Opt Quant Electron (2018) 50:170
https://doi.org/10.1007/s11082-018-1436-x

On nonlocal complex Maxwell equations and wave motion in electrodynamics and dielectric media

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Received: 5 February 2018 / Accepted: 14 March 2018 / Published online: 20 March 2018

Abstract A formulation of Maxwell's equations on the complex domain is presented in this paper which is based on the extension of the nonlocal-in-time kinetic energy approach recently introduced by Suykens. New wave equations with time-dependent source terms are obtained in a uniform optical medium. A number of physical effects were raised and their implications in classical electrodynamics besides to the propagation of waves in dielectric media are analyzed and discussed accordingly.

Keywords Nonlocal-in-time kinetic energy approach · Complex Maxwell equations · Electrodynamics · Dielectric media

1 Introduction

Nonlocal effects play an important role in the description of electrodynamics of continuous media, e.g. memory-dependent phenomena (Landau and Lifshitz 1960), vacuum electrodynamics of accelerated systems (Mashhoon 2003), normal and superconducting films (Vestgarden et al. 2013; Vestgarden and Johansen 2012), superconductor with spatially varying gap parameter (Brandt 1972), planar Josephson junctions (Boris et al. 2013), electrodynamics of rotating systems (Mashhoon 2005) among others. Nonlocal effects arise also in quantum electrodynamics and have important effects, e.g. EPR-like quantum correlation (Rai Dastidar and Rai Dastidar 1998), quantum optics (Rai Dastidar 1992) and also in quantum field electrodynamics phenomenologies (Efimov 1972; Phat 1973; Addazi and Esposito 2015). In fact, nonlocal effects result into nonlocal conservation laws and nonlocal symmetries which give rise to additional constants of motion not obtainable from local conservation and symmetry laws (Anco and Bluman 1997; Pohjanpelto 1995). Since

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classical and quantum electrodynamics are primary based on Maxwell equations, we turn our attention to these central equations in the present paper. Our main aim is to obtain nonlocal Maxwell's equations based on the nonlocal-in-time (NLT) framework.

In recent years, there have rapid developments in NLT theories. Time nonlocality arises in a large number of physical phenomena including extended classical mechanics (El-Nabulsi 2014, 2017a), nonconservative dynamics (Li et al. 2009), self-diffusion process (Stecki 1971), noncommutative geometry (Gomis et al. 2001), discrete quantum mechanics (El-Nabulsi 2016a), quantum field theory (El-Nabulsi 2016b, 2017b; Kamalov 2006, 2007, 2009, 2010, 2013) etc. NLT occurs as well in the theory of parabolic and hyperbolic differential equations (Gordeziani 1989, 1991, 1994; Gordeziani and Grigalashvili 1993). A more recent interesting approach dealing with NLT aspect was introduced by Suykens (2009). This innovative and trouble-free approach was motivated in reality from Feynman's observation of the kinetic energy functional written as $\frac{1}{2}m\frac{x_{k+1}-x_k}{\varepsilon}\frac{x_k-x_{k-1}}{\varepsilon}$ in place of $\frac{1}{2}mvv$ with $\varepsilon = t_{i+1} - t_i$, i.e. particle positions are shifted backward and forward in time (m being the mass of the body and $v = \frac{dx}{dt} \equiv \dot{x}$ its velocity) (Feynman 1948). Suykens used the shifting-coordinates tactic and rewrite the kinetic energy as $K = \frac{1}{2}mv\frac{\Delta}{2}$ where $\Delta = \dot{x}(t+\tau) + \dot{x}(t-\tau)$ and τ is to some extent a relatively tiny parameter entitled the "nonlocal time parameter". This effortless scheme leads to a number of inspiring properties at all scales which were discussed in a series of research papers (El-Nabulsi 2017a, c, 2018). Since in general $\Delta = \dot{x}(t+\tau) + \dot{x}(t-\tau)$ can be expanded in Taylor series using the series expansions $x(t+\tau) = x(t) + \sum_{k=1}^{n}\frac{\tau^k}{k!}x^{(k)}(t)$ and $x(t-\tau) = x(t) + \sum_{k=1}^{n}\frac{(-\tau)^k}{k!}x^{(k)}(t)$, higher-order derivative emerge naturally in the theory. In reality, higher-order derivative theories were shunned in the past since they allow states of negative norms to emerge in the theory under analysis. An infinite number of higher-order temporal derivatives of the coordinates arise in the Lagrange function which in general disagrees with the quantum formalism (Kamalov 2006, 2007, 2009, 2010, 2013). Nevertheless, after a large number of studies, it was observed that higher-order derivatives hold a number of generic outcomes and they constitute an indispensable mathematical tool nowadays in theoretical physics and sciences, e.g. in Abraham–Lorentz electrodynamics theory which describes the equation of motion for charged particles taking into account radiative effects (Jackson 1975). There exist quite a lot of methodologies to deal with higher-order derivative theories, e.g. the method of perturbative constraints introduced in Simon (1990) which is used to study dynamical systems characterized by equations of motion depending on more than one moment in time in addition to the backward and forward shifting coordinates/positions in time. This later was used by Nelson 50 years ago in his influential paper (Nelson 1966) which aim to derive the Schrödinger equation from stochastic aspects, i.e. stochastic models of quantum mechanics. Moreover, shifting coordinates motivated Laurent Nottale to construct a fractal theory of spacetime which today is recognized as scale relativity, a theory which puts quantum mechanics and special relativity in a single box (Nottale 1993). Shifting coordinates were more recently used in fluid dynamics mainly in nanotubes (El-Nabulsi 2017d) where several features were revealed. In this work, we are in particular concerned with Suykens's methodology due to its uncomplicated formalism. We will use this approach to construct a nonlocal version of Maxwell's equations. Nevertheless, in this paper we will extend Suykens's approach by replacing $\Delta = \dot{x}(t+\tau) + \dot{x}(t-\tau)$ by its complex counterpart $\Delta_{\mathsf{i}} = \frac{\dot{x}(t+\tau)+\dot{x}(t-\tau)}{2} + \mathsf{i}\gamma\frac{\dot{x}(t+\tau)-\dot{x}(t-\tau)}{2}$ where $\mathsf{i} = \sqrt{-1} \in \mathbb{C}$ and γ is a real or a complex parameter. More precisely, we will replace all velocities components in the Lagrangian by Δ_{i}, i.e. the kinetic energy $K = \frac{1}{2}mv\frac{\Delta}{2}$ will be replaced by $K = \frac{1}{2}m\frac{\Delta_{\mathsf{i}}}{2}\frac{\Delta_{\mathsf{i}}}{2}$. In fact when $\gamma = 0$, both $K_1 = \frac{1}{2}mv\frac{\Delta}{2}$ and $K_2 = \frac{1}{2}m\frac{\Delta}{2}\frac{\Delta}{2}$ leads to higher-order derivatives upper than in those obtained in Suykens's formalism, i.e. it is easy to verify that series expansion gives $K_1 = \frac{m\dot{x}^2}{2} + \frac{m\dot{x}}{4}\sum_{k=1}^{n}\frac{1+(-1)^k}{k!}\tau^k x^{(k+1)}$ and $K_2 = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\sum_{k=1}^{n}\frac{1+(-1)^k}{k!}\tau^k x^{(k+2)}(t) + \mathrm{O}(\tau^{2k})$. We will show in this paper that this transition will lead to many interesting properties not found in the classical electrodynamics theory.

It was observed in Newman (1973, Newman 2004) and Gsponer (2001) that solutions of Maxwell's equations in the complex domain correspond to the world-lines of a charged particle holding an intrinsic magnetic-dipole moment. An extension of electrodynamics to the complex Lagrangian domain mainly the Finsler geometry was developed in Munteanu (2007) and complexified electrodynamics was also studied in Friedman and Danziger (2008). It was observed in Arbab (2013a) that complex Maxwell's equations are more convenient to study duality transformations. In Arbab (2014a, 2015) by assuming that both the electric and magnetic fields for massive photons are complexified, a set of Maxwell's equations is obtained in free space and in a medium accordingly. This set of equations suggests that photons are massive in a medium and massless outside. More generally, complex electromagnetic theory based on complex vector algebra was constructed in Muralidhar (2014, 2015) and the analysis showed that existence of spin transforms a classical oscillator into a quantum oscillator and that the classical mechanics combined with zero point field leads to quantum mechanics. Complex Maxwell's equations and complex electrodynamics were also discussed in Gordeziani and Grigalashvili (1993), Friedman and Ostapenko (2010), Friedman (2013), Aste (2012), Arbab (2009, 2013b, 2014b) through different contexts and outcomes showed that massive photons may be included in Maxwell's electromagnetic theory and that the electromagnetic fields travel with speed of light in the presence or absence of free charges. However, most of the theories do not take into account the nonlocality in their formulation. In Landau and Lifshitz (1960), Mashhoon (2003), Vestgarden et al. (2013), Vestgarden and Johansen (2012), Brandt (1972), Boris et al. (2013), Mashhoon (2005), Rai Dastidar and Rai Dastidar (1998), Rai Dastidar (1992), Efimov (1972), Phat (1973), Addazi and Esposito (2015), Anco and Bluman (1997), Pohjanpelto (1995), Cho (1991), nonlocal Maxwell's equations were constructed in the real domain and not in the complex domain. To the best of our knowledge, this work represents the first attempt to construct "nonlocal complex Maxwell's equations" based on Suykens's nonlocal-in-time kinetic energy approach.

The paper is organized as follows: in Sect. 2, we introduce the basic setups of our model: we start from the extended NLT Lagrangian of a particle moving in an electromagnetic field in order to derive the modified electromagnetic field tensor which will be used to derive the NLT complex Maxwell's equations; in Sect. 3, we derive the complex NLT Maxwell's equations; in Sect. 4 we derive the complex Lorentz force and discuss some of its consequences; Sect. 5 is devoted to the implications of NLT complex Maxwell's equations in classical and quantum electrodynamics; in Sect. 6, we explore the propagation of electric field in dielectric medium; finally conclusions and perspectives are given in Sect. 7. Through this work, spacetime coordinates are expressed in terms of $(ct, x^i), i = 1, 2, 3$. We will refer to these variables as $(x_0, \vec{x}) = \mathsf{x}$. Einstein's summation notation is used as well. In tensor notation, we use the 4-vectors notation from special relativity: $X^\mu = (t, \vec{X})$, $X_\mu = (-t, \vec{X})$ and the four dimensional differential operators $\partial^\mu = (-\partial/\partial t, \vec{\nabla})$ and $\partial_\mu = (\partial/\partial t, \vec{\nabla})(\mu, \nu = 0, 1, 2, 3)$. Finally, we work in units $\hbar = c = 1$.

2 Basic setups of the theory: derivation of the NLT electromagnetic field tensor

In general, Maxwell's equations are set of partial differential equations which describe the evolution of electromagnetic fields on the real domain. In general, the electric field $\mathsf{E} = (E_i)$ and the magnetic fields $\mathsf{B} = (B_i)$ are assumed to be functions $f, g : \mathbb{R}^{1+3} \to \mathbb{R}$ and are expressed in terms of the 4-vector potential $A^i = (\phi, -\mathsf{A})$ by the following relations: $\mathsf{E} = -\vec{\nabla}\phi - \partial_{x_0}\mathsf{A}$ and $\mathsf{B} = \vec{\nabla} \times \mathsf{A}$. Here ϕ is the scalar potential and A is the vector potential (Landau and Lifschitz 1993). We consider at present a particle of mass m and electric charge e moving in the electromagnetic field. In the standard electrodynamics formalism, the Lagrangian is given by (Goldstein 1950; Yu 2012):

$$\mathcal{L} = \frac{1}{2}m\partial_{x_0}\mathsf{x}\cdot\partial_{x_0}\mathsf{x} + e\sum_{i=0}^{3}A_i\partial_{x_0}x_i \equiv \frac{1}{2}m\mathsf{v}(t)\cdot\mathsf{v}(t) + e\sum_{i=0}^{3}A_i\partial_{x_0}x_i, \tag{1}$$

where $\mathsf{v}(t) = \partial_{x_0}\mathsf{x} = \partial_t\mathsf{x} \equiv \dot{\mathsf{x}}$ is the velocity of the particle. In this paper, we will generalize Eq. (1) by globally replacing the velocity vector in Eq. (1) by Δ_{i} as follows:

$$\begin{aligned}\mathcal{L} = \frac{1}{2}m&\left(\frac{\mathsf{v}(t+\tau)+\mathsf{v}(t-\tau)}{2} + \mathrm{i}\gamma\frac{\mathsf{v}(t+\tau)-\mathsf{v}(t-\tau)}{2}\right)\cdot\\ &\left(\frac{\mathsf{v}(t+\tau)+\mathsf{v}(t-\tau)}{2} + \mathrm{i}\gamma\frac{\mathsf{v}(t+\tau)-\mathsf{v}(t-\tau)}{2}\right)\\ &+ e\sum_{i=0}^{3}A_i\left(\frac{\dot{x}_i(t+\tau)+\dot{x}_i(t-\tau)}{2} + \mathrm{i}\gamma\frac{\dot{x}_i(t+\tau)-\dot{x}_i(t-\tau)}{2}\right).\end{aligned} \tag{2}$$

By taking the following Taylor series expansions of $\mathsf{x}(t+\tau)$, $\mathsf{x}(t-\tau)$, $\mathsf{v}(t+\tau) = d\mathsf{x}(t+\tau)/dt \equiv \dot{\mathsf{x}}(t+\tau)$, $\mathsf{v}(t-\tau) = d\mathsf{x}(t-\tau)/dt \equiv \dot{\mathsf{x}}(t-\tau)$, $\dot{x}_i(t+\tau)$ and $\dot{x}_i(t-\tau)$:

$$\mathsf{x}(t+\tau) = \mathsf{x}(t) + \sum_{k=1}^{n}\frac{\tau^k}{k!}\mathsf{x}^{(k)}(t), \tag{3}$$

$$\mathsf{x}(t-\tau) = \mathsf{x}(t) + \sum_{k=1}^{n}\frac{(-1)^k\tau^k}{k!}\mathsf{x}^{(k)}(t), \tag{4}$$

$$\mathsf{v}(t+\tau) = \dot{\mathsf{x}}(t) + \sum_{k=1}^{n}\frac{\tau^k}{k!}\mathsf{x}^{(k+1)}(t), \tag{5}$$

$$\mathsf{v}(t-\tau) = \dot{\mathsf{x}}(t) + \sum_{k=1}^{n}\frac{(-1)^k\tau^k}{k!}\mathsf{x}^{(k+1)}(t), \tag{6}$$

$$\dot{x}_i(t+\tau) = \dot{x}_i(t) + \sum_{k=1}^{n}\frac{\tau^k}{k!}x_i^{(k+1)}(t), \tag{7}$$

$$\dot{x}_i(t-\tau) = \dot{x}_i(t) + \sum_{k=1}^{n} \frac{(-1)^k \tau^k}{k!} x_i^{(k+1)}(t), \tag{8}$$

we can write Lagrangian as:

$$\begin{aligned} \boldsymbol{\mathcal{L}}_{\tau,n} = & \underbrace{\frac{1}{2} m \left(\dot{\mathsf{x}}(t) + \frac{1+\mathrm{i}\gamma}{2} \sum_{k=1}^{n} \frac{\tau^k}{k!} \mathsf{x}^{(k+1)}(t) + \frac{1-\mathrm{i}\gamma}{2} \sum_{k=1}^{n} \frac{(-1)^k \tau^k}{k!} \mathsf{x}^{(k+1)}(t) \right)^2}_{\boldsymbol{\mathcal{T}}_{\tau,n}} \\ & \underbrace{+ e \sum_{i=0}^{3} A_i \left(\dot{x}_i(t) + \frac{1+\mathrm{i}\gamma}{2} \sum_{k=1}^{n} \frac{\tau^k}{k!} x_i^{(k+1)}(t) + \frac{1-\mathrm{i}\gamma}{2} \sum_{k=1}^{n} \frac{(-1)^k \tau^k}{k!} x_i^{(k+1)}(t) \right)}_{-\boldsymbol{\mathcal{V}}_{\tau,n}}. \end{aligned} \tag{9}$$

