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Analytical treatment of nonlinear conformable time-fractional Boussinesq equations by three integration methods

Mehrdad Lakestani[1] · **Jalil Manafian**[1]

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Abstract Under investigation in this paper is a nonlinear conformable time-fractional Boussinesq equations as an important class of fractional differential equations in mathematical physics. The extended trial equation method, the $\exp(-\Omega(\eta))$-expansion method and the $\tan(\phi(\eta)/2)$-expansion method are used in examining the analytical solution of the nonlinear fractional equations. The proposed methods are based on the integration method and a wave transformation. The fractional derivative in the sense of conformable time-fractional derivative is defined. Fractional complex transform is implemented to change fractional differential equations into ordinary differential equations in this paper. In addition, explicit new exact solutions are derived in different form such as dark solitons, bright solitons, solitary wave, periodic solitary wave, rational function, and elliptic function solutions of nonlinear conformable time-fractional Boussinesq equations.

Keywords Conformable time-fractional Boussinesq equations · Conformable time-fractional derivative · The extended trial equation method · The $\exp(-\Omega(\eta))$-expansion method · The $\tan(\phi(\eta)/2)$-expansion method

1 Introduction

Fractional calculus is considered as a novel topic (Podlubny 1999), it has gained considerable popularity and also much more importance during the past three decades. Fractional partial differential equations (FPDEs) involve unknown multivariable functions and their fractional partial operators. Several physical phenomena have been dressed up in FPDEs in recent years such as viscoelasticity, plasma, solid mechanics, optical fibers, signal processing, electromagnetic waves, biomedical sciences, Diffusion processes and vibration with fractional damping. Seeking exact solutions to fractional differential equations

✉ Jalil Manafian
j_manafianheris@tabrizu.ac.ir

[1] Department of Applied Mathematics, Mathematical Sciences, University of Tabriz, Tabriz, Iran

(FDEs) is an important task of many researchers. A wide range of straightforward and powerful methods have been introduced to obtain exact solutions of nonlinear PDEs, such as, the Exp-function method (Ekici et al. 2017b; Manafian and Lakestani 2015; Manafian 2015), the generalized Kudryashov method (Zhou et al. 2005), the extended Jacobi elliptic function expansion method (Ekici et al. 2017c; Mirzazadeh et al. 2016), the improve $\tan(\phi/2)$-expansion method (Manafian 2016, 2017; Manafian and Lakestani 2016a, b; Manafian et al. 2016), the $\exp(-\Omega(\eta))$-expansion method (Abdelrahman et al. 2014; Zhou et al. 2016) the G'/G-expansion method (Manafian and Lakestani 2015, 2017; Sindi and Manafian 2016) the generalized G'/G-expansion method (Zinati and Manafian 2017) the Bernoulli sub-equation function method (Baskonus 2017; Baskonus and Bulut 2016; Baskonus et al. 2016; Bulut and Baskonus 2016) the sine-Gordon expansion method (Baskonus et al. 2016; Baskonus and Bulut 2016; Baskonus 2016; Yel et al. 2017), the Ricatti equation expansion (Zhou 2016; Inc et al. 2016) the formal linearization method (Mirzazadeh and Eslami 2015) the Lie symmetry (Tchier et al. 2017) and so on.

Extended trial equation method is one of the robust techniques to look for the exact solutions of nonlinear partial differential equations that has received special interest owing to its fairly great performance. For example, Mohyud-Din and Irshad (2017) explored new exact solitary wave solutions of some nonlinear PDEs arising in electronics using the extended trial equation method. Mirzazadeh et al. (2017) adopted he extended trial equation method to obtain analytical solutions to the generalized resonant dispersive nonlinear Schrödinger's equation with power law nonlinearity. Ekici et al. (2017a) found the exact soliton solutions to magneto-optic waveguides that appear with Kerr, power and log-law nonlinearities using the extended trial equation method.

Consider the following nonlinear conformable time-fractional Boussinesq equation (Demiray 2014; Hosseini and Ansari 2017; Hosseini et al. 2017) as

$$D_t^{2\alpha}u - u_{xx} - (u^2)_{xx} + u_{xxxx} = 0, \quad 0 < \alpha \leq 1, \tag{1.1}$$

and also consider the following coupled conformable time-fractional Boussinesq equations (Hosseini and Ansari 2017; Kheiri et al. 2013; Hosseini et al. 2017) as

$$\begin{array}{ll} D_t^{\alpha}u + v_x = 0, & 0 < \alpha \leq 1 \\ D_t^{\alpha}v + \mu(u^2)_x - \lambda u_{xxx} = 0, & 0 < \alpha \leq 1, \end{array} \tag{1.2}$$

in which $D_t^{\alpha}u = \frac{\partial^{\alpha}u}{\partial t^{\alpha}}$ and $D_t^{\alpha}v = \frac{\partial^{\alpha}v}{\partial t^{\alpha}}$.

This paper will adopt three integration schemes that are known as the extended trial equation method, the $\exp(-\Omega(\eta))$-expansion method and the $\tan(\phi(\eta)/2)$-expansion method that will reveal soliton solutions as well as other solutions. The equations for the model studied here to investigate exact solution structures. We note that these equations have not yet been studied using the aforementioned methods.

The rest of the paper is ordered as follows: Conformable fractional derivative and its properties is presented in Sect. 2. In Sect. 3, analytical methods are given. In Sects. 4 and 5, applications of nonlinear conformable time-fractional Boussinesq equation by help of the extended trial equation method, the $\exp(-\Omega(\eta))$-expansion method and the $\tan(\phi(\eta)/2)$-expansion method are investigated. Finally, the conclusion is given in Sect. 6.

2 Conformable fractional derivative and its properties

The conformable fractional derivative proposed by Khalil et al. (2014), which can amend the shortcomings of the previous definitions. Moreover, a number of definitions and properties of the conformable fractional derivative are presented in Abdeljawad (2015) and Eslami and Rezazadeh (2016).

Definition 1 Assume $f : (0, \infty) \rightarrow R$ be a function. The conformable fractional derivative of f of order α is defined as

$$T_{\alpha}(f)(t) = \lim_{\theta \rightarrow 0} \frac{f(t + \theta t^{1-\alpha}) - f(t)}{\theta}, \tag{2.1}$$

in which $0 < \alpha \leq 1$.

Theorem 1 *Assume* $\alpha \in (0, 1]$, *and f and g be* α*-differentiable at* $t > 0$. *Then*

1. $T_{\alpha}(af + bg) = aT_{\alpha}(f) + bT_{\alpha}(g), \ \forall a, b \in R.$
2. $T_{\alpha}(t^{\vartheta}) = \vartheta t^{\vartheta - \alpha}, \ \forall \vartheta \in R.$
3. $T_{\alpha}(fg) = fT_{\alpha}(g) + gT_{\alpha}(f).$
4. $T_{\alpha}\left(\frac{f}{g}\right) = \frac{gT_{\alpha}(f) - fT_{\alpha}(g)}{g^2}.$
5. *If f is differentiable; then* $T_{\alpha}(f)(t) = t^{1-\alpha} \frac{df}{dt}(t).$

Theorem 2 *Suppose* $f : (0, \infty) \rightarrow R$ *be a function such that f is differentiable and also* α*-differentiable. Let g be a function defined in the range of f and also differentiable. Then*

$$T_{\alpha}(fog)(t) = t^{1-\alpha} g'(t) f'(g(t)).$$

3 Analytical methods

3.1 Extended trial equation method

The current method described here is the extended trial equation method utilized to find traveling wave solutions of the conformable time-fractional Boussinesq equations which can be understood through the following steps:

Step 1 We assume that the given nonlinear PDE with the conformable time-fractional derivative as

$$\mathcal{N}\left(u, \frac{\partial^{\alpha} u}{\partial t^{\alpha}}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \ldots\right) = 0. \tag{3.1}$$

Utilizing the wave transformation

$$u(x, t) = u(\eta), \eta = x - \frac{kt^{\alpha}}{\alpha}, \tag{3.2}$$

where $k \neq 0$. Substituting (3.2) into Eq. (3.1) yields a nonlinear ordinary differential equation,

$$\mathcal{Q}(u, -ku', u', u'', \ldots) = 0, \tag{3.3}$$

where prime shows the derivation with respect to η.

Step 2 Take the transformation and trial equation as follows:

$$u(\eta) = \sum_{i=0}^{\delta} \tau_i \Gamma^i, \tag{3.4}$$

where

$$(\Gamma')^2 = \Omega(\Gamma) = \frac{\Phi(\Gamma)}{\Psi(\Gamma)} = \frac{\xi_\theta \Gamma^\theta + \cdots + \xi_1 \Gamma + \xi_0}{\zeta_\epsilon \Gamma^\epsilon + \cdots + \zeta_1 \Gamma + \zeta_0}. \tag{3.5}$$

Using the Eqs. (3.4) and (3.5), we can find

$$(u')^2 = \frac{\Phi(\Gamma)}{\Psi(\Gamma)} \left(\sum_{i=0}^{\delta} i\tau_i \Gamma^{i-1} \right)^2, \tag{3.6}$$

$$u'' = \frac{\Phi'(\Gamma)\Psi(\Gamma) - \Phi(\Gamma)\Psi'(\Gamma)}{2\Psi^2(\Gamma)} \left(\sum_{i=0}^{\delta} i\tau_i \Gamma^{i-1} \right) + \frac{\Phi(\Gamma)}{\Psi(\Gamma)} \left(\sum_{i=0}^{\delta} i(i-1)\tau_i \Gamma^{i-2} \right), \tag{3.7}$$

where $\Phi(\Gamma)$ and $\Psi(\Gamma)$ are polynomials. Substituting these terms into Eq. (3.1) yields an equation of polynomial $\Lambda(\Gamma)$ of Γ:

$$\Lambda(\Gamma) = \varrho_s \Gamma^s + \cdots + \varrho_1 \Gamma + \varrho_0 = 0. \tag{3.8}$$

By utilizing the balance principle on (3.8), we can determine a relation of θ, ϵ and δ. We can take some values of θ, ϵ and δ.

Step 3 Setting each coefficient of polynomial $\Lambda(\Gamma)$ to zero to derive a system of algebraic equations:

$$\varrho_i = 0, \quad i = 1, 2, \ldots, s. \tag{3.9}$$

By solving the system (3.9), we will obtain the values of $\xi_0, \xi_1, \ldots, \xi_\theta$, $\zeta_0, \zeta_1, \ldots, \zeta_\sigma$ and $\tau_0, \tau_1, \ldots, \tau_\delta$.

Step 4 In the following step, we obtain the elementary form of the integral by reduction of Eq. (3.5), as follows

$$\pm(\eta - \eta_0) = \int \frac{d\Gamma}{\sqrt{\Omega(\Gamma)}} = \int \sqrt{\frac{\Psi(\Gamma)}{\Phi(\Gamma)}} d\Gamma, \tag{3.10}$$

where η_0 is an arbitrary constant.

3.2 The $\exp(-\Omega(\eta))$-expansion method

In this subsection, we describe the $\exp(-\Omega(\eta))$-expansion method which was utilized to find traveling wave solutions of nonlinear partial differential equations. This approach is based on the $\exp(-\Omega(\eta))$-expansion method (Khan and Akbar 2014; Rayhanul Islam et al. 2015). We consider the following steps:

Step 1 We assume that the given nonlinear PDE with the conformable time-fractional derivative as

$$\mathcal{N}\left(u, \frac{\partial^\alpha u}{\partial t^\alpha}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \ldots\right) = 0. \tag{3.11}$$

which can be converted to an ODE

$$\mathcal{Q}(u, -ku', u', u'', \ldots) = 0, \tag{3.12}$$

by the transformation $\eta = x - \frac{kt^\alpha}{\alpha}$ is the wave variable. Also, k is constant to be determined later.

Step 2 We suppose the solution of nonlinear equation (3.12) can be expressed by a rational polynomial in $F(\eta)$ as the following:

$$u(\eta) = \sum_{j=-M}^{M} \xi_j F^j(\eta), \tag{3.13}$$

where $F(\eta) = \exp(-\Omega(\eta))$ and $\xi_j(-M \le j \le M)$, are constants to be determined, such that $\xi_M \neq 0$, and, $\Omega = \Omega(\eta)$ satisfies the following ordinary differential;

$$\Omega' = \mu F^{-1}(\eta) + F(\eta) + \lambda. \tag{3.14}$$

The following exact analytical solutions (Hafez et al. 2014, 2015) can be considered from Eq. (3.14):

Solution-1 If $\mu \neq 0$ and $\lambda^2 - 4\mu > 0$, then we have

$$\Omega(\eta) = \ln\left(-\frac{\sqrt{\lambda^2 - 4\mu}}{2\mu}\tanh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}(\eta + E)\right) - \frac{\lambda}{2\mu}\right), \tag{3.15}$$

where E is integral constant.

Solution-2 If $\mu \neq 0$ and $\lambda^2 - 4\mu < 0$, then we have

$$\Omega(\eta) = \ln\left(\frac{\sqrt{-\lambda^2 + 4\mu}}{2\mu}\tan\left(\frac{\sqrt{-\lambda^2 + 4\mu}}{2}(\eta + E)\right) - \frac{\lambda}{2\mu}\right). \tag{3.16}$$

Solution-3 If $\mu = 0$, $\lambda \neq 0$, and $\lambda^2 - 4\mu > 0$, then we get

$$\Omega(\eta) = -\ln\left(\frac{\lambda}{\exp(\lambda(\eta + E)) - 1}\right). \tag{3.17}$$

Solution-4 If $\mu \neq 0$, $\lambda \neq 0$, and $\lambda^2 - 4\mu = 0$, then we get

$$\Omega(\eta) = \ln\left(-\frac{2\lambda(\eta + E) + 4}{\lambda^2(\eta + E)}\right). \tag{3.18}$$

Solution-5 If $\mu = 0$, $\lambda = 0$, and $\lambda^2 - 4\mu = 0$, then we get

$$\Omega(\eta) = \ln(\eta + E), \tag{3.19}$$

where $\xi_j(0 \le j \le M)$, λ and μ are constants to be determined. The value M can be identified by taking the balance principle which is based on the relationship between the highest order derivatives and the highest degree of the nonlinear terms occurring in Eq. (3.12).

