Optical and Quantum Electronics (2018) 50:340
https://doi.org/10.1007/s11082-018-1597-7

Conservation laws, modulation instability and solitons interactions for a nonlinear Schrödinger equation with the sextic operators in an optical fiber

Zhong-Zhou Lan[1,2] · **Bo-Ling Guo**[1]

Received: 7 June 2018 / Accepted: 11 August 2018 / Published online: 27 August 2018

Abstract
Under investigation in this paper is a sextic nonlinear Schrödinger equation, which describes the pulses propagating along an optical fiber. Based on the symbolic computation, Lax pair and infinitely-many conservation laws are derived. Via the modiied Hirota method, bilinear forms and multi-soliton solutions are obtained. Propagation and interactions of the solitons are illustrated graphically: Initial position and velocity of the soliton are related to the coefficient of the sixth-order dispersion, while the amplitude of the soliton is not affected by it. Head-on, overtaking and oscillating interactions between the two solitons are displayed. Through the asymptotic analysis, interaction between the two solitons is proved to be elastic. Based on the linear stability analysis, the modulation instability condition for the soliton solutions is obtained.

Keywords Optical fiber · Nonlinear Schrödinger equation · Modified Hirota method · Solitons · Infinitely-many conservation laws · Modulation instability

1 Introduction

As a nonlinear wave, solitons have been a hot area of research in the integrable systems (Lü et al. 2016; Wazwaz 2016, 2017a; Liu et al. 2011; Lü and Ma 2016; Osman and Wazwaz 2018). For instance, optical solitons have been extensively studied in the telecommunication systems because of their potential applications in the longdistance optical fiber communication and all-optical ultrafast switching devices (Wang et al. 2015a, b, 2016a, b). Optical solitons have been formed in the balance between the group velocity dispersion and self-phase modulation (Szmytkowski 2012; Zhou et al. 2013; Guo and Zhao 2016; Liu et al. 2014, 2017; Lü and Lin 2016; Guo et al. 2015; Wazwaz 2016; Cai et al. 2017), so they could propagate over a long distance in a fiber without either attenuation or change of shape (Hasegawa and

✉ Zhong-Zhou Lan
zhongzhou-lan@buaa.edu.cn

1 Institute of Applied Physics and Computational Mathematics, Beijing 100088, China

2 School of Computer Information Management, Inner Mongolia University of Finance and Economics, Hohhot 010070, China

Tappert 1973a, b; Dai et al. 2017, 2016). To describe the propagation of the picosecond pulses in an optical fiber, the nonlinear Schrödinger (NLS) equation (Nakatsuka et al. 1981; Lan and Gao 2017; Zhao et al. 2016, 2017),

$$iu_z + u_{\tau\tau} + 2|u|^2 u = 0, \tag{1}$$

has been proposed, where $i^2 = -1$, u is the slowly-varying electric field function with respect to the scaled space coordinate z and time coordinate τ, while the subscripts mean the corresponding derivatives.

NLS equation has been regarded as the basic model to describe the phenomena in optical fibers, plasmas, cold atoms and oceans (Sun et al. 2015). Nonetheless, Eq. (1) only includes the basic effects such as the lowest-order dispersion and nonlinearity (Chai et al. 2015). When the intensity of the optical field gets stronger and the pulses get shorter in optics, one should consider the higher-order effects (Ankiewicz et al. 2014; Liu et al. 2016; Nakazawa et al. 2000; Lakoba and Kaup 1998; Bourkoff et al. 1987; Oliveira and Moura 1998). In fact, some high-order effects and dark solitons for the NLS equations have been discussed (Chowdury et al. 2015; Daniel et al. 1999; Guo et al. 2016; Wazwaz 2017b; Li et al. 2018; Zhang et al. 2017; Lan et al. 2016).

In this paper, a sextic NLS equation has been proposed to model the pulses propagating along a fiber (Ankiewicz et al. 2016; Sun 2017), i.e.,

$$\begin{aligned} iq_x &+ \frac{1}{2}(q_{tt} + 2q|q|^2) + \delta\{q_{tttttt} + q^2[60|q_t|^2 q^* + 50q_{tt}(q^*)^2 \\ &+ 2q^*_{tttt}] + q[12q_{tttt}q^* + 8q_t q^*_{ttt} + 22|q_{tt}|^2 + 18q_{ttt}q^*_t + 70q_t^2(q^*)^2] \\ &+ 20q_t^2 q^*_{tt} + 10q_t(5q_{tt}q^*_t + 3q_{ttt}q^*) + 20q_{tt}^2 q^* + 10q^3[(q^*_t)^2 + 2q^* q^*_{tt}] + 20q|q|^6\} = 0, \end{aligned} \tag{2}$$

where x and t are respectively the scaled space and time coordinates, $q(x, t)$ is the envelope of the waves, $*$ represents the complex conjugation, and δ is a real parameter denoting the coefficient of the sextic-order dispersion. When $\delta = 0$, Eq. (2) becomes the basic NLS equation to describe the different nonlinear waves in the Heisenberg ferromagnetic spin chain, Bose–Einstein condensation and nonlinear optics (Sun 2017). Equation (2) has been presented in Ref. Ankiewicz et al. (2016) for the first time. Some different nonlinear waves (Ankiewicz et al. 2016) and breather-to-soliton transitions (Sun 2017) for Eq. (2) have been discussed. Here, the bilinear forms and analytic solutions for Eq. (2) will be obtained by using the modified Hirota bilinear method and symbolic computation.

However, to our knowledge, infinitely-many conservation laws, bilinear forms, soliton solutions and linear stability analysis for Eq. (2) have not been discussed through the modified Hirota method and symbolic computation. In Sect. 2, infinitely-many conservation laws for Eq. (2) will be constructed by virtue of the symbolic computation. In Sect. 3, bilinear forms, multi-soliton solutions for Eq. (2) will be obtained via the modified Hirota method. In Sect. 4, propagations and interactions between the solitons will be illustrated and discussed graphically. In Sect. 5, linear stability analysis will be presented. Conclusions will be given in Sect. 6.

