6.18}$$

Hence, finally, the 1-soliton solution to the fractional CNLE is given by

$$U(x,t)=\sqrt{\frac{\omega+k^2}{3\delta(2\gamma+1)}}\,e^{i\left(-\frac{k}{\beta}x^{\beta}-\frac{\omega}{\alpha}x^{\alpha}+\theta\right)}\,\text{sech}\left[\frac{1}{\mu}\sqrt{-\frac{\omega+k^2}{6}}\left(\mu\left(\frac{1}{\beta}x^{\beta}+\frac{2k}{\alpha}x^{\alpha}\right)+C\right)\right]. \tag{6.19}$$

Also, a bright soliton wave solution can be found at

6.1.2 Case II

$$u(\xi)=\frac{F}{\cosh^2(G\xi)},\quad v(\xi)=\frac{F}{\cosh^2(G\xi)}, \tag{6.20}$$

where F and G are free constants. Upon inserting (6.20) into (6.12) and carrying out the integration gives

$$J_1=\frac{4}{15}\mu^2F^2G+\frac{4}{35}\frac{\delta F^4}{G}+\frac{8}{35}\frac{\delta\gamma F^4}{G}-\frac{1}{18}\frac{F^2(\omega+k^2)}{G}. \tag{6.21}$$

Making J_1 stationary with A and B supplies

$$\frac{\partial J_1(F,G)}{\partial F}=\frac{8}{15}\mu^2FG+\frac{16}{35}\frac{\delta F^3}{G}+\frac{32}{35}\frac{\delta\gamma F^3}{G}-\frac{1}{9}\frac{F(\omega+k^2)}{G}=0, \tag{6.22}$$

$$\frac{\partial J_1(F,G)}{\partial G} = \frac{4}{15}\mu^2 F^2 - \frac{4}{35}\frac{\delta F^4}{G^2} - \frac{8}{35}\frac{\delta\gamma F^4}{G^2} + \frac{1}{18}\frac{F^2(\omega+k^2)}{G^2} = 0. \tag{6.23}$$

Solving Eqs. (6.22) and (6.23), one can obtain

$$F = \sqrt{\frac{35(\omega+k^2)}{108\delta(2\gamma+1)}}, \quad G = \frac{1}{6\mu}\sqrt{-\frac{5(\omega+k^2)}{2}}. \tag{6.24}$$

The domain of definition of above relations is:

$$\omega + k^2 < 0, \quad \delta(2\gamma+1) < 0. \tag{6.25}$$

Thus, finally, the bright soliton to the fractional CNLE is given by

$$U(x,t) = \sqrt{\frac{35(\omega+k^2)}{108\delta(2\gamma+1)}}\, e^{i\left(-\frac{k}{\beta}x^\beta - \frac{\omega}{\alpha}x^\alpha + \theta\right)} \operatorname{sech}^2\left[\frac{1}{6\mu}\sqrt{-\frac{5(\omega+k^2)}{2}}\left(\mu\left(\frac{1}{\beta}x^\beta + \frac{2k}{\alpha}x^\alpha\right) + C\right)\right]. \tag{6.26}$$

Moreover, another singular wave solution can be considered as

6.1.3 Case III

$$u(\xi) = \frac{F}{\sqrt{\cosh(G\xi)}}, \quad v(\xi) = \frac{F}{\sqrt{\cosh(G\xi)}}, \tag{6.27}$$

where F and G are free constants. Upon inserting (6.27) into (6.12) and carrying out the integration gives

$$J_1 = \frac{1}{32}\pi\mu^2 F^2 G + \frac{1}{4}\frac{\delta F^4}{G} + \frac{1}{2}\frac{\delta\gamma F^4}{G} - \frac{1}{24}\frac{\pi F^2(\omega+k^2)}{G}. \tag{6.28}$$

Making J_1 stationary with A and B supplies

$$\frac{\partial J_1(F,G)}{\partial F} = \frac{1}{16}\pi\mu^2 FG + \frac{\delta F^3}{G} + 2\frac{\delta\gamma F^3}{G} - \frac{1}{12}\frac{\pi F(\omega+k^2)}{G} = 0, \tag{6.29}$$

$$\frac{\partial J_1(F,G)}{\partial G} = \frac{1}{32}\pi\mu^2 F^2 - \frac{1}{4}\frac{\delta F^4}{G^2} - \frac{1}{2}\frac{\delta\gamma F^4}{G^2} + \frac{1}{24}\frac{\pi F^2(\omega+k^2)}{G^2} = 0. \tag{6.30}$$

Solving Eqs. (6.29) and (6.30), one can get

$$F = \sqrt{\frac{\pi(\omega+k^2)}{9\delta(2\gamma+1)}}, \quad G = \frac{2}{3\mu}\sqrt{-\omega-k^2}. \tag{6.31}$$