Step 3 Substituting (3.13) into Eq. (3.12) with the value of M obtained in Step 2. collecting the coefficients of $F(\eta)$, then setting each coefficient to zero, we can get a set of over-determined equations for $\xi_0, \xi_1, \ldots, \xi_M$ λ, and μ with the aid of symbolic computation Maple. Solving the algebraic equations including coefficients of $\xi_0, \xi_1, \ldots, \xi_M$ λ, and μ into (3.13) we get to exact solution of considered problem.

3.3 Description of the improved tan(ϕ(η)/2)-expansion method

The $\tan(\phi/2)$-expansion method is a well-known analytical method. In this paper we propose to develop this method, but prior to that we give a detailed description of the method throughout the following steps:

Step 1 We assume that the given nonlinear PDE with the conformable time-fractional derivative as

$$\mathcal{N}\left(u, \frac{\partial^\alpha u}{\partial t^\alpha}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \ldots\right) = 0. \tag{3.20}$$

which can be converted to an ODE

$$\mathcal{Q}(u, -ku', u', u'', \ldots) = 0, \tag{3.21}$$

by the transformation $\eta = x - \frac{kt^\alpha}{\alpha}$ is the wave variable. Also, k is constant to be determined later.

Step 2 Suppose the traveling wave solution of Eq. (3.21) can be expressed as follows:

$$u(\eta) = \sum_{k=0}^{M} A_k \tan^k\left(\frac{\Phi(\eta)}{2}\right) + \sum_{k=1}^{M} B_k \cot^k\left(\frac{\Phi(\eta)}{2}\right), \tag{3.22}$$

where $A_k (0 \leq k \leq M), B_k (1 \leq k \leq M)$ are constants to be determined, such that $A_M \neq 0, B_M \neq 0$, and $\phi = \phi(\xi)$ satisfies the following ordinary differential equation:

$$\phi'(\eta) = a \sin(\phi(\eta)) + b \cos(\phi(\eta)) + c. \tag{3.23}$$

Step 3 To determine M. This, usually, can be accomplished by balancing the linear term(s) of highest order with the highest-order nonlinear term(s) in Eq. (3.21). But, the positive integer M can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in Eq. (3.21). Moreover, precisely, we define the degree of $u(\eta)$ as $D(u(\eta)) = M$, which gives rise to degree of another expression as follows:

$$D\left(\frac{d^q u}{d\eta^q}\right) = M + q, \quad D\left(u^p \left(\frac{d^q u}{d\eta^q}\right)^s\right) = Mp + s(M + q). \tag{3.24}$$

Step 4 Substituting (3.22) into Eq. (3.21) with the value of M obtained in Step 3. collecting the coefficients of $\tan(\phi/2)^k$, $\cot(\phi/2)^k (k = 0, 1, 2, \ldots, M)$, then setting each coefficient to zero, we can get a set of over-determined equations for $A_0, A_k, B_k (k = 1, 2, \ldots,)$ a, b, and c with the aid of symbolic computation Maple. Solving the algebraic equations, then substituting $A_0, A_1, \ldots, A_M, B_1, \ldots, B_M, k$ in (3.22).

Consider the following special solutions of equation (3.23):

Family 1 When $\Delta = a^2 + b^2 - c^2 < 0$ and $b - c \neq 0$, then $\phi(\eta) = 2 \arctan \left[\frac{a}{b-c} - \frac{\sqrt{-\Delta}}{b-c} \tan\left(\frac{\sqrt{-\Delta}}{2} \widetilde{\eta}\right)\right]$.

Family 2 When $\Delta = a^2 + b^2 - c^2 > 0$ and $b - c \neq 0$, then $\phi(\eta) = 2\arctan\left[\frac{a}{b-c} + \frac{\sqrt{\Delta}}{b-c}\tanh\left(\frac{\sqrt{\Delta}}{2}\widetilde{\eta}\right)\right]$.

Family 3 When $a^2 + b^2 - c^2 > 0$, $b \neq 0$ and $c = 0$, then $\phi(\eta) = 2\arctan\left[\frac{a}{b} + \frac{\sqrt{b^2+a^2}}{b}\tanh\left(\frac{\sqrt{b^2+a^2}}{2}\widetilde{\eta}\right)\right]$.

Family 4 When $a^2 + b^2 - c^2 < 0$, $c \neq 0$ and $b = 0$, then $\phi(\eta) = 2\arctan\left[-\frac{a}{c} + \frac{\sqrt{c^2-a^2}}{c}\tan\left(\frac{\sqrt{c^2-a^2}}{2}\widetilde{\eta}\right)\right]$.

Family 5 When $a^2 + b^2 - c^2 > 0$, $b - c \neq 0$ and $a = 0$, then $\phi(\eta) = 2\arctan\left[\sqrt{\frac{b+c}{b-c}}\tanh\left(\frac{\sqrt{b^2-c^2}}{2}\widetilde{\eta}\right)\right]$.

Family 6 When $a = 0$ and $c = 0$, then $\phi(\eta) = \arctan\left[\frac{e^{2b\widetilde{\eta}}-1}{e^{2b\widetilde{\eta}}+1}, \frac{2e^{b\widetilde{\eta}}}{e^{2b\widetilde{\eta}}+1}\right]$.

Family 7 When $b = 0$ and $c = 0$, then $\phi(\eta) = \arctan\left[\frac{2e^{a\widetilde{\eta}}}{e^{2a\widetilde{\eta}}+1}, \frac{e^{2a\widetilde{\eta}}-1}{e^{2a\widetilde{\eta}}+1}\right]$.

Family 8 When $a^2 + b^2 = c^2$, then $\phi(\eta) = -2\arctan\left[\frac{(b+c)(a\widetilde{\eta}+2)}{a^2\widetilde{\eta}}\right]$.

Family 9 When $c = a$, then $\phi(\eta) = -2\arctan\left[\frac{(a+b)e^{b\widetilde{\eta}}-1}{(a-b)e^{b\widetilde{\eta}}-1}\right]$.

Family 10 When $c = -a$, then $\phi(\eta) = 2\arctan\left[\frac{e^{b\widetilde{\eta}}+b-a}{e^{b\widetilde{\eta}}-b-a}\right]$.

Family 11 When $b = -c$, then $\phi(\eta) = -2\arctan\left[\frac{ae^{a\widetilde{\eta}}}{ce^{a\widetilde{\eta}}-1}\right]$.

Family 12 When $b = 0$ and $a = c$, then $\phi(\eta) = -2\arctan\left[\frac{c\widetilde{\eta}+2}{c\widetilde{\eta}}\right]$.

Family 13 When $a = 0$ and $b = c$, then $\phi(\eta) = 2\arctan[c\widetilde{\eta}]$.

Family 14 When $a = 0$ and $b = -c$, then $\phi(\eta) = -2\arctan\left[\frac{1}{c\widetilde{\eta}}\right]$,

Family 15 When $a = 0$ and $b = 0$, then $\phi(\eta) = c\widetilde{\eta} + C$.

Family 16 When $b = c$ then $\phi(\eta) = 2\arctan\left[\frac{e^{a\widetilde{\eta}}-c}{a}\right]$,

where $\widetilde{\eta} = \eta + C$.

4 Application of conformable time-fractional Boussinesq equation

In this section, we present three methods based on the integration methods for solving the conformable time-fractional Boussinesq equation.

4.1 ETEM

In this subsection, consider ETEM for Eq. (5.4) which is given as

$$(k^2 - 1)u'' - (u^2)'' + u^{(4)} = 0, \tag{4.1}$$

where the prime indicates the derivation with respect to η. Integrating Eq. (4.1) twice and setting the constants of integration equal to zero, results in

$$(k^2 - 1)u - u^2 + u'' = 0. \tag{4.2}$$

We can determine values of δ, θ, and ϵ, by balancing u^2 and u'' in Eq. (4.2) as follows:

$$\delta = \theta - \epsilon - 2. \tag{4.3}$$

For different values of δ, θ, and ϵ, we have the following cases:

Case I $\delta = 1, \theta = 4, \text{and}\, \epsilon = 1$.

If we take $\delta = 1, \theta = 4$, and $\epsilon = 1$ for Eqs. (3.4) and (3.5), then we obtain

$$u(\eta) = \tau_0 + \tau_1 \Gamma, \tag{4.4}$$

$$(u'(\eta))^2 = \frac{\tau_1^2(\xi_4\Gamma^4 + \xi_3\Gamma^3 + \xi_2\Gamma^2 + \xi_1\Gamma + \xi_0)}{\zeta_0 + \zeta_1\Gamma}, \tag{4.5}$$

where $\xi_4 \neq 0$ and $\zeta_1 \neq 0$. Solving the algebraic equation system (3.9) yields

- *First set of parameters*

$$\begin{aligned} \xi_0 =& \frac{\zeta_0(\tau_1\xi_1 + 2\zeta_0 k^2\tau_0 - 2\tau_0\zeta_0 - 2\zeta_0\tau_0^2)}{\tau_1\zeta_1}, \quad \xi_1 = \xi_1, \\ \xi_2 =& -\frac{k^2\tau_1\zeta_0 - 2\tau_0\zeta_1 - 2\tau_0\tau_1\zeta_0 - 2\tau_0^2\zeta_1 - \tau_1\zeta_0 + 2k^2\tau_0\zeta_1}{\tau_1}, \\ \xi_3 =& \frac{2}{3}\tau_1\zeta_0 + \zeta_1 + 2\tau_0\zeta_1 - k^2\zeta_1, \\ \xi_4 =& \frac{2}{3}\tau_1\zeta_1, \quad \zeta_0 = \zeta_0, \quad \zeta_1 = \zeta_1, \quad k = k, \quad \tau_0 = \tau_0, \quad \tau_1 = \tau_1. \end{aligned} \tag{4.6}$$

Substituting these results into Eqs. (3.5) and (3.10), we get

$$\pm(\eta - \eta_0) = \int \frac{\sqrt{\frac{\zeta_0}{\xi_4} + \frac{\zeta_1}{\xi_4}\Gamma}\, d\Gamma}{\sqrt{\Gamma^4 + \frac{\xi_3}{\xi_4}\Gamma^3 + \frac{\xi_2}{\xi_4}\Gamma^2 + \frac{\xi_1}{\xi_4}\Gamma + \frac{\xi_0}{\xi_4}}} = \int \sqrt{\frac{\frac{\zeta_0}{\xi_4} + \frac{\zeta_1}{\xi_4}\Gamma}{\Gamma^4 + \frac{\xi_3}{\xi_4}\Gamma^3 + \frac{\xi_2}{\xi_4}\Gamma^2 + \frac{\xi_1}{\xi_4}\Gamma + \frac{\xi_0}{\xi_4}}}\, d\Gamma, \tag{4.7}$$

we have

$$\begin{aligned} &\frac{\frac{\zeta_0}{\xi_4} + \frac{\zeta_1}{\xi_4}\Gamma}{\Gamma^4 + \frac{\xi_3}{\xi_4}\Gamma^3 + \frac{\xi_2}{\xi_4}\Gamma^2 + \frac{\xi_1}{\xi_4}\Gamma + \frac{\xi_0}{\xi_4}} \\ =& \frac{-3\tau_1\zeta_1}{-2\tau_1^2\zeta_1\Gamma^3 + 3\tau_1\zeta_1(k^2 - 2\tau_0 - 1)\Gamma^2 + 6\tau_0(k^2 - \tau_0 - 1)(\zeta_1\Gamma - \zeta_0) - 3\tau_1\xi_1}, \end{aligned} \tag{4.8}$$

and

$$\begin{aligned} &-2\tau_1^2\zeta_1\Gamma^3 + 3\tau_1\zeta_1(k^2 - 2\tau_0 - 1)\Gamma^2 + 6\tau_0(k^2 - \tau_0 - 1)(\zeta_1\Gamma - \zeta_0) - 3\tau_1\xi_1 \\ &= (\Gamma - \alpha_1)(\Gamma - \alpha_2)(\Gamma - \alpha_3), \end{aligned} \tag{4.9}$$

$$\pm(\eta-\eta_0) = \int \sqrt{\frac{-3\tau_1\zeta_1}{-2\tau_1^2\zeta_1\Gamma^3+3\tau_1\zeta_1(k^2-2\tau_0-1)\Gamma^2+6\tau_0(k^2-\tau_0-1)(\zeta_1\Gamma-\zeta_0)-3\tau_1\xi_1}}\,d\Gamma. \tag{4.10}$$

Integrating (4.13), we obtain the solutions to the Eq. (1.1) as follows:

First solution

$$\pm(\eta-\eta_0)=\frac{2\sqrt{-3\tau_1\zeta_1}}{\sqrt{\alpha_1-\alpha_3}}F(\varphi,l), \quad \alpha_1>\alpha_2>\alpha_3, \tag{4.11}$$

where

$$F(\varphi,l)=\int_0^{\varphi}\frac{d\psi}{\sqrt{1-l^2\sin^2\psi}}, \varphi=\arcsin\sqrt{\frac{\Gamma-\alpha_3}{\alpha_2-\alpha_3}}, l^2=\frac{\alpha_2-\alpha_3}{\alpha_2-\alpha_3}, \tag{4.12}$$

$$\begin{aligned}\Gamma =&\alpha_3+(\alpha_2-\alpha_3)sn^2\left[\mp\frac{\sqrt{\alpha_1-\alpha_3}}{2\sqrt{-3\tau_1\zeta_1}}(\eta-\eta_0),\frac{\alpha_2-\alpha_3}{\alpha_1-\alpha_3}\right],\\ u(\eta) =&\tau_0+\alpha_3\tau_1+\tau_1(\alpha_2-\alpha_3)sn^2\left[\mp\frac{\sqrt{\alpha_1-\alpha_3}}{2\sqrt{-3\tau_1\zeta_1}}(\eta-\eta_0),\frac{\alpha_2-\alpha_3}{\alpha_1-\alpha_3}\right].\end{aligned} \tag{4.13}$$

Therefore, the solution for the conformable time-fractional Boussinesq equation will be as

$$u_1(x,t)=\tau_0+\alpha_3\tau_1+\tau_1(\alpha_2-\alpha_3)sn^2\left[\mp\frac{\sqrt{\alpha_1-\alpha_3}}{2\sqrt{-3\tau_1\zeta_1}}\left(x-\frac{kt^{\alpha}}{\alpha}-\eta_0\right),\frac{\alpha_2-\alpha_3}{\alpha_1-\alpha_3}\right]. \tag{4.14}$$

Remark 1 If the modulus $l\to 1$, then the solution for the conformable time-fractional Boussinesq equation can be reduced to the solitary wave solution

$$u_2(x,t)=\tau_0+\alpha_3\tau_1+\tau_1(\alpha_2-\alpha_3)\tanh^2\left[\mp\frac{\sqrt{\alpha_2-\alpha_3}}{2\sqrt{-3\tau_1\zeta_1}}\left(x-\frac{kt^{\alpha}}{\alpha}-\eta_0\right)\right]. \tag{4.15}$$

where $\alpha_1=\alpha_2$, and $v=\frac{k}{\alpha}$ represents the velocity of the dark soliton.