2 Infinitely-many conservation laws for Eq. (2)

In this section, we will derive the Lax pair and infinitely-many conservation laws for Eq. (2). Based on the Ablowitz–Kaup–Newell–Segur system (Ablowitz et al. 1973), Lax pair for Eq. (2) can be written into the following form,

$$\Phi_t = M\Phi, \;\; \Phi_x = N\Phi, \tag{3}$$

where $\Phi = (\varphi_1, \varphi_2)^T$ is a vector eigenfunction for Lax Pair (3), φ_1 and φ_2 are both the complex functions of x and t, while the superscript T signifies the vector transpose, the 2×2 matrices M and N can be defined by,

$$M = i\begin{pmatrix} \lambda & r \\ q & -\lambda \end{pmatrix}, \;\; N = i\begin{pmatrix} A & B \\ C & -A \end{pmatrix}, \tag{4}$$

where r, A, B and C are the complex functions of x and t, λ is a complex eigenvalue parameter. Based on the compatibility condition $\Phi_{tx} = \Phi_{xt}$ and Lax Pair (4), we can get the zero-curvature equation as follows:

$$M_t - N_x + MN - NM = 0. \tag{5}$$

Substituting Matrices (4) into Eq. (5), we can get

$$A_t = -i\,q\,B + i\,r\,C, \;\; B_t = r_x - 2\,i\,r\,A + 2\,i\lambda\,B, \;\; C_t = q_x - 2\,i\lambda\,C + 2\,i\lambda\,q\,A. \tag{6}$$

In order to facilitate the calculation of Lax Pair (3), A, B and C are expended into the forms as (Chai et al. 2015; Wang et al. 2015c),

$$A = \sum_{j=0}^{6} i\lambda^j A_j, \;\; B = \sum_{j=0}^{6} i\lambda^j B_j, \;\; C = \sum_{j=0}^{6} i\lambda^j C_j, \tag{7}$$

where A_j's, B_j's and C_j's are the complex functions of x and t. Substituting Expressions (7) into Eqressions (6) and equating the coefficients of the same powers of λ, we have

$$\begin{aligned}
A_0 =& -\frac{1}{2}|q|^2 - \delta\big\{10|q|^6 + 5q_t^2(q^*)^2 + q_{tt}q_{tt}^* \\
&+ 5q^2[(q^*)^2 + q^*q_{tt}^*] - q_{ttt}q_t^* - q_tq_{ttt}^* + q_{tttt}q^* + q[10q_{tt}(q^*)^2 + q_{tttt}^*]\big\},\\
A_1 =& -2i\delta\big\{q_{tt}q_t^* + 6q|q|^2q_t^* - q_tq_{tt}^* - q_{ttt}q^* - q[6q_t(q^*)^2 - q_{ttt}^*]\big\}, \;\; A_3 = -8i\delta(q_tq^* - qq^*),\\
A_2 =& 1 + 4\delta(3|q|^4 - |q_t|^2 + q_{tt}q^* + qq_{tt}^*), \;\; A_4 = -16\delta|q|^2, \;\; A_5 = 0, \;\; A_6 = 32\delta,\\
B_0 =& -i\frac{1}{2}q_t^* - i\delta\big\{30|q|^4q^* + 10q_{tt}q_t^*q^* + 10q_t[(q_t^*)^2 + q^*q_{tt}^*] + 10q(2q_t^*q_{tt}^* + q^*q_{ttt}^*) + q_{ttttt}^*\big\},\\
B_1 =& q^* + 2\delta\big\{6|q|^4q^* + 4|q_t|^2q^* + 2q_{tt}(q^*)^2 + q[6(q_t^*)^2 + 8q^*q_{tt}^*] + q_{tttt}^*\big\}, \quad B_4 = -16i\delta q_t^*,\\
B_2 =& 4i\delta(6|q|^2q_t^* + q_{ttt}^*), \;\; B_3 = -8\delta(2|q|^2q^* + q_{tt}^*), \;\; B_5 = 32\delta q^*, \;\; B_6 = 0, \;\; C_j = B_j^*, \;\; r = q^*.
\end{aligned}$$

Our calculation indicates that Compatibility Condition (5) leads to Eq. (2). Thus, Eq. (2) is integrable in the Lax sense.

Next, based on Lax Pair (3), we will show how to derive the infinitely-many conservation laws for Eq. (2). Introducing three complex functions (Chai et al. 2015; Wang et al. 2015c)

$$\Lambda_1 = \frac{\varphi_{1,t}}{\varphi_1}, \;\; \Lambda_2 = \frac{\varphi_{1,x}}{\varphi_1}, \;\; \Lambda_3 = \frac{\varphi_2}{\varphi_1}, \tag{8}$$

and applying the compatibility condition $\Lambda_{1,x} = \Lambda_{2,t}$ from Lax Pair (3), we have

$$\Lambda_{3,t} = i\,q - 2\,i\lambda\,\Lambda_3 - i\,r\Lambda_3^2, \;\; (\lambda + r\Lambda_3)_x = (A + B\Lambda_3)_t. \tag{9}$$

Then, expanding the expansion of Λ_3 with respect to λ as follows:

$$\Lambda_3 = \sum_{j=1}^{\infty} \frac{\chi_j}{\lambda^j}, \tag{10}$$

where $\chi_j{}'$s (j =1,2...) are all the complex functions of x and t. Substituting Expressions (7) and (10) into Eq. (9) and collecting the coefficients of the same powers of λ to be equal to zero, we get

$$\chi_1 = \frac{1}{2}q, \quad \chi_2 = \frac{i}{4}q_t, \; \cdots, \quad \chi_{j+1} = \frac{i}{2}\left(\chi_{j,t} + iq^* \sum_{o=1}^{j-1} \chi_o \chi_{j-o} \right), \tag{11}$$

and the infinite-many conservation laws for Eq. (2) as

$$\frac{\partial \Gamma_j}{\partial t} + \frac{\partial \Theta_j}{\partial x} = 0, \quad \Gamma_j = q\chi_j, \quad \Theta_j = -i \sum_{o=0}^{7} B_o \chi_{j+o}, \tag{12}$$

where $\Gamma_j{}'$s and $\Theta_j{}'$s are all the complex functions of x and t, $\Gamma_j{}'$s represent the conserved densities and $\Theta_j{}'$s represent the fluxes.