This Lagrangian $\boldsymbol{\mathcal{L}}_{\tau,n} = \boldsymbol{\mathcal{L}}_{\tau,n}(\mathsf{x}, \dot{\mathsf{x}}, \ldots, \mathsf{x}^{(N)})(N = n+1)$ contains higher-order derivatives terms and takes the form $\boldsymbol{\mathcal{L}}_{\tau,n} = \boldsymbol{\mathcal{T}}_{\tau,n} - \boldsymbol{\mathcal{V}}_{\tau,n}$. The equation of motion is derived from the following higher-order Euler–Lagrange equation by considering independent variables $\mathsf{X}_i(t)$ such that $\mathsf{X}_i = \dot{\mathsf{X}}_{i-1}, i = 1, 2, \ldots, N-1$, $\mathsf{X}_0 = \mathsf{x}$ and $\mathsf{X} = \mathsf{X}_0$ (Suykens 2009):

$$\frac{\partial \boldsymbol{\mathcal{L}}_{\tau,n}}{\partial \mathsf{X}_i} - \partial_t \frac{\partial \boldsymbol{\mathcal{L}}_{\tau,n}}{\partial \dot{\mathsf{X}}_i} + \partial_{tt} \frac{\partial \boldsymbol{\mathcal{L}}_{\tau,n}}{\partial \ddot{\mathsf{X}}_i} - \partial_{ttt} \frac{\partial \boldsymbol{\mathcal{L}}_{\tau,n}}{\partial \dddot{\mathsf{X}}_i} + \cdots \equiv \sum_{j=0}^{N} (-1)^j \partial_{t^j} \frac{\partial \boldsymbol{\mathcal{L}}_{\tau,n}}{\partial \mathsf{X}_i^{(j)}} = 0, \tag{10}$$

which is the stationary solution to the action functional $\mathsf{S} = \int_{t_0}^{t_f} \boldsymbol{\mathcal{L}}_{\tau,n} dt$ under the postulations that the action functional is subject to given boundary conditions $\delta \mathsf{x}^{(j)}(t_0) = \delta \mathsf{x}^{(j)}(t_f) = 0, j = 0, 1, 2, \ldots, N-1$ (Simon 1990). In Eq. (10) the following notations hold: $\partial \mathsf{X}_i^{(0)} \equiv \partial \mathsf{X}_i$, $\partial \mathsf{X}_i^{(1)} \equiv \partial \dot{X}_i = \partial(\partial \mathsf{X}_i / \partial t)$ and so on.

Remark 2.1 It should be stressed that the Lagrangian (9) is complex valued. In fact, recent studies proved that complexification of real Lagrangian and Hamiltonian may throw some light in explaining a number of properties in specific dynamical systems (Bender et al. 2007a, b; Bender 2007; Sbitnev 2009; Alber and Marsden 1996; Kaushal 2009; El-Nabulsi 2012; El-Nabulsi et al. 2012). We can always express the Lagrangian in terms of the complex variable $z = x + \mathrm{i}y$ where $x = \frac{x(t+\tau)+x(t-\tau)}{2}$ and $y = \frac{x(t+\tau)-x(t-\tau)}{2}$ and obtain a holomorphic Lagrangian (Markushevich 1965) yet in this work we use the variables X for convenience.

We restrict our analysis up to $n = 1$ since the nonlocal time parameter is tiny, i.e. the higher derivatives (related to the terms $n > 1$) are small. Consequently the Lagrangian we will admit in what follows for $(X_0, \vec{X}) = \mathsf{X}$ is:

$$\boldsymbol{\mathcal{L}}_{\tau,1} = \frac{1}{2} m \left(\dot{\mathsf{X}} + \mathrm{i}\gamma\tau\ddot{\mathsf{X}} \right) \cdot \left(\dot{\mathsf{X}} + \mathrm{i}\gamma\tau\ddot{\mathsf{X}} \right) + e \sum_{i=0}^{3} A_i \left(\dot{X}_i + \mathrm{i}\gamma\tau\ddot{X}_i \right) + \mathsf{O}(\tau^2), \tag{11}$$

and the corresponding equation of motion is derived from the following Euler–Lagrange equation:

$$\frac{\partial \boldsymbol{\mathcal{L}}_{\tau,1}}{\partial X_i} - \partial_t \left(\frac{\partial \boldsymbol{\mathcal{L}}_{\tau,1}}{\partial \dot{X}_i} \right) + \partial_{tt} \left(\frac{\partial \boldsymbol{\mathcal{L}}_{\tau,1}}{\partial \ddot{X}_i} \right) = 0. \tag{12}$$

For the sake of simplicity, we will not repeat the written of $\mathsf{O}(\tau^2)$ in the rest of equations since our analysis is limited to $n = 1$ in the rest of the paper.

Remark 2.2 It is notable that the canonical conjugate momentum in the NLT approach is $\mathsf{p}^i = m\mathsf{i}\gamma\tau\ddot{X}^i + m\dot{X}^i + eA^i$ and is different from the canonical conjugate momentum obtained in the standard approach. The new momentum is complexified and holds an acceleration term.

Using Einstein's summation notation, the following partial derivatives hold $(i, j = 0, 1, 2, 3)$:

$$\frac{\partial \mathcal{L}_{\tau,2}}{\partial X_i} = e\frac{\partial A_j}{\partial X_i}\left(\dot{X}_i + \mathsf{i}\gamma\tau\ddot{X}_i\right), \tag{13}$$

$$\partial_t\left(\frac{\partial \mathcal{L}_{\tau,2}}{\partial \dot{X}_i}\right) = e\dot{A}_i + m\ddot{X}_i + \mathsf{i}\gamma\tau m\dddot{X}_i, \tag{14}$$

$$\partial_{tt}\left(\frac{\partial \mathcal{L}_{\tau,2}}{\partial \ddot{X}_i}\right) = \mathsf{i}\gamma\tau m\dddot{X} + \mathsf{i}\gamma\tau e\ddot{A}_i. \tag{15}$$

The equation of motion takes therefore the following form:

$$m\ddot{X}_i = e\frac{\partial A_j}{\partial X_i}\left(\dot{X}_j + \mathsf{i}\gamma\tau\ddot{X}_j\right) - e\dot{A}_i + \mathsf{i}\gamma\tau e\ddot{A}_i. \tag{16}$$

Now using the fact that:

$$\dot{A}_i = \frac{\partial A_i}{\partial X_j}\dot{X}_j, \tag{17}$$

and

$$\ddot{A}_i = \frac{\partial}{\partial t}\left(\frac{\partial A_i}{\partial X_j}\dot{X}_j\right) = \frac{\partial}{\partial t}\left(\frac{\partial A_i}{\partial X_j}\right)\dot{X}_j + \frac{\partial A_i}{\partial X_j}\ddot{X}_j, \tag{18}$$

we can rewrite Eq. (16) as:

$$m\ddot{X}_i = e\frac{\partial A_j}{\partial X_i}\left(\dot{X}_j + \mathsf{i}\gamma\tau\ddot{X}_j\right) - e\frac{\partial A_i}{\partial X_j}\dot{X}_j + \mathsf{i}\gamma\tau e\left(\frac{\partial}{\partial t}\left(\frac{\partial A_i}{\partial X_j}\right)\dot{X}_j + \frac{\partial A_i}{\partial X_j}\ddot{X}_j\right). \tag{19}$$

Introducing the standard covariant electromagnetic field tensor:

$$\mathsf{F}_{ij} \triangleq \frac{\partial A_j}{\partial X_i} - \frac{\partial A_i}{\partial X_j}, \tag{20}$$

we can rewrite Eq. (19) as:

$$m\ddot{X}_i = e\mathsf{F}_{ij}\dot{X}_j + \mathsf{i}\gamma\tau e\left(2\frac{\partial A_j}{\partial X_i}\ddot{X}_j + \frac{\partial}{\partial t}\left(\frac{\partial A_i}{\partial X_j}\right)\dot{X}_j\right), \tag{21}$$

or as:

$$\ddot{X}_i = \frac{e}{m}\mathbb{F}_{ij}\dot{X}_j, \tag{22}$$

where

$$\mathbb{F}_{ij} = \frac{\mathsf{F}_{ij} + \mathrm{i}\gamma\tau\frac{\partial}{\partial t}\left(\frac{\partial A_i}{\partial X_j}\right)}{1 - \frac{2\mathrm{i}\gamma\tau e}{m}\frac{\partial A_i}{\partial X_j}}, \tag{23}$$

is the nonlocal covariant electromagnetic field tensor. Obviously when $\tau = 0$, Eq. (22) is reduced to the standard equation of motion. Equation (22) represents the NLT complex equation of motion for a particle in an electromagnetic field. Since in units $c = 1$ and in particular for $\tau << 1$ the factor $\tau e/m << 1$, we can simplify Eq. (23) to:

$$\mathbb{F}_{ij} \approx \frac{\partial A_j}{\partial X_i} - \frac{\partial A_i}{\partial X_j} + \mathrm{i}\gamma\tau\frac{\partial}{\partial t}\left(\frac{\partial A_i}{\partial X_j}\right). \tag{24}$$

The following statement holds accordingly:

Statement 1 *In the extended nonlocal-in-time kinetic energy approach where the classical velocity is replaced by* $\Delta_\mathrm{i} = \frac{\dot{x}(t+\tau)+\dot{x}(t-\tau)}{2} + \mathrm{i}\gamma\frac{\dot{x}(t+\tau)-\dot{x}(t-\tau)}{2}$, *the covariant electromagnetic field tensor is complexified and takes up to* $n = 1$ *the form*:

$$\mathbb{F}_{ij} \approx \frac{\partial A_j}{\partial X_i} - \frac{\partial A_i}{\partial X_j} + \mathrm{i}\gamma\tau\frac{\partial}{\partial t}\left(\frac{\partial A_i}{\partial X_j}\right).$$

In the next section we will use Eq. (24) to construct NLT complex Maxwell's equations and discuss their implications in electrodynamics, classical and quantum optics.

3 NLT complex Maxwell's equations

In tensor notation, we use the terminology $A^\mu = (\phi, \mathsf{A})$ and $A_\mu = (-\phi, \mathsf{A})$ (Griffiths 1999). Therefore we can write the nonlocal complex electromagnetic field strength in terms of covariant space–time derivatives of the four vector potential field A^μ as:

$$\mathbb{F}^{\mu\nu} \approx \boldsymbol{\mathcal{F}}^{\mu\nu} = \frac{\partial A^\nu}{\partial X_\mu} - \frac{\partial A^\mu}{\partial X_\nu} + \mathrm{i}\gamma\tau\frac{\partial}{\partial t}\left(\frac{\partial A^\mu}{\partial X_\nu}\right). \tag{25}$$

Using the relations $\mathsf{E} = -\vec{\nabla}\phi - \dot{\mathsf{A}}$ and $\mathsf{B} = \vec{\nabla} \times \mathsf{A}$, it is easy to check that:

$$\mathbb{F}_{0i} = -E_i - \mathrm{i}\gamma\tau\partial_i\dot{\phi}, \tag{26}$$

$$\mathbb{F}_{i0} = E_i + \mathrm{i}\gamma\tau\ddot{A}_i, \tag{27}$$

$$\mathbb{F}_{ij} = \varepsilon_{ijk}B_k + \mathrm{i}\gamma\tau\partial_j\dot{A}_i. \tag{28}$$

Here ε_{ijk} is the Levi–Civita anti-symmetric tensor. Therefore the following matrix holds:

$$\mathbb{F}_{\mu\nu} = \begin{pmatrix} -\mathrm{i}\gamma\tau\ddot{\phi} & -E_1 - \mathrm{i}\gamma\tau\nabla_x\dot{\phi} & -E_1 - \mathrm{i}\gamma\tau\nabla_y\dot{\phi} & -E_1 - \mathrm{i}\gamma\tau\nabla_z\dot{\phi} \\ E_1 + \mathrm{i}\gamma\tau\ddot{A}_1 & \mathrm{i}\gamma\tau\nabla_x\dot{A}_1 & B_3 + \mathrm{i}\gamma\tau\nabla_y\dot{A}_2 & -B_2 + \mathrm{i}\gamma\tau\nabla_z\dot{A}_3 \\ E_2 + \mathrm{i}\gamma\tau\ddot{A}_2 & -B_3 + \mathrm{i}\gamma\tau\nabla_x\dot{A}_2 & \mathrm{i}\gamma\tau\nabla_y\dot{A}_2 & B_1 + \mathrm{i}\gamma\tau\nabla_z\dot{A}_2 \\ E_3 + \mathrm{i}\gamma\tau\ddot{A}_3 & B_2 + \mathrm{i}\gamma\tau\dot{A}_{3,x} & -B_1 + \mathrm{i}\gamma\tau\nabla_y\dot{A}_3 & \mathrm{i}\gamma\tau\nabla_z\dot{A}_3 \end{pmatrix}, \tag{29}$$

where $\nabla_0 \equiv \partial/\partial t, \nabla_x \equiv \partial/\partial x$ and so on.

Let us at the present evaluate the following term: $\frac{\partial \mathbb{F}_{\mu\nu}}{\partial X^\lambda} + \frac{\partial \mathbb{F}_{\nu\lambda}}{\partial X^\mu} + \frac{\partial \mathbb{F}_{\lambda\mu}}{\partial X^\nu}$. We find:

$$\frac{\partial \mathbb{F}_{\mu\nu}}{\partial X^\lambda} + \frac{\partial \mathbb{F}_{\nu\lambda}}{\partial X^\mu} + \frac{\partial \mathbb{F}_{\lambda\mu}}{\partial X^\nu} = \mathrm{i}\gamma\tau\frac{\partial}{\partial t}\left(\frac{\partial^2 A_\mu}{\partial X^\nu \partial X^\lambda} + \frac{\partial^2 A_\nu}{\partial X^\lambda \partial X^\mu} + \frac{\partial^2 A_\mu}{\partial X^\lambda \partial X^\nu}\right), \tag{30}$$

and therefore $\frac{\partial \mathbb{F}_{\mu\nu}}{\partial X^\lambda} + \frac{\partial \mathbb{F}_{\nu\lambda}}{\partial X^\mu} + \frac{\partial \mathbb{F}_{\lambda\mu}}{\partial X^\nu} = 0$ unless

$$\frac{\partial^2 A_\mu}{\partial X^\nu \partial X^\lambda} + \frac{\partial^2 A_\nu}{\partial X^\lambda \partial X^\mu} + \frac{\partial^2 A_\mu}{\partial X^\lambda \partial X^\nu} = 0, \tag{31}$$

or a certain constant. It is notable that if we replace A^μ by

$$\tilde{A}^\mu = A^\mu + \frac{\partial \lambda}{\partial X_\mu}, \tag{32}$$

in $\mathbb{F}^{\mu\nu}$ we obtain:

$$\begin{aligned} \tilde{\mathbb{F}}^{\mu\nu} &\approx \frac{\partial \tilde{A}^\mu}{\partial X_\nu} - \frac{\partial \tilde{A}^\nu}{\partial X_\mu} + \mathrm{i}\gamma\tau\frac{\partial}{\partial t}\left(\frac{\partial \tilde{A}^\nu}{\partial X_\mu}\right), \\ &= \frac{\partial A^\mu}{\partial X_\nu} - \frac{\partial A^\nu}{\partial X_\mu} + \mathrm{i}\gamma\tau\frac{\partial}{\partial t}\left(\frac{\partial}{\partial X_\mu}\left(A^\nu + \frac{\partial \lambda}{\partial X_\nu}\right)\right), \end{aligned} \tag{33}$$

which gives $\tilde{\mathbb{F}}^{\mu\nu} = \mathbb{F}^{\mu\nu}$ unless:

$$\frac{\partial^2 \lambda}{\partial X_\mu \partial X_\nu} = k, \tag{34}$$

where k is a certain constant, i.e. $\frac{\partial \lambda}{\partial X_\mu} = kX_\nu$.