The domain of definition of above relations is:

$$\omega + k^2 < 0, \quad \delta(2\gamma+1) < 0. \tag{6.32}$$

Thus, finally, the singular soliton solution to the fractional CNLE is given by

$$U(x,t)=e^{i\left(-\frac{k}{\beta}x^{\beta}-\frac{\omega}{\alpha}x^{\alpha}+\theta\right)}\frac{\sqrt{\frac{\pi(\omega+k^2)}{9\delta(2\gamma+1)}}}{\sqrt{sech\left[\frac{2}{3\mu}\sqrt{-\omega-k^2}\left(\mu\left(\frac{1}{\beta}x^{\beta}+\frac{2k}{\alpha}x^{\alpha}\right)+C\right)\right]}}. \tag{6.33}$$

Finally, another singular wave solution can be considered as

6.1.4 Case IV

$$u(\xi)=\frac{F}{D+\cosh(G\xi)},\quad v(\xi)=\frac{F}{D+\cosh(G\xi)}, \tag{6.34}$$

where F and G are free constants. Upon putting (6.34) into (6.12) and carrying out the integration gives

$$J=\frac{1}{12}\frac{D\,\text{arctanh}\left(\frac{D-1}{\sqrt{D^2-1}}\right)(3\delta F^4(2D^2+3)(2\gamma+1)-6\mu^2F^2G^2(D^2-1)-2(D^2-1)^2(k^2+w)F^2)}{\sqrt{D^2-1}(D^2-1)^3G}. \tag{6.35}$$

Making J stationary with F and G yields

$$\frac{\partial J(F,G)}{\partial F}=\frac{1}{12}\frac{D\,\text{arctanh}\left(\frac{D-1}{\sqrt{D^2-1}}\right)(12\delta F^3(2D^2+3)(2\gamma+1)-12\mu^2FG^2(D^2-1)-4(D^2-1)^2(k^2+w)F)}{\sqrt{D^2-1}(D^2-1)^3G}=0, \tag{6.36}$$

$$\begin{aligned}\frac{\partial J(F,G)}{\partial G}=&\frac{1}{12}\frac{D\,\text{arctanh}\left(\frac{D-1}{\sqrt{D^2-1}}\right)(-12\mu^2F^2G(D^2-1))}{\sqrt{D^2-1}(D^2-1)^3G}\\&-\frac{1}{12}\frac{D\,\text{arctanh}\left(\frac{D-1}{\sqrt{D^2-1}}\right)(3\delta F^4(2D^2+3)(2\gamma+1)-6\mu^2F^2G^2(D^2-1)-2(D^2-1)^2(k^2+w)F^2)}{\sqrt{D^2-1}(D^2-1)^3G^2}=0.\end{aligned} \tag{6.37}$$

Solving Eqs. (6.36) and (6.37), one can catch

$$F=(D^2-1)\sqrt{\frac{4(\omega+k^2)}{9\delta(2D^2+3)(2\gamma+1)}},\quad G=\frac{1}{3\mu}\sqrt{(D^2-1)(\omega+k^2)}. \tag{6.38}$$

The domain of definition of above relations is:

$$(D^2-1)(\omega+k^2)>0,\quad \delta(2D^2+3)(2\gamma+1)(\omega+k^2)>0. \tag{6.39}$$

Thus, finally, the singular soliton solution to the fractional CNLE is given by

$$U(x,t)=e^{i\left(-\frac{k}{\beta}x^{\beta}-\frac{\omega}{\alpha}x^{\alpha}+\theta\right)}\frac{(D^2-1)\sqrt{\frac{4(\omega+k^2)}{9\delta(2D^2+3)(2\gamma+1)}}}{D+\cosh\left[\frac{1}{3\mu}\sqrt{(D^2-1)(\omega+k^2)}\left(\mu\left(\frac{1}{\beta}x^{\beta}+\frac{2k}{\alpha}x^{\alpha}\right)+C\right)\right]}. \tag{6.40}$$

7 Conclusion

Under investigation in this work is a $(2+1)$-dimensional the space–time fractional coupled nonlinear Schrödinger equations, i.e., system (1.1), which describes the amplitudes of circularly-polarized waves in a nonlinear optical fiber. Based on the conformable fractional derivative and the fractional wave transformation, we derive the analytical soliton solutions where have been given in the form of rational soliton, periodic soliton, hyperbolic soliton solutions by four integration method, namely, the extended trial equation method, the $\exp(-\Omega(\eta))$-expansion method and the improved $\tan(\phi(\eta)/2)$-expansion method and semi-inverse variational principle method.

With the aid of the extended trial equation method, we derived the several types of solutions including singular, kink-singular, bright, solitary wave, compacton and elliptic function solutions. Under certain condition, the 1-soliton, bright, singular solutions are driven by semi-inverse variational principle method. Based on the analytical methods, we found that the solutions give birth to the dark solitons, the bright solitons, combine dark-singular, kink, kink-singular solutions with fractional order for nonlinear fractional partial differential equations arise in nonlinear optics. The aforementioned methods are powerful and efficient mathematical tool such as Maple in exploring search for the solutions of the various nonlinear fractional equations arising in the various field of nonlinear sciences.

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