3 Bilinear forms and soliton solutions for Eq. (2)

In order to detect the form of linearizable representation of Eq. (2), a transformation

$$q(x,t) = \frac{g}{f}, \tag{13}$$

will be introduced, where f is a real function of x and t, while g is a complex one. Substituting Expression (13) into Eq. (2) and letting $D_t^2 f \cdot f - 2|g|^2 = 0$, the bilinear forms for Eq. (2) will be obtained as

$$\begin{aligned} &D_t^2 g \cdot g - hf = 0, \quad 30g^*(D_t g \cdot g_{3t} + f_t h_t) - 15 f_t g_t^* h - f\,\rho\,\varrho = 0, \\ &D_t^2 f \cdot f - 2|g|^2 = 0, \quad \left(iD_x + \frac{1}{2}D_t^2 + \delta D_t^6 \right) g \cdot f + \delta(5g_t^* h_t - 10 g_{2t}^* + 5g^* h_{2t} + \rho\varrho) = 0, \end{aligned} \tag{14}$$

where the Hirota D-operator is defined as (Hirota 1991),

$$D_x^{n_1} D_t^{n_2} p \cdot q = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^{n_1} \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^{n_2} p(x,y,t) q\left(x', t'\right) \big|_{x'=x, t'=t}, \tag{15}$$

$p(x, t)$ is a function of x and t, $q(x', t')$ is a function of the formal variables x' and t', while n_1 and n_2 are all the non-negative integers.

In order to obtain the soliton solutions for Eq. (2), f, g, h, ρ and ϱ are expended with respect to a small parameter ε as

$$\begin{aligned} &f = 1 + \varepsilon^2 f_2 + \varepsilon^4 f_4 + \dots, \quad g = \varepsilon g_1 + \varepsilon^3 g_3 + \varepsilon^5 g_5 + \dots, \\ &h = h_0 + \varepsilon^2 h_2 + \varepsilon^4 h_4 + \dots, \quad \rho = \varepsilon \rho_1 + \varepsilon^3 \rho_3 + \varepsilon^5 \rho_5 + \dots, \quad \varrho = \varrho_0 + \varepsilon^2 \varrho_2 + \varepsilon^4 \varrho_4 + \dots, \end{aligned} \tag{16}$$

where f_ℓ's ($\ell = 2, 4, ...$), $\rho_\jmath$'s ($\jmath$ =1,3,...), h_k's and $\varrho_\imath$'s ($\imath$ =0,2,...) are the real functions of x and t, while $g_\jmath$'s are the complex ones.

Truncating Expressions (16) as

$$f = 1 + \varepsilon^2 f_2, \;\; g = \varepsilon g_1, \;\; \rho = \varepsilon \rho_1, \;\; h = h_0 + \varepsilon^2 h_2, \;\; \varrho = \varrho_0 + \varepsilon^2 \varrho_2, \tag{17}$$

and substituting Expressions (17) into Bilinear Forms (14) with $\varepsilon = 1$, the one-soliton solutions are obtained as

$$q = \frac{g_1}{1 + f_2}, \tag{18}$$

where

$$f_2 = Ae^{\xi+\xi^*}, \;\; g_1 = e^{\xi}, \;\; \xi = \frac{i}{2}\omega^2(1 + 2\delta\omega^4)x + \omega t, \;\; A = \frac{1}{(\omega + \omega^*)^2}, \;\; h_0 = h_2 = \varrho_0 = \varrho_2 = 0,$$

with ω's as the complex constants.

Next, truncating Expressions (16) as

$$\begin{aligned} &f = 1 + \varepsilon^2 f_2 + \varepsilon^4 f_4, \;\; g = \varepsilon g_1 + \varepsilon^3 g_3, \\ &\rho = \varepsilon \rho_1 + \varepsilon^3 \rho_3, \;\; h = h_0 + \varepsilon^2 h_2 + \varepsilon^4 h_4, \;\; \varrho = \varrho_0 + \varepsilon^2 \varrho_2 + \varepsilon^4 \varrho_4, \end{aligned} \tag{19}$$

and substituting Expressions (19) into Bilinear Forms (14) with $\varepsilon = 1$, the two-soliton solutions are obtained as

$$q = \frac{g_1 + g_3}{1 + f_2 + f_4}, \tag{20}$$

where

$$\begin{aligned} &g_1 = e^{\xi_1} + e^{\xi_2}, \;\; g_3 = A_{123}e^{\xi_1+\xi_2+\xi_1^*} + A_{124}e^{\xi_1+\xi_2+\xi_2^*}, \;\; h_2 = 2A_{12}e^{\xi_1+\xi_2}, \\ &f_2 = A_{13}e^{\xi_1+\xi_1^*} + A_{23}e^{\xi_2+\xi_1^*} + A_{14}e^{\xi_1+\xi_2^*} + A_{24}e^{\xi_2+\xi_2^*}, \;\; f_4 = A_{1234}e^{\xi_1+\xi_1^*+\xi_2+\xi_2^*}, \\ &\varrho_2 = B_{12}e^{\xi_1+\xi_2}, \;\; \rho_1 = B_{34}(e^{\xi_1^*} + e^{\xi_2^*}), \;\; \rho_3 = B_{134}e^{\xi_1+\xi_1^*+\xi_2^*} + B_{234}e^{\xi_2+\xi_1^*+\xi_2^*}, \;\; \xi_j = k_j x + \omega_j t, \\ &k_j = \frac{i}{2}\omega_j^2(1 + 2\delta\omega_j^4), \;\; A_{12} = \omega_1 - \omega_2, \;\; A_{34} = \omega_1^* - \omega_2^*, \;\; A_{j,l+2} = \frac{1}{(\omega_j + \omega_j^*)^2}, \;\; (j = 1, 2), \\ &A_{123} = A_{12}A_{13}A_{23}, \;\; A_{124} = A_{12}A_{14}A_{24}, \;\; A_{1234} = A_{12}A_{13}A_{14}A_{23}A_{24}A_{34} \;\; (l = 1, 2), \end{aligned}$$

with k_l and ω_l as the complex constants.