Consider first $(\mu, \nu, \lambda) = (0, i, j)$: Eq. (30) gives for the case of a source free region:

$$\partial_0 F_{ij} + \partial_i F_{j0} + \partial_j F_{0i} = \mathrm{i}\gamma\tau\left(\partial_i\partial_j\dot{A}_0 + \partial_j\ddot{A}_i + \partial_i\ddot{A}_j\right), \tag{35}$$

and using Eqs. (26)–(28), we find:

$$\partial_0\left(\varepsilon_{ijk}B_k + \mathrm{i}\gamma\tau\partial_j\dot{A}_i\right) + \partial_i\left(E_j + \mathrm{i}\gamma\tau\ddot{A}_j\right) + \partial_j\left(-E_i - \mathrm{i}\gamma\tau\partial_i\dot{\phi}\right) = \mathrm{i}\gamma\tau\left(\partial_i\partial_j\dot{A}_0 + \partial_j\ddot{A}_i + \partial_i\ddot{A}_j\right). \tag{36}$$

After simple arrangements, we can write Eq. (36) as:

$$\varepsilon_{ijk}\dot{B}_k + \partial_i E_j - \partial_j E_i + \mathrm{i}\gamma\tau\left(\partial_j\ddot{A}_i + \partial_i\ddot{A}_j + \partial_j\partial_i\dot{\phi} - \partial_i\partial_j\dot{\phi} - \partial_j\ddot{A}_i - \partial_i\ddot{A}_j\right) = 0. \tag{37}$$

Contracting by ε_{ijl} and using $\varepsilon_{ijm}\varepsilon_{kjm} = 2\delta_{ik}$ where δ_{ik} is the Kronecker symbol, we can reduce this equation after some algebra to:

$$\dot{B}_l + \varepsilon_{ijl}\partial_i E_j = 0. \tag{38}$$

This is just the standard equation:

$$\vec{\nabla} \times \mathsf{E} = -\dot{\mathsf{B}}. \tag{39}$$

$\square = \Delta - \partial^2/\partial t^2 = \partial^\mu \partial_\mu$ the d'Alembertian operator and Δ is the Laplacian operator.
Now for $(\mu, v, \lambda) = (i, j, k)$, Eq. (30) gives:

$$\partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij} = \mathrm{i}\gamma\tau\left(\partial_j \partial_k \dot{A}_i + \partial_i \partial_k \dot{A}_j + \partial_i \partial_j \dot{A}_k\right). \tag{40}$$

Contracting by ε_{ijk} we find using Eq. (28):

$$\varepsilon_{ijk}\varepsilon_{ijk}\partial_i B_k + \mathrm{i}\gamma\tau\varepsilon_{ijk}\partial_i\partial_j\dot{A}_k = \mathrm{i}\gamma\tau\varepsilon_{ijk}\partial_i\partial_j\dot{A}_k, \tag{41}$$

which is equivalent to $\partial_i B_i = 0$ or

$$\vec{\nabla} \cdot \mathsf{B} = 0. \tag{42}$$

The NLT electromagnetic theory is therefore free from magnetic monopoles. Physically, for $\mu = 0$, we can write:

$$\begin{aligned} \partial^v \mathbb{F}_{0v} &= -\vec{\nabla} \cdot \mathsf{E} - \mathrm{i}\gamma\tau\left(\nabla_{xx}\dot{\phi} + \nabla_{yy}\dot{\phi} + \nabla_{zz}\phi - \ddot{\phi}\right), \\ &= -\vec{\nabla} \cdot \mathsf{E} - \mathrm{i}\gamma\tau\square\dot{\phi}. \end{aligned} \tag{43}$$

Since in the absence of a source $\partial^v \mathbb{F}_{0v} = 0$ (Kok 2016), we obtain from Eq. (43):

$$\vec{\nabla} \cdot \mathsf{E} = -\mathrm{i}\gamma\tau\square\dot{\phi}, \tag{44}$$

which is the NLT complex Gauss' law. Moreover, one may check from $\partial_\mu \mathbb{F}^{\mu j} = \partial_0 \mathbb{F}^{0j} + \partial_i \mathbb{F}^{ij} = 0$ that the following relation holds:

$$\vec{\nabla} \times \mathsf{B} + \mathrm{i}\gamma\tau\square\dot{\mathsf{A}} = \dot{\mathsf{E}}. \tag{45}$$

The subsequent 2nd statement then holds:

Statement 2 *In the extended nonlocal-in-time kinetic energy approach, the set of NLT complex Maxwell's equations is the following*:

NLT complexified Gauss's law $\vec{\nabla} \cdot \mathsf{E} = -\mathrm{i}\gamma\tau\square\dot{\phi}$,
NLT complexified Maxwell's law $\vec{\nabla} \times \mathsf{B} + \mathrm{i}\gamma\tau\square\dot{\mathsf{A}} = \dot{\mathsf{E}}$,
No magnetic charge law $\vec{\nabla} \cdot \mathsf{B} = 0$,
NLT complexified Faraday's law $\vec{\nabla} \times \mathsf{E} = -\dot{\mathsf{B}}$.

In the presence of the volume charge density ρ and the current density $\vec{\mathbb{J}}$, one may check that the NLT complex Maxwell's equations take the following general form:

NLT complexified Gauss's law $\vec{\nabla} \cdot \mathsf{E} = \mu_0\rho - \mathrm{i}\gamma\tau\square\dot{\phi}$,
NLT complexified Maxwell's law $\vec{\nabla} \times \mathsf{B} + \mathrm{i}\gamma\tau\square\dot{\mathsf{A}} = \dot{\mathsf{E}} + \mu_0\vec{\mathbb{J}}$,
No magnetic charge law $\vec{\nabla} \cdot \mathsf{B} = 0$,
NLT complexified Faraday's law $\vec{\nabla} \times \mathsf{E} = -\dot{\mathsf{B}}$.
Recall that the vector A is the spatial part of the 4-vector $A^i = (\phi, -\mathsf{A})$.

We can also write these equations in terms of the displacement field $\mathsf{D} = \varepsilon_0 \mathsf{E}$ and the magnetizing field $\mathsf{H} = \mu_0^{-1} \mathsf{B}$ where ε_0 is the permittivity of the vacuum and μ_0 is the permeability constant. We obtain accordingly:

$$\vec{\nabla} \cdot \mathsf{D} = -\mathrm{i}\gamma\tau\varepsilon_0 \Box \dot{\phi}, \tag{46}$$

$$\varepsilon_0 \mu_0 \vec{\nabla} \times \mathsf{H} + \mathrm{i}\gamma\tau\varepsilon_0 \Box \dot{\mathsf{A}} = \dot{\mathsf{D}}, \tag{47}$$

$$\vec{\nabla} \cdot \mathsf{H} = 0, \tag{48}$$

$$\vec{\nabla} \times \mathsf{D} = -\varepsilon_0 \mu_0 \dot{\mathsf{H}}. \tag{49}$$

Here $\varepsilon_0 \mu_0 = 1$ in unit $c = 1$. Some consequences of these equations are the followings: First, we can write Eq. (39) as $\vec{\nabla} \times \mathsf{E} = -\mu_0 \dot{\mathsf{H}}$ and taking the curl of both sides gives:

$$\vec{\nabla} \times \vec{\nabla} \times \mathsf{E} = -\mu_0 \partial_0 \underbrace{\left(\vec{\nabla} \times \mathsf{H}\right)}_{\text{equation}(47)} = -\ddot{\mathsf{E}}. \tag{50}$$

Since $\vec{\nabla} \times \vec{\nabla} \times \mathsf{E} = \vec{\nabla}(\vec{\nabla} \cdot \mathsf{E}) - \Delta \mathsf{E}$, we find using Eq. (44):

$$\Box \mathsf{E} = 0, \tag{51}$$

which is the wave equation describing the propagation of the electric field in a medium. Besides, one can write:

$$\begin{aligned} \vec{\nabla} \times \vec{\nabla} \times \mathsf{H} &= \vec{\nabla} \underbrace{\vec{\nabla} \cdot \mathsf{H}}_{=0} - \vec{\nabla}^2 \mathsf{H} = -\vec{\nabla}^2 \mathsf{H} = \frac{\partial}{\partial t}\left(\vec{\nabla} \times \mathsf{D}\right) - \mathrm{i}\gamma\tau\varepsilon_0 \underbrace{\vec{\nabla} \times \Box \dot{\mathsf{A}}}_{=0}, \\ &= \varepsilon_0 \frac{\partial}{\partial t}\left(\vec{\nabla} \times \mathsf{E}\right) = -\ddot{\mathsf{H}}, \end{aligned} \tag{52}$$

and therefore the wave equation of motion for the field H is:

$$\Box \mathsf{H} = 0. \tag{53}$$

Moreover, we can write:

$$\begin{aligned} \mu_0 \vec{\nabla} \times \mathsf{H} = \vec{\nabla} \times \mathsf{B} = \vec{\nabla} \times \vec{\nabla} \times \mathsf{A} = \vec{\nabla}\vec{\nabla} \cdot \mathsf{A} - \vec{\nabla}^2 \mathsf{A} = \dot{\mathsf{E}} - \mathrm{i}\gamma\tau \Box \dot{\mathsf{A}}, \\ = -\vec{\nabla}\dot{\phi} - \ddot{\mathsf{A}} - \mathrm{i}\gamma\tau \Box \dot{\mathsf{A}}, \end{aligned} \tag{54}$$

and as a result we have:

$$\Box \mathsf{A} = \vec{\nabla}\dot{\phi} + \vec{\nabla}\vec{\nabla} \cdot \mathsf{A} + \mathrm{i}\gamma\tau \Box \dot{\mathsf{A}}. \tag{55}$$

Let us at the end evaluate $\vec{\nabla} \cdot \mathsf{E}$ as follows:

$$\vec{\nabla} \cdot \mathsf{E} = \vec{\nabla} \cdot \left(-\vec{\nabla}\phi - \dot{\mathsf{A}}\right) = -\Delta\phi - \vec{\nabla} \cdot \dot{\mathsf{A}} = -\mathrm{i}\gamma\tau \Box \dot{\phi}. \tag{56}$$

However, in order that ϕ obeys a wave equation, we need to set $\vec{\nabla} \cdot \mathsf{A} = -\dot{\phi}$ which is the well-known Lorentz gauge condition and therefore we can write Eq. (56) as:

$$\square\phi = \mathrm{i}\gamma\tau\square\dot{\phi}, \tag{57}$$

which is reduced after simple integration to $\square\phi = e^{-\mathrm{i}t/\gamma\tau}$. The following 3rd statement consequently holds:

Statement 3 *In the extended nonlocal-in-time kinetic energy approach, the wave equations for* $(\mathsf{E},\mathsf{H},\mathsf{A},\phi)$ *take respectively the following forms*:

$$\square\mathsf{E} = 0, \tag{58}$$

$$\square\mathsf{H} = 0, \tag{59}$$

$$\square\mathsf{A} = e^{-\mathrm{i}t/\gamma\tau}, \tag{60}$$

$$\square\phi = e^{-\mathrm{i}t/\gamma\tau}. \tag{61}$$

All these equations are reduced to their standard forms when $t\to\infty$. One may interpret these equations as wave equations with time-dependent source terms, i.e. the propagation of electromagnetic waves upon a time-dependent complex force although these equations were obtained in the case of $\rho = 0$ and in the absence of the current. In fact, one may check that if we choose the following NLT complexified gauge condition $\vec{\nabla}\cdot\mathsf{A} = -\dot{\phi} + \mathrm{i}\gamma\tau\square\phi$, we find $\square\phi = 0$ and $\square\mathsf{A} = \mathrm{i}\gamma\tau\square\dot{\mathsf{A}}$.

Let us at the end of this section calculate the following product:

$$\begin{aligned}\nabla\cdot(\mathsf{E}\times\mathsf{B}) &= \mathsf{B}\cdot\underbrace{(\nabla\times\mathsf{E})}_{\text{equation (39)}} - \mathsf{E}\cdot\underbrace{(\nabla\times\mathsf{B})}_{\substack{\text{equation (45)}\\ \text{in the presence of } \vec{\mathbb{J}}}}, \\ &= \mathsf{B}\cdot\left(-\dot{\mathsf{B}}\right) - \mathsf{E}\cdot\left(\dot{\mathsf{E}} - \mathrm{i}\gamma\tau\square\dot{\mathsf{A}}\right) - \mu_0\mathsf{E}\cdot\vec{\mathbb{J}}.\end{aligned} \tag{62}$$

However since $\mathsf{E}\cdot\square\dot{\mathsf{A}}\to 0$ at very large time, then:

$$\nabla\cdot(\mathsf{E}\times\mathsf{B}) = -\mathsf{B}\cdot\dot{\mathsf{B}} - \mathsf{E}\cdot\dot{\mathsf{E}} - \mu_0\mathsf{E}\cdot\mathbb{J}. \tag{63}$$

Introducing the usual Poynting vector $\mathsf{S} = (\mathsf{E}\times\mathsf{B})/\mu_0$ we can write Eq. (63) as:

$$\nabla\cdot\mathsf{S} + \mathsf{E}\cdot\vec{\mathbb{J}} + \frac{1}{2\mu_0}\frac{\partial}{\partial t}\left(\mathsf{B}\cdot\dot{\mathsf{B}} + \mathsf{E}\cdot\dot{\mathsf{E}}\right) = 0. \tag{64}$$

One can integrate this equation over an arbitrary volume V bounded by a surface S and prove that the Poynting's theorem of conservation of energy is not violated at very large time in the NLT complex approach to Maxwell's equations (Peatross and Ware 2015).