Case II $\delta=2, \theta=4, \text{and } \epsilon=0.$

If we take $\delta=2, \theta=4$, and $\epsilon=0$ for Eqs. (3.4) and (3.5), then we obtain

$$u(\eta)=\tau_0+\tau_1\Gamma+\tau_2\Gamma^2, \tag{4.16}$$

$$(u'(\eta))^2=\frac{(2\tau_2\Gamma+\tau_1)^2(\xi_4\Gamma^4+\xi_3\Gamma^3+\xi_2\Gamma^2+\xi_1\Gamma+\xi_0)}{\zeta_0}, \tag{4.17}$$

where $\xi_4\neq 0$ and $\zeta_0\neq 0$. Solving the algebraic equation system (3.9) yields

- *First set of parameters*

$$\xi_0 = -\frac{\zeta_0\left(6\tau_1^2\tau_2(1-k^2+2\tau_0)-\tau_1^4-48\tau_0\tau_2^2(1+\tau_0-k^2)\right)}{96\tau_2^3},$$
$$\xi_1 = -\frac{\zeta_0\tau_1\left(6k^2\tau_2-12\tau_0\tau_2+\tau_1^2-6\tau_2\right)}{2\tau_2^2},$$
$$\xi_2 = -\frac{\zeta_0\left(-4\tau_0\tau_2+2k^2\tau_2-\tau_1^2-2\tau_2\right)}{8\tau_2},$$
$$\xi_3 = \frac{1}{3}\tau_1\zeta_0, \quad \xi_4 = \frac{1}{6}\tau_2\zeta_0, \quad \zeta_0=\zeta_0, \quad k=k, \quad \tau_0=\tau_0, \quad \tau_1=\tau_1, \quad \tau_2=\tau_2. \tag{4.18}$$

Substituting these results into Eqs. (3.5) and (3.10), we get

$$\pm(\eta-\eta_0) = \int \frac{\sqrt{\frac{\zeta_0}{\xi_4}}\, d\Gamma}{\sqrt{\Gamma^4+\frac{\xi_3}{\xi_4}\Gamma^3+\frac{\xi_2}{\xi_4}\Gamma^2+\frac{\xi_1}{\xi_4}\Gamma+\frac{\xi_0}{\xi_4}}}. \tag{4.19}$$

Integrating Eq. (4.19), we obtain the solutions to the Eq. (1.1) as follows:
First solution

$$\pm(\eta-\eta_0) = \frac{2\sqrt{\frac{\zeta_0}{\xi_4}}}{\sqrt{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}}F(\varphi,l), \quad \alpha_1>\alpha_2>\alpha_3>\alpha_4, \tag{4.20}$$

where

$$F(\varphi,l)=\int_0^{\varphi}\frac{d\psi}{\sqrt{1-l^2\sin^2\psi}}, \quad \varphi=\arcsin\sqrt{\frac{(\alpha_2-\alpha_4)(\Gamma-\alpha_1)}{(\alpha_1-\alpha_4)(\Gamma-\alpha_2)}}, \quad l^2=\frac{(\alpha_2-\alpha_3)(\alpha_1-\alpha_4)}{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)},$$
$$\Rightarrow \Gamma=\alpha_2+\frac{(\alpha_1-\alpha_2)(\alpha_4-\alpha_2)}{\alpha_4-\alpha_2+(\alpha_1-\alpha_4)sn^2\left[\mp\frac{\sqrt{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}}{2\Pi}(\eta-\eta_0),\frac{(\alpha_2-\alpha_3)(\alpha_1-\alpha_4)}{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}\right]}, \tag{4.21}$$

$$\Gamma^4+\frac{2\tau_1}{\tau_2}\Gamma^3-\frac{3\left(-4\tau_0\tau_2+2k^2\tau_2-\tau_1^2-2\tau_2\right)}{4\tau_2^2}\Gamma^2+\frac{3\tau_1\left(6k^2\tau_2-12\tau_0\tau_2+\tau_1^2-6\tau_2\right)}{\tau_2^3}\Gamma$$
$$-\frac{6\tau_1^2\tau_2(1-k^2+2\tau_0)-\tau_1^4-48\tau_0\tau_2^2(1+\tau_0-k^2)}{16\tau_2^4}=(\Gamma-\alpha_1)(\Gamma-\alpha_2)(\Gamma-\alpha_3)(\Gamma-\alpha_4)\Rightarrow \tag{4.22}$$

$$\alpha_j=-\frac{\tau_1}{2\tau_2}\pm\frac{\sqrt{6\tau_1^2+12k^2\tau_2-24\tau_0\tau_2-12\tau_2\pm2\sqrt{\Omega}}}{4\tau_2}, \quad j=1,2,3,4, \tag{4.23}$$

where

$$\Omega=3\left(4\tau_2\tau_0-\tau_1^2-2\tau_2+2\tau_2k^2\right)\left(-4\tau_2\tau_0+\tau_1^2-6\tau_2+6\tau_2k^2\right).$$

Therefore, the solution for the conformable time-fractional Boussinesq equation will be as

$$u_3(x,t) = \tau_0 + \tau_0\alpha_2 + \frac{\tau_1(\alpha_1-\alpha_2)(\alpha_4-\alpha_2)}{\alpha_4-\alpha_2+(\alpha_1-\alpha_4)sn^2\left[\mp\frac{\sqrt{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}}{2\sqrt{\frac{6}{\tau_2}}}\left(x-\frac{kt^\alpha}{\alpha}-\eta_0\right), \frac{(\alpha_2-\alpha_3)(\alpha_1-\alpha_4)}{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}\right]}. \tag{4.24}$$

Remark 1 If the modulus $l \to 1$, then the solution for the conformable time-fractional Boussinesq equation can be reduced to the solitary wave solution

$$u_4(x,t) = \tau_0 + \tau_0\alpha_2 + \frac{\tau_1(\alpha_1-\alpha_2)(\alpha_4-\alpha_2)}{\alpha_4-\alpha_2+(\alpha_1-\alpha_4)\tanh^2\left[\mp\frac{\sqrt{(\alpha_1-\alpha_4)(\alpha_2-\alpha_4)}}{2\sqrt{\frac{6}{\tau_2}}}\left(x-\frac{kt^\alpha}{\alpha}-\eta_0\right)\right]}, \tag{4.25}$$

where $\alpha_3 = \alpha_4$.

Remark 2 If the modulus $l \to 0$, then the solution for the conformable time-fractional Boussinesq equation can be reduced to the solitary wave solution

$$u_5(x,t) = \tau_0 + \tau_0\alpha_3 + \frac{\tau_1(\alpha_1-\alpha_3)(\alpha_4-\alpha_3)}{\alpha_4-\alpha_3+(\alpha_1-\alpha_4)\sin^2\left[\mp\frac{\sqrt{(\alpha_1-\alpha_3)(\alpha_3-\alpha_4)}}{2\sqrt{\frac{6}{\tau_2}}}\left(x-\frac{kt^\alpha}{\alpha}-\eta_0\right)\right]}, \tag{4.26}$$

where $\alpha_2 = \alpha_3$.

Case III $\delta = 1, \theta = 5,$ and $\epsilon = 2$.
If we take $\delta = 1, \theta = 5$, and $\epsilon = 2$ for Eqs. (3.4) and (3.5), then we obtain

$$u(\eta) = \tau_0 + \tau_1\Gamma, \tag{4.27}$$

$$(u'(\eta))^2 = \frac{\tau_1^2(\xi_5\Gamma^5+\xi_4\Gamma^4+\xi_3\Gamma^3+\xi_2\Gamma^2+\xi_1\Gamma+\xi_0)}{\zeta_0+\zeta_1\Gamma+\zeta_2\Gamma^2}, \tag{4.28}$$

where $\xi_5 \neq 0$ and $\zeta_2 \neq 0$. Solving the algebraic equation system (3.9) yields

- *First set of parameters*

$$\begin{aligned}
&\tau_0 = \tau_0, \quad \tau_1 = \tau_1, \quad k = k, \quad \xi_0 = \frac{\zeta_0\left(-\tau_1\zeta_0+2k^2\tau_0\zeta_1-2\tau_0\zeta_1-2\tau_0\tau_1\zeta_0+k^2\tau_1\zeta_0-2\tau_0^2\zeta_1+\tau_1\xi_2\right)}{\tau_1\zeta_2},\\
&\xi_1 = -\frac{2\tau_0\tau_1\zeta_0\zeta_1-k^2\tau_1\zeta_0\zeta_1+\tau_1\zeta_0\zeta_1+2k^2\tau_0\zeta_0\zeta_2-2\tau_0\zeta_0\zeta_2-2\tau_0^2\zeta_0\zeta_2+2\tau_0\zeta_1^2+2\tau_0^2\zeta_1^2-\tau_1\xi_2\zeta_1-2k^2\tau_0\zeta_1^2}{\tau_1\zeta_2},\\
&\xi_2 = \xi_2, \quad \xi_3 = -\frac{-6\tau_0\tau_1\zeta_1+3k^2\tau_1\zeta_1-3\tau_1\zeta_1-2\tau_1^2\zeta_0+6\zeta_2\tau_0k^2-6\zeta_2\tau_0-6\zeta_2\tau_0^2}{3\tau_1},\\
&\xi_4 = \frac{2}{3}\tau_1\zeta_1+2\zeta_2\tau_0-k^2\zeta_2+\zeta_2, \quad \xi_5 = \frac{2}{3}\tau_1\zeta_2, \quad \zeta_0 = \zeta_0, \quad \zeta_1 = \zeta_1, \quad \zeta_2 = \zeta_2.
\end{aligned} \tag{4.29}$$

Substituting these results into Eqs. (3.5) and (3.10), we get

$$\pm(\eta-\eta_0)=\int\frac{\sqrt{\frac{\zeta_0}{\xi_5}+\frac{\zeta_1}{\xi_5}\Gamma+\frac{\zeta_2}{\xi_5}\Gamma^2}\,d\Gamma}{\sqrt{\Gamma^5+\frac{\xi_4}{\xi_5}\Gamma^4+\frac{\xi_3}{\xi_5}\Gamma^3+\frac{\xi_2}{\xi_5}\Gamma^2+\frac{\xi_1}{\xi_5}\Gamma+\frac{\xi_0}{\xi_5}}}$$
$$=\int\frac{\sqrt{-3\tau_1\zeta_2}\,d\Gamma}{\sqrt{(\Gamma-\alpha_1)(\Gamma-\alpha_2)(\Gamma-\alpha_3)}},(\Gamma-\alpha_1)(\Gamma-\alpha_2)(\Gamma-\alpha_3)$$
$$=-2\tau_1^2\zeta_2\Gamma^3+3\tau_1(\zeta_2\Gamma^2-\zeta_0)(k^2-1-2\tau_0)+6\tau_0(\zeta_2\Gamma-\zeta_1)(k^2-1-\tau_0)-3\tau_1\xi_2, \tag{4.30}$$

$$\pm(\eta-\eta_0)=\int\sqrt{\frac{-3\tau_1\zeta_2}{-2\tau_1^2\zeta_2\Gamma^3+3\tau_1(\zeta_2\Gamma^2-\zeta_0)(k^2-1-2\tau_0)+6\tau_0(\zeta_2\Gamma-\zeta_1)(k^2-1-\tau_0)-3\tau_1\xi_2}}\,d\Gamma. \tag{4.31}$$

Integrating (4.31), we obtain the solutions to the Eq. (1.1) as follows:

First solution

$$\pm(\eta-\eta_0)=\frac{2\sqrt{-3\tau_1\zeta_1}}{\sqrt{\alpha_1-\alpha_3}}F(\varphi,l),\quad \alpha_1>\alpha_2>\alpha_3, \tag{4.32}$$

where

$$F(\varphi,l)=\int_0^{\varphi}\frac{d\psi}{\sqrt{1-l^2\sin^2\psi}},\quad \varphi=\arcsin\sqrt{\frac{\Gamma-\alpha_3}{\alpha_2-\alpha_3}},\quad l^2=\frac{\alpha_2-\alpha_3}{\alpha_2-\alpha_3},$$
$$\Gamma=\alpha_3+(\alpha_2-\alpha_3)sn^2\left[\mp\frac{\sqrt{\alpha_1-\alpha_3}}{2\sqrt{-3\tau_1\zeta_2}}(\eta-\eta_0),\frac{\alpha_2-\alpha_3}{\alpha_1-\alpha_3}\right], \tag{4.33}$$

$$u(\eta)=\tau_0+\alpha_3\tau_1+\tau_1(\alpha_2-\alpha_3)sn^2\left[\mp\frac{\sqrt{\alpha_1-\alpha_3}}{2\sqrt{-3\tau_1\zeta_2}}(\eta-\eta_0),\frac{\alpha_2-\alpha_3}{\alpha_1-\alpha_3}\right]. \tag{4.34}$$

Therefore, the solution for the conformable time-fractional Boussinesq equation will be as

$$u_6(x,t)=\tau_0+\alpha_3\tau_1+\tau_1(\alpha_2-\alpha_3)sn^2\left[\mp\frac{\sqrt{\alpha_1-\alpha_3}}{2\sqrt{-3\tau_1\zeta_2}}\left(x-\frac{kt^{\alpha}}{\alpha}-\eta_0\right),\frac{\alpha_2-\alpha_3}{\alpha_1-\alpha_3}\right]. \tag{4.35}$$

Remark 1 If the modulus $l\to1$, then the solution for the conformable time-fractional Boussinesq equation can be reduced to the solitary wave solution

$$u_7(x,t)=\tau_0+\alpha_3\tau_1+\tau_1(\alpha_2-\alpha_3)\tanh^2\left[\mp\frac{\sqrt{\alpha_2-\alpha_3}}{2\sqrt{-3\tau_1\zeta_2}}\left(x-\frac{kt^{\alpha}}{\alpha}-\eta_0\right)\right], \tag{4.36}$$

where $\alpha_1=\alpha_2$, and $v=\frac{k}{\alpha}$ represents the velocity of the dark soliton.