4 Discussions on the soliton solutions

In this section, the solitons for Eq. (2) will be analyzed. From One-Soliton Solutions (18), q could be rewritten as,

$$|q| = |\omega_R| \operatorname{sech}\Big(\xi_R + \frac{1}{2}\ln A\Big), \tag{21}$$

where the subscripts I and R are the imaginary and real parts, respectively. Here, the concept of characteristic line for the solitons' propagation (Yang et al. 2016) will be introduced to determine the velocity. Letting $\omega = a_m + ib_m$ with a_m and b_m being the real constants ($m = 1, 2, \cdots$), the soliton amplitude $\Delta = a_m$ will be derived. Through the method in Ref. Yang et al. (2016), the characteristic-line equation for each soliton is deduced to

$$-a_m b_m \left[1 + 2\delta(3a_m^4 - 10a_m^2 b_m^2 + 3b_m^4)\right]x + a_m t + \frac{\ln A}{2} = \text{constant}.$$

Then differentiating it on both sides with respect to x, the velocity of each soliton is

$$v = \frac{1}{b_m\left[1 + 2\delta(3a_m^4 - 10a_m^2 b_m^2 + 3b_m^4)\right]}. \tag{22}$$

Based on the above analysis, we find that the initial position and velocity of the soliton are related to the parameter δ, but the soliton's amplitude is not affected by it. As seen in Fig. 1, we find that the one solitons propagate stably with the same amplitude and shape but different velocities and a certain degree of broadening or compressing.

Based on the asymptotic analysis, the two solitons will be analyzed as follows: When $x \to -\infty$ (before the interaction),

$$\begin{aligned} q \to q^{1-} &= |\omega_{1R}| e^{i\xi_{1I}} \mathrm{sech}\left(\xi_{1R} + \frac{1}{2}\ln A_{11}\right), (\xi_1 + \xi_1^* \to 0, \xi_2 + \xi_2^* \to -\infty), \\ q \to q^{2-} &= |\omega_{2R}| e^{i\xi_{2I}} \mathrm{sech}\left(\xi_{2R} + \frac{1}{2}\ln \frac{A_{1234}}{A_{11}}\right), (\xi_2 + \xi_2^* \to 0, \xi_1 + \xi_1^* \to +\infty), \end{aligned} \tag{23}$$

where q^{1-} and q^{2-} denote the two solitons before the interaction. When $t \to +\infty$ (after the interaction),

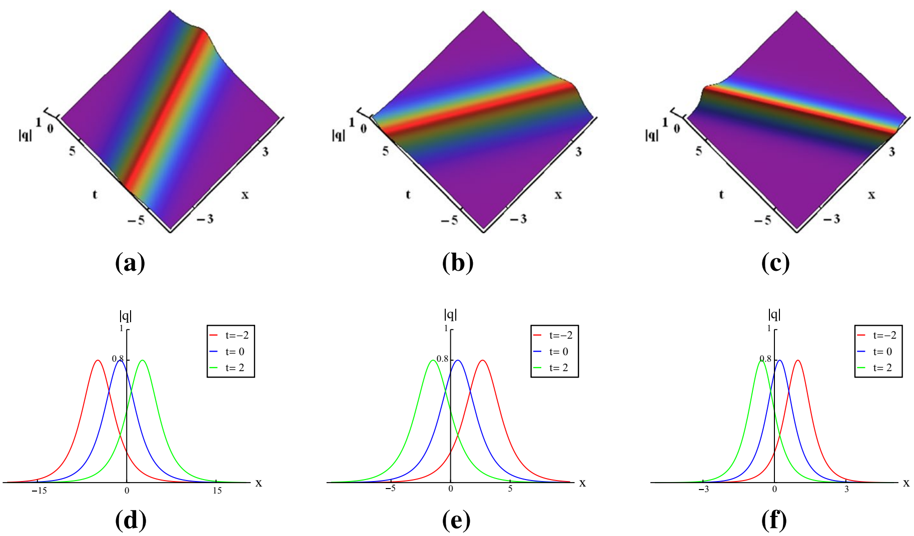

Fig. 1 One soliton via solutions (18) with $\omega = 0.8 + 0.7i$: **a** $\delta = 0.1$; **b** $\delta = 1$; **c** $\delta = 2$. **d–f** Trajectories of **a–c** at $t = -2$, $t = 0$ and $t = 2$, respectively

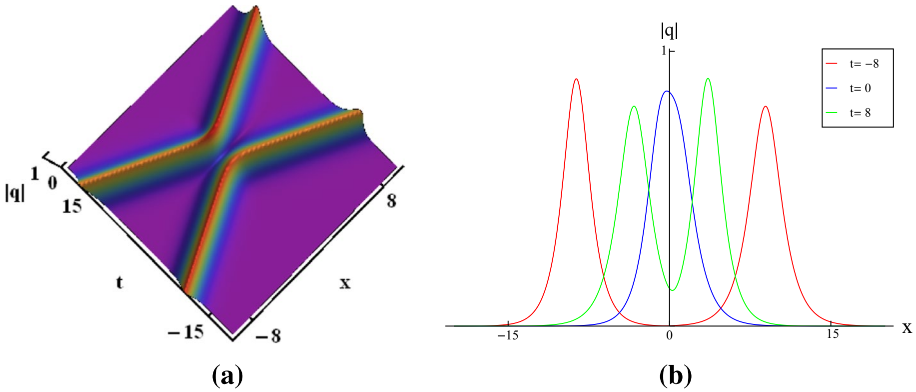

Fig. 2 **a** Head-on interaction between the two solitons via solutions (20) with $\omega_1 = 0.8 + 0.7i$, $\omega_2 = 0.9 + 0.4i$ and $\delta = 1$. **b** Trajectories of (a) at $t = -8$, 0 and 8