Remark 3.1 In materials, molecules become polarized in response to an applied electric field. The associated current is usually given by $\vec{\mathbb{J}}_{\mathbf{P}} = \dot{\mathbf{P}}$ where $\mathbf{P}(\mathsf{r})$ is the dipole distribution function or the polarization and the resulting charge density is $\rho = -\vec{\nabla}\cdot\mathbf{P}$. Physically, an effective current density occurs when the dipoles change their direction as a function of time in some coordinated way (Field 2006). Therefore, the NLT complex Maxwell's equations become:

NLT complexified Gauss's law $\vec{\nabla}\cdot\mathsf{E} = -\mathrm{i}\gamma\tau\square\dot{\phi} - \vec{\nabla}\cdot\mathbf{P}$, NLT complexified Maxwell's law $\vec{\nabla} \times \mathsf{B} + \mathrm{i}\gamma\tau\Box\dot{\mathsf{A}} = \dot{\mathsf{E}} + \mu_0\dot{\mathbf{P}} + \mu_0\vec{\mathbb{J}}$,

No magnetic charge law $\vec{\nabla} \cdot \mathsf{B} = 0$,

NLT complexified Faraday's law $\vec{\nabla} \times \mathsf{E} = -\dot{\mathsf{B}}$.

Using the similar methodology used previously, the resulting wave equation for E is given by:

$\Box\mathsf{E} = \mu_0\ddot{\mathbf{P}} + \mu_0\dot{\vec{\mathbb{J}}} - \mathrm{i}\gamma\tau\Box\ddot{\mathsf{A}} - \mathrm{i}\gamma\tau\Box\nabla\dot{\phi} - \vec{\nabla}\left(\vec{\nabla} \cdot \mathbf{P}\right)$.

The wave equations for $(\mathsf{H}, \mathsf{A}, \phi)$ may also be obtained using the same method. The 1st, 2nd and 5th terms on the RHS of the previous wave equation describe respectively dipole oscillations, electric currents and polarization currents. The 3rd and 4th terms arise from the extended NLT approach and describe new currents connected respectively to the variations of the vector and scalar potentials. In Sect. 6, we will use this extended wave equation to derive the modified index of refraction in dielectrics.

4 Complex Lorentz force

For a particle of charge q and vector velocity v, the NLT complex Lorentz force is obtained as follows: we first introduce the NLT complex electric field $E^i = \partial^i A^0 - \partial^0 A^i + \mathrm{i}\gamma\tau\partial^0\dot{A}^i$ and then derive the force from the generalized Newton's 2nd law of motion $\mathsf{F} = d\mathsf{P}/d\tau$, F being the 4-vector force, P the 4-vector momentum and $\tau = t'$ is the proper time in a frame R' moving with respect to the rest frame R with velocity v such that $dt = \gamma d\tau \equiv d\tau \Big/ \sqrt{1-\beta^2}$ with $\beta = v$ (in units $c = 1$). Accordingly using $U(\beta) \equiv (\gamma; \gamma\beta, 0, 0)$ we find (Field 2006):

$$\begin{aligned}
\mathsf{F}^i &= q\left(\partial^i A^\alpha - \partial^\alpha A^i + \mathrm{i}\gamma\tau\partial^\alpha \dot{A}^i\right)U(\beta)_\alpha, \\
&= \gamma q\left(\partial^i A^0 - \partial^0 A^i + \mathrm{i}\gamma\tau\partial^0\dot{A}^i - \beta_j\left(\partial^i A^j - \partial^j A^i + \mathrm{i}\gamma\tau\partial^j\dot{A}^i\right) - \beta_k\left(\partial^i A^k - \partial^k A^i + \mathrm{i}\gamma\tau\partial^k\dot{A}^i\right)\right), \\
&= q\left(E^i + \beta_j B^k - \beta_j B^k - \mathrm{i}\gamma\tau\left(\beta_j\partial^j\dot{A}^i + \beta_k\partial^k\dot{A}^i\right)\right), \\
&= q\left(E^i + \left(\vec{\beta} \times \vec{B}\right)^i - 2\mathrm{i}\gamma\tau\beta\nabla\dot{\vec{A}}^i\right),
\end{aligned} \tag{65}$$

and therefore we obtain:

$$\mathsf{F} = q\left(\mathsf{E} + \boldsymbol{\beta} \times \mathsf{B} - 2\mathrm{i}\gamma\tau\boldsymbol{\beta} \cdot \vec{\nabla}\dot{\mathsf{A}}\right), \tag{66}$$

where we have used the fact $B^k = -\varepsilon_{ijk}(\partial^i A^j - \partial^j A^i) = (\vec{\nabla} \times \mathsf{A})^k$.

One direct consequence of the NLT complex Lorentz force is the following: consider a continuous charge distribution in motion. We can write the NLT complex Lorentz force as:

$$d\mathsf{F} = dq\left(\mathsf{E} + \mathsf{v} \times \mathsf{B} - 2\mathrm{i}\gamma\tau\mathsf{v} \cdot \vec{\nabla}\dot{\mathsf{A}}\right), \tag{67}$$

where $d\mathsf{F}$ is the force acting on the elementary charge dq. If we denote by $\mathsf{f} = d\mathsf{F}/dV$ the force density per unit volume and the current density by $\vec{\mathbb{J}} = \rho\mathsf{v}$, Eq. (67) takes the form:

$$\mathsf{f} = \rho \mathsf{E} + \vec{\mathbb{J}} \times \mathsf{B} - 2\mathrm{i}\gamma\tau \vec{\mathbb{J}} \cdot \vec{\nabla}\dot{\mathsf{A}}, \tag{68}$$

Using the first two NLT complex Maxwell equations in the presence of ρ and $\vec{\mathbb{J}}$ which may be written respectively as:

$$\rho = \frac{1}{\mu_0}\vec{\nabla} \cdot \mathsf{E} + \frac{\mathrm{i}\gamma\tau}{\mu_0}\Box\dot{\phi}, \tag{69}$$

$$\vec{\mathbb{J}} = \frac{1}{\mu_0}\vec{\nabla} \times \mathsf{B} + \frac{\mathrm{i}\gamma\tau}{\mu_0}\Box\dot{\mathsf{A}} - \frac{1}{\mu_0}\dot{\mathsf{E}}, \tag{70}$$

we can write Eq. (68) and in particular for very large time as:

$$\mathsf{f}_\infty = \left(\frac{1}{\mu_0}\vec{\nabla} \cdot \mathsf{E}\right)\mathsf{E} + \left(\frac{1}{\mu_0}\vec{\nabla} \times \mathsf{B} - \frac{1}{\mu_0}\dot{\mathsf{E}}\right) \times \mathsf{B} - 2\mathrm{i}\gamma\tau\left(\frac{1}{\mu_0}\vec{\nabla} \times \mathsf{B} - \frac{1}{\mu_0}\dot{\mathsf{E}}\right) \cdot \vec{\nabla}\dot{\mathsf{A}}. \tag{71}$$

Making use of the following vector calculus identities:

$$\dot{\mathsf{E}} \times \mathsf{B} = \frac{\partial}{\partial t}(\mathsf{E} \times \mathsf{B}) - \mathsf{E} \times \dot{\mathsf{B}}, \tag{72}$$

$$\left(\vec{\nabla} \times \mathsf{B}\right) \times \mathsf{B} = \left(\mathsf{B} \cdot \vec{\nabla}\right)\mathsf{B} + \left(\vec{\nabla} \cdot \mathsf{B}\right)\mathsf{B} - \frac{1}{2}\vec{\nabla}\mathsf{B}^2, \tag{73}$$

together with $\vec{\nabla} \cdot \mathsf{B} = 0$ and $\varepsilon_0\mu_0 = 1$ we can reduce Eq. (71) after some algebra to:

$$\begin{aligned}
\mathsf{f}_\infty &= \varepsilon_0\left(\left(\vec{\nabla} \cdot \mathsf{E}\right)\mathsf{E} + \left(\mathsf{E} \cdot \vec{\nabla}\right)\mathsf{E}\right) + \frac{1}{\mu_0}\left(\left(\vec{\nabla} \cdot \mathsf{B}\right)\mathsf{B} + \left(\mathsf{B} \cdot \vec{\nabla}\right)\mathsf{B}\right) - \vec{\nabla}\left(\frac{1}{\mu_0}\mathsf{B} \cdot \mathsf{B} + \varepsilon_0\mathsf{E} \cdot \mathsf{E}\right) \\
&\quad - \frac{\partial \mathsf{S}}{\partial t} - 2\mathrm{i}\gamma\tau\left(\dot{\mathsf{E}} + \mathsf{E}\right) \cdot \vec{\nabla}\dot{\mathsf{A}} + 2\mathrm{i}\gamma\tau\frac{\partial}{\partial t}\left(\mathsf{E} \cdot \vec{\nabla}\mathsf{A}\right), \\
&= \varepsilon_0\left(\left(\vec{\nabla} \cdot \mathsf{E}\right)\mathsf{E} + \left(\mathsf{E} \cdot \vec{\nabla}\right)\mathsf{E}\right) + \frac{1}{\mu_0}\left(\left(\vec{\nabla} \cdot \mathsf{B}\right)\mathsf{B} + \left(\mathsf{B} \cdot \vec{\nabla}\right)\mathsf{B}\right) - \vec{\nabla}\left(\frac{1}{\mu_0}\mathsf{B} \cdot \mathsf{B} + \varepsilon_0\mathsf{E} \cdot \mathsf{E}\right) \\
&\quad - \frac{\partial \mathtt{S}}{\partial t} - 2\mathrm{i}\gamma\tau\left(\dot{\mathsf{E}} + \mathsf{E}\right) \cdot \vec{\nabla}\dot{\mathsf{A}},
\end{aligned} \tag{74}$$

where $\mathsf{S} = (\mathsf{E} \times \mathsf{B})/\mu_0$ is the usual Poynting vector (Griffiths 1999) and $\mathtt{S} = \mathsf{S} - 2\mathrm{i}\gamma\tau(\mathsf{E} \cdot \vec{\nabla}\mathsf{A})$. In tensor notation, we can write this equation as:

$$\begin{aligned}
\mathtt{f}_j &= \partial_i \underbrace{\left(\varepsilon_0\left(E_iE_j - \frac{\delta_{ij}}{2}E^2\right) + \frac{1}{\mu_0}\left(B_iB_j - \frac{\delta_{ij}}{2}B^2\right)\right)}_{T_{ij}} - \frac{\partial \mathtt{S}_j}{\partial t}, \\
&= \partial_i T_{ij} - \frac{\partial S_j}{\partial t},
\end{aligned} \tag{75}$$

where

$$\mathtt{f}_j = \mathsf{f}_j + 2\mathrm{i}\gamma\tau\frac{1}{\mu_0}\left(\frac{\partial E_i}{\partial t} + E_i\right) \cdot \partial_i\dot{A}_j, \tag{76}$$

T_{ij} is the Maxwell stress tensor and $\mathtt{f}_j$ is the complexified force due to NLT and higher-order derivative effects. Equation (75) is thus the extended NLT complex Lorentz force

Table 1 The main outcomes of complex NLT Maxwell's equations

	Extended NLT approach
Lagrangians	$\boldsymbol{\mathcal{L}} = \frac{1}{2}m\frac{\Delta_{\mathrm{i}}}{2}\frac{\Delta_{\mathrm{i}}}{2} + e\sum_{i=0}^{3} A_i\frac{\Delta_{\mathrm{i}}}{2}$ $\Delta_{\mathrm{i}} = \frac{\dot{x}(t+\tau)+\dot{x}(t-\tau)}{2} + \mathrm{i}\gamma\frac{\dot{x}(t+\tau)-\dot{x}(t-\tau)}{2}$
Covariant electromagnetic field tensor up to $n = 1$	$\mathbb{F}_{ij} \approx \frac{\partial A_j}{\partial X_i} - \frac{\partial A_i}{\partial X_j} + \mathrm{i}\gamma\tau\frac{\partial}{\partial t}\left(\frac{\partial A_i}{\partial X_j}\right)$
NLT complex Maxwell's equations in the absence of the volume charge density ρ and the current density $\vec{\mathbb{J}}$	NLT complexified Gauss's law $\vec{\nabla}\cdot\mathsf{E} = -\mathrm{i}\gamma\tau\Box\dot{\phi}$ NLT complexified Maxwell's law $\vec{\nabla}\times\mathsf{B} + \mathrm{i}\gamma\tau\Box\dot{\mathsf{A}} = \dot{\mathsf{E}}$ No magnetic charge law $\vec{\nabla}\cdot\mathsf{B} = 0$ NLT complexified Faraday's law $\vec{\nabla}\times\mathsf{E} = -\dot{\mathsf{B}}$
NLT complex Maxwell's equations in the presence of the volume charge density ρ and the current density $\vec{\mathbb{J}}$	NLT complexified Gauss's law $\vec{\nabla}\cdot\mathsf{E} = \mu_0\rho - \mathrm{i}\gamma\tau\Box\dot{\phi}$ NLT complexified Maxwell's law $\vec{\nabla}\times\mathsf{B} + \mathrm{i}\gamma\tau\Box\dot{\mathsf{A}} = \dot{\mathsf{E}} + \mu_0\vec{\mathbb{J}}$ No magnetic charge law $\vec{\nabla}\cdot\mathsf{B} = 0$ NLT complexified Faraday's law $\vec{\nabla}\times\mathsf{E} = -\dot{\mathsf{B}}$
Wave equations for $(\mathsf{E},\mathsf{H},\mathsf{A},\phi)$	$\Box\mathsf{E} = 0, \Box\mathsf{H} = 0, \Box\mathsf{A} = e^{-\mathrm{i}t/\gamma\tau}, \Box\phi = e^{-\mathrm{i}t/\gamma\tau}$
NLT complex Lorentz force	$\mathsf{f} = \rho\mathsf{E} + \vec{\mathbb{J}}\times\mathsf{B} - 2\mathrm{i}\gamma\tau\vec{\mathbb{J}}\cdot\vec{\nabla}\dot{\mathsf{A}}$
NLT complex Lorentz force in tensor notation	$\mathbf{f}_j = \mathsf{f}_j + 2\mathrm{i}\gamma\tau\frac{1}{\mu_0}\left(\frac{\partial E_i}{\partial t} + E_i\right)\cdot\partial_i\dot{A}_j$

that is associated with the NLT complex Maxwell's equations. The consequences of the NLT complex Lorentz force in classical and quantum electrodynamics will be discussed in detail in an upcoming paper. We summarize in Table 1 the main outcomes of the extended nonlocal-in-time kinetic energy approach.

There exists one observation that deserves to be mentioned at the end of this section. Although the present formalism is based on the extended nonlocal-in-time kinetic energy approach which is characterized by the emergence of higher-order derivatives, the entire equations obtained are extended to include new terms added to the standard ones, e.g. an extra force $-2\mathrm{i}\gamma\tau\vec{\mathbb{J}}\cdot\vec{\nabla}\dot{\mathsf{A}}$ emerged in NLT complex Lorentz force. Since the dot product of two vectors is a scalar, then we denote $\boldsymbol{\mathcal{S}}\triangleq\vec{\mathbb{J}}\cdot\vec{\nabla}\dot{\mathsf{A}}$ by the corresponding scalar which affects the field and force equations and accordingly the complex Lorentz force may be written as $\mathsf{f} = \rho\mathsf{E} + \vec{\mathbb{J}}\times\mathsf{B} - 2\mathrm{i}\gamma\boldsymbol{\mathcal{S}}$. It is remarkable to see that for a constant A, the complex Lorenz force is reduced to its standard form. We expect that these outcomes may have interesting impacts in quantum theory and quantum electronics.