Case IV $\delta=2,\theta=5,$ and $\epsilon=1$.

If we take $\delta=2,\theta=5,$ and $\epsilon=1$ for Eqs. (3.4) and (3.5), then we obtain

$$u(\eta)=\tau_0+\tau_1\Gamma+\tau_2\Gamma^2, \tag{4.37}$$

$$(u'(\eta))^2=\frac{(\tau_1+2\tau_2\Gamma)^2(\xi_5\Gamma^5+\xi_4\Gamma^4+\xi_3\Gamma^3+\xi_2\Gamma^2+\xi_1\Gamma+\xi_0)}{\zeta_0+\zeta_1\Gamma}, \tag{4.38}$$

where $\xi_5 \neq 0$ and $\zeta_1 \neq 0$. Solving the algebraic equation system (3.9) yields

- *First set of parameters*

$$\begin{aligned}
\tau_0 =&\tau_0, \quad \tau_1 = \tau_1, \quad \tau_2 = \tau_2, \quad k = k,\\
\xi_0 =& -\frac{\zeta_0(12\tau_0\tau_2(-4\tau_2 - 4\tau_2\tau_0 + 4\tau_2k^2 + \tau_1^2) - 6\tau_1^2\tau_2(k^2-1) - \tau_1^4)}{96\tau_2^3},\\
\xi_1 =& -\frac{4\tau_1\tau_2\zeta_0(6\tau_2k^2 + \tau_1^2 - 12\tau_2\tau_0 - 6\tau_2) + \zeta_1(-6\tau_1^2k^2\tau_2 - \tau_1^4 + 12\tau_1^2\tau_0\tau_2 + 6\tau_1^2\tau_2 - 48\tau_0\tau_2^2 + 48k^2\tau_0\tau_2^2 - 48\tau_0^2\tau_2^2)}{96\tau_2^3},\\
\xi_2 =& -\frac{3\zeta_0\tau_2(2\tau_2k^2 - 4\tau_2\tau_0 - 2\tau_2 - \tau_1^2) + \tau_1\zeta_1(6\tau_2k^2 + \tau_1^2 - 12\tau_2\tau_0 - 6\tau_2)}{24\tau_2^2},\\
\xi_3 =& \frac{8\tau_1\tau_2\zeta_0 - 3\zeta_1(2\tau_2k^2 - 4\tau_2\tau_0 - 2\tau_2 - \tau_1^2)}{24\tau_2},\\
\xi_4 =& \frac{1}{6}\tau_2\zeta_0 + \frac{1}{3}\zeta_1\tau_1, \quad \xi_5 = \frac{1}{6}\tau_2\zeta_1, \quad \zeta_0 = \zeta_0, \quad \zeta_1 = \zeta_1.
\end{aligned} \tag{4.39}$$

Substituting these results into Eqs. (3.5) and (3.10), we get

$$\begin{aligned}
\pm(\eta - \eta_0) =& \int \frac{\sqrt{\frac{\zeta_0}{\xi_5} + \frac{\zeta_1}{\xi_5}\Gamma}\, d\Gamma}{\sqrt{\Gamma^5 + \frac{\xi_4}{\xi_5}\Gamma^4 + \frac{\xi_3}{\xi_5}\Gamma^3 + \frac{\xi_2}{\xi_5}\Gamma^2 + \frac{\xi_1}{\xi_5}\Gamma + \frac{\xi_0}{\xi_5}}}\\
=& \int \frac{\sqrt{-96\tau_2^3}\, d\Gamma}{\sqrt{(\Gamma-\alpha_1)(\Gamma-\alpha_2)(\Gamma-\alpha_3)(\Gamma-\alpha_4)}}, (\Gamma-\alpha_1)(\Gamma-\alpha_2)(\Gamma-\alpha_3)(\Gamma\\
&-\alpha_4)\\
=& \Omega_4\Gamma^4 + \Omega_3\Gamma^3 + \Omega_2\Gamma^2 + \Omega_1\Gamma + \Omega_0,
\end{aligned} \tag{4.40}$$

where

$$\begin{aligned}
\Omega_4 =& -16\tau_2^4, \quad \Omega_3 = -32\tau_2^3\tau_1, \quad \Omega_2 = 12\tau_2^2(2\tau_2k^2 - 4\tau_0\tau_2 - 2\tau_2 - \tau_1^2),\\
\Omega_1 =& 4\tau_1\tau_2(\tau_1^2 - 6\tau_2 + 6\tau_2k^2 - 12\tau_0\tau_2),\\
\Omega_0 =& -6\tau_1^2k^2\tau_2 + 12\tau_1^2\tau_0\tau_2 + 6\tau_1^2\tau_2 - \tau_1^4 - 48\tau_0\tau_2^2 - 48\tau_0^2\tau_2^2 + 48k^2\tau_0\tau_2^2,\\
\pm(\eta - \eta_0) =& \int \sqrt{\frac{-96\tau_2^3}{\Omega_4\Gamma^4 + \Omega_3\Gamma^3 + \Omega_2\Gamma^2 + \Omega_1\Gamma + \Omega_0}}\, d\Gamma.
\end{aligned} \tag{4.41}$$

Integrating (4.41), we obtain the solutions to the Eq. (1.1) as follows:

First solution

$$\pm(\eta - \eta_0) = \frac{2\sqrt{-96\tau_2^3}}{\sqrt{(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_4)}}F(\varphi, l), \quad \alpha_1 > \alpha_2 > \alpha_3 > \alpha_4, \tag{4.42}$$

where

$$F(\varphi,l)=\int_0^{\varphi}\frac{d\psi}{\sqrt{1-l^2\sin^2\psi}},\quad \varphi=\arcsin\sqrt{\frac{(\alpha_2-\alpha_4)(\Gamma-\alpha_1)}{(\alpha_1-\alpha_4)(\Gamma-\alpha_2)}},\quad l^2=\frac{(\alpha_2-\alpha_3)(\alpha_1-\alpha_4)}{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)},$$

$$\Rightarrow \Gamma=\alpha_2+\frac{(\alpha_1-\alpha_2)(\alpha_4-\alpha_2)}{\alpha_4-\alpha_2+(\alpha_1-\alpha_4)sn^2\left[\mp\frac{\sqrt{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}}{2\sqrt{-96\tau_2^3}}(\eta-\eta_0),\frac{(\alpha_2-\alpha_3)(\alpha_1-\alpha_4)}{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}\right]},\tag{4.43}$$

$$\Gamma^4+\frac{2\tau_1}{\tau_2}\Gamma^3-\frac{3(-4\tau_0\tau_2+2k^2\tau_2-\tau_1^2-2\tau_2)}{4\tau_2^2}\Gamma^2+\frac{3\tau_1(6k^2\tau_2-12\tau_0\tau_2+\tau_1^2-6\tau_2)}{\tau_2^3}\Gamma-$$
$$\frac{6\tau_1^2\tau_2(1-k^2+2\tau_0)-\tau_1^4-48\tau_0\tau_2^2(1+\tau_0-k^2)}{16\tau_2^4}=(\Gamma-\alpha_1)(\Gamma-\alpha_2)(\Gamma-\alpha_3)(\Gamma-\alpha_4)\Rightarrow\tag{4.44}$$

$$\alpha_j=-\frac{\tau_1}{2\tau_2}\pm\frac{\sqrt{6\tau_1^2+12k^2\tau_2-24\tau_0\tau_2-12\tau_2\pm2\sqrt{\Omega}}}{4\tau_2},\quad j=1,2,3,4,\tag{4.45}$$

where

$$\Omega=3\left(4\tau_2\tau_0-\tau_1^2-2\tau_2+2\tau_2k^2\right)\left(-4\tau_2\tau_0+\tau_1^2-6\tau_2+6\tau_2k^2\right).$$

Therefore, the solution for the conformable time-fractional Boussinesq equation will be as

$$u_8(x,t)=\tau_0+\tau_0\alpha_2+\frac{\tau_1(\alpha_1-\alpha_2)(\alpha_4-\alpha_2)}{\alpha_4-\alpha_2+(\alpha_1-\alpha_4)sn^2\left[\mp\frac{\sqrt{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}}{2\sqrt{-96\tau_2^3}}\left(x-\frac{kt^{\alpha}}{\alpha}-\eta_0\right),\frac{(\alpha_2-\alpha_3)(\alpha_1-\alpha_4)}{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}\right]}.\tag{4.46}$$

Remark 1 If the modulus $l\to1$, then the solution for the conformable time-fractional Boussinesq equation can be reduced to the solitary wave solution

$$u_9(x,t)=\tau_0+\tau_0\alpha_2+\frac{\tau_1(\alpha_1-\alpha_2)(\alpha_4-\alpha_2)}{\alpha_4-\alpha_2+(\alpha_1-\alpha_4)\tanh^2\left[\mp\frac{\sqrt{(\alpha_1-\alpha_4)(\alpha_2-\alpha_4)}}{2\sqrt{-96\tau_2^3}}\left(x-\frac{kt^{\alpha}}{\alpha}-\eta_0\right)\right]},\tag{4.47}$$

where $\alpha_3=\alpha_4$.

Remark 2 If the modulus $l\to0$, then the solution for the conformable time-fractional Boussinesq equation can be reduced to the solitary wave solution

$$u_{10}(x,t)=\tau_0+\tau_0\alpha_3+\frac{\tau_1(\alpha_1-\alpha_3)(\alpha_4-\alpha_3)}{\alpha_4-\alpha_3+(\alpha_1-\alpha_4)\sin^2\left[\mp\frac{\sqrt{(\alpha_1-\alpha_3)(\alpha_3-\alpha_4)}}{2\sqrt{-96\tau_2^3}}\left(x-\frac{kt^{\alpha}}{\alpha}-\eta_0\right)\right]},\tag{4.48}$$

where $\alpha_2=\alpha_3$.

4.2 EEM

In this subsection, consider EEM for Eq. (4.2) which is given in before subsection. By balancing u^2 and u'' in Eq. (4.2) we obtain the balance number $m = 2$, and the exact solution becomes,

$$u(\eta) = \sum_{j=-2}^{2} \xi_j F^j(\eta), \quad F(\eta) = \exp(-\Omega(\eta)). \tag{4.49}$$

Substituting (4.49) into Eq. (4.2) and comparing the terms, we will reach a system of nonlinear algebraic equations, and by solving system of the nonlinear equations, yields

Case 1

$$\begin{aligned} &k = k, \quad \lambda = 0, \quad \mu = \frac{1}{16}(k^2 - 1), \quad \xi_{-2} = \frac{3}{128}(k^2 - 1)^2, \\ &\xi_{-1} = 0, \quad \xi_0 = \frac{3}{4}(k^2 - 1), \quad \xi_1 = 0, \quad \xi_2 = 6. \end{aligned} \tag{4.50}$$

By using (3.15) and (4.50), the exact solution for the conformable time-fractional Boussinesq equation becomes,

$$\begin{aligned} u_1(x,t) = &-\frac{3}{8}(k^2 - 1)\tanh^2\left(\frac{\sqrt{1-k^2}}{4}\left(x - \frac{kt^\alpha}{\alpha} + E\right)\right) + \frac{3}{4}(k^2 - 1) \\ &-\frac{3}{8}(k^2 - 1)\coth^2\left(\frac{\sqrt{1-k^2}}{4}\left(x - \frac{kt^\alpha}{\alpha} + E\right)\right), \end{aligned} \tag{4.51}$$

when $-1 < k < 1$.

By using (3.16) and (4.50), the exact solution for the conformable time-fractional Boussinesq equation yields,

$$\begin{aligned} u_2(x,t) = &\frac{3}{8}(k^2 - 1)\tan^2\left(\frac{\sqrt{k^2-1}}{4}\left(x - \frac{kt^\alpha}{\alpha} + E\right)\right) + \frac{3}{4}(k^2 - 1) \\ &+\frac{3}{8}(k^2 - 1)\cot^2\left(\frac{\sqrt{k^2-1}}{4}\left(x - \frac{kt^\alpha}{\alpha} + E\right)\right), \end{aligned} \tag{4.52}$$

when $k > 1$ or $k < -1$.

By using (3.19) and (4.50), the exact solution for the conformable time-fractional Boussinesq equation becomes,

$$u_3(x,t) = \frac{6}{\left(x - \frac{t^\alpha}{\alpha} + E\right)^2}, \tag{4.53}$$

when $k = 1$.