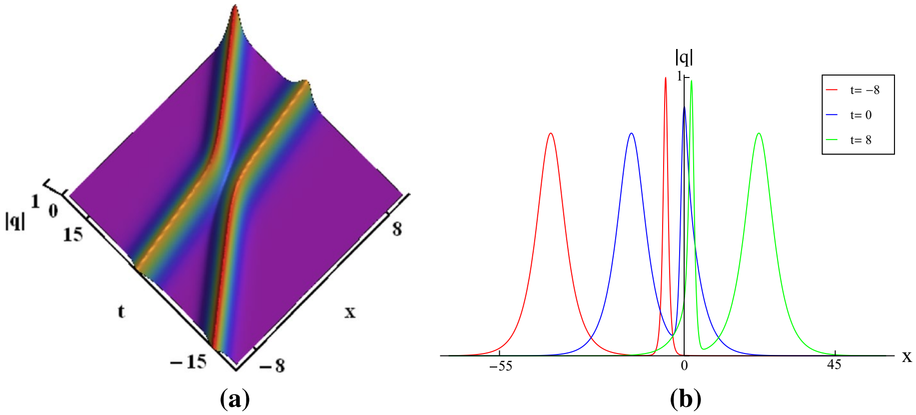

Fig. 3 **a** Overtaking interaction between the two solitons via solutions (20) with $\omega_1 = 0.8 + 0.1i$, $\omega_2 = 1.0 + 0.3i$ and $\delta = 1$. **b** Trajectories of **a** at $t = -8$, 0 and 8

$$
\begin{aligned}
&q \to q^{1+} = |\omega_{1R}| e^{i\xi_{1I}} \operatorname{sech}\left(\xi_{1R} + \frac{1}{2}\ln\frac{A_{1234}}{A_{22}}\right), (\xi_1 + \xi_1^* \to 0, \xi_2 + \xi_2^* \to +\infty),\\
&q \to q^{2+} = |\omega_{2R}| e^{i\xi_{2I}} \operatorname{sech}\left(\xi_{2R} + \frac{1}{2}\ln A_{22}\right), (\xi_1 + \xi_1^* \to 0, \xi_1 + \xi_1^* \to -\infty),
\end{aligned}
\tag{24}
$$

where q^{1+} and q^{2+} denote the two solitons after the interaction. Through the asymptotic analysis, we find that the interaction between the two solitons is elastic, which means that their amplitudes and shapes keep invariant after each interaction except for certain phase shifts, and this phenomenon can be confirmed in Figs. 2, 3, 4.

As shown in Fig. 2a, when the velocities for the two solitons satisfy the different signs, the head-on interaction happens. While the velocities of them are the same sign, the overtaking interaction is observed in Fig. 3a, where the soliton with a smaller amplitude moves faster and overtakes the larger one. Comparing Figs. 4a with 2a, we find

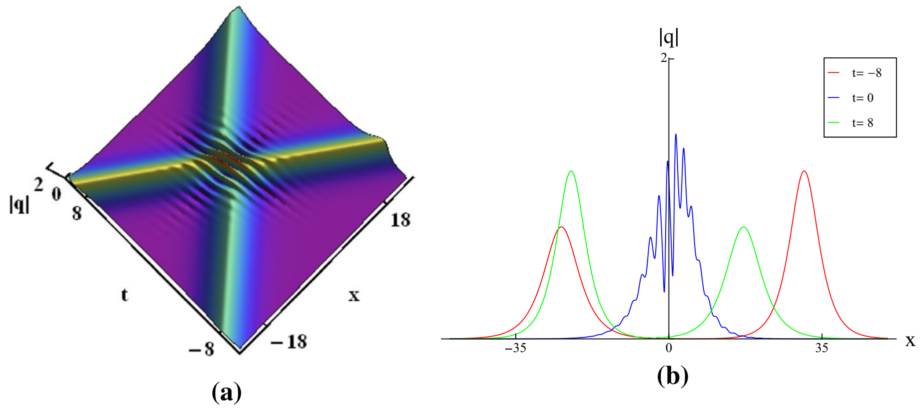

Fig. 4 **a** Oscillation interaction between the two solitons via solutions (20) with $\omega_1 = 1.2 - 0.731i$, $\omega_2 = 0.8 + 0.532i$ and $\delta = 1/2$. **b** Trajectories of **a** at $t = -8$, 0 and 8

that the two solitons present oscillation interaction when the distance between them decreases to a certain value, and the oscillation phenomenon is especially obvious at $t = 0$. Meanwhile, as seen in Figs. 2b–4b, we notice that those interactions are elastic. Bound state for the two solitons is formed as they have the equal velocity, and they attract and repulse each other periodically in Fig. 5.

5 Modulation instability

Based on the linear stability analysis (Zhao et al. 2016), we will investigate the Modulation instability (MI) of the stationary solutions for Eq. (2). The stationary solution for Eq. (2) has the following form,

$$q = q_0 e^{i(\kappa t + \theta x)}, \tag{25}$$

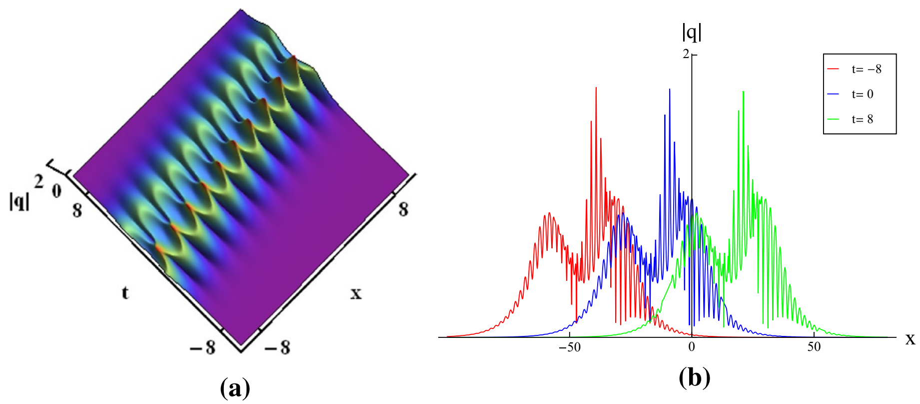

Fig. 5 **a** Bound state via solutions (20) with $\omega_1 = 1.2 + 0.0124i$, $\omega_2 = 0.8 + 0.0486i$ and $\delta = 1$. **b** Trajectories of **a** at $t = -8$, 0 and 8