5 Consequences and applications of the NLT complexified Maxwell equations in electrodynamics

5.1 Classical electrodynamics

As a first implication of the previous results we discuss the plane wave problem. We choose z to define direction of propagation, i.e. $(\partial/\partial x, \partial/\partial y = 0)$. The plane waves are transverse therefore from $\vec{\nabla}\cdot\mathsf{E} = -\mathrm{i}\gamma\tau\Box\dot{\phi}$ and $\Box\phi = e^{-\mathrm{i}t/\gamma\tau}$ we find:

$$\frac{\partial E_z}{\partial z} = -e^{-\mathrm{i}t/\gamma\tau}, \tag{77}$$

and from $\vec{\nabla}\cdot\mathsf{B} = 0$ we find:

$$\frac{\partial B_z}{\partial z} = 0. \tag{78}$$

From $\vec{\nabla}\times\mathsf{E} = -\dot{\mathsf{B}}$ we obtain:

$$\frac{\partial B_z}{\partial t} = 0, \tag{79}$$

and from $\vec{\nabla}\times\mathsf{B} + \mathrm{i}\gamma\tau\Box\dot{\mathsf{A}} = \dot{\mathsf{E}}$ and $\Box\mathsf{A} = e^{-\mathrm{i}t/\gamma\tau}$ we get:

$$\frac{\partial E_z}{\partial t} = e^{-\mathrm{i}t/\gamma\tau}. \tag{80}$$

Equations (79) and (80) give $B_z = c_1 - \mathrm{i}\gamma\tau e^{-\mathrm{i}t/\gamma\tau}$ and $E_z = c_2 + \mathrm{i}\gamma\tau e^{-\mathrm{i}t/\gamma\tau}$ where $c_i, i = 1, 2, \ldots$ are constants of integration. These show that B_z is constant, yet E_z is time-dependent and tend toward a constant at very large time. The (x, y) components of $\vec{\nabla}\times\mathsf{E} = -\dot{\mathsf{B}} - e^{-\mathrm{i}t/\gamma\tau}$ are:

$$\frac{\partial E_y}{\partial z} = \frac{\partial B_x}{\partial t} + e^{-\mathrm{i}t/\gamma\tau}, \tag{81}$$

$$\frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t} - e^{-\mathrm{i}t/\gamma\tau}, \tag{82}$$

whereas the (x, y) components of $\vec{\nabla}\times\mathsf{B} + \mathrm{i}\gamma\tau\Box\dot{\mathsf{A}} = \dot{\mathsf{E}}$ are:

$$-\frac{\partial B_y}{\partial z} + \mathrm{i}\gamma\tau\frac{\partial}{\partial t}\underbrace{\left(\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_z}{\partial z^2} - \frac{\partial^2 A_x}{\partial t^2}\right)}_{=e^{-\mathrm{i}t/\gamma\tau}} = \frac{\partial E_x}{\partial t}, \tag{83}$$

$$\frac{\partial B_x}{\partial z} + \mathrm{i}\gamma\tau\frac{\partial}{\partial t}\underbrace{\left(\frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} + \frac{\partial^2 A_z}{\partial z^2} - \frac{\partial^2 A_z}{\partial t^2}\right)}_{=e^{-\mathrm{i}t/\gamma\tau}} = \frac{\partial E_y}{\partial t}. \tag{84}$$

Assuming solutions of the forms $E(z, t) = E_0 e^{\mathrm{i}(\omega t \mp kz)} + g(z)e^{-\mathrm{i}t/\gamma\tau}$ and $B(z, t) = B_0 e^{\mathrm{i}(\omega t \mp kz)} + f(z)e^{-\mathrm{i}t/\gamma\tau}$ with frequency ω and wave number k along x-direction with $g(z)$ and $f(z)$ are arbitrary functions of z, Eq. (82) gives:

$$\mp \mathrm{i}kE_0 e^{\mathrm{i}(\omega t \mp kz)} + \frac{\partial g}{\partial z} e^{-\mathrm{i}t/\gamma\tau} = -\mathrm{i}B_0 \omega e^{\mathrm{i}(\omega t \mp kz)} - \frac{\mathrm{i}}{\gamma\tau} f e^{-\mathrm{i}t/\gamma\tau} - e^{-\mathrm{i}t/\gamma\tau}. \tag{85}$$

A possible solution is obtained if $\omega = -1/\gamma\tau$ and therefore following partial differential equation holds:

$$\frac{\partial g}{\partial z} + \frac{\mathrm{i}}{\gamma\tau} f = \mathrm{i}(-\omega B_0 \pm kE_0) e^{\mp \mathrm{i}kz} - 1. \tag{86}$$

From Eq. (84) we get consequently:

$$\frac{\partial f}{\partial z} + \frac{\mathrm{i}}{\gamma\tau} g = \mathrm{i}(\omega E_0 \pm kB_0) e^{\mp \mathrm{i}kz} - 1. \tag{87}$$

Differentiating Eq. (86) and using Eq. (87) we find the following 2nd order differential equation:

$$\frac{\partial^2 f}{\partial z^2} + \frac{1}{\gamma^2\tau^2} f = \left(\pm k\,\omega E_0 \pm k^2 B_0 - \frac{B_0 \omega}{\gamma\tau} \pm \frac{kE_0}{\gamma\tau} \right) e^{\mp \mathrm{i}kz} + \frac{\mathrm{i}}{\gamma\tau}. \tag{88}$$

The solution of Eq. (88) is given by:

$$\begin{aligned} f(z) = {} & c_3 \sin\left(\frac{t}{\gamma\tau}\right) + c_4 \cos\left(\frac{t}{\gamma\tau}\right) \\ & + \underbrace{\frac{\gamma^4\tau^4 \left(\pm k\omega E_0 \pm k^2 B_0 - \frac{B_0\omega}{\gamma\tau} \pm \frac{kE_0}{\gamma\tau} \right) e^{\mp \mathrm{i}kz} - \mathrm{i}\gamma^5\tau^5 k^2 + \mathrm{i}\gamma^3\tau^3}{1 - k^2\gamma^2\tau^2}}_{\approx 0}. \end{aligned} \tag{89}$$

From Eq. (87) we find:

$$g(z) = \pm \frac{B_0\omega}{k} e^{\mp \mathrm{i}kz} \mp E_0 e^{\mp \mathrm{i}kz} - z - \frac{\mathrm{i}}{\gamma\tau} \left(c_3 \sin\left(\frac{t}{\gamma\tau}\right) + c_4 \cos\left(\frac{t}{\gamma\tau}\right) \right) z + c_5, \tag{90}$$

and therefore the electric and magnetic fields along x-direction are respectively given by:

$$E(z,t) = \left(E_0 e^{\mp \mathrm{i}kz} \pm \frac{B_0\omega}{k} e^{\mp \mathrm{i}kz} \mp E_0 e^{\mp \mathrm{i}kz} - z - \frac{\mathrm{i}}{\gamma\tau} \left(c_3 \sin\left(\frac{t}{\gamma\tau}\right) + c_4 \cos\left(\frac{t}{\gamma\tau}\right) \right) z + c_5 \right) e^{-\mathrm{i}t/\gamma\tau}, \tag{91}$$

$$B(z,t) = \left(B_0 e^{\mp \mathrm{i}kz} + c_3 \sin\left(\frac{t}{\gamma\tau}\right) + c_4 \cos\left(\frac{t}{\gamma\tau}\right) \right) e^{-\mathrm{i}t/\gamma\tau}. \tag{92}$$

For illustration purpose, we assume the initial conditions $E(0,0) = 0$, $\dot{E}(0,0) = 0$, $E'(0,0) = 0$, $\dot{E}(1,0) = 0$, $B(0,0) = 0$ and $\dot{B}(0,0) = 0$. Equations (91) and (92) are reduced respectively for the case of waves traveling towards the positive direction to:

$$E(z,t) = \left(\frac{B_0\omega}{k} \left(e^{-\mathrm{i}kz} - 1 \right) - z + \cos\left(\frac{t}{\gamma\tau}\right) z + \mathrm{i}B_0\omega \cos\left(\frac{t}{\gamma\tau}\right) z \right) e^{-\mathrm{i}t/\gamma\tau}, \tag{93}$$

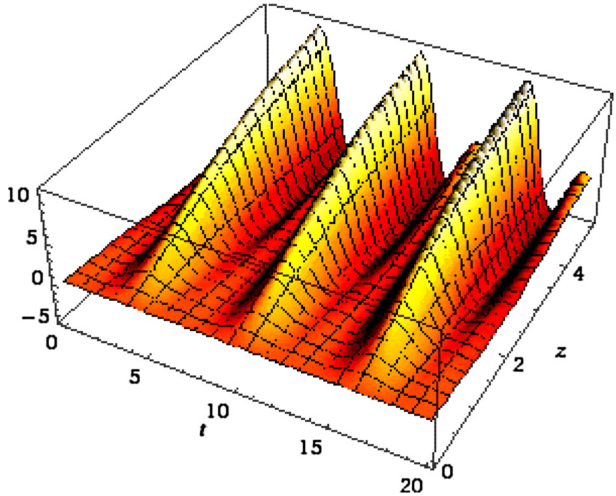

Fig. 1 Variations of the electrical field for $\tau = 1$ and $\gamma = -1/2$

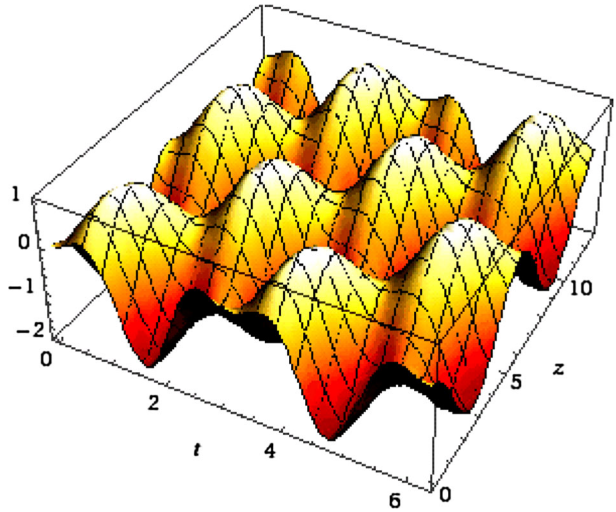

Fig. 2 Variations of the magnetic field for $\tau = 1$ and $\gamma = -1/2$

$$B(z,t) = B_0\left(e^{-\mathrm{i}kz} - \cos\left(\frac{t}{\gamma\tau}\right)\right)e^{-\mathrm{i}t/\gamma\tau}. \tag{94}$$

We plot in Figs. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13 and 14 the variations of the real parts of the electric and magnetic fields for $k = \omega = \tau = B_0 = E_0 = 1$ (for illustrations purpose) and for different values of γ and τ.

We observe that for lower values of τ the electrical and magnetic fields oscillate in a disordered way, i.e. strongly disordered electromagnetic dynamics. Disordered/chaotic magnetic and electrical fields were observed in fusion physics, astrophysics, relaxors and

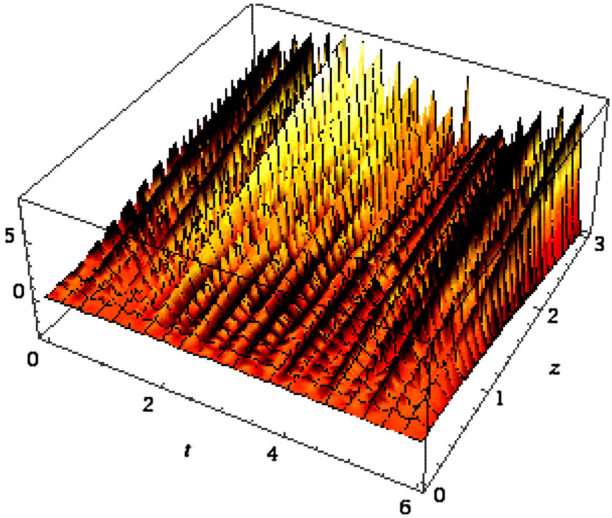

Fig. 3 Variations of the electrical field for $\tau = 0.1$ and $\gamma = -1/2$

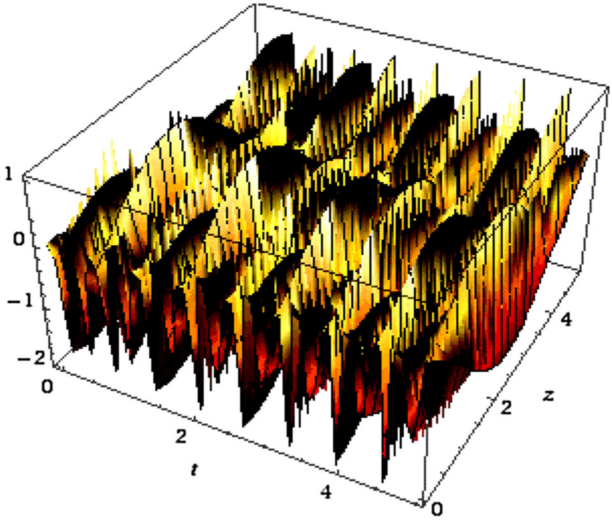

Fig. 4 Variations of the magnetic field for $\tau = 0.1$ and $\gamma = -1/2$

periodically driven superlattices (Portela et al. 2003; Viana 2000; Ram and Dasgupta 2007; Phelan et al. 2014; Bulashenko et al. 1996).

