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Traveling wave solution of conformable fractional generalized reaction Duffing model by generalized projective Riccati equation method

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Abstract The generalized projective Riccati equation method is proposed to establish exact solutions for generalized form of the reaction Duffing model in fractional sense namely, Khalil's derivative. The compatible traveling wave transform converts the governing equation to a non linear ODE. The predicted solution is a series of two new variables that solve a particular ODE system. Coefficients of terms in the series are calculated by solving an algebraic system that comes into existence by substitution of the predicted solution into the ODE which is the result of the wave transformation of the governing equation. Returning original variables give exact solutions to the governing equation in various forms.

Keywords Generalized reaction duffing model · Generalized projective Riccati equations method · Conformable fractional derivative

1 Introduction

A diversity of powerful methods to solve non linear PDEs have been derived in last decades. Even though most of them can be categorized as finite power series of various particular and complicated functions, some are completely different forms and can not be

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included in any category. Different from simple hyperbolic function (tanh $(\cdot)$, $sech(\cdot)$, $csch(\cdot)$) ansatz methods (Korkmaz 2017; Guner et al. 2017; Eslami 2016) or first integral approach (Eslami and Rezazadeh 2016; Eslami et al. 2017; Ekici et al. 2016), exact solutions in finite series forms covering variations of Kudryashov method (Hosseini et al. 2017; Ege and Misirli 2014; Korkmaz 2017), (G'/G)-expansion method (Bekir and Guner 2013; Khan and Akbar 2014; Younis and Zafar 2014), sub equation methods (Aminikhah et al. 2016; Khodadad et al. 2017), functional variable method (Eslami et al. 2017) and exponential function methods (He 2013; Rezazadeh et al. 2018). The projective Riccati equations method in general form (GPREM) can be categorized in the finite series type solutions family. The first form of the method appeared in Conte and Musette (1992) by Conte and Musette to present a general ansatz for seeking more new solitary wave solutions of some non linear PDEs that can be expressed as a polynomial in two elementary functions that are the solutions of projective Riccati equation. Later on, Yan developed further Conte and Musette's method and derived the GPREM (Yan 2003), and was successfully studied in a lot of problems (Chen and Li 2004; Rui-Min et al. 2007; Zayed and Alurrfi 2014; Li and Chen 2003; Gomez and Salas 2006) and so on. The description, details, implementations and significant points of the GPREM are summarized in the next sections.

Moreover, we derive some exact solutions by the proposed approach to conformable time fractional gRDM defined as

$$D_t^{2\gamma}u + pu_{xx} + qu + \omega u^2 + su^3 = 0, \quad t > 0, \quad 0 < \gamma \leq 1. \tag{1}$$

where p, q, ω and s are all constants, $u = u(x, t)$ and $D_t^{2\gamma} = D_t^{\gamma}D_t^{\gamma}$ is the 2γth order derivative operator in conformable derivative sense defined in the next section. Equation (1) reduces many well-known non linear conformable time fractional (CTF) wave equations such as

(i)   CTF Klein–Gordon equation

$$D_t^{2\gamma}u - u_{xx} - au - bu^3 = 0, \quad t > 0, \quad 0 < \gamma \leq 1.$$

(ii)   CTF Landau-Ginzburg-Higgs equation

$$D_t^{2\gamma}u - u_{xx} - m^2 u + g^2 u^3 = 0, \quad t > 0, \quad 0 < \gamma \leq 1.$$

(iii)   CTF φ^4 equation

$$D_t^{2\gamma}u - u_{xx} + u - u^3 = 0, \quad t > 0, \quad 0 < \gamma \leq 1.$$

(iv)   CTF duffing equation

$$D_t^{2\gamma}u + au + bu^3 = 0, \quad t > 0, \quad 0 < \gamma \leq 1.$$

(v)   CTF Sine–Gordon equation

$$D_t^{2\gamma}u - u_{xx} + u - \frac{1}{6}u^3 = 0, \quad t > 0, \quad 0 < \gamma \leq 1.$$

Yan and Zhang (1999) solved a particular form of the gRDM using a new ansatz and expressed the solutions in explicit forms. A family of solutions covering some shock or bell-shaped solitonic solutions, and complex valued solutions were constructed by generalized hyperbolic function method (Tian and Gao 2002). Auxilary function method is another effective method to derive the solutions to the gRDM (Kim and Hong 2004).

Several families of exact solutions including some bell-type and kink-type solitary wave solutions, periodic solutions in triangular wave forms and singular-type solutions of Eq. (1) when $\gamma = 1$ have been reported in Huang and Zhang (2005). The Jumarie's-space–time fractional form of the gRDM was solved by Guner et al. (2017) various hyperbolic function ansatzes.

Sub equation approach based on first kind elliptic functions (Huang and Zhang 2005), fractional sub equation technique (Zheng and Wen 2013) and first integral approach (Eslami et al. 2014) are other powerful tools to solve the gRDM in both fractional or non fractional cases.