where $\theta = q_0^2 + 20\delta q_0^6 - \kappa^2(\frac{1}{2} + 90\delta q_0^4) + 30\delta\kappa^4 q_0^2 - \delta\kappa^6$, q_0 and κ are the real constants. In order to perform the linear stability analysis, we set Solutions (25) with a small perturbation term as follows (Zhao et al. 2016; Chai et al. 2015; Wang et al. 2015c):

$$q = (q_0 + \chi\tilde{q})e^{i(\kappa t+\varpi x)}, \tag{26}$$

where χ is a perturbation parameter, $\tilde{q}$ is a function of x and t. In general, $\tilde{q}$ can be set as (Chai et al. 2015; Wang et al. 2015c)

$$\tilde{q} = q_1 e^{i(\kappa_1 t-\theta x)} + q_2 e^{-i(\kappa_1 t-\theta x)}, \tag{27}$$

where q_1 and q_2 are both the coefficients of the linear combination, κ_1 is a real disturbance wave numbers, while θ is a real disturbance frequency. Substituting Eqs. (26) and (27) into Eq. (2), we have linear equations about q_1 and q_2 as follows:

$$\begin{aligned} \Delta_{11}q_1 + \Delta_{12}q_2 = 0,\\ \Delta_{21}q_1 + \Delta_{22}q_2 = 0, \end{aligned} \tag{28}$$

with

$$\begin{aligned} \Delta_{11} =& -2q_0^2\left\{1 + 2\delta\left[\kappa_1^4 - 5\kappa_1^2(2q_0^2 - 3\varpi^2) + 15(2q_0^4 - 6q_0^2\varpi^2 + \varpi^4)\right]\right\},\\ \Delta_{12} =& \kappa_1^2 + 2\delta\kappa_1^6 - 24\kappa_1^4 q_0^2\delta + 100\kappa_1^2 q_0^4\delta - 160q_0^6\delta\\ &+ 2\kappa_1[1 + 6\delta\varpi(\kappa_1^4 - 10\kappa_1^2 q_0^2 + 30q_0^4)] + \varpi^2[1 + 30\delta(\kappa_1^4 - 8\kappa_1^2 q_0^2 + 18q_0^4)]\\ &+ 40\delta\kappa_1\varpi^3(\kappa_1^2 - 6q_0^2) + 30\delta\varpi^4(\kappa_1^2 - 4q_0^2) + 12\delta\kappa_1\varpi^5 + 2\delta\varpi^6 - 2(2q_0^2 + \theta) + 2\kappa,\\ \Delta_{21} =& \kappa_1^2 - 4q_0^2 + 2\delta\kappa_1^6 - 24\kappa_1^4 q_0^2\delta + 100\kappa_1^2 q_0^4\delta - 160q_0^6\delta\\ &- 2\kappa_1[1 + 6\delta\varpi(\kappa_1^4 - 10\kappa_1^2 q_0^2 + 30q_0^4)] + \varpi^2[1 + 30\delta(\kappa_1^4 - 8\kappa_1^2 q_0^2 + 18q_0^4)]\\ &- 40\delta\kappa_1\varpi^3(\kappa_1^2 - 6q_0^2) + 30\delta\varpi^4(\kappa_1^2 - 4q_0^2) - 12\delta\kappa_1\varpi^5 + 2\delta\varpi^6 + 2\theta + 2\kappa,\\ \Delta_{22} =& -2q_0^2\left\{1 + 2\delta\left[\kappa_1^4 - 5\kappa_1^2(2q_0^2 - 3\varpi^2) + 15(2q_0^4 - 6q_0^2\varpi^2 + \varpi^4)\right]\right\}. \end{aligned}$$

Equation (28) have a nontrivial solutions if and only if the determinant

$$\begin{vmatrix} \Delta_{11} & \Delta_{12}\\ \Delta_{21} & \Delta_{22} \end{vmatrix} = 0. \tag{29}$$

Based on the above expression, we get the following dispersion relation as

$$\theta = \kappa_1\varpi[1 + 6\delta\kappa_1^4 - 20\delta\kappa_1^2(3q_o^2 - \varpi^2) + 6\delta(30q_0^4 - 20q_0^2\varpi^2 + \varpi^4)] \pm \frac{1}{2}\sqrt{\Lambda}, \tag{30}$$

with

$$\begin{aligned} \Lambda =& \{\kappa_1^2 - 6q_0^2 + 2\delta\kappa_1^6 - 28\delta\varpi^4 q_0^2 + 140\delta\varpi^2 q_0^4 - 280\delta q_0^6\\ &+ \varpi^2[1 + 30\delta(\kappa_1^4 - 10\kappa_1^2 q_0^2 + 30q_0^4)] + 30\delta\varpi^4(\kappa_1^2 - 6q_0^2) + 2\delta\varpi^6 + 2\kappa\}\\ &\times \{\kappa_1^2 - 2q_0^2 + 2\delta\kappa_1^6 - 20\delta\varpi^4 q_0^2 + 60\delta\varpi^2 q_0^4 - 40\delta q_0^6\\ &+ \varpi^2[1 + 30\delta(\kappa_1^4 - 6\kappa_1^2 q_0^2 + 6q_0^4)] + 30\delta\varpi^4(\kappa_1^2 - 2q_0^2) + 2\delta\varpi^6 + 2\kappa\}. \end{aligned}$$

If $\Lambda \geqslant 0$, θ is always a real number. Based on the case, the intensity of the small perturbation $\tilde{q}$ will keep invariable along the fiber, which implies that the envelopes for Eq. (2) are stable against the disturbance of small perturbation. On the contrary, if $\Lambda < 0$, the θ will be a complex one. Then the $\tilde{q}$ will exponentially increase along the fiber, causing that the modulation instability will take place.