5.2 Quantum electrodynamics

As a second implication of the NLT complex Maxwell equations, we consider the Lagrangian of a charge particle moving in an electromagnetic field. The NLT complex Lagrangian in our approach is given by Eq. (1) which may be written as:

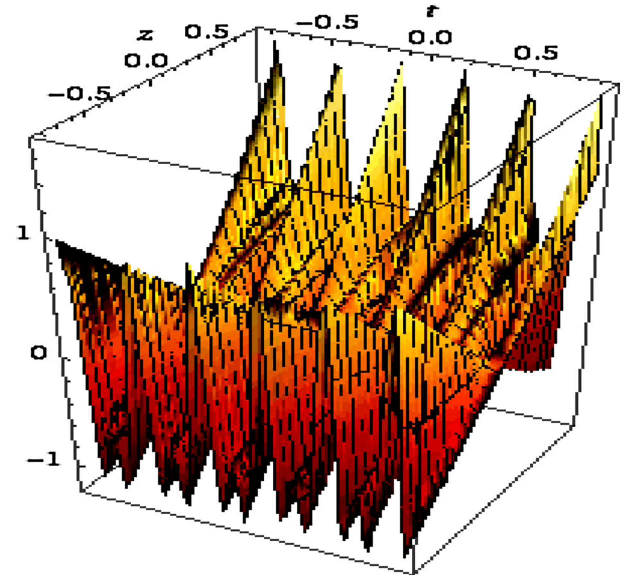

Fig. 5 Variations of the electrical field for $\tau = 0.1$ and $\gamma = 1/2$

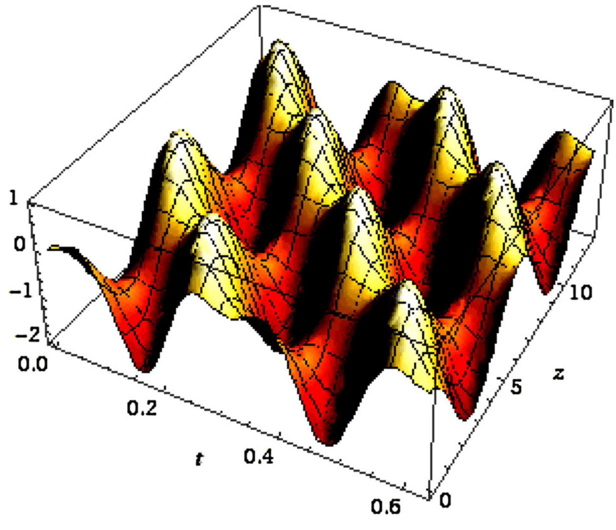

Fig. 6 Variations of the magnetic field for $\tau = 0.1$ and $\gamma = 1/2$

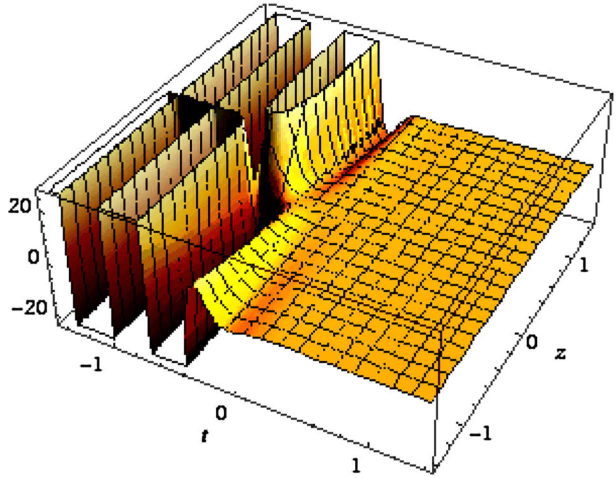

Fig. 7 Variations of the electrical field for $\tau = 0.1$ and $\gamma = 1 + \mathrm{i}$

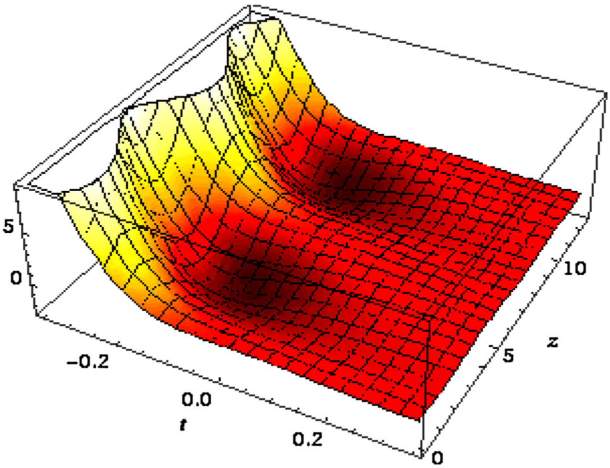

Fig. 8 Variations of the magnetic field for $\tau = 0.1$ and $\gamma = 1 + \mathrm{i}$

$$\begin{aligned}\mathcal{L}_{\tau,n} &= \frac{1}{2}m\left(\frac{\mathsf{v}(t+\tau)+\mathsf{v}(t-\tau)}{2}+\mathrm{i}\gamma\frac{\mathsf{v}(t+\tau)-\mathsf{v}(t-\tau)}{2}\right)\cdot \\ &\quad\left(\frac{\mathsf{v}(t+\tau)+\mathsf{v}(t-\tau)}{2}+\mathrm{i}\gamma\frac{\mathsf{v}(t+\tau)-\mathsf{v}(t-\tau)}{2}\right) \\ &\quad - e\left(\phi-\left(\frac{\mathsf{v}(t+\tau)+\mathsf{v}(t-\tau)}{2}+\mathrm{i}\gamma\frac{\mathsf{v}(t+\tau)-\mathsf{v}(t-\tau)}{2}\right)\cdot\mathsf{A}\right), \\ &= \frac{1}{2}m(\mathsf{v}+\mathrm{i}\gamma\tau\mathsf{a})\cdot(\mathsf{v}+\mathrm{i}\gamma\tau\mathsf{a})-e(\phi-(\mathsf{v}+\mathrm{i}\gamma\tau\mathsf{a})\cdot\mathsf{A}),\end{aligned} \tag{95}$$

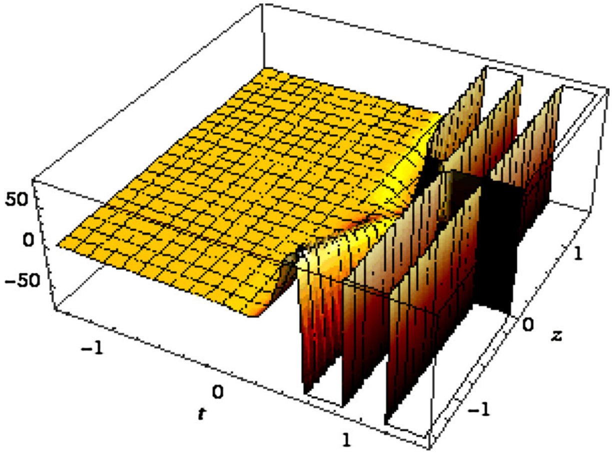

Fig. 9 Variations of the electrical field for $\tau = 0.1$ and $\gamma = 1 - \mathrm{i}$

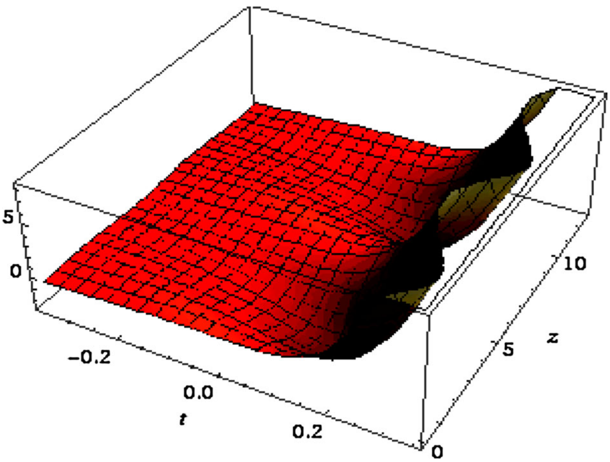

Fig. 10 Variations of the magnetic field for $\tau = 0.1$ and $\gamma = 1 - \mathrm{i}$

From Remark 2.1, the extended complex momentum conjugate is given by $\mathsf{p} = m\mathrm{i}\gamma\tau\mathsf{a} + m\mathsf{v} + e\mathsf{A}$ with a being the acceleration of the particle and therefore the velocity is given by:

$$\mathsf{v} = \frac{\mathsf{p} - \mathrm{i}\gamma\tau\mathsf{a}m - e\mathsf{A}}{m}. \tag{96}$$

Substituting of Eq. (96) into the Hamiltonian $\boldsymbol{\mathcal{H}}(\mathsf{p}, \mathsf{x}) = \mathsf{p}\cdot\mathsf{v} - \boldsymbol{\mathcal{L}}$ gives for $n = 1$:

$$\boldsymbol{\mathcal{H}}(\mathsf{p}, \mathsf{x}) = \frac{1}{2m}(\mathsf{p} - e\mathsf{A})(\mathsf{p} - e\mathsf{A}) - \mathrm{i}\gamma\tau\mathsf{a}\mathsf{p} + e\phi. \tag{97}$$

Passing now from the classical to the quantum mechanics we are led to the NLT complex Schrödinger equation $\hat{\boldsymbol{\mathcal{H}}}\boldsymbol{\Psi} = \mathrm{i}\dot{\boldsymbol{\Psi}}$ where $\boldsymbol{\Psi}$ is the wave function. Using the momentum operator $\hat{p} = \hbar\nabla/\mathrm{i}$ (Shankar 1994), we can write Eq. (97) as:

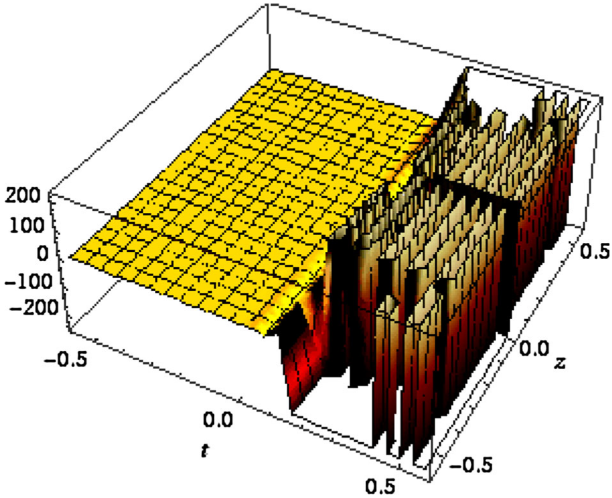

Fig. 11 Variations of the electrical field for $\tau = 0.01$ and $\gamma = 2 - \mathrm{i}$

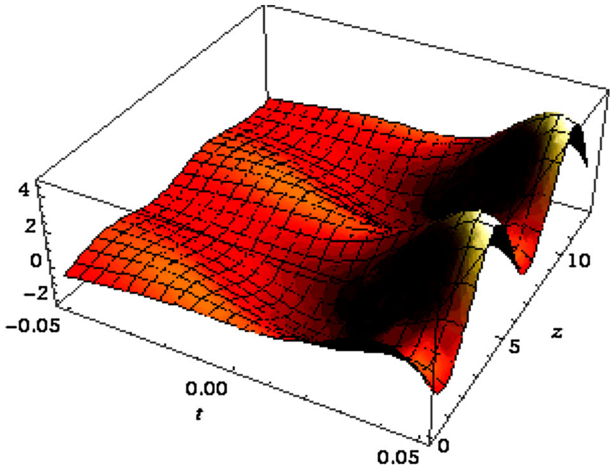

Fig. 12 Variations of the magnetic field for $\tau = 0.01$ and $\gamma = 2 - \mathrm{i}$

$$-\frac{1}{2m}(\nabla - \mathrm{i}g\mathsf{A})(\nabla - \mathrm{i}g\mathsf{A})\boldsymbol{\Psi} + \mathrm{i}\gamma\tau\mathsf{a}\nabla\boldsymbol{\Psi} = \mathrm{i}\left(\frac{\partial}{\partial t} + \mathrm{i}g\phi\right)\boldsymbol{\Psi}, \tag{98}$$

which is the NLT complex Schrödinger equation with $g = e/\hbar$. The term $\mathsf{a}\nabla\boldsymbol{\Psi}$ violates the gauge covariance of the quantum theory. However, the theory is free from violation unless the velocity of the particle is constant. To prove this, we multiply Eq. (89) by $e^{-\mathrm{i}g\chi}$ and we use the "shift-rule" which states that $e^{-f(u)}\frac{\partial}{\partial u}\bullet \equiv (\frac{\partial}{\partial u} + \frac{\partial f}{\partial u})e^{-f(u)}\bullet$ (Wheeler 2002) we find:

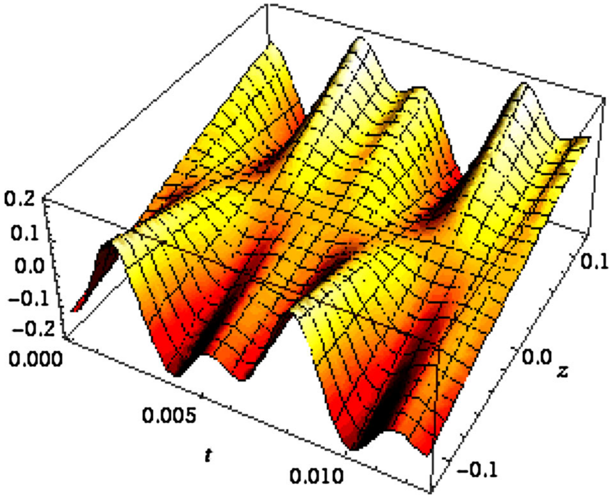

Fig. 13 Variations of the electrical field for $\tau = 0.001$ and $\gamma = 1$

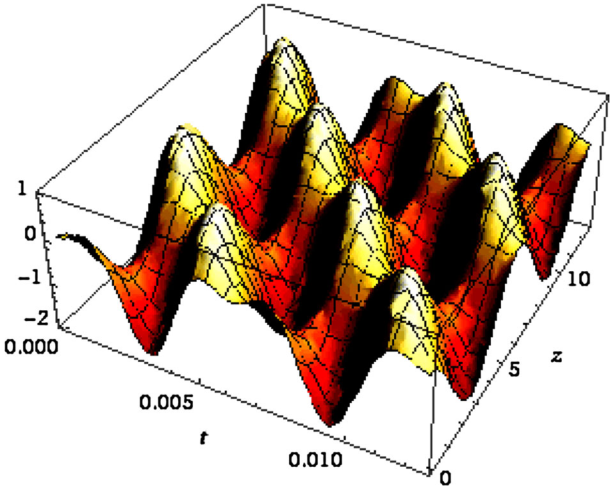

Fig. 14 Variations of the magnetic field for $\tau = 0.001$ and $\gamma = 1$

$$
\begin{aligned}
&-\frac{1}{2m}(\nabla + \mathrm{i}g\nabla\boldsymbol{\chi} - \mathrm{i}g\mathsf{A})(\nabla + \mathrm{i}g\nabla\boldsymbol{\chi} - \mathrm{i}g\mathsf{A})e^{-\mathrm{i}g\chi}\boldsymbol{\Psi} + \mathrm{i}\gamma\tau\mathsf{a}(\nabla + \mathrm{i}g\nabla\boldsymbol{\chi})e^{-\mathrm{i}g\chi}\boldsymbol{\Psi} \\
&\quad = \mathrm{i}\left(\frac{\partial}{\partial t} + \mathrm{i}g\frac{\partial\boldsymbol{\chi}}{\partial t} + \mathrm{i}g\phi\right)e^{-\mathrm{i}g\chi}\boldsymbol{\Psi}.
\end{aligned} \tag{99}
$$

If we let the usual transformation $\mathtt{A} = \mathsf{A} - \nabla\boldsymbol{\chi}$, $\boldsymbol{\varphi} = \phi + \frac{\partial\chi}{\partial t}$ and $\boldsymbol{\psi} = e^{-\mathrm{i}g\chi}\boldsymbol{\Psi}$ hold, we find:

$$-\frac{1}{2m}(\nabla - \mathrm{i}g\mathtt{A})(\nabla - \mathrm{i}g\mathtt{A})\boldsymbol{\Psi} + \mathrm{i}\gamma\tau\mathsf{a}(\nabla + \mathrm{i}g\nabla\boldsymbol{\chi})\boldsymbol{\Psi} = \mathrm{i}\left(\frac{\partial}{\partial t} + \mathrm{i}g\boldsymbol{\varphi}\right)\boldsymbol{\Psi}, \tag{100}$$

which is not gauge-covariant. However, if define new space coordinates such that $\nabla = \nabla + \mathrm{i}g\nabla\boldsymbol{\chi}$, we obtain a gauge-covariant NLT complex quantum mechanics.