The study is organized as below. In Sect. 2, we present some fundamental definitions and significant properties of conformable fractional derivative. Section 3 gives the description of the GPREM for solving a conformable fractional PDE in general form. Then, in Sect. 4, we implement the proposed method to set solutions for the CFT-gRDM. A brief report on the solution is given in the last section.

2 Conformable fractional derivative

Here, preliminaries of the basic calculus and tools of conformable fractional theory are summarized (Khalil et al. 2014; Abdeljawad 2015).

Definition 2.1 Let $\omega : [0, \infty) \to \mathbb{R}$, then, the conformable fractional derivative of ω of order γ is defined as

$$D_t^\gamma \omega(t) = \lim_{\varepsilon \to 0} \frac{\omega(t + \varepsilon t^{1-\gamma}) - \omega(t)}{\varepsilon}, \tag{2}$$

for all $t > 0$, $\gamma \in (0, 1)$.

The new definition satisfies the properties which given in the following theorem.

Theorem 1 *Let $\gamma \in (0, 1]$, and ω, g be γ-differentiable at a point t, then*

(i) $D_t^\gamma(a\omega + bg) = aD_t^\gamma \omega + bD_t^\gamma g$, *for all $a, b \in \mathbb{R}$.*
(ii) $D_t^\gamma(t^\mu) = \mu t^{\mu - \gamma}$, *for all $\mu \in \mathbb{R}$.*
(iii) $D_t^\gamma(\omega g) = D_t^\gamma g + gD_t^\gamma \omega$.
(iv) $D_t^\gamma\left(\frac{\omega}{g}\right) = \frac{gD_t^\gamma(\omega) - \omega D_t^\gamma(g)}{g^2}$.

In addition, if ω is differentiable, then $D_t^\gamma(\omega)(t) = t^{1-\gamma}\frac{d\omega}{dt}$.

In Abdeljawad (2015), the chain rule for fractional derivative in conformable sense was established as:

Theorem 2 *Let $\omega : [0, \infty) \to \mathbb{R}$ be a function such that ω is γ-differentiable in $\gamma \in (0, 1]$ and the differentiable function $g = g(t)$ be defined in the range of ω. then, we write*

$$D_t^\gamma(\omega o g)(t) = t^{1-\gamma} g'(t) \omega'(g(t)). \tag{3}$$

3 The GPREM

The main steps of the GPREM are described below to determine exact solutions of conformable time fractional PDEs in generalized reaction Duffing family.

Consider a given non linear CFT-PDE not involving independent variables explicitly such that

$$P\left(u, D_t^\gamma u, u_x, \, D_t^{2\gamma} u, u_{xx}, \dots\right) = 0, \quad 0 < \gamma \leq 1, \tag{4}$$

Then, the first step of the implementation of GPREM is to introduce the compatible wave transform defined as

$$u(x,t) = v(\xi), \quad \xi = x - \lambda \frac{t^\gamma}{\gamma}, \tag{5}$$

where nonzero λ is constant. This transform reduces the governing Eq. (4) to an ODE for one variable function $v(\xi)$

$$P(v, v', v'', \dots) = 0. \tag{6}$$

The next fundamental step is to introduce new variables $\theta(\xi)$, $\tau(\xi)$ solving the projective Riccati system (PRS)

$$\begin{cases} \theta'(\xi) = \varepsilon\theta(\xi)\tau(\xi), \\ \tau'(\xi) = \varepsilon\tau^2(\xi) - m\theta(\xi) + R. \end{cases} \tag{7}$$

The first integral of PRS can be expressed in the form

$$\theta^2(\xi) = -\varepsilon\left[R - 2m\theta(\xi) + \frac{m^2 - 1}{R}\theta^2(\xi)\right]. \tag{8}$$

Various particular solutions of this equation can be formed as:
Case I If $R = m = 0$ then

$$\tau_1(\xi) = \frac{1}{\varepsilon\xi}, \quad \theta_1(\xi) = \frac{C}{\xi}. \tag{9}$$

Case II If $\varepsilon = 1$ and $R \neq 0$

$$\begin{cases} \tau_2(\xi) = \dfrac{\sqrt{R}\, tan_{\mu\eta}(\sqrt{R}\xi)}{m\, sec_{\mu\eta}(\sqrt{R}\xi) + 1}, & \theta_2(\xi) = \dfrac{R\, sec_{\mu\eta}(\sqrt{R}\xi)}{m\, sec_{\mu\eta}(\sqrt{R}\xi) + 1}, \\ \tau_3(\xi) = -\dfrac{\sqrt{R}\, cot_{\mu\eta}(\sqrt{R}\xi)}{m\, csc_{\mu\eta}(\sqrt{R}\xi) + 1}, & \theta_3(\xi) = \dfrac{R\, csc_{\mu\eta}(\sqrt{R}\xi)}{m\, csc_{\mu\eta}(\sqrt{R}\xi) + 1}. \end{cases} \tag{10}$$