Case 2

$$\begin{aligned} &k = k, \quad \lambda = \lambda, \quad \mu = \frac{1}{4}(\lambda^2 - k^2 + 1), \quad \xi_{-2} = \frac{3}{8}(\lambda^2 - k^2 + 1)^2, \\ &\xi_{-1} = \frac{3}{2}\lambda(\lambda^2 - k^2 + 1), \quad \xi_0 = \frac{1}{2}(3\lambda^2 - k^2 + 1), \quad \xi_1 = 0, \quad \xi_2 = 0. \end{aligned} \tag{4.54}$$

By using (3.15) and (4.54), the exact solution for the conformable time-fractional Boussinesq equation becomes,

$$\begin{aligned} u_4(x,t) =& \frac{3}{2}\left[\sqrt{k^2-1}\tanh\left(\frac{\sqrt{k^2-1}}{2}\left(x-\frac{kt^\alpha}{\alpha}+E\right)\right)-\lambda\right]^2+\frac{1}{2}(3\lambda^2-k^2+1)\\ &+3\lambda\left[\sqrt{k^2-1}\tanh\left(\frac{\sqrt{k^2-1}}{2}\left(x-\frac{kt^\alpha}{\alpha}+E\right)\right)-\lambda\right], \end{aligned} \tag{4.55}$$

when $k>1$ or $k<-1$.

By using (3.16) and (4.54), the exact solution for the conformable time-fractional Boussinesq equation yields,

$$\begin{aligned} u_5(x,t) =& \frac{3}{2}\left[\sqrt{1-k^2}\tan\left(\frac{\sqrt{1-k^2}}{2}\left(x-\frac{kt^\alpha}{\alpha}+E\right)\right)-\lambda\right]^2+\frac{1}{2}(3\lambda^2-k^2+1)\\ &+3\lambda\left[\sqrt{1-k^2}\tan\left(\frac{\sqrt{1-k^2}}{2}\left(x-\frac{kt^\alpha}{\alpha}+E\right)\right)-\lambda\right], \end{aligned} \tag{4.56}$$

when $-1<k<1$.

By using (3.18) and (4.54), the exact solution for the conformable time-fractional Boussinesq equation becomes,

$$u_6(x,t) = \frac{3}{8}\lambda^4\left[\frac{4-2\lambda\left(x-\frac{t^\alpha}{\alpha}+E\right)}{\lambda^2\left(x-\frac{t^\alpha}{\alpha}+E\right)}\right]^2+\frac{3}{2}\lambda^3\left[\frac{4-2\lambda\left(x-\frac{t^\alpha}{\alpha}+E\right)}{\lambda^2\left(x-\frac{t^\alpha}{\alpha}+E\right)}\right]+\frac{3}{2}\lambda^2, \tag{4.57}$$

when $k=1$.

4.3 ITEM

By applying the ITEM for Eq. (4.2) and by balancing u^2 and u'' in Eq. (4.2) we obtain the balance number $m=2$, and the exact solution yields

$$u(\eta) = \sum_{j=0}^{2} A_j \tan^j(\phi/2) + \sum_{j=1}^{2} B_j \cot^j(\phi/2). \tag{4.58}$$

Substituting (4.58) into Eq. (4.2) and comparing the terms, we will reach a system of nonlinear algebraic equations, and by solving system of the nonlinear equations, yields

Case 1

$$k=k,\quad a=a,\quad b=b,\quad c=c,\quad A_0=0,\quad A_1=-B_1,\quad A_2=-B_2,\quad B_1=B_1,\quad B_2=B_2. \tag{4.59}$$

By using (3.23) and (4.59), the exact solution for the conformable time-fractional Boussinesq equation becomes,

$$u_1(x,t)=-B_1\left[\frac{a}{b-c}-\frac{\sqrt{-\Delta}}{b-c}\tan\left(\frac{\sqrt{-\Delta}}{2}\left(x-\frac{kt^\alpha}{\alpha}\right)\right)\right]-B_2\left[\frac{a}{b-c}-\frac{\sqrt{-\Delta}}{b-c}\tan\left(\frac{\sqrt{-\Delta}}{2}\left(x-\frac{kt^\alpha}{\alpha}\right)\right)\right]^2+B_1\left[\frac{a}{b-c}-\frac{\sqrt{-\Delta}}{b-c}\tan\left(\frac{\sqrt{-\Delta}}{2}\left(x-\frac{kt^\alpha}{\alpha}\right)\right)\right]^{-1}+B_2\left[\frac{a}{b-c}-\frac{\sqrt{-\Delta}}{b-c}\tan\left(\frac{\sqrt{-\Delta}}{2}\left(x-\frac{kt^\alpha}{\alpha}\right)\right)\right]^{-2}, \tag{4.60}$$

when $\Delta=a^2+b^2-c^2<0$ (*Family 1*).

$$u_2(x,t)=-B_1\left[\frac{a}{b-c}+\frac{\sqrt{\Delta}}{b-c}\tanh\left(\frac{\sqrt{\Delta}}{2}\left(x-\frac{kt^\alpha}{\alpha}\right)\right)\right]-B_2\left[\frac{a}{b-c}+\frac{\sqrt{\Delta}}{b-c}\tanh\left(\frac{\sqrt{\Delta}}{2}\left(x-\frac{kt^\alpha}{\alpha}\right)\right)\right]^2+B_1\left[\frac{a}{b-c}+\frac{\sqrt{\Delta}}{b-c}\tanh\left(\frac{\sqrt{\Delta}}{2}\left(x-\frac{kt^\alpha}{\alpha}\right)\right)\right]^{-1}+B_2\left[\frac{a}{b-c}+\frac{\sqrt{\Delta}}{b-c}\tanh\left(\frac{\sqrt{\Delta}}{2}\left(x-\frac{kt^\alpha}{\alpha}\right)\right)\right]^{-2}, \tag{4.61}$$

when $\Delta=a^2+b^2-c^2>0$ (*Family 2*).

$$u_3(x,t)=-B_1\left[\frac{e^{\left(b\left(x-\frac{kt^\alpha}{\alpha}\right)\right)}+b-a}{e^{\left(b\left(x-\frac{kt^\alpha}{\alpha}\right)\right)}-b-a}\right]-B_2\left[\frac{e^{\left(b\left(x-\frac{kt^\alpha}{\alpha}\right)\right)}+b-a}{e^{\left(b\left(x-\frac{kt^\alpha}{\alpha}\right)\right)}-b-a}\right]^2+B_1\left[\frac{e^{\left(b\left(x-\frac{kt^\alpha}{\alpha}\right)\right)}+b-a}{e^{\left(b\left(x-\frac{kt^\alpha}{\alpha}\right)\right)}-b-a}\right]^{-1}+B_2\left[\frac{e^{\left(b\left(x-\frac{kt^\alpha}{\alpha}\right)\right)}+b-a}{e^{\left(b\left(x-\frac{kt^\alpha}{\alpha}\right)\right)}-b-a}\right]^{-2}, \tag{4.62}$$

when $c=-a$ (*Family 10*).

$$u_4(x,t)=-B_1\left(c\left(x-\frac{kt^\alpha}{\alpha}\right)\right)-B_2\left(c\left(x-\frac{kt^\alpha}{\alpha}\right)\right)^2+B_1\left(c\left(x-\frac{kt^\alpha}{\alpha}\right)\right)^{-1}+B_2\left(c\left(x-\frac{kt^\alpha}{\alpha}\right)\right)^{-2}, \tag{4.63}$$

when $a=0$ and $b=c$ (*Family 13*).

Case 2

$$k=k,\quad a=a,\quad b=b,\quad c=c,\quad A_0=k^2-1,\quad A_1=-B_1,\quad A_2=-B_2,\quad B_1=B_1,\quad B_2=B_2. \tag{4.64}$$

By using (3.23) and (4.64), the exact solution for the conformable time-fractional Boussinesq equation becomes,

$$\begin{aligned}u_5(x,t) =&k^2-1-B_1\left[\frac{a}{b-c}-\frac{\sqrt{-\Delta}}{b-c}\tan\left(\frac{\sqrt{-\Delta}}{2}\left(x-\frac{kt^\alpha}{\alpha}\right)\right)\right]\\&-B_2\left[\frac{a}{b-c}-\frac{\sqrt{-\Delta}}{b-c}\tan\left(\frac{\sqrt{-\Delta}}{2}\left(x-\frac{kt^\alpha}{\alpha}\right)\right)\right]^2\\&+B_1\left[\frac{a}{b-c}-\frac{\sqrt{-\Delta}}{b-c}\tan\left(\frac{\sqrt{-\Delta}}{2}\left(x-\frac{kt^\alpha}{\alpha}\right)\right)\right]^{-1}\\&+B_2\left[\frac{a}{b-c}-\frac{\sqrt{-\Delta}}{b-c}\tan\left(\frac{\sqrt{-\Delta}}{2}\left(x-\frac{kt^\alpha}{\alpha}\right)\right)\right]^{-2},\end{aligned}\tag{4.65}$$

when $\Delta = a^2+b^2-c^2<0$ (*Family 1*).

$$\begin{aligned}u_6(x,t) =&k^2-1-B_1\left[\frac{a}{b-c}+\frac{\sqrt{\Delta}}{b-c}\tanh\left(\frac{\sqrt{\Delta}}{2}\left(x-\frac{kt^\alpha}{\alpha}\right)\right)\right]\\&-B_2\left[\frac{a}{b-c}+\frac{\sqrt{\Delta}}{b-c}\tanh\left(\frac{\sqrt{\Delta}}{2}\left(x-\frac{kt^\alpha}{\alpha}\right)\right)\right]^2\\&+B_1\left[\frac{a}{b-c}+\frac{\sqrt{\Delta}}{b-c}\tanh\left(\frac{\sqrt{\Delta}}{2}\left(x-\frac{kt^\alpha}{\alpha}\right)\right)\right]^{-1}\\&+B_2\left[\frac{a}{b-c}+\frac{\sqrt{\Delta}}{b-c}\tanh\left(\frac{\sqrt{\Delta}}{2}\left(x-\frac{kt^\alpha}{\alpha}\right)\right)\right]^{-2},\end{aligned}\tag{4.66}$$

when $\Delta = a^2+b^2-c^2>0$ (*Family 2*).

$$\begin{aligned}u_7(x,t) = k^2-1-B_1\left[\frac{e^{\left(b\left(x-\frac{kt^\alpha}{\alpha}\right)\right)}+b-a}{e^{\left(b\left(x-\frac{kt^\alpha}{\alpha}\right)\right)}-b-a}\right]-B_2\left[\frac{e^{\left(b\left(x-\frac{kt^\alpha}{\alpha}\right)\right)}+b-a}{e^{\left(b\left(x-\frac{kt^\alpha}{\alpha}\right)\right)}-b-a}\right]^2\\+B_1\left[\frac{e^{\left(b\left(x-\frac{kt^\alpha}{\alpha}\right)\right)}+b-a}{e^{\left(b\left(x-\frac{kt^\alpha}{\alpha}\right)\right)}-b-a}\right]^{-1}+B_2\left[\frac{e^{\left(b\left(x-\frac{kt^\alpha}{\alpha}\right)\right)}+b-a}{e^{\left(b\left(x-\frac{kt^\alpha}{\alpha}\right)\right)}-b-a}\right]^{-2},\end{aligned}\tag{4.67}$$

when $c=-a$ (*Family 10*).

$$\begin{aligned}u_8(x,t) = k^2-1-B_1\left(c\left(x-\frac{kt^\alpha}{\alpha}\right)\right)-B_2\left(c\left(x-\frac{kt^\alpha}{\alpha}\right)\right)^2+B_1\left(c\left(x-\frac{kt^\alpha}{\alpha}\right)\right)^{-1}\\+B_2\left(c\left(x-\frac{kt^\alpha}{\alpha}\right)\right)^{-2},\end{aligned}\tag{4.68}$$

when $a=0$ and $b=c$ (*Family 13*).

5 Application of coupled conformable time-fractional Boussinesq equation

In this section, we present three methods based on integration methods for solving the coupled conformable time-fractional Boussinesq equation.