Let us make two quick comments concerning Eq. (100): first in the absence of the electromagnetic field, one observes the presence of the gradient term which indicates that the energy–momentum of the particle is deformed and takes the form $\mathtt{E} = \alpha\mathsf{p}^2 + \beta\mathsf{p}$, α, β are constants; second we can write Eq. (98) as:

$$-\frac{1}{2m}(\nabla - \mathrm{i}g\mathtt{A})(\nabla - \mathrm{i}g\mathsf{A})\boldsymbol{\Psi} + \mathrm{i}\gamma\tau\mathsf{a}(\nabla - \mathrm{i}g\mathsf{A})\boldsymbol{\Psi} = \mathrm{i}\left(\frac{\partial}{\partial t} + \mathrm{i}g(\phi - \gamma\tau\mathsf{a}\mathsf{A})\right)\boldsymbol{\Psi}. \tag{101}$$

For the case of a constant acceleration, we can define a new scalar potential $\boldsymbol{\phi} = \phi - \gamma\tau\mathsf{a}\mathsf{A}$ and rewrite Eq. (101) as:

$$-\frac{1}{2m}(\nabla - \mathrm{i}g\mathsf{A})(\nabla - \mathrm{i}g\mathsf{A})\boldsymbol{\Psi} + \mathrm{i}\gamma\tau\mathsf{a}(\nabla - \mathrm{i}g\mathsf{A})\boldsymbol{\Psi} = \mathrm{i}\left(\frac{\partial}{\partial t} + \mathrm{i}g\boldsymbol{\phi}\right)\boldsymbol{\Psi}. \tag{102}$$

After multiplying Eq. (102) by $e^{-\mathrm{i}g\chi}$ and using the usual transformation $\mathtt{A} = \mathsf{A} - \nabla\boldsymbol{\chi}$, $\boldsymbol{\varphi} = \boldsymbol{\phi} + \frac{\partial\chi}{\partial t}$ and $\boldsymbol{\psi} = e^{-\mathrm{i}g\chi}\boldsymbol{\Psi}$, a gauge-covariant NLT complex quantum mechanics hold in a uniformly accelerated reference frame. It is notable that the deformed energy–momentum equation $\mathtt{E} = \alpha\mathsf{p}^2 + \beta\mathsf{p}$ deforms as well the Heisenberg's uncertainty principle and may lead to new insights in quantum theory. We summarize in Table 2 the main outcomes of the implications of NLT complex Maxwell equations in electrodynamics.

Remark 5.1 Based on nonlocal-in-time kinetic energy approach and on maximal acceleration argument, an acceleratum operator of the form $\hat{\mathsf{a}} = \frac{\hat{\mathsf{p}}}{\alpha \mathrm{i} m}\nabla$ is obtained in El-Nabulsi (2017b). Here $\hat{\mathsf{p}} = -\mathrm{i}\nabla$ is the quantum momentum operator and α is a real parameter. Therefore, we can write Eq. (98) as:

$$-\frac{1}{2m}(\nabla - \mathrm{i}g\mathsf{A})(\nabla - \mathrm{i}g\mathsf{A})\boldsymbol{\Psi} - \mathrm{i}\gamma\tau\frac{1}{\alpha m}\nabla^3\boldsymbol{\Psi} = \mathrm{i}\left(\frac{\partial}{\partial t} + \mathrm{i}g\phi\right)\boldsymbol{\Psi},$$

which is a higher-order derivatives NLT Schrödinger equation. The common feature of this equation is the presence of higher degrees higher than two. This equation will be

Table 2 Implications of complex NLT Maxwell equations in electrodynamics

	Classical electrodynamics	Quantum electrodynamics
Dynamics	Complexified + strongly disordered electromagnetic dynamics for lower values of τ	Complexified
Nonlocality	Strongly affected by nonlocality	Strongly affected by nonlocality
Velocity	Classical	Complexified and quantized $\mathsf{v} = \frac{\mathsf{p} - \mathrm{i}\gamma\tau\mathsf{a}m - e\mathsf{A}}{m}$
Equations of motion	Governed by extended complex NLT Maxwell equations	Governed by the modified Schrödinger equation $-\frac{1}{2m}(\nabla - \mathrm{i}g\mathsf{A})(\nabla - \mathrm{i}g\mathsf{A})\boldsymbol{\Psi} + \mathrm{i}\gamma\tau\mathsf{a}\nabla\boldsymbol{\Psi} = \mathrm{i}\left(\frac{\partial}{\partial t} + \mathrm{i}g\phi\right)\boldsymbol{\Psi}$ + deformed energy–momentum relation $\mathtt{E} = \alpha\mathsf{p}^2 + \beta\mathsf{p}$

useful to explore the propagation of ultrashort femtosecond pulses in optical fibers (Cao 2013).

6 Extended index of refraction in dielectric media

Let us in this section study how in the complex NLT approach plane waves behave in dielectric media, e.g. glass in particular assumed to be isotropic, homogeneous and non-conducting ($\vec{\mathbb{J}} = 0$ and $\vec{\nabla} \cdot \mathbf{P} = 0$) (Griffiths 1999). We set in this section $\mu_0 = \varepsilon_0 = c = 1$ for convenience. In the absence of charge, the wave equation for E may be approximated to (Remark 3.1):

$$\Box E = \ddot{\mathbf{P}} - \mathrm{i}\gamma\tau\Box\ddot{\mathsf{A}}. \tag{103}$$

We assume solutions of the form $(\mathsf{E}, \mathbf{P}, \mathsf{A}, \phi) = (\mathsf{E}_0, \mathbf{P}_0, \mathsf{A}_0, \phi_0)e^{\mathrm{i}(\vec{\kappa}\cdot\mathsf{r}-\omega t)}$. Substitution of these trial sinusoidal solutions in Eq. (103) gives:

$$\left(-k^2 + \omega^2\right)\left(\mathsf{E}_0(\omega) - \mathrm{i}\gamma\tau\omega^2\mathsf{A}_0(\omega)\right) = -\mathbf{P}_0(\omega)\omega^2. \tag{104}$$

In a linear medium, e.g. basically any material characterized by reasonable electric field strength, the polarization amplitude $\mathbf{P}_0(\omega)$ is proportional to the strength of the electric field applied to the medium. In other words: $\mathbf{P}_0(\omega) \triangleq \chi_1(\omega)\mathsf{E}_0(\omega)$, $\chi_1(\omega) = a_1 + \mathrm{i}b_1$ being the susceptibility associated with the polarization (which in general is complex). However, in our formalism, we introduce the following extended polarization amplitude $\mathbf{P}_0(\omega) \triangleq (a_1 + \mathrm{i}b_1)\mathsf{E}_0(\omega) + (a_2 + \mathrm{i}b_2)\mathsf{A}_0(\omega)$ where $\chi_2(\omega) = a_2 + \mathrm{i}b_2$ is the complex susceptibility associated with the vector potential. Here (a_1, a_2) represent the real parts of susceptibilities and (b_1, b_2) represent the imaginary parts. With these, we can write Eq. (104) as:

$$\left(\mathsf{E}_0(\omega) - \mathrm{i}\gamma\tau\omega^2\mathsf{A}_0(\omega)\right)k^2 = \omega^2\left((a_1+1)\mathsf{E}_0(\omega) + a_2\mathsf{A}_0(\omega) + \mathrm{i}\left(b_2 - \gamma\tau\omega^2\right)\mathsf{A}_0(\omega) + \mathrm{i}b_1\mathsf{E}_0(\omega)\right). \tag{105}$$

The corresponding dispersion relation is therefore given by:

$$k = \omega\sqrt{\frac{(a_1+1)\mathsf{E}_0(\omega) + a_2\mathsf{A}_0(\omega) + \mathrm{i}((b_2 - \gamma\tau\omega^2)\mathsf{A}_0(\omega) + b_1\mathsf{E}_0(\omega))}{\mathsf{E}_0(\omega) - \mathrm{i}\gamma\tau\omega^2\mathsf{A}_0(\omega)}}. \tag{106}$$

Obviously, the speed of the sinusoidal wave in the dielectric material is therefore given by:

$$v = \sqrt{\frac{\mathsf{E}_0(\omega) - \mathrm{i}\gamma\tau\omega^2\mathsf{A}_0(\omega)}{(a_1+1)\mathsf{E}_0(\omega) + a_2\mathsf{A}_0(\omega) + \mathrm{i}((b_2 - \gamma\tau\omega^2)\mathsf{A}_0(\omega) + b_1\mathsf{E}_0(\omega))}}. \tag{107}$$

The index of refraction takes accordingly the following form:

$$n(\omega) = \sqrt{\frac{(a_1+1)\mathsf{E}_0(\omega) + a_2\mathsf{A}_0(\omega) + \mathrm{i}((b_2 - \gamma\tau\omega^2)\mathsf{A}_0(\omega) + b_1\mathsf{E}_0(\omega))}{\mathsf{E}_0(\omega) - \mathrm{i}\gamma\tau\omega^2\mathsf{A}_0(\omega)}}. \tag{108}$$

We can reduce this equation by setting $(a_1, b_2) = (-1, \gamma\tau\omega^2)$ to:

$$\begin{aligned} n(\omega) &= \sqrt{\frac{(a_2 - \gamma\tau\omega^2 b_1)\mathsf{E}_0(\omega)\mathsf{A}_0(\omega) + \mathrm{i}\left(b_1 \mathsf{E}_0^2(\omega) + \gamma\tau\omega^2 a_2 \mathsf{A}_0^2(\omega)\right)}{\mathsf{E}_0^2(\omega) + \gamma^2\tau^2\omega^4 \mathsf{A}_0^2(\omega)}}, \\ &\approx \sqrt{\frac{(a_2 - \gamma\tau\omega^2 b_1)\mathsf{E}_0(\omega)\mathsf{A}_0(\omega) + \mathrm{i}\left(b_1 \mathsf{E}_0^2(\omega) + \gamma\tau\omega^2 a_2 \mathsf{A}_0^2(\omega)\right)}{\mathsf{E}_0^2(\omega)}}. \end{aligned} \tag{109}$$

It is easy to check that by setting $a_2 = \gamma\tau\omega^2 b_1 = \gamma\omega$, the index of refraction (linearly polarized light) is simplified to:

$$n(\omega) = \sqrt{\frac{b_1\left(\mathsf{E}_0^2(\omega) + \gamma^2\tau^2\omega^4\mathsf{A}_0^2(\omega)\right)}{2\mathsf{E}_0^2(\omega)}}(1+\mathrm{i}) \approx \sqrt{\frac{b_1}{2}}(1+\mathrm{i}) \equiv N + \mathrm{i}K. \tag{110}$$

The magnitude of the wave function may be written as: $k = \sqrt{\frac{b_1}{2}}(1+\mathrm{i})\omega = \sqrt{\frac{a_2}{2\gamma\tau}}(1+\mathrm{i}) = \sqrt{\frac{\omega}{2\tau}}(1+\mathrm{i})$ and hence:

$$\begin{aligned} \mathsf{E}(\mathsf{r},t) &= \mathsf{E}_0 e^{-\sqrt{\frac{a_2}{2\gamma\tau}}\hat{u}\cdot\mathsf{r}} e^{\mathrm{i}\left(\sqrt{\frac{a_2}{2\gamma\tau}}\hat{u}\cdot\mathsf{r} - \omega t\right)}, \\ &= \mathsf{E}_0 e^{-\sqrt{\frac{a_2}{2\gamma\tau}}\hat{u}\cdot\mathsf{r}}\left(\cos\left(\sqrt{\frac{a_2}{2\gamma\tau}}\hat{u}\cdot\mathsf{r} - \omega t\right) + \mathrm{i}\sin\left(\sqrt{\frac{a_2}{2\gamma\tau}}\hat{u}\cdot\mathsf{r} - \omega t\right)\right), \\ &= \mathsf{E}_0 e^{-\sqrt{\frac{\omega}{2\tau}}\hat{u}\cdot\mathsf{r}}\left(\cos\left(\sqrt{\frac{\omega}{2\tau}}\hat{u}\cdot\mathsf{r} - \omega t\right) + \mathrm{i}\sin\left(\sqrt{\frac{\omega}{2\tau}}\hat{u}\cdot\mathsf{r} - \omega t\right)\right). \end{aligned} \tag{111}$$

Here $\hat{u}$ is a real unit vector identifying the direction of the wave vector. Then the real part of Eq. (111) is:

$$\Re_e\{\mathsf{E}(\mathsf{r},t)\} = \mathsf{E}_0 e^{-\sqrt{\frac{\omega}{2\tau}}\hat{u}\cdot\mathsf{r}}\cos\left(\sqrt{\frac{\omega}{2\tau}}\hat{u}\cdot\mathsf{r} - \omega t\right). \tag{112}$$

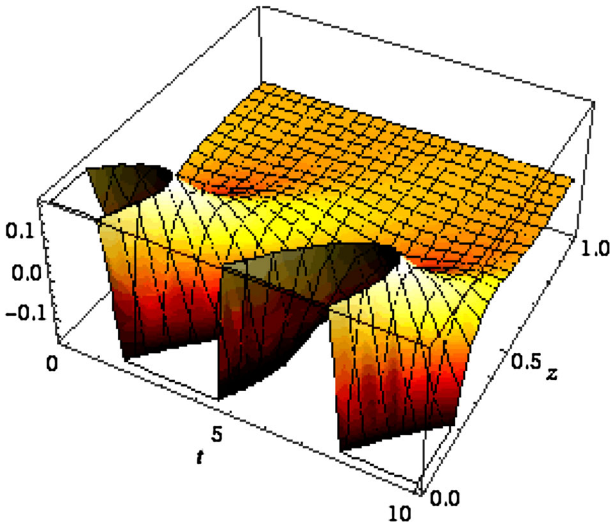

Fig. 15 Variations of Eq. (112) for $\tau = 0.01$

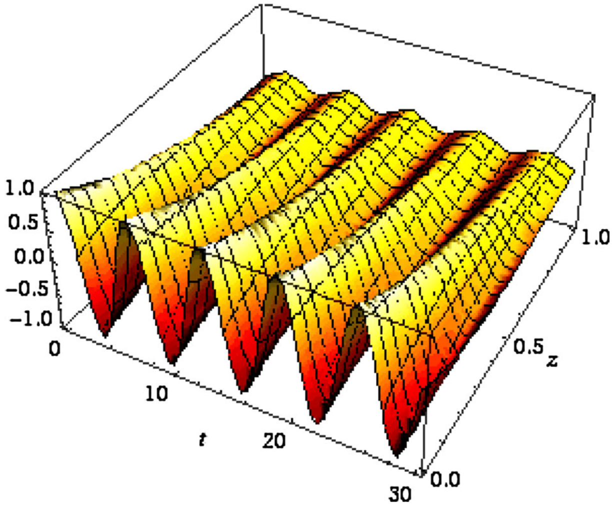

Fig. 16 Variations of Eq. (112) for $\tau = 0.1$

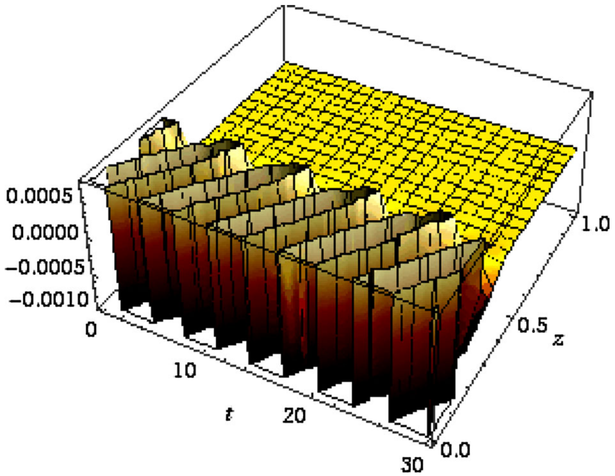

Fig. 17 Variations of Eq. (112) for $\tau = 0.001$

We plot in Figs. 15, 16, 17, 18, 19, 20, 21, 22, 23 and 24 for different values of τ the variations of Eq. (112) at different scales after assuming that the direction of propagation is in the z-direction and after fixing $\mathsf{E}_0 = \omega = 1$.