Case III If $\varepsilon = -1$ and $R \neq 0$

$$\begin{cases} \tau_4(\xi) = \dfrac{\sqrt{R}\, tanh_{\mu\eta}(\sqrt{R}\xi)}{m\, sech_{\mu\eta}(\sqrt{R}\xi) + 1}, & \theta_4(\xi) = \dfrac{R\, sech_{\mu\eta}(\sqrt{R}\xi)}{m\, sech_{\mu\eta}(\sqrt{R}\xi) + 1}, \\ \tau_5(\xi) = \dfrac{\sqrt{R}\, coth_{\mu\eta}(\sqrt{R}\xi)}{m\, csch_{\mu\eta}(\sqrt{R}\xi) + 1}, & \theta_5(\xi) = \dfrac{R\, csch_{\mu\eta}(\sqrt{R}\xi)}{m\, csch_{\mu\eta}(\sqrt{R}\xi) + 1}. \end{cases} \tag{11}$$

The predicted solution to (4) is formed as

$$u(x,t) = v(\xi) = a_0 + \sum_{i=1}^{M} \theta^{i-1}(\xi)(a_i\,\theta(\xi) + b_i\,\tau(\xi)), \qquad (12)$$

where $\theta(\xi)$, $\tau(\xi)$ satisfy the system (7). The standard balance procedure widely equating highest order derivative and the non linear terms to each other gives the balance constant M. Substituting, the predicted solution in finite series form given in (12), and collecting all terms with the same power in $\theta^i(\xi)\tau^i(\xi)$ ($i = 0, 1, ...; j = 0, 1$) gives a polynomial equation in terms of $\theta(\xi)$ and $\tau(\xi)$. Using polynomial equality, that is the right hand side is zero, we reach an algebraic system. Solving this system for m, R, λ, a_i, $b_i(i = 1, 2, ..., M)$ gives the unknown data to construct the explicit exact solutions. Thus, according to (9)–(11) and the values of m, R, λ, a_i, $b_i(i = 1, 2, ..., M)$, many families of exact solutions in traveling wave forms (4) are constructed. Without loss of generality ε can be chosen as 1. Generalized hyperbolic (GH) and triangular function in general forms are defined as Ren and Zhang (2006), Liu and Jiang (2002):

The GH sine function is

$$\sinh_{\mu\eta}(\xi) = \frac{\mu e^{\xi} - \eta e^{-\xi}}{2},$$

the GH cosine function is

$$\cosh_{\mu\eta}(\xi) = \frac{\mu e^{\xi} + \eta e^{-\xi}}{2},$$

the GH tangent function is

$$\tanh_{\mu\eta}(\xi) = \frac{\mu e^{\xi} - \eta e^{-\xi}}{\mu e^{\xi} + \eta e^{-\xi}},$$

the GH cotangent function is

$$\coth_{\mu\eta}(\xi) = \frac{\mu e^{\xi} + \eta e^{-\xi}}{\mu e^{\xi} - \eta e^{-\xi}},$$

the GH secant function is

$$sech_{\mu\eta}(\xi) = \frac{2}{\mu e^{\xi} + \eta e^{-\xi}},$$

the GH cosecant function is

$$csch_{\mu\eta}(\xi) = \frac{2}{\mu e^{\xi} - \eta e^{-\xi}},$$

where ξ is independent variable, μ and η are arbitrary positive deformation constants.

In a similar manner.

The GT sine function is

$$\sin_{\mu\eta}(\xi) = \frac{\mu e^{i\xi} - \eta e^{-i\xi}}{2i},$$

the GT cosine function is

$$\cos_{\mu\eta}(\xi) = \frac{\mu e^{i\xi} + \eta e^{-i\xi}}{2},$$

the GT tangent function is

$$\tan_{\mu\eta}(\xi) = -i\frac{\mu e^{i\xi} - \eta e^{-i\xi}}{\mu e^{i\xi} + \eta e^{-i\xi}},$$

the GT cotangent function is

$$\cot_{\mu\eta}(\xi) = i\frac{\mu e^{i\xi} + \eta e^{-i\xi}}{\mu e^{i\xi} - \eta e^{-i\xi}},$$

the GT secant function is

$$\sec_{\mu\eta}(\xi) = \frac{2}{\mu e^{i\xi} + \eta e^{-i\xi}},$$

the GT cosecant function is

$$\csc_{\mu\eta}(\xi) = \frac{2i}{\mu e^{i\xi} - \eta e^{-i\xi}},$$

4 Application

The suggested method described in Sect. 3 is implemented to construct the explicit exact solutions in traveling wave forms of the CTF-gRDM. To begin with, we take the traveling wave transform

$$u(x,t) = v(\xi), \quad \xi = x - \lambda\frac{t^{\gamma}}{\gamma}, \tag{13}$$

then Eq. (1) is reduced into an easily solvable non linear ODE

$$\lambda^2 v_{\xi\xi} + p v_{\xi\xi} + q v + \omega v^2 + s v^3 = 0. \tag{14}$$

Balancing $v_{\xi\xi}$ with v^3 in (14), we get $M = 1$. Consequently, we get

$$v(\xi) = a_0 + a_1 \theta(\xi) + b_1 \tau(\xi)), \tag{15}$$

where the constant coefficients a_0, a_1 and b_1 are determined later.