6 Conclusions

In this paper, Eq. (2) has been focused, which describes the pulses propagating along an optical fiber. With the help of the symbolic computation, we have derived Infinite-Many Conservation Laws (12) for Eq. (2). By virtue of the Hirota method and three auxiliary functions, we have obtained Bilinear Forms (14), One-Soliton Solutions (18) and Two-Soliton Solutions (20). From Expressions (21) and (22), we have found that the initial position and velocity of the soliton are related to the parameter δ in Eq. (2), but the soliton's amplitude is not affected by it. In Fig. 1, with the different values of δ, we have found that the one solitons propagate stably with the same amplitude and shape but different velocities and a certain degree of broadening or compressing. Figures 2, 3, 4 and 5 have illustrated the interactions between the two solitons: Head-on, overtaking, oscillating interactions and the bound state, respectively. Through the asymptotic analysis, interaction between the two solitons has been proved to be elastic. Based on the Linear stability analysis, we have derived the modulation instability condition for the soliton solutions.

Acknowledgements We express our sincere thanks to all the members of our discussion group for their valuable comments. We thanks Prof. Y. T. Gao and Dr. J. J. Su for the timely and valuable comments. This work has been supported by the Science Research Project of Higher Education in Inner Mongolia Autonomous Region under Grant No. NJZZ18117, by the Natural Science Foundation of Inner Mongolia Autonomous Region under Grant No. 2018BS01004, and by the National Natural Science Foundation of China under Grant No. 11772017.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

References

Ablowitz, M.J., Kaup, D.J., Newell, A.C., Segur, H.: Nonlinear-evolution equations of physical significance. Phys. Rev. Lett. **31**, 125–127 (1973)

Ankiewicz, A., Wang, Y., Wabnitz, S., N, Akhmediev: Extended nonlinear Schrödinger equation with higher-order odd and even terms and its rogue wave solutions. Phys. Rev. E **89**, 012907 (2014)

Ankiewicz, A., Kedziora, D.J., Chowdury, A., Bandelow, U., Akhmediev, N.: Infinite hierarchy of nonlinear Schrödinger equations and their solutions. Phys. Rev. E **93**, 012206 (2016)

Bourkoff, E., Christodoulides, D.N., Zhao, W., Joseph, R.I., Christodoulides, D.N.: Evolution of femtosecond pulses in single-mode fibers having higher-order nonlinearity and dispersion. Opt. Lett. **12**, 272–274 (1987)

Cai, L.Y., Wang, X., Wang, L., Li, M., Liu, Y., Shi, Y.Y.: Nonautonomous multi-peak solitons and modulation instability for a variable-coefficient nonlinear Schrödinger equation with higher-order effects. Nonlinear Dyn. **90**, 2221–2230 (2017)

Chai, J., Tian, B., Zhen, H.L., Sun, W.R.: Conservation laws, bilinear forms and solitons for a fifth-order nonlinear Schrödinger equation for the attosecond pulses in an optical fiber. Ann. Phys. **359**, 371–384 (2015)

Chowdury, A., Kedziora, D.J., Ankiewicz, A., Akhmediev, N.: Breather solutions of the integrable quintic nonlinear Schrödinger equation and their interactions. Phys. Rev. E **91**, 022919 (2015)

Dai, C.Q., Fan, Y., Zhou, G.Q., Zheng, J., Chen, L.: Vector spatiotemporal localized structures in (3+1)-dimensional strongly nonlocal nonlinear media. Nonlinear Dyn. **86**, 999–1005 (2016)

Dai, C.Q., Zhang, C.Q., Fan, Y., Chen, L.: Localized modes of the (n+1)dimensional Schrödinger equation with powerlaw nonlinearities in PT-symmetric potentials. Commun. Nonlinear Sci. Numer. Simul. **43**, 239–250 (2017)

Daniel, M., Kavitha, L., Amuda, R.: Soliton spin excitations in an anisotropic Heisenberg ferromagnet with octupole-dipole interaction. Phys. Rev. B **59**, 13774–13781 (1999)

de Oliveira, J.R., de Moura, M.A.: Analytical solution for the modified nonlinear Schrödinger equation describing optical shock formation. Phys. Rev. E **57**, 4751–4756 (1998)

Guo, R., Zhao, X.J.: Discrete Hirota equation: discrete Darboux transformation and new discrete soliton solutions. Nonlinear Dyn. **84**, 1901–1907 (2016)

Guo, R., Liu, Y.F., Hao, H.Q., Qi, F.H.: Coherently coupled solitons, breathers and rogue waves for polarized optical waves in an isotropic medium. Nonlinear Dyn. **80**, 1221–1230 (2015)

Guo, R., Zhao, H.H., Wang, Y.: A higher-order coupled nonlinear Schrödinger system: solitons, breathers, and rogue wave solutions. Nonlinear Dyn. **83**, 2475–2484 (2016)

Hasegawa, A., Tappert, F.: Transmission of stationary nonlinear optical pulses in dispersive dielectric fibers. I. Anomalous dispersion. Appl. Phys. Lett. **23**, 142–144 (1973a)

Hasegawa, A., Tappert, F.: Transmission of stationary nonlinear optical pulses in dispersive dielectric fibers. II. Normal dispersion. Appl. Phys. Lett. **23**, 171–172 (1973b)

Hirota, R.: The Direct Method in Soliton Theory. Cambridge University Press, New York (1991)

Lakoba, T.I., Kaup, D.J.: Hermite–Gaussian expansion for pulse propagation in strongly dispersion managed fibers. Phys. Rev. E **58**, 6728–6741 (1998)

Lan, Z., Gao, B.: Solitons, breather and bound waves for a generalized higher-order nonlinear Schrödinger equation in an optical fiber or a planar waveguide. Eur. Phys. J. Plus **132**, 512 (2017)

Lan, Z.Z., Gao, Y.T., Zhao, C., Yang, J.W., Su, C.Q.: Dark soliton interactions for a fifth-order nonlinear Schrödinger equation in a Heisenberg ferromagnetic spin chain. Superlattices Microstruct. **100**, 191–197 (2016)

Li, P., Wang, L., Kong, L.Q., Wang, X., Xie, Z.Y.: Nonlinear waves in the modulation instability regime for the fifth-order nonlinear Schrödinger equation. Appl. Math. Lett. **85**, 110–117 (2018)

Liu, W.J., Tian, B., Jiang, Y., Sun, K., Wang, P., Li, M., Qu, Q.X.: Soliton solutions and Bäcklund transformation for the complex Ginzburg–Landau equation. Appl. Math. Comput. **217**, 4369–4376 (2011)