We observe that the variations of the real part of the electric field exhibits some kind of disordered motion in particular for tiny values of the nonlocal time parameter τ. The associated wavelength is given by:

$$\lambda = \frac{\lambda_{vacuum}}{N} = \sqrt{\frac{2}{b_1}}\lambda_{vacuum} = \sqrt{2\tau\omega}\lambda_{vacuum}. \tag{113}$$

and depends on the nonlocal time parameter. The real part of the index of refraction is here $N = \frac{1}{\sqrt{2\tau\omega}}$ and the extended polarization amplitude is:

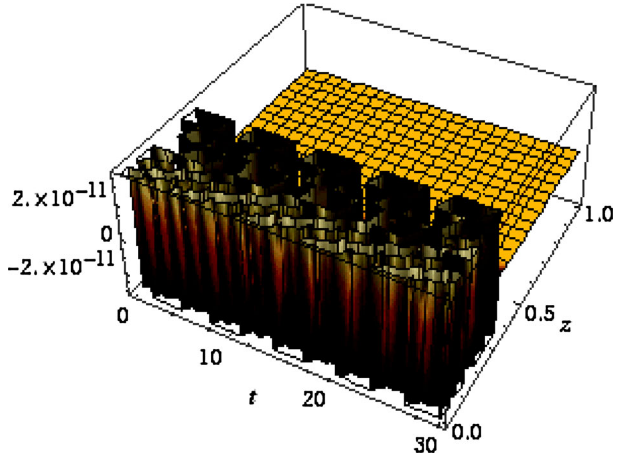

Fig. 18 Variations of Eq. (112) for $\tau = 1$

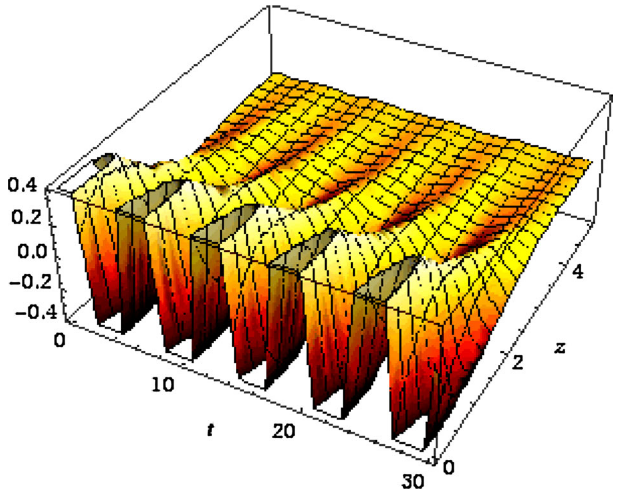

Fig. 19 Variations of Eq. (112) for $\tau = 0.5$

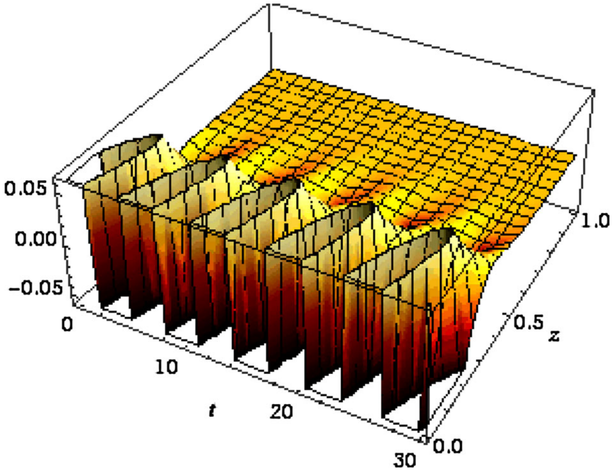

Fig. 20 Variations of Eq. (116) for $\tau = 0.05$

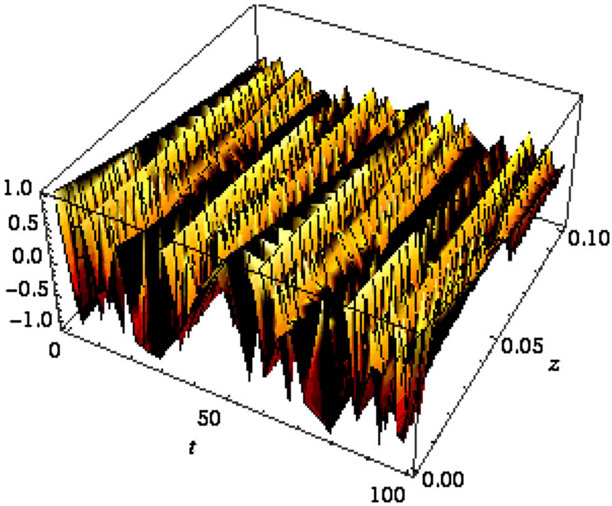

Fig. 21 Variations of Eq. (112) for $\tau = 0.01$ and tiny scales

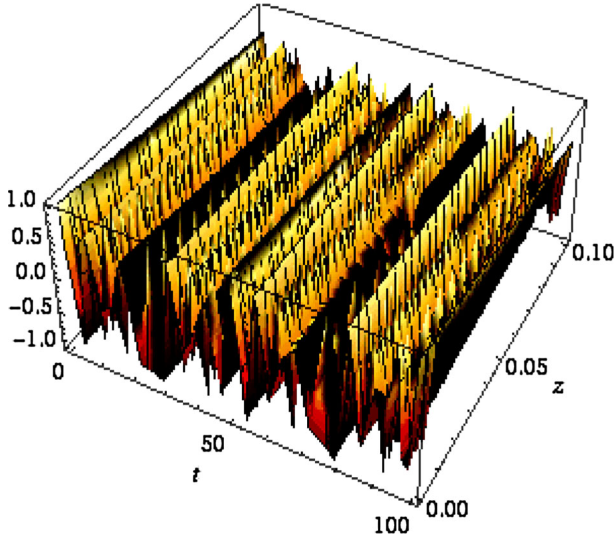

Fig. 22 Variations of Eq. (112) for $\tau = 0.1$ and tiny scales

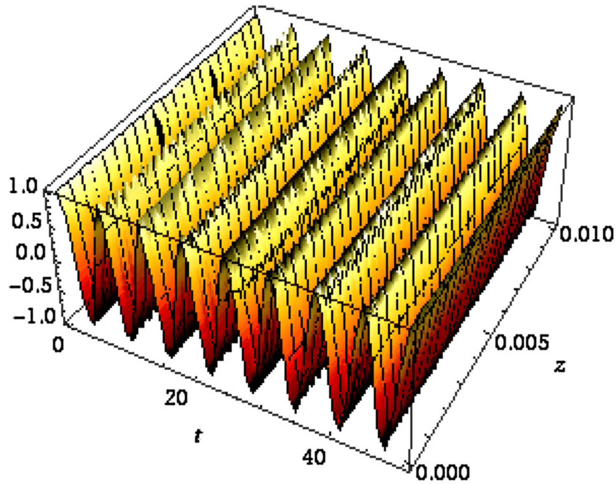

Fig. 23 Variations of Eq. (112) for $\tau = 0.5$ and tiny scales

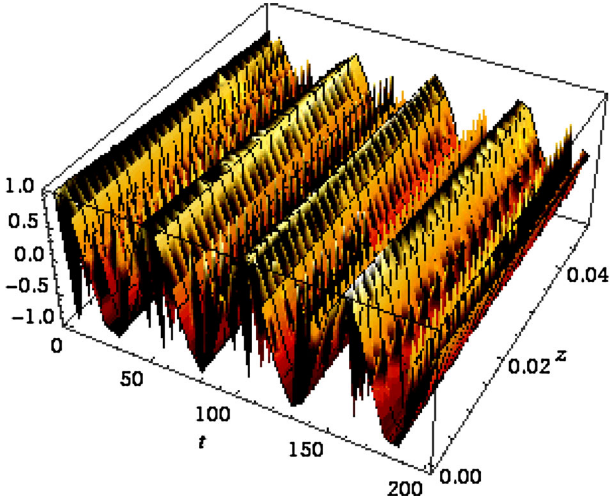

Fig. 24 Variations of Eq. (112) for $\tau = 0.1$ and tiny scales

$$\mathbf{P}_0(\omega) = \left(-1 + \frac{\mathsf{i}}{\tau\omega}\right)\mathsf{E}_0(\omega) + \gamma\omega(1 + \mathsf{i}\tau\omega)\mathsf{A}_0(\omega), \tag{114}$$

and depends on the amplitudes of both the electric field and the vector potential. More generally, we can write Eq. (109) as:

$$\begin{aligned} n(\omega) &= \sqrt{\frac{(a_2 - \gamma\tau\omega^2 b_1)\mathsf{E}_0(\omega)\mathsf{A}_0(\omega) + \mathsf{i}\left(b_1\mathsf{E}_0^2(\omega) + \gamma\tau\omega^2 a_2\mathsf{A}_0^2(\omega)\right)}{\mathsf{E}_0^2(\omega)}} \\ &\approx \underbrace{\frac{1}{\sqrt{2}}\sqrt{\sqrt{\frac{a_2^2\mathsf{A}_0^2(\omega) + b_1^2\mathsf{E}_0^2(\omega)}{\mathsf{E}_0^2(\omega)}} + \frac{(a_2 - \gamma\tau\omega^2 b_1)\mathsf{A}_0(\omega)}{\mathsf{E}_0(\omega)}}}_{\Omega_1} \\ &\quad + \mathsf{i}\underbrace{\frac{sign\,\zeta}{\sqrt{2}}\sqrt{\sqrt{\frac{a_2^2\mathsf{A}_0^2(\omega) + b_1^2\mathsf{E}_0^2(\omega)}{\mathsf{E}_0^2(\omega)}} - \frac{(a_2 - \gamma\tau\omega^2 b_1)\mathsf{A}_0(\omega)}{\mathsf{E}_0(\omega)}}}_{\Omega_2}, \end{aligned} \tag{115}$$

where

$$\zeta = \frac{b_1\mathsf{E}_0^2(\omega) + \gamma\tau\omega^2 a_2\mathsf{A}_0^2(\omega)}{\mathsf{E}_0^2(\omega)}. \tag{116}$$

Only terms a_2 and b_1 are raised in the solutions. The magnitude of the wave function is then given by: $k = (\Omega_1 + \mathsf{i}\Omega_2)\omega$ and the electrical field takes the form:

$$\mathsf{E}(\mathsf{r}, t) = \mathsf{E}_0 e^{-\Omega_2\hat{u}\cdot\mathsf{r}} e^{\mathsf{i}(\Omega_1\hat{u}\cdot\mathsf{r} - \omega t)}. \tag{117}$$

and its real part is therefore:

$$\Re_e\{\mathsf{E}(\mathsf{r},t)\} = \mathsf{E}_0 e^{-\Omega_2 \hat{u}\cdot \mathsf{r}} \cos(\Omega_1 \hat{u}\cdot \mathsf{r} - \omega t). \tag{118}$$

The wavelength is consequently given by:

$$\lambda = \frac{\lambda_{vacuum}}{\Omega_1} = \frac{\sqrt{2}}{\sqrt{\sqrt{\frac{a_2^2 \mathsf{A}_0^2(\omega) + b_1^2 \mathsf{E}_0^2(\omega)}{\mathsf{E}_0^2(\omega)}} + \frac{(a_2 - \gamma\tau\omega^2 b_1)\mathsf{A}_0(\omega)}{\mathsf{E}_0(\omega)}}} \lambda_{vacuum}, \tag{119}$$

and the extended polarization amplitude is:

$$\mathbf{P}_0(\omega) = (a_1 + \mathrm{i}b_1)\mathsf{E}_0(\omega) + (a_2 + \mathrm{i}b_2)\mathsf{A}_0(\omega) \underset{a_1=b_2=0}{=} \mathrm{i}b_1 \mathsf{E}_0(\omega) + a_2 \mathsf{A}_0(\omega). \tag{120}$$

From these calculations, we can deduce that the polarization effect is modified since $\mathbf{P}_0(\omega) = f(\mathsf{E}_0(\omega), \mathsf{A}_0(\omega))$ whereas in the standard approach $\mathbf{P}_0(\omega) = f(\mathsf{E}_0(\omega))$. This is interesting since polarization in this case refers to orientation of the electric field but it is affected by $\mathsf{A}_0(\omega)$. Together with the electric potential, the magnetic vector potential is used to specify the polarization amplitude within the nonlocal-in-time kinetic energy approach. One is able mathematically to reduce $\mathbf{P}_0(\omega) = f(\mathsf{E}_0(\omega), \mathsf{A}_0(\omega)) \to f(\mathsf{E}_0(\omega))$ by means of Maxwell's equations yet the resulting polarization amplitude will be nonlinear, e.g. in general if a strong light (laser) acts on the nonlinear medium, then $\mathbf{P}_0(\omega) = f(\mathsf{E}_0(\omega))$ is nonlinear (Li et al. 2017). It will be of interest to explore in a future work the implications of these outcomes in radiation theory and photonics theory.

7 Conclusions and perspectives

In the present paper we have constructed complex NLT Maxwell equations by extending the nonlocal-in-time kinetic energy approach recently introduced by Suykens. It was observed that these equations provide new features like the occurrence of a NLT complex electromagnetic tensor; the emergence of electromagnetic wave equations with time-dependent source terms even though the current and density are absent; the occurrence of disordered electric and magnetic fields; the emergence of a gauge-covariant NLT complexified quantum mechanics characterized by a higher-order derivatives Schrödinger equation and the emergence of a new polarization amplitude $\mathbf{P}_0(\omega) = f(\mathsf{E}_0(\omega), \mathsf{A}_0(\omega))$ in dielectrics with linearly polarized light and characterized by a disordered electric field and a wavelength which depends on $\mathsf{E}_0(\omega), \mathsf{A}_0(\omega)$ and the nonlocal time parameter. In fact, all scenarios illustrated in this work show that equations and solutions are affected by the value of the nonlocal time parameter. The disorder of fields increases as the values of the nonlocal time parameter decreases. It is very plausible that these modifications could describe phenomena that cannot be captured by the standard Maxwell's electrodynamics theory. We believe that the models constructed in this paper require more analytical analysis in addition to its connection with experimental results obtained in different fields like quantum optics and quantum electronics. Nevertheless the consequences obtained in this work prove the importance of nonlocality-in-time in theoretical physics and real-world applications. A number of applications are under construction and in particular the significance of the NLT complexified electromagnetic potential in the quantum theory and its connection to the Aharonov–Bohm effect (Aharonov 1959), massive photons in superconductivity (de Bruyn 2017) besides their impacts in quantum optics, materials science and biology (electromagnetic modeling of biological cells (Fear and Stuchly 1998)) and in glass science (Nemilov 2014) deserve to be explored in details.

Acknowledgements The author would like to thank the anonymous referees for their useful comments and valuable suggestions.

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