Substituting Eq. (15) along with Eqs. (7) and (8) into Eq. (14), the left-hand side of Eq. (14) becomes a polynomial in $\theta(\xi)$ and $\tau(\xi)$. Necessary assumption of coefficients of this resultant polynomial as zero yields the following system of algebraic equations

$$\theta^3 : Rsa_1^3 - a_1m^2\lambda^2 - a_1m^2p + a_1\lambda^2 + a_1p - 3sa_1b_1^2(m^2 - 1) - a_1(m^2 - 1)\lambda^2$$
$$\quad - a_1(m^2 - 1)p,$$
$$\theta^2 : 4a_1Rmp + 6Rsa_1b_1^2m + 4a_1Rm\lambda^2 + \omega Ra_1^2 + 3Rsa_0a_1^2 - a_1Rmp - a_1Rm\lambda^2$$
$$\quad - b_1^2m^2\omega - 3b_1^2m^2sa_0 + b_1^2\omega + 3b_1^2sa_0,$$
$$\theta^2\tau : b_1(-2m^2p - 2m^2\lambda^2 + 2\lambda^2 + 3Rsa_1^2 + 2p - b_1^2m^2s + b_1^2s),$$
$$\theta : a_1R\lambda^2 + a_1Rp - 2a_1Rp + a_1q + 2a_1\omega a_0 - 2a_1R\lambda^2 - 3a_1sRb_1^2 + 3a_1sa_0^2$$
$$\quad + 2b_1^2m\omega + 6b_1^2msa_0,$$
$$\theta\tau : b_1(2\omega a_1 + 2m\lambda^2 - m\lambda^2 - mp + 2mp + 6sa_0a_1 + 2sb_1^2m),$$
$$\tau : -b_1(sRb_1^2 - 2\omega a_0 - q - 3sa_0^2),$$
$$Con : qa_0 + \omega a_0^2 + sa_0^3 - \omega Rb_1^2 - 3sRa_0b_1^2.$$

Symbolic solutions of this system via software leads the following results.
Case 1 We have

$$R = \frac{q}{\lambda^2 + p}, \quad a_0 = 0, \quad m = \omega\sqrt{\frac{-2}{(-2\omega^2 + 9sq)}}, \quad a_1 = \frac{-3m}{\omega}(\lambda^2 + p), \quad b_1 = 0.$$
$$(16)$$

From (10), (15), (13) and (16), we deduce the following exact solutions

$$u_1(x,t) = \frac{-3\sqrt{\frac{-2}{(-2\omega^2+9sq)}}q\sec_{\mu\eta}\left(\sqrt{\frac{q}{\lambda^2+p}}\left(x - \frac{t^\gamma}{\gamma}\right)\right)}{\sqrt{\frac{-2}{(-2\omega^2+9sq)}}\omega\sec_{\mu\eta}\left(\sqrt{\frac{q}{\lambda^2+p}}\left(x - \frac{t^\gamma}{\gamma}\right)\right) + 1}, \tag{17}$$

$$u_2(x,t) = \frac{-3\sqrt{\frac{-2}{(-2\omega^2+9sq)}}q\csc_{\mu\eta}\left(\sqrt{\frac{q}{\lambda^2+p}}\left(x - \frac{t^\gamma}{\gamma}\right)\right)}{\sqrt{\frac{-2}{(-2\omega^2+9sq)}}\omega\csc_{\mu\eta}\left(\sqrt{\frac{q}{\lambda^2+p}}\left(x - \frac{t^\gamma}{\gamma}\right)\right) + 1}. \tag{18}$$

Case 2 We have

$$R = \frac{1}{4}\frac{q}{\lambda^2 + p}, \quad a_0 = \frac{1}{2}\sqrt{\frac{2q}{s}}, \quad a_1 = 0, \quad m = 0, \quad b_1 = \sqrt{-\frac{2\lambda^2 + 2p}{s}}, \quad \omega = -3s\,a_0.$$
$$(19)$$

Substituting the result above into Eq. (15), and combining with Eq. (10), then by use of (13), we deduce the following exact solutions

$$u_3(x,t) = \frac{1}{2}\sqrt{\frac{2q}{s}} + \frac{1}{2}\sqrt{\frac{-2q}{s}}\tan_{\mu\eta}\left(\frac{1}{2}\sqrt{\frac{q}{\lambda^2+p}}\left(x - \frac{t^\gamma}{\gamma}\right)\right), \tag{20}$$

$$u_4(x,t) = \frac{1}{2}\sqrt{\frac{2q}{s}} - \frac{1}{2}\sqrt{\frac{-2q}{s}}\cot_{\mu\eta}\left(\frac{1}{2}\sqrt{\frac{q}{\lambda^2+p}}\left(x - \frac{t^\gamma}{\gamma}\right)\right). \tag{21}$$