Liu, W.J., Pan, N., Huang, L.G., Lei, M.: Soliton interactions for coupled nonlinear Schrödinger equations with symbolic computation. Nonlinear Dyn. **78**, 755–770 (2014)

Liu, W.J., Pang, L.H., Yan, H., Ma, G.L., Lei, M., Wei, Z.Y.: High-order solitons transmission in hollow-core photonic crystal fibers. EPL **116**, 64002 (2016)

Liu, W., Yu, W., Yang, C., Liu, M., Zhang, Y., Lei, M.: Analytic solutions for the generalized complex Ginzburg–Landau equation in fiber lasers. Nonlinear Dyn. **89**, 2933–2939 (2017)

Lü, X., Lin, F.: Soliton excitations and shape-changing collisions in alphahelical proteins with interspine coupling at higher order. Commun. Nonlinear Sci. Numer. Simul. **32**, 241–261 (2016)

Lü, X., Ma, W.X.: Study of lump dynamics based on a dimensionally reduced Hirota bilinear equation. Nonlinear Dyn. **85**, 1217–1222 (2016)

Lü, X., Chen, S.T., Ma, W.X.: Constructing lump solutions to a generalized Kadomtsev–Petviashvili–Boussinesq equation. Nonlinear Dyn. **86**, 523–534 (2016)

Nakatsuka, H., Grischkowsky, D., Balant, A.C.: Nonlinear picosecond-pulse propagation through optical fibers with positive group velocity dispersion. Phys. Rev. Lett. **47**, 910–913 (1981)

Nakazawa, M., Kubota, H., Suzuki, K., Yamada, E., Sahara, A.: Recent progress in soliton transmission technology. Chaos **10**, 486–514 (2000)

Osman, M.S., Wazwaz, A.M.: An efficient algorithm to construct multi-soliton rational solutions of the (2+1)-dimensional KdV equation with variable coefficients. Appl. Math. Comput. **321**, 282–289 (2018)

Sun, W.R.: Breather-to-soliton transitions and nonlinear wave interactions for the nonlinear Schrödinger equation with the sextic operators in optical fibers. Ann. Phys. (Berlin) **529**, 1–2 (2017)

Sun, W.R., Tian, B., Zhen, H.L., Sun, Y.: Breathers and rogue waves of the fifth-order nonlinear Schrödinger equation in the Heisenberg ferromagnetic spin chain. Nonlinear Dyn. **81**, 725–732 (2015)

Szmytkowski, R.: Alternative approach to the solution of the momentum-space Schrödinger equation for bound states of the N-dimensional coulomb problem. Ann. Phys. (Berlin) **524**, 345–352 (2012)

Wang, L., Li, X., Qi, F.H., Zhang, L.L.: Breather interactions and higher-order nonautonomous rogue waves for the inhomogeneous nonlinear Schrödinger Maxwell-Bloch equations. Ann. Phys. **359**, 97–114 (2015a)

Wang, L., Zhu, Y.J., Qi, F.H., Li, M., Guo, R.: Modulational instability, higher-order localized wave structures, and nonlinear wave interactions for a nonautonomous Lenells-Fokas equation in inhomogeneous fibers. Chaos **25**, 063111 (2015b)

Wang, Q.M., Gao, Y.T., Su, C.Q., Mao, B.Q., Gao, Z., Yang, J.W.: Dark solitonic interaction and conservation laws for a higher-order (2+1)-dimensional nonlinear Schrödinger-type equation in a Heisenberg ferromagnetic spin chain with bilinear and biquadratic interaction. Ann. Phys. **363**, 440–456 (2015c)

Wang, L., Zhang, J.H., Wang, Z.Q., Liu, C., Li, M., Qi, F.H., Guo, R.: Breather-to-soliton transitions, nonlinear wave interactions, and modulational instability in a higher-order generalized nonlinear Schrödinger equation. Phys. Rev. E **93**, 012214 (2016a)

Wang, L., Zhu, Y.J., Wang, Z.Q., Xu, T., Qi, F.H., Xue, Y.S.: Asymmetric Rogue waves, breather-to-soliton conversion, and nonlinear wave interactions in the Hirota-Maxwell-Bloch system. J. Phys. Soc. Jpn. **85**, 024001 (2016b)

Wazwaz, A.M.: Gaussian solitary wave solutions for nonlinear evolution equations with logarithmic nonlinearities. Nonlinear Dyn. **83**, 591–596 (2016)

Wazwaz, A.M.: A two-mode modified KdV equation with multiple soliton solutions. Appl. Math. Lett. **70**, 1–6 (2017a)

Wazwaz, A.M.: Two-mode fifth-order KdV equations: necessary conditions for multiplesoliton solutions to exist. Nonlinear Dyn. **87**, 1685–1691 (2017b)

Yang, J.W., Gao, Y.T., Wang, Q.M., Su, C.Q., Feng, Y.J., Yu, X.: Bilinear forms and soliton solutions for a fourth-order variable-coefficient nonlinear Schrödinger equation in an inhomogeneous Heisenberg ferromagnetic spin chain or an alpha helical protein. Phys. B **481**, 148–155 (2016)

Zhang, J.H., Wang, L., Liu, C.: Superregular breathers, characteristics of nonlinear stage of modulation instability induced by higher-order effects. Proc. R. Soc. A **473**, 20160681 (2017)

Zhao, X.H., Tian, B., Liu, D.Y., Wu, X.Y., Chai, J., Guo, Y.J.: Dark solitons interaction for a (2+1)-dimensional nonlinear Schrödinger equation in the Heisenberg ferromagnetic spin chain. Superlattices Microstruct. **100**, 587–595 (2016)

Zhao, X.H., Tian, B., Guo, Y.J.: Solitons, Lax pair and infinitely-many conservation laws for a higher-order nonlinear Schrödinger equation in an optical fiber. Optik **132**, 417–426 (2017)

Zhou, Q., Yao, D., Chen, F.: Analytical study of optical solitons in media with Kerr and parabolic law nonlinearities. J. Mod. Opt. **60**, 1652–1657 (2013)