5.1 ETEM

In this subsection, consider ETEM for Eq. (1.2) which is given as

$$\begin{aligned} -\ ku' + v' &= 0, \\ -\ kv' + \mu(u^2)' - \lambda u''' &= 0, \end{aligned} \tag{5.1}$$

where the prime indicates the derivation with respect to η. By integrating Eq. (5.1) once and setting the constant of integration equal to zero, results in

$$v = ku, \tag{5.2}$$

$$-kv + \mu u^2 - \lambda u'' = 0. \tag{5.3}$$

By inserting (5.2) into (5.3), yields

$$k^2 u - \mu u^2 + \lambda u'' = 0. \tag{5.4}$$

Once again, we can determine values of δ, θ, and ϵ, by balancing u^2 and u'' in Eq. (5.4) as follows:

$$\delta = \theta - \epsilon - 2. \tag{5.5}$$

For different values of δ, θ, and ϵ, we have the following cases:

Case I $\delta = 2, \theta = 4, \text{and } \epsilon = 0.$

If we take $\delta = 2, \theta = 4$, and $\epsilon = 0$ for Eqs. (3.4) and (3.5), then we obtain

$$u(\eta) = \tau_0 + \tau_1 \Gamma + \tau_2 \Gamma^2, \tag{5.6}$$

$$(u'(\eta))^2 = \frac{(2\tau_2\Gamma + \tau_1)^2 \left(\xi_4\Gamma^4 + \xi_3\Gamma^3 + \xi_2\Gamma^2 + \xi_1\Gamma + \xi_0\right)}{\zeta_0}, \tag{5.7}$$

where $\xi_4 \neq 0$ and $\zeta_0 \neq 0$. Solving the algebraic equation system (3.9) yields

- *First set of parameters*

$$\begin{aligned} \xi_0 &= -\frac{\zeta_0\left(12\tau_1^2\mu\tau_0\tau_2 - 6\tau_1^2k^2\tau_2 - \tau_1^4\mu + 48k^2\tau_0\tau_2^2 - 48\mu\tau_0^2\tau_2^2\right)}{96\lambda\tau_2^3}, \\ \xi_1 &= -\frac{\zeta_0\tau_1\left(-12\mu\tau_0\tau_2 + 6k^2\tau_2 + \mu\tau_1^2\right)}{24\lambda\tau_2^2}, \\ \xi_2 &= -\frac{\zeta_0\left(2k^2\tau_2 - \mu\tau_1^2 - 4\mu\tau_0\tau_2\right)}{8\lambda\tau_2}, \\ \xi_3 &= \frac{1}{3\lambda}\mu\tau_1\zeta_0, \quad \xi_4 = \frac{1}{6\lambda}\mu\tau_2\zeta_0, \quad \zeta_0 = \zeta_0, \quad k = k, \quad \tau_0 = \tau_0, \quad \tau_1 = \tau_1, \quad \tau_2 = \tau_2. \end{aligned} \tag{5.8}$$

Substituting these results into Eqs. (3.5) and (3.10), we get

$$\pm(\eta-\eta_0)=\int\frac{\sqrt{\frac{\zeta_0}{\xi_4}}\,d\Gamma}{\sqrt{\Gamma^4+\frac{\xi_3}{\xi_4}\Gamma^3+\frac{\xi_2}{\xi_4}\Gamma^2+\frac{\xi_1}{\xi_4}\Gamma+\frac{\xi_0}{\xi_4}}}. \tag{5.9}$$

Integrating Eq. (5.9), we obtain the solutions to the Eq. (1.2) as follows:

First solution

$$\pm(\eta-\eta_0)=\frac{2\sqrt{\frac{\zeta_0}{\xi_4}}}{\sqrt{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}}F(\varphi,l),\quad \alpha_1>\alpha_2>\alpha_3>\alpha_4, \tag{5.10}$$

where

$$F(\varphi,l)=\int_0^{\varphi}\frac{d\psi}{\sqrt{1-l^2\sin^2\psi}},\quad \varphi=\arcsin\sqrt{\frac{(\alpha_2-\alpha_4)(\Gamma-\alpha_1)}{(\alpha_1-\alpha_4)(\Gamma-\alpha_2)}},\quad l^2=\frac{(\alpha_2-\alpha_3)(\alpha_1-\alpha_4)}{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)},$$
$$\Rightarrow\Gamma=\alpha_2+\frac{(\alpha_1-\alpha_2)(\alpha_4-\alpha_2)}{\alpha_4-\alpha_2+(\alpha_1-\alpha_4)sn^2\left[\mp\frac{\sqrt{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}}{2\Pi}(\eta-\eta_0),\frac{(\alpha_2-\alpha_3)(\alpha_1-\alpha_4)}{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}\right]}, \tag{5.11}$$

$$\Gamma^4+\frac{2\tau_1}{\tau_2}\Gamma^3-\frac{3\left(-4\mu\tau_0\tau_2+2k^2\tau_2-\mu\tau_1^2\right)}{4\mu\tau_2^2}\Gamma^2-\frac{\tau_1\left(6k^2\tau_2-12\mu\tau_0\tau_2+\mu\tau_1^2\right)}{4\mu\tau_2^3}\Gamma-$$
$$\frac{12\tau_1^2\mu\tau_0\tau_2-6\tau_1^2k^2\tau_2-\tau_1^4\mu+48k^2\tau_0\tau_2^2-48\mu\tau_0^2\tau_2^2}{16\mu\tau_2^4}=(\Gamma-\alpha_1)(\Gamma-\alpha_2)(\Gamma-\alpha_3)(\Gamma-\alpha_4)\Rightarrow \tag{5.12}$$

$$\alpha_j=-\frac{\tau_1}{2\tau_2}\pm\frac{\sqrt{6\mu^2\tau_1^2+12\mu k^2\tau_2-24\mu^2\tau_0\tau_2\pm2\sqrt{\Omega}}}{4\mu\tau_2},\quad j=1,2,3,4, \tag{5.13}$$

where

$$\Omega=3\left(4\mu\tau_2\tau_0-\mu\tau_1^2+2\tau_2k^2\right)\left(-4\mu\tau_2\tau_0+\mu\tau_1^2+6\tau_2k^2\right).$$

Therefore, the solution for the coupled conformable time-fractional Boussinesq equation will be as

$$u_1(x,t)=\tau_0+\tau_0\alpha_2+\frac{\tau_1(\alpha_1-\alpha_2)(\alpha_4-\alpha_2)}{\alpha_4-\alpha_2+(\alpha_1-\alpha_4)sn^2\left[\mp\frac{\sqrt{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}}{2\sqrt{\frac{6}{\tau_2}}}\left(x-\frac{kt^{\alpha}}{\alpha}-\eta_0\right),\frac{(\alpha_2-\alpha_3)(\alpha_1-\alpha_4)}{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}\right]}, \tag{5.14}$$

$$v_1(x,t)=k\tau_0+k\tau_0\alpha_2+\frac{k\tau_1(\alpha_1-\alpha_2)(\alpha_4-\alpha_2)}{\alpha_4-\alpha_2+(\alpha_1-\alpha_4)sn^2\left[\mp\frac{\sqrt{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}}{2\sqrt{\frac{6}{\tau_2}}}\left(x-\frac{kt^{\alpha}}{\alpha}-\eta_0\right),\frac{(\alpha_2-\alpha_3)(\alpha_1-\alpha_4)}{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}\right]}. \tag{5.15}$$

Remark 1 If the modulus $l \to 1$, then the solution for the coupled conformable time-fractional Boussinesq equation can be reduced to the solitary wave solution

$$u_2(x,t) = \tau_0 + \tau_0\alpha_2 + \frac{\tau_1(\alpha_1-\alpha_2)(\alpha_4-\alpha_2)}{\alpha_4-\alpha_2+(\alpha_1-\alpha_4)\tanh^2\left[\mp\frac{\sqrt{(\alpha_1-\alpha_4)(\alpha_2-\alpha_4)}}{2\sqrt{\frac{6}{\tau_2}}}\left(x-\frac{kt^\alpha}{\alpha}-\eta_0\right)\right]}, \tag{5.16}$$

$$v_2(x,t) = k\tau_0 + k\tau_0\alpha_2 + \frac{k\tau_1(\alpha_1-\alpha_2)(\alpha_4-\alpha_2)}{\alpha_4-\alpha_2+(\alpha_1-\alpha_4)\tanh^2\left[\mp\frac{\sqrt{(\alpha_1-\alpha_4)(\alpha_2-\alpha_4)}}{2\sqrt{\frac{6}{\tau_2}}}\left(x-\frac{kt^\alpha}{\alpha}-\eta_0\right)\right]}, \tag{5.17}$$

where $\alpha_3 = \alpha_4$.

Remark 2 If the modulus $l \to 0$, then the solution for the coupled conformable time-fractional Boussinesq equation can be reduced to the solitary wave solution

$$u_3(x,t) = \tau_0 + \tau_0\alpha_3 + \frac{\tau_1(\alpha_1-\alpha_3)(\alpha_4-\alpha_3)}{\alpha_4-\alpha_3+(\alpha_1-\alpha_4)\sin^2\left[\mp\frac{\sqrt{(\alpha_1-\alpha_3)(\alpha_3-\alpha_4)}}{2\sqrt{\frac{6}{\tau_2}}}\left(x-\frac{kt^\alpha}{\alpha}-\eta_0\right)\right]}, \tag{5.18}$$

$$v_3(x,t) = k\tau_0 + k\tau_0\alpha_3 + \frac{k\tau_1(\alpha_1-\alpha_3)(\alpha_4-\alpha_3)}{\alpha_4-\alpha_3+(\alpha_1-\alpha_4)\sin^2\left[\mp\frac{\sqrt{(\alpha_1-\alpha_3)(\alpha_3-\alpha_4)}}{2\sqrt{\frac{6}{\tau_2}}}\left(x-\frac{kt^\alpha}{\alpha}-\eta_0\right)\right]}, \tag{5.19}$$

where $\alpha_2 = \alpha_3$.

Case II $\delta = 2, \theta = 5,$ and $\epsilon = 1$.

If we take $\delta = 2, \theta = 5,$ and $\epsilon = 1$ for Eqs. (3.4) and (3.5), then we obtain

$$u(\eta) = \tau_0 + \tau_1\Gamma + \tau_2\Gamma^2, \tag{5.20}$$

$$(u'(\eta))^2 = \frac{(\tau_1+2\tau_2\Gamma)^2\left(\xi_5\Gamma^5+\xi_4\Gamma^4+\xi_3\Gamma^3+\xi_2\Gamma^2+\xi_1\Gamma+\xi_0\right)}{\zeta_0+\zeta_1\Gamma}, \tag{5.21}$$

where $\xi_5 \neq 0$ and $\zeta_1 \neq 0$. Solving the algebraic equation system (3.9) yields

- *First set of parameters*

$$\begin{aligned}
\tau_0 =&\tau_0, \quad \tau_1=\tau_1, \quad \tau_2=\tau_2, \quad k=k,\\
\xi_0 =& -\frac{\zeta_0\left(12\tau_1^2\mu\tau_0\tau_2-\tau_1^4\mu-6\tau_1^2k^2\tau_2+48k^2\tau_0\tau_2^2-48\mu\tau_0^2\tau_2^2\right)}{96\lambda\tau_2^3},\\
\xi_1 =& -\frac{4\tau_1\tau_2\zeta_0\left(6\tau_2k^2+\mu\tau_1^2-12\mu\tau_2\tau_0\right)+\zeta_1\left(-6\tau_1^2k^2\tau_2-\mu\tau_1^4+12\mu\tau_1^2\tau_0\tau_2-48\mu\tau_0^2\tau_2^2+48k^2\tau_0\tau_2^2\right)}{96\lambda\tau_2^3},\\
\xi_2 =& -\frac{3\zeta_0\tau_2\left(2\tau_2k^2-4\mu\tau_2\tau_0-\mu\tau_1^2\right)+\tau_1\zeta_1\left(6\tau_2k^2+\mu\tau_1^2-12\mu\tau_2\tau_0\right)}{24\lambda\tau_2^2},\\
\xi_3 =&\frac{8\mu\tau_1\tau_2\zeta_0-3\zeta_1\left(2\tau_2k^2-4\mu\tau_2\tau_0-\mu\tau_1^2\right)}{24\lambda\tau_2},\\
\xi_4 =&\frac{1}{6\lambda}\mu(2\tau_1\zeta_1+\tau_2\zeta_0), \quad \xi_5=\frac{1}{6\lambda}\mu\tau_2\zeta_1, \quad \zeta_0=\zeta_0, \quad \zeta_1=\zeta_1.
\end{aligned} \tag{5.22}$$

Substituting these results into Eqs. (3.5) and (3.10), we get

$$\begin{aligned}
\pm(\eta-\eta_0) =& \int\frac{\sqrt{\frac{\zeta_0}{\xi_5}+\frac{\zeta_1}{\xi_5}\Gamma}\,d\Gamma}{\sqrt{\Gamma^5+\frac{\xi_4}{\xi_5}\Gamma^4+\frac{\xi_3}{\xi_5}\Gamma^3+\frac{\xi_2}{\xi_5}\Gamma^2+\frac{\xi_1}{\xi_5}\Gamma+\frac{\xi_0}{\xi_5}}}\\
=& \int\frac{\sqrt{-96\tau_2^3}\,d\Gamma}{\sqrt{(\Gamma-\alpha_1)(\Gamma-\alpha_2)(\Gamma-\alpha_3)(\Gamma-\alpha_4)}},(\Gamma-\alpha_1)(\Gamma-\alpha_2)(\Gamma-\alpha_3)(\Gamma-\alpha_4)\\
=& \Omega_4\Gamma^4+\Omega_3\Gamma^3+\Omega_2\Gamma^2+\Omega_1\Gamma+\Omega_0,
\end{aligned} \tag{5.23}$$

where

$$\begin{aligned}
\Omega_4 =& -16\mu\tau_2^4, \quad \Omega_3=-32\mu\tau_2^3\tau_1, \quad \Omega_2=12\tau_2^2\left(2\tau_2k^2-4\mu\tau_0\tau_2-\mu\tau_1^2\right),\\
\Omega_1 =& 4\tau_1\tau_2\left(\mu\tau_1^2+6\tau_2k^2-12\mu\tau_0\tau_2\right), \quad \Omega_0=-6\tau_1^2k^2\tau_2+12\mu\tau_1^2\tau_0\tau_2\\
&-\mu\tau_1^4-48\mu\tau_0^2\tau_2^2+48k^2\tau_0\tau_2^2,
\end{aligned} \tag{5.24}$$

$$\pm(\eta-\eta_0)=\int\sqrt{\frac{-96\lambda\tau_2^3}{\Omega_4\Gamma^4+\Omega_3\Gamma^3+\Omega_2\Gamma^2+\Omega_1\Gamma+\Omega_0}}\,d\Gamma.$$

Integrating (5.24), we obtain the solutions to the Eq. (1.2) as follows:

First solution

$$\pm(\eta-\eta_0)=\frac{2\sqrt{-96\tau_2^3}}{\sqrt{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}}F(\varphi,l), \quad \alpha_1>\alpha_2>\alpha_3>\alpha_4, \tag{5.25}$$

where

$$\begin{aligned}
&F(\varphi,l)=\int_0^{\varphi}\frac{d\psi}{\sqrt{1-l^2\sin^2\psi}}, \quad \varphi=\arcsin\sqrt{\frac{(\alpha_2-\alpha_4)(\Gamma-\alpha_1)}{(\alpha_1-\alpha_4)(\Gamma-\alpha_2)}}, l^2=\frac{(\alpha_2-\alpha_3)(\alpha_1-\alpha_4)}{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)},\\
&\Rightarrow\Gamma=\alpha_2+\frac{(\alpha_1-\alpha_2)(\alpha_4-\alpha_2)}{\alpha_4-\alpha_2+(\alpha_1-\alpha_4)sn^2\left[\mp\frac{\sqrt{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}}{2\sqrt{-96\tau_2^3}}(\eta-\eta_0),\frac{(\alpha_2-\alpha_3)(\alpha_1-\alpha_4)}{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}\right]},
\end{aligned} \tag{5.26}$$

$$\Omega_4\Gamma^4 + \Omega_3\Gamma^3 + \Omega_2\Gamma^2 + \Omega_1\Gamma + \Omega_0 = (\Gamma - \alpha_1)(\Gamma - \alpha_2)(\Gamma - \alpha_3)(\Gamma - \alpha_4) \Rightarrow \tag{5.27}$$

$$\alpha_j = -\frac{\tau_1}{2\tau_2} \pm \frac{\sqrt{6\mu^2\tau_1^2 + 12\mu k^2\tau_2 - 24\mu^2\tau_0\tau_2 \pm 2\sqrt{\Omega}}}{4\tau_2}, \quad j = 1, 2, 3, 4, \tag{5.28}$$

where

$$\Omega = 3\left(4\mu\tau_2\tau_0 - \mu\tau_1^2 + 2\tau_2 k^2\right)\left(-4\mu\tau_2\tau_0 + \mu\tau_1^2 + 6\tau_2 k^2\right).$$

Therefore, the solution for the coupled conformable time-fractional Boussinesq equation will be as

$$u_4(x,t) = \tau_0 + \tau_0\alpha_2 + \frac{\tau_1(\alpha_1 - \alpha_2)(\alpha_4 - \alpha_2)}{\alpha_4 - \alpha_2 + (\alpha_1 - \alpha_4) sn^2\left[\mp\frac{\sqrt{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}}{2\sqrt{-96\tau_2^3}}\left(x - \frac{kt^\alpha}{\alpha} - \eta_0\right), \frac{(\alpha_2-\alpha_3)(\alpha_1-\alpha_4)}{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}\right]}, \tag{5.29}$$

$$v_4(x,t) = k\tau_0 + k\tau_0\alpha_2 + \frac{k\tau_1(\alpha_1 - \alpha_2)(\alpha_4 - \alpha_2)}{\alpha_4 - \alpha_2 + (\alpha_1 - \alpha_4) sn^2\left[\mp\frac{\sqrt{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}}{2\sqrt{-96\tau_2^3}}\left(x - \frac{kt^\alpha}{\alpha} - \eta_0\right), \frac{(\alpha_2-\alpha_3)(\alpha_1-\alpha_4)}{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}\right]}. \tag{5.30}$$

Remark 1 If the modulus $l \rightarrow 1$, then the solution for the coupled conformable time-fractional Boussinesq equation can be reduced to the solitary wave solution

$$u_5(x,t) = \tau_0 + \tau_0\alpha_2 + \frac{\tau_1(\alpha_1 - \alpha_2)(\alpha_4 - \alpha_2)}{\alpha_4 - \alpha_2 + (\alpha_1 - \alpha_4)\tanh^2\left[\mp\frac{\sqrt{(\alpha_1-\alpha_4)(\alpha_2-\alpha_4)}}{2\sqrt{-96\tau_2^3}}\left(x - \frac{kt^\alpha}{\alpha} - \eta_0\right)\right]}, \tag{5.31}$$

$$v_5(x,t) = k\tau_0 + k\tau_0\alpha_2 + \frac{k\tau_1(\alpha_1 - \alpha_2)(\alpha_4 - \alpha_2)}{\alpha_4 - \alpha_2 + (\alpha_1 - \alpha_4)\tanh^2\left[\mp\frac{\sqrt{(\alpha_1-\alpha_4)(\alpha_2-\alpha_4)}}{2\sqrt{-96\tau_2^3}}\left(x - \frac{kt^\alpha}{\alpha} - \eta_0\right)\right]}, \tag{5.32}$$

where $\alpha_3 = \alpha_4$.

Remark 2 If the modulus $l \rightarrow 0$, then the solution for the coupled conformable time-fractional Boussinesq equation can be reduced to the solitary wave solution

$$u_6(x,t) = \tau_0 + \tau_0\alpha_3 + \frac{\tau_1(\alpha_1 - \alpha_3)(\alpha_4 - \alpha_3)}{\alpha_4 - \alpha_3 + (\alpha_1 - \alpha_4)\sin^2\left[\mp\frac{\sqrt{(\alpha_1-\alpha_3)(\alpha_3-\alpha_4)}}{2\sqrt{-96\tau_2^3}}\left(x - \frac{kt^\alpha}{\alpha} - \eta_0\right)\right]}, \tag{5.33}$$

$$v_6(x,t) = k\tau_0 + k\tau_0\alpha_3 + \frac{k\tau_1(\alpha_1 - \alpha_3)(\alpha_4 - \alpha_3)}{\alpha_4 - \alpha_3 + (\alpha_1 - \alpha_4)\sin^2\left[\mp\frac{\sqrt{(\alpha_1-\alpha_3)(\alpha_3-\alpha_4)}}{2\sqrt{-96\tau_2^3}}\left(x - \frac{kt^\alpha}{\alpha} - \eta_0\right)\right]}, \tag{5.34}$$

where $\alpha_2 = \alpha_3$.

5.2 EEM

In this subsection, consider EEM for Eq. (5.4) which is given in before subsection. By balancing u^2 and u'' in Eq. (5.4) we obtain the balance number $m = 2$, and the exact solution becomes,

$$u(\eta) = \sum_{j=-2}^{2} \xi_j F^j(\eta), \quad F(\eta) = \exp(-\Omega(\eta)). \tag{5.35}$$

Substituting (5.35) into Eq. (5.4) and comparing the terms, we will reach a system of nonlinear algebraic equations, and by solving system of the nonlinear equations, yields

Case 1

$$k = k, \quad \lambda = 0, \quad \mu = \frac{k^2}{16a}, \xi_{-2} = \frac{3k^4}{128ab}, \quad \xi_{-1} = 0, \quad \xi_0 = \frac{3k^2}{4b}, \xi_1 = 0, \quad \xi_2 = \frac{6a}{b}. \tag{5.36}$$

By using (3.15) and (5.36), the exact solution for the coupled conformable time-fractional Boussinesq equation becomes,

$$u_1(x,t) = -\frac{3k^2}{8b}\tanh^2\left(\frac{k}{8\sqrt{-a}}\left(x - \frac{kt^\alpha}{\alpha} + E\right)\right) + \frac{3k^2}{8b} - \frac{3k^2}{8b}\coth^2\left(\frac{k}{8\sqrt{-a}}\left(x - \frac{kt^\alpha}{\alpha} + E\right)\right), \tag{5.37}$$

$$v_1(x,t) = -\frac{3k^3}{8b}\tanh^2\left(\frac{k}{8\sqrt{-a}}\left(x - \frac{kt^\alpha}{\alpha} + E\right)\right) + \frac{3k^3}{4b} - \frac{3k^3}{8b}\coth^2\left(\frac{k}{8\sqrt{-a}}\left(x - \frac{kt^\alpha}{\alpha} + E\right)\right), \tag{5.38}$$

when $a < 0$.

By using (3.16) and (5.36), the exact solution for the coupled conformable time-fractional Boussinesq equation yields,

$$u_2(x,t) = \frac{3k^2}{8b}\tanh^2\left(\frac{k}{8\sqrt{-a}}\left(x - \frac{kt^\alpha}{\alpha} + E\right)\right) + \frac{3k^2}{4b} + \frac{3k^2}{8b}\coth^2\left(\frac{k}{8\sqrt{-a}}\left(x - \frac{kt^\alpha}{\alpha} + E\right)\right), \tag{5.39}$$

$$v_2(x,t) = \frac{3k^3}{8b}\tan^2\left(\frac{k}{8\sqrt{a}}\left(x - \frac{kt^\alpha}{\alpha} + E\right)\right) + \frac{3k^3}{4b} + \frac{3k^3}{8b}\cot^2\left(\frac{k}{8\sqrt{a}}\left(x - \frac{kt^\alpha}{\alpha} + E\right)\right), \tag{5.40}$$

when $a > 0$.

Case 2

$$\begin{aligned} &k = k, \quad \lambda = \lambda, \quad \mu = \frac{a\lambda^2 + k^2}{4a}, \xi_{-2} = \frac{3(a\lambda^2 + k^2)^2}{8ab}, \quad \xi_{-1} = \frac{3\lambda(a\lambda^2 + k^2)}{2b}, \\ &\xi_0 = \frac{3(a\lambda^2 + k^2)}{2b}, \xi_1 = 0, \quad \xi_2 = 0. \end{aligned} \tag{5.41}$$

By using (3.15) and (5.41), the exact solution for the coupled conformable time-fractional Boussinesq equation becomes,

$$u_3(x,t) = \frac{3a}{2b}\left[\frac{k}{\sqrt{-a}}\tanh\left(\frac{k}{2\sqrt{-a}}\left(x-\frac{kt^\alpha}{\alpha}+E\right)\right)-\lambda\right]^2+\frac{3\lambda(a\lambda^2+k^2)}{2b} + \frac{3a\lambda}{b}\left[\frac{k}{\sqrt{-a}}\tanh\left(\frac{k}{2\sqrt{-a}}\left(x-\frac{kt^\alpha}{\alpha}+E\right)\right)-\lambda\right], \tag{5.42}$$

$$v_3(x,t) = \frac{3ak}{2b}\left[\frac{k}{\sqrt{-a}}\tanh\left(\frac{k}{2\sqrt{-a}}\left(x-\frac{kt^\alpha}{\alpha}+E\right)\right)-\lambda\right]^2+\frac{3k\lambda(a\lambda^2+k^2)}{2b} + \frac{3ak\lambda}{b}\left[\frac{k}{\sqrt{-a}}\tanh\left(\frac{k}{2\sqrt{-a}}\left(x-\frac{kt^\alpha}{\alpha}+E\right)\right)-\lambda\right], \tag{5.43}$$

when $a<0$.

By using (3.16) and (5.41), the exact solution for the coupled conformable time-fractional Boussinesq equation yields,

$$u_4(x,t) = \frac{3a}{2b}\left[\frac{k}{\sqrt{a}}\tan\left(\frac{k}{2\sqrt{a}}\left(x-\frac{kt^\alpha}{\alpha}+E\right)\right)-\lambda\right]^2+\frac{3\lambda(a\lambda^2+k^2)}{2b} + \frac{3a\lambda}{b}\left[\frac{k}{\sqrt{a}}\tan\left(\frac{k}{2\sqrt{a}}\left(x-\frac{kt^\alpha}{\alpha}+E\right)\right)-\lambda\right], \tag{5.44}$$

$$v_4(x,t) = \frac{3ak}{2b}\left[\frac{k}{\sqrt{a}}\tan\left(\frac{k}{2\sqrt{a}}\left(x-\frac{kt^\alpha}{\alpha}+E\right)\right)-\lambda\right]^2+\frac{3k\lambda(a\lambda^2+k^2)}{2b} + \frac{3ak\lambda}{b}\left[\frac{k}{\sqrt{a}}\tan\left(\frac{k}{2\sqrt{a}}\left(x-\frac{kt^\alpha}{\alpha}+E\right)\right)-\lambda\right], \tag{5.45}$$

when $a>0$.

5.3 ITEM

As last applied method, namely, the ITEM for Eq. (5.4) and by balancing u^2 and u'' in Eq. (5.4) we obtain the balance number $m=2$, and the exact solution will be as

$$u(\eta) = \sum_{j=0}^{2} A_j \tan^j(\phi/2) + \sum_{j=1}^{2} B_j \cot^j(\phi/2). \tag{5.46}$$

Substituting (5.46) into Eq. (5.4) and comparing the terms, we will reach a system of nonlinear algebraic equations, and by solving system of the nonlinear equations, yields

Case 1

$$k=k,\quad a=a,\quad b=b,\quad c=c,\quad A_0=0,\quad A_1=-B_1,\quad A_2=-B_2,\quad B_1=B_1,\quad B_2=B_2. \tag{5.47}$$

By using (3.23) and (5.47), the exact solution for the coupled conformable time-fractional Boussinesq equation becomes,

$$
\begin{aligned}
u_1(x,t) = & - B_1\left[\frac{a}{b-c} - \frac{\sqrt{-\Delta}}{b-c}\tan\left(\frac{\sqrt{-\Delta}}{2}\left(x - \frac{kt^{\alpha}}{\alpha}\right)\right)\right] \\
& - B_2\left[\frac{a}{b-c} - \frac{\sqrt{-\Delta}}{b-c}\tan\left(\frac{\sqrt{-\Delta}}{2}\left(x - \frac{kt^{\alpha}}{\alpha}\right)\right)\right]^2 \\
& + B_1\left[\frac{a}{b-c} - \frac{\sqrt{-\Delta}}{b-c}\tan\left(\frac{\sqrt{-\Delta}}{2}\left(x - \frac{kt^{\alpha}}{\alpha}\right)\right)\right]^{-1} \\
& + B_2\left[\frac{a}{b-c} - \frac{\sqrt{-\Delta}}{b-c}\tan\left(\frac{\sqrt{-\Delta}}{2}\left(x - \frac{kt^{\alpha}}{\alpha}\right)\right)\right]^{-2},
\end{aligned} \tag{5.48}
$$

$$
\begin{aligned}
v_1(x,t) = & - kB_1\left[\frac{a}{b-c} - \frac{\sqrt{-\Delta}}{b-c}\tan\left(\frac{\sqrt{-\Delta}}{2}\left(x - \frac{kt^{\alpha}}{\alpha}\right)\right)\right] \\
& - kB_2\left[\frac{a}{b-c} - \frac{\sqrt{-\Delta}}{b-c}\tan\left(\frac{\sqrt{-\Delta}}{2}\left(x - \frac{kt^{\alpha}}{\alpha}\right)\right)\right]^2 \\
& + kB_1\left[\frac{a}{b-c} - \frac{\sqrt{-\Delta}}{b-c}\tan\left(\frac{\sqrt{-\Delta}}{2}\left(x - \frac{kt^{\alpha}}{\alpha}\right)\right)\right]^{-1} \\
& + kB_2\left[\frac{a}{b-c} - \frac{\sqrt{-\Delta}}{b-c}\tan\left(\frac{\sqrt{-\Delta}}{2}\left(x - \frac{kt^{\alpha}}{\alpha}\right)\right)\right]^{-2},
\end{aligned} \tag{5.49}
$$

when $\Delta = a^2 + b^2 - c^2 < 0$ (*Family 1*).