Case 3 We have

$$R = -\frac{2q}{\lambda^2 + p}, \quad a_0 = 0, \quad a_1 = 0, \quad m = 1, \quad b_1 = \sqrt{-\frac{\lambda^2 + p}{2s}}, \quad \omega = 0. \tag{22}$$

From (10), (15), (13), and (22), we deduce the following exact solutions

$$u_5(x, t) = \frac{\frac{q}{s} \tan_{\mu\eta}\left(\sqrt{\frac{-2q}{\lambda^2 + p}}\left(x - \frac{t^\gamma}{\gamma}\right)\right)}{\sec_{\mu\eta}\left(\sqrt{\frac{-2q}{\lambda^2 + p}}\left(x - \frac{t^\gamma}{\gamma}\right)\right) + 1}, \tag{23}$$

$$u_6(x, t) = -\frac{\frac{q}{s} \cot_{\mu\eta}\left(\sqrt{\frac{-2q}{\lambda^2 + p}}(x - \frac{t^\gamma}{\gamma})\right)}{\csc_{\mu\eta}\left(\sqrt{\frac{-2q}{\lambda^2 + p}}(x - \frac{t^\gamma}{\gamma})\right) + 1}. \tag{24}$$

Case 4 We have

$$R = \frac{q}{\lambda^2 + p}, \quad a_0 = \frac{1}{2}\sqrt{\frac{2q}{s}}, \quad a_1 = \sqrt{-\frac{1}{2sq}(\lambda^2 + p)},$$

$$m = 0, \quad b_1 = \sqrt{-\frac{\lambda^2 + p}{2s}}, \quad \omega = -3s\, a_0. \tag{25}$$

Substituting the result above into Eq. (15), and combining with Eq. (10), then by use of (13), we deduce the following exact solutions

$$u_7(x, t) = \sqrt{\frac{q}{2s}} + \sqrt{-\frac{q}{2s}} \sec_{\mu\eta}\left(\sqrt{\frac{q}{\lambda^2 + p}}\left(x - \frac{t^\gamma}{\gamma}\right)\right)$$
$$+ \sqrt{-\frac{q}{2s}} \tan_{\mu\eta}\left(\sqrt{\frac{q}{\lambda^2 + p}}\left(x - \frac{t^\gamma}{\gamma}\right)\right), \tag{26}$$

$$u_8(x, t) = \sqrt{\frac{q}{2s}} + \sqrt{-\frac{q}{2s}} \csc_{\mu\eta}\left(\sqrt{\frac{q}{\lambda^2 + p}}(x - \frac{t^\gamma}{\gamma})\right) - \sqrt{-\frac{q}{2s}} \cot_{\mu\eta}\left(\sqrt{\frac{q}{\lambda^2 + p}}(x - \frac{t^\gamma}{\gamma})\right). \tag{27}$$

In addition, when $p = -1$, $q = -a$, $\omega = 0$ and $s = -b$, Eq. (1) is expressed as CTF Klein–Gordon equation and its exact solutions

$$u_1(x, t) = a\sqrt{\frac{-2}{ab}} \sec_{\mu\eta}\left(\sqrt{\frac{-a}{\lambda^2 - 1}}\left(x - \frac{t^\gamma}{\gamma}\right)\right), \tag{28}$$

$$u_2(x, t) = a\sqrt{\frac{-2}{ab}} \csc_{\mu\eta}\left(\sqrt{\frac{-a}{\lambda^2 - 1}}\left(x - \frac{t^\gamma}{\gamma}\right)\right), \tag{29}$$

$$u_3(x, t) = \frac{1}{2}\sqrt{\frac{2a}{b}} + \frac{1}{2}\sqrt{\frac{-2a}{b}} \tan_{\mu\eta}\left(\frac{1}{2}\sqrt{\frac{-a}{\lambda^2 - 1}}\left(x - \frac{t^\gamma}{\gamma}\right)\right), \tag{30}$$

$$u_4(x,t) = \frac{1}{2}\sqrt{\frac{2a}{b}} - \frac{1}{2}\sqrt{\frac{-2a}{b}}\cot_{\mu\eta}\left(\frac{1}{2}\sqrt{\frac{-a}{\lambda^2-1}}\left(x - \frac{t^\gamma}{\gamma}\right)\right), \tag{31}$$

$$u_5(x,t) = \frac{\frac{a}{b}\tan_{\mu\eta}\left(\sqrt{\frac{2a}{\lambda^2-1}}\left(x - \frac{t^\gamma}{\gamma}\right)\right)}{\sec_{\mu\eta}\left(\sqrt{\frac{2a}{\lambda^2-1}}\left(x - \frac{t^\gamma}{\gamma}\right)\right) + 1}, \tag{32}$$

$$u_6(x,t) = -\frac{\frac{a}{b}\cot_{\mu\eta}\left(\sqrt{\frac{2a}{\lambda^2-1}}\left(x - \frac{t^\gamma}{\gamma}\right)\right)}{\csc_{\mu\eta}\left(\sqrt{\frac{2a}{\lambda^2-1}}\left(x - \frac{t^\gamma}{\gamma}\right)\right) + 1}, \tag{33}$$

$$u_7(x,t) = \sqrt{\frac{a}{2b}} + \sqrt{-\frac{a}{2b}}\sec_{\mu\eta}\left(\sqrt{\frac{-a}{\lambda^2-1}}\left(x - \frac{t^\gamma}{\gamma}\right)\right)$$
$$+ \sqrt{-\frac{a}{2b}}\tan_{\mu\eta}\left(\sqrt{\frac{-a}{\lambda^2-1}}\left(x - \frac{t^\gamma}{\gamma}\right)\right), \tag{34}$$

$$u_8(x,t) = \sqrt{\frac{a}{2b}} + \sqrt{-\frac{a}{2b}}\csc_{\mu\eta}\left(\sqrt{\frac{-a}{\lambda^2-1}}\left(x - \frac{t^\gamma}{\gamma}\right)\right)$$
$$- \sqrt{-\frac{a}{2b}}\cot_{\mu\eta}\left(\sqrt{\frac{-a}{\lambda^2-1}}\left(x - \frac{t^\gamma}{\gamma}\right)\right). \tag{35}$$

Similarly, when $p = -1$, $q = -m^2$, $\omega = 0$ and $s = g^2$, Eq. (1) is expressed as CTF Landau–Ginzburg–Higgs equation and its exact solutions

$$u_1(x,t) = -\sqrt{2}\frac{m}{g}\sec_{\mu\eta}\left(\sqrt{\frac{m^2}{1-\lambda^2}}\left(x - \frac{t^\gamma}{\gamma}\right)\right), \tag{36}$$

$$u_2(x,t) = -\sqrt{2}\frac{m}{g}\csc_{\mu\eta}\left(\sqrt{\frac{m^2}{1-\lambda^2}}\left(x - \frac{t^\gamma}{\gamma}\right)\right), \tag{37}$$

$$u_3(x,t) = \sqrt{-2}\frac{m}{2g} + \sqrt{2}\frac{m}{2g}\tan_{\mu\eta}\left(\frac{1}{2}\sqrt{\frac{m^2}{1-\lambda^2}}\left(x - \frac{t^\gamma}{\gamma}\right)\right), \tag{38}$$

$$u_4(x,t) = \sqrt{-2}\frac{m}{2g} - \sqrt{2}\frac{m}{2g}\cot_{\mu\eta}\left(\frac{1}{2}\sqrt{\frac{m^2}{1-\lambda^2}}\left(x - \frac{t^\gamma}{\gamma}\right)\right), \tag{39}$$

$$u_5(x,t) = \frac{\frac{-m^2}{g^2}\tan_{\mu\eta}\left(\sqrt{\frac{2m^2}{\lambda^2-1}}\left(x - \frac{t^\gamma}{\gamma}\right)\right)}{\sec_{\mu\eta}\left(\sqrt{\frac{2m^2}{\lambda^2-1}}\left(x - \frac{t^\gamma}{\gamma}\right)\right) + 1}, \tag{40}$$

$$u_6(x,t) = -\frac{\frac{-m^2}{g^2}\cot_{\mu\eta}\left(\sqrt{\frac{2m^2}{\lambda^2-1}}\left(x - \frac{t^\gamma}{\gamma}\right)\right)}{\csc_{\mu\eta}\left(\sqrt{\frac{2m^2}{\lambda^2-1}}\left(x - \frac{t^\gamma}{\gamma}\right)\right) + 1}, \tag{41}$$

$$u_7(x,t) = \frac{\sqrt{-2}}{2}\frac{m}{g} + \frac{\sqrt{2}}{2}\frac{m}{g}\sec_{\mu\eta}\left(\sqrt{\frac{m^2}{1-\lambda^2}}\left(x - \frac{t^\gamma}{\gamma}\right)\right) + \frac{\sqrt{2}}{2}\tan_{\mu\eta}\left(\sqrt{\frac{m^2}{1-\lambda^2}}\left(x - \frac{t^\gamma}{\gamma}\right)\right),$$

(42)

$$u_8(x,t) = \frac{\sqrt{-2}}{2}\frac{m}{g} + \frac{\sqrt{2}}{2}\frac{m}{g}\csc_{\mu\eta}\left(\sqrt{\frac{m^2}{1-\lambda^2}}\left(x - \frac{t^\gamma}{\gamma}\right)\right)$$
$$- \frac{\sqrt{2}}{2}\frac{m}{g}\cot_{\mu\eta}\left(\sqrt{\frac{m^2}{1-\lambda^2}}\left(x - \frac{t^\gamma}{\gamma}\right)\right).$$

(43)

When $p = -1$, $q = 1$, $\omega = 0$ and $s = -1$, Eq. (1) is expressed as CTF φ^4 equation and its exact solutions

$$u_1(x,t) = -\sqrt{2}\sec_{\mu\eta}\left(\sqrt{\frac{1}{\lambda^2-1}}\left(x - \frac{t^\gamma}{\gamma}\right)\right),$$

(44)

$$u_2(x,t) = -\sqrt{2}\csc_{\mu\eta}\left(\sqrt{\frac{1}{\lambda^2-1}}\left(x - \frac{t^\gamma}{\gamma}\right)\right),$$

(45)

$$u_3(x,t) = \frac{\sqrt{-2}}{2} + \frac{\sqrt{2}}{2}\tan_{\mu\eta}\left(\frac{1}{2}\sqrt{\frac{1}{\lambda^2-1}}\left(x - \frac{t^\gamma}{\gamma}\right)\right),$$

(46)

$$u_4(x,t) = \frac{\sqrt{-2}}{2} - \frac{\sqrt{2}}{2}\cot_{\mu\eta}\left(\frac{1}{2}\sqrt{\frac{1}{\lambda^2-1}}\left(x - \frac{t^\gamma}{\gamma}\right)\right),$$

(47)

$$u_5(x,t) = -\frac{\tan_{\mu\eta}\left(\sqrt{\frac{2}{1-\lambda^2}}\left(x - \frac{t^\gamma}{\gamma}\right)\right)}{\sec_{\mu\eta}\left(\sqrt{\frac{2}{1-\lambda^2}}\left(x - \frac{t^\gamma}{\gamma}\right)\right) + 1},$$

(48)

$$u_6(x,t) = \frac{\cot_{\mu\eta}\left(\sqrt{\frac{2}{1-\lambda^2}}\left(x - \frac{t^\gamma}{\gamma}\right)\right)}{\csc_{\mu\eta}\left(\sqrt{\frac{2}{1-\lambda^2}}\left(x - \frac{t^\gamma}{\gamma}\right)\right) + 1},$$

(49)

$$u_7(x,t) = \frac{\sqrt{-2}}{2} + \frac{\sqrt{2}}{2}\sec_{\mu\eta}\left(\sqrt{\frac{1}{\lambda^2-1}}\left(x - \frac{t^\gamma}{\gamma}\right)\right) + \frac{\sqrt{2}}{2}\tan_{\mu\eta}\left(\sqrt{\frac{1}{\lambda^2-1}}\left(x - \frac{t^\gamma}{\gamma}\right)\right),$$

(50)

$$u_8(x,t) = \frac{\sqrt{-2}}{2} + \frac{\sqrt{2}}{2}\csc_{\mu\eta}\left(\sqrt{\frac{1}{\lambda^2-1}}\left(x - \frac{t^\gamma}{\gamma}\right)\right) - \frac{\sqrt{2}}{2}\cot_{\mu\eta}\left(\sqrt{\frac{1}{\lambda^2-1}}\left(x - \frac{t^\gamma}{\gamma}\right)\right)$$ (51)

Similary when $p = 0$, $q = a$, $\omega = 0$ and $s = b$, Eq. (1) is expressed as CTP duffing equation and its exact solutions

$$u_1(x,t) = -a\sqrt{\frac{-2}{ba}}\sec_{\mu\eta}\left(\sqrt{\frac{a}{\lambda^2}}\left(x - \frac{t^\gamma}{\gamma}\right)\right), \tag{52}$$

$$u_2(x,t) = -a\sqrt{\frac{-2}{ba}}\csc_{\mu\eta}\left(\sqrt{\frac{a}{\lambda^2}}\left(x - \frac{t^\gamma}{\gamma}\right)\right), \tag{53}$$

$$u_3(x,t) = \frac{1}{2}\sqrt{\frac{2a}{b}} + \frac{1}{2}\sqrt{\frac{-2a}{b}}\tan_{\mu\eta}\left(\frac{1}{2}\sqrt{\frac{a}{\lambda^2}}\left(x - \frac{t^\gamma}{\gamma}\right)\right), \tag{54}$$

$$u_4(x,t) = \frac{1}{2}\sqrt{\frac{2a}{b}} - \frac{1}{2}\sqrt{\frac{-2a}{b}}\cot_{\mu\eta}\left(\frac{1}{2}\sqrt{\frac{a}{\lambda^2}}\left(x - \frac{t^\gamma}{\gamma}\right)\right), \tag{55}$$

$$u_5(x,t) = \frac{\frac{a}{b}\tan_{\mu\eta}\left(\sqrt{\frac{-2a}{\lambda^2}}\left(x - \frac{t^\gamma}{\gamma}\right)\right)}{\sec_{\mu\eta}\left(\sqrt{\frac{-2a}{\lambda^2}}\left(x - \frac{t^\gamma}{\gamma}\right)\right) + 1}, \tag{56}$$

$$u_6(x,t) = -\frac{\frac{a}{b}\cot_{\mu\eta}\left(\sqrt{\frac{-2a}{\lambda^2}}\left(x - \frac{t^\gamma}{\gamma}\right)\right)}{\csc_{\mu\eta}\left(\sqrt{\frac{-2a}{\lambda^2}}\left(x - \frac{t^\gamma}{\gamma}\right)\right) + 1}, \tag{57}$$

$$u_7(x,t) = \sqrt{\frac{a}{2b}} + \sqrt{-\frac{a}{2b}}\sec_{\mu\eta}\left(\sqrt{\frac{a}{\lambda^2}}\left(x - \frac{t^\gamma}{\gamma}\right)\right) + \sqrt{-\frac{a}{2b}}\tan_{\mu\eta}\left(\sqrt{\frac{a}{\lambda^2}}\left(x - \frac{t^\gamma}{\gamma}\right)\right), \tag{58}$$

$$u_8(x,t) = \sqrt{\frac{a}{2b}} + \sqrt{-\frac{a}{2b}}\csc_{\mu\eta}\left(\sqrt{\frac{a}{\lambda^2}}\left(x - \frac{t^\gamma}{\gamma}\right)\right) - \sqrt{-\frac{a}{2b}}\cot_{\mu\eta}\left(\sqrt{\frac{a}{\lambda^2}}\left(x - \frac{t^\gamma}{\gamma}\right)\right). \tag{59}$$

When $p = -1$, $q = 1$, $\omega = 0$ and $s = -\frac{1}{6}$, Eq. (1) is expressed as CTF Klein–Gordon equation and its exact solutions

$$u_1(x,t) = -\sqrt{12}\sec_{\mu\eta}\left(\sqrt{\frac{1}{\lambda^2 - 1}}\left(x - \frac{t^\gamma}{\gamma}\right)\right), \tag{60}$$

$$u_2(x,t) = -\sqrt{12}\csc_{\mu\eta}\left(\sqrt{\frac{1}{\lambda^2 - 1}}\left(x - \frac{t^\gamma}{\gamma}\right)\right), \tag{61}$$

$$u_3(x,t) = \sqrt{-3} + \sqrt{3}\tan_{\mu\eta}\left(\frac{1}{2}\sqrt{\frac{1}{\lambda^2 - 1}}\left(x - \frac{t^\gamma}{\gamma}\right)\right), \tag{62}$$

$$u_4(x,t) = \sqrt{-3} - \sqrt{3}\cot_{\mu\eta}\left(\frac{1}{2}\sqrt{\frac{1}{\lambda^2 - 1}}\left(x - \frac{t^\gamma}{\gamma}\right)\right), \tag{63}$$

$$u_5(x,t) = -6\tan_{\mu\eta}\left(\sqrt{\frac{2}{1 - \lambda^2}}\left(x - \frac{t^\gamma}{\gamma}\right)\right)\sec_{\mu\eta}\left(\sqrt{\frac{2a}{1 - \lambda^2}}\left(x - \frac{t^\gamma}{\gamma}\right)\right) + 1, \tag{64}$$

$$u_6(x,t) = 6\cot_{\mu\eta}\left(\sqrt{\frac{2}{1-\lambda^2}}\left(x-\frac{t^\gamma}{\gamma}\right)\right)\csc_{\mu\eta}\left(\sqrt{\frac{2}{1-\lambda^2}}\left(x-\frac{t^\gamma}{\gamma}\right)\right) + 1, \quad (65)$$

$$u_7(x,t) = \sqrt{-12} + \sqrt{12}\sec_{\mu\eta}\left(\sqrt{\frac{1}{\lambda^2-1}}\left(x-\frac{t^\gamma}{\gamma}\right)\right) + \sqrt{12}\tan_{\mu\eta}\left(\sqrt{\frac{1}{\lambda^2-1}}\left(x-\frac{t^\gamma}{\gamma}\right)\right),$$
$$(66)$$

$$u_8(x,t) = \sqrt{-12} + \sqrt{12}\csc_{\mu\eta}\left(\sqrt{\frac{1}{\lambda^2-1}}\left(x-\frac{t^\gamma}{\gamma}\right)\right) - \sqrt{12}\cot_{\mu\eta}\left(\sqrt{\frac{1}{\lambda^2-1}}\left(x-\frac{t^\gamma}{\gamma}\right)\right).$$
$$(67)$$

5 Conclusion

The GPREM was proposed to determine explicit exact solutions to some conformable time fractional PDEs in gRDM family. The accordance of the traveling wave transform reduces the target governing gRDM equation to an ODE with integer order. The predicted power series solution of some functions satisfying Riccati system (7) was substituted into the resultant ODE. The coefficients of the predicted solution was determined by solving some algebraic system. The results are reported explicitly in powers of some trigonometric functions or rational forms of trigonometric function series.

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