$$
\begin{aligned}
u_2(x,t) = & - B_1\left[\frac{a}{b-c} + \frac{\sqrt{\Delta}}{b-c}\tanh\left(\frac{\sqrt{\Delta}}{2}\left(x - \frac{kt^{\alpha}}{\alpha}\right)\right)\right] \\
& - B_2\left[\frac{a}{b-c} + \frac{\sqrt{\Delta}}{b-c}\tanh\left(\frac{\sqrt{\Delta}}{2}\left(x - \frac{kt^{\alpha}}{\alpha}\right)\right)\right]^2 \\
& + B_1\left[\frac{a}{b-c} + \frac{\sqrt{\Delta}}{b-c}\tanh\left(\frac{\sqrt{\Delta}}{2}\left(x - \frac{kt^{\alpha}}{\alpha}\right)\right)\right]^{-1} \\
& + B_2\left[\frac{a}{b-c} + \frac{\sqrt{\Delta}}{b-c}\tanh\left(\frac{\sqrt{\Delta}}{2}\left(x - \frac{kt^{\alpha}}{\alpha}\right)\right)\right]^{-2},
\end{aligned} \tag{5.50}
$$

$$
\begin{aligned}
v_2(x,t) = & -kB_1\left[\frac{a}{b-c}+\frac{\sqrt{\Delta}}{b-c}\tanh\left(\frac{\sqrt{\Delta}}{2}\left(x-\frac{kt^{\alpha}}{\alpha}\right)\right)\right] \\
& -kB_2\left[\frac{a}{b-c}+\frac{\sqrt{\Delta}}{b-c}\tanh\left(\frac{\sqrt{\Delta}}{2}\left(x-\frac{kt^{\alpha}}{\alpha}\right)\right)\right]^2 \\
& +kB_1\left[\frac{a}{b-c}+\frac{\sqrt{\Delta}}{b-c}\tanh\left(\frac{\sqrt{\Delta}}{2}\left(x-\frac{kt^{\alpha}}{\alpha}\right)\right)\right]^{-1} \\
& +kB_2\left[\frac{a}{b-c}+\frac{\sqrt{\Delta}}{b-c}\tanh\left(\frac{\sqrt{\Delta}}{2}\left(x-\frac{kt^{\alpha}}{\alpha}\right)\right)\right]^{-2},
\end{aligned} \tag{5.51}
$$

when $\Delta = a^2+b^2-c^2>0$ (*Family 2*).

$$
\begin{aligned}
u_3(x,t) = & -B_1\left[\frac{e^{b\left(x-\frac{kt^{\alpha}}{\alpha}\right)}+b-a}{e^{b\left(x-\frac{kt^{\alpha}}{\alpha}\right)}-b-a}\right]-B_2\left[\frac{e^{b\left(x-\frac{kt^{\alpha}}{\alpha}\right)}+b-a}{e^{b\left(x-\frac{kt^{\alpha}}{\alpha}\right)}-b-a}\right]^2+B_1\left[\frac{e^{b\left(x-\frac{kt^{\alpha}}{\alpha}\right)}+b-a}{e^{b\left(x-\frac{kt^{\alpha}}{\alpha}\right)}-b-a}\right]^{-1} \\
& +B_2\left[\frac{e^{b\left(x-\frac{kt^{\alpha}}{\alpha}\right)}+b-a}{e^{b\left(x-\frac{kt^{\alpha}}{\alpha}\right)}-b-a}\right]^{-2},
\end{aligned} \tag{5.52}
$$

$$
\begin{aligned}
v_3(x,t) = & -kB_1\left[\frac{e^{b\left(x-\frac{kt^{\alpha}}{\alpha}\right)}+b-a}{e^{b\left(x-\frac{kt^{\alpha}}{\alpha}v\right)}-b-a}\right]-kB_2\left[\frac{e^{b\left(x-\frac{kt^{\alpha}}{\alpha}\right)}+b-a}{e^{b\left(x-\frac{kt^{\alpha}}{\alpha}\right)}-b-a}\right]^2+Bk_1\left[\frac{e^{b\left(x-\frac{kt^{\alpha}}{\alpha}\right)}+b-a}{e^{b\left(x-\frac{kt^{\alpha}}{\alpha}\right)}-b-a}\right]^{-1} \\
& +kB_2\left[\frac{e^{b\left(x-\frac{kt^{\alpha}}{\alpha}\right)}+b-a}{e^{b\left(x-\frac{kt^{\alpha}}{\alpha}\right)}-b-a}\right]^{-2},
\end{aligned} \tag{5.53}
$$

when $c=-a$ (*Family 10*).

$$
\begin{aligned}
u_4(x,t) = & -B_1\left(c\left(x-\frac{kt^{\alpha}}{\alpha}\right)\right)-B_2\left(c\left(x-\frac{kt^{\alpha}}{\alpha}\right)\right)^2+B_1\left(c\left(x-\frac{kt^{\alpha}}{\alpha}\right)\right)^{-1} \\
& +B_2\left(c\left(x-\frac{kt^{\alpha}}{\alpha}\right)\right)^{-2},
\end{aligned} \tag{5.54}
$$

$$
\begin{aligned}
v_4(x,t) = & -kB_1\left(c\left(x-\frac{kt^{\alpha}}{\alpha}\right)\right)-kB_2\left(c\left(x-\frac{kt^{\alpha}}{\alpha}\right)\right)^2+kB_1\left(c\left(x-\frac{kt^{\alpha}}{\alpha}\right)\right)^{-1} \\
& +kB_2\left(c\left(x-\frac{kt^{\alpha}}{\alpha}\right)\right)^{-2},
\end{aligned} \tag{5.55}
$$

when $a=0$ and $b=c$ (*Family 13*).

Case 2

$$
k=k,\ \ a=a,\ \ b=b,\ \ c=c,\ \ A_0=\frac{k^2}{\mu},\ \ A_1=-B_1,\ \ A_2=-B_2, B_1=B_1,\ \ B_2=B_2. \tag{5.56}
$$

By using (3.23) and (5.56), the exact solution for the coupled conformable time-fractional Boussinesq equation yields,

$$\begin{aligned} u_5(x,t) =& \frac{k^2}{\mu} - B_1\left[\frac{a}{b-c} - \frac{\sqrt{-\Delta}}{b-c}\tan\left(\frac{\sqrt{-\Delta}}{2}\left(x-\frac{kt^\alpha}{\alpha}\right)\right)\right] \\ &- B_2\left[\frac{a}{b-c} - \frac{\sqrt{-\Delta}}{b-c}\tan\left(\frac{\sqrt{-\Delta}}{2}\left(x-\frac{kt^\alpha}{\alpha}\right)\right)\right]^2 \\ &+ B_1\left[\frac{a}{b-c} - \frac{\sqrt{-\Delta}}{b-c}\tan\left(\frac{\sqrt{-\Delta}}{2}\left(x-\frac{kt^\alpha}{\alpha}\right)\right)\right]^{-1} \\ &+ B_2\left[\frac{a}{b-c} - \frac{\sqrt{-\Delta}}{b-c}\tan\left(\frac{\sqrt{-\Delta}}{2}\left(x-\frac{kt^\alpha}{\alpha}\right)\right)\right]^{-2}, \end{aligned} \tag{5.57}$$

$$\begin{aligned} v_5(x,t) =& \frac{k^3}{\mu} - kB_1\left[\frac{a}{b-c} - \frac{\sqrt{-\Delta}}{b-c}\tan\left(\frac{\sqrt{-\Delta}}{2}\left(x-\frac{kt^\alpha}{\alpha}\right)\right)\right] \\ &- kB_2\left[\frac{a}{b-c} - \frac{\sqrt{-\Delta}}{b-c}\tan\left(\frac{\sqrt{-\Delta}}{2}\left(x-\frac{kt^\alpha}{\alpha}\right)\right)\right]^2 \\ &+ kB_1\left[\frac{a}{b-c} - \frac{\sqrt{-\Delta}}{b-c}\tan\left(\frac{\sqrt{-\Delta}}{2}\left(x-\frac{kt^\alpha}{\alpha}\right)\right)\right]^{-1} \\ &+ kB_2\left[\frac{a}{b-c} - \frac{\sqrt{-\Delta}}{b-c}\tan\left(\frac{\sqrt{-\Delta}}{2}\left(x-\frac{kt^\alpha}{\alpha}\right)\right)\right]^{-2}, \end{aligned} \tag{5.58}$$

when $\Delta = a^2 + b^2 - c^2 < 0$ (*Family 1*).

$$\begin{aligned} u_6(x,t) =& \frac{k^2}{\mu} - B_1\left[\frac{a}{b-c} + \frac{\sqrt{\Delta}}{b-c}\tanh\left(\frac{\sqrt{\Delta}}{2}\left(x-\frac{kt^\alpha}{\alpha}\right)\right)\right] \\ &- B_2\left[\frac{a}{b-c} + \frac{\sqrt{\Delta}}{b-c}\tanh\left(\frac{\sqrt{\Delta}}{2}\left(x-\frac{kt^\alpha}{\alpha}\right)\right)\right]^2 \\ &+ B_1\left[\frac{a}{b-c} + \frac{\sqrt{\Delta}}{b-c}\tanh\left(\frac{\sqrt{\Delta}}{2}\left(x-\frac{kt^\alpha}{\alpha}\right)\right)\right]^{-1} \\ &+ B_2\left[\frac{a}{b-c} + \frac{\sqrt{\Delta}}{b-c}\tanh\left(\frac{\sqrt{\Delta}}{2}\left(x-\frac{kt^\alpha}{\alpha}\right)\right)\right]^{-2}, \end{aligned} \tag{5.59}$$

$$
\begin{aligned}
v_6(x,t) = & \frac{k^3}{\mu} - kB_1\left[\frac{a}{b-c} + \frac{\sqrt{\Delta}}{b-c}\tanh\left(\frac{\sqrt{\Delta}}{2}\left(x - \frac{kt^\alpha}{\alpha}\right)\right)\right] \\
& - kB_2\left[\frac{a}{b-c} + \frac{\sqrt{\Delta}}{b-c}\tanh\left(\frac{\sqrt{\Delta}}{2}\left(x - \frac{kt^\alpha}{\alpha}\right)\right)\right]^2 \\
& + kB_1\left[\frac{a}{b-c} + \frac{\sqrt{\Delta}}{b-c}\tanh\left(\frac{\sqrt{\Delta}}{2}\left(x - \frac{kt^\alpha}{\alpha}\right)\right)\right]^{-1} \\
& + kB_2\left[\frac{a}{b-c} + \frac{\sqrt{\Delta}}{b-c}\tanh\left(\frac{\sqrt{\Delta}}{2}\left(x - \frac{kt^\alpha}{\alpha}\right)\right)\right]^{-2},
\end{aligned} \tag{5.60}
$$

when $\Delta = a^2 + b^2 - c^2 > 0$ (*Family 2*).

$$
\begin{aligned}
u_7(x,t) = & \frac{k^2}{\mu} - B_1\left[\frac{e^{b\left(x-\frac{kt^\alpha}{\alpha}\right)} + b - a}{e^{b\left(x-\frac{kt^\alpha}{\alpha}\right)} - b - a}\right] - B_2\left[\frac{e^{b\left(x-\frac{kt^\alpha}{\alpha}\right)} + b - a}{e^{b\left(x-\frac{kt^\alpha}{\alpha}\right)} - b - a}\right]^2 + B_1\left[\frac{e^{b\left(x-\frac{kt^\alpha}{\alpha}\right)} + b - a}{e^{b\left(x-\frac{kt^\alpha}{\alpha}\right)} - b - a}\right]^{-1} \\
& + B_2\left[\frac{e^{b\left(x-\frac{kt^\alpha}{\alpha}\right)} + b - a}{e^{b\left(x-\frac{kt^\alpha}{\alpha}\right)} - b - a}\right]^{-2},
\end{aligned} \tag{5.61}
$$

$$
\begin{aligned}
v_7(x,t) = & \frac{k^3}{\mu} - kB_1\left[\frac{e^{b\left(x-\frac{kt^\alpha}{\alpha}\right)} + b - a}{e^{b\left(x-\frac{kt^\alpha}{\alpha}\right)} - b - a}\right] - kB_2\left[\frac{e^{b\left(x-\frac{kt^\alpha}{\alpha}\right)} + b - a}{e^{b\left(x-\frac{kt^\alpha}{\alpha}\right)} - b - a}\right]^2 + kB_1\left[\frac{e^{b\left(x-\frac{kt^\alpha}{\alpha}\right)} + b - a}{e^{b\left(x-\frac{kt^\alpha}{\alpha}\right)} - b - a}\right]^{-1} \\
& + kB_2\left[\frac{e^{b\left(x-\frac{kt^\alpha}{\alpha}\right)} + b - a}{e^{b\left(x-\frac{kt^\alpha}{\alpha}\right)} - b - a}\right]^{-2},
\end{aligned} \tag{5.62}
$$

when $c = -a$ (*Family 10*).

$$
\begin{aligned}
u_8(x,t) = & \frac{k^2}{\mu} - B_1\left(c\left(x - \frac{kt^\alpha}{\alpha}\right)\right) - B_2\left(c\left(x - \frac{kt^\alpha}{\alpha}\right)\right)^2 + B_1\left(c\left(x - \frac{kt^\alpha}{\alpha}\right)\right)^{-1} \\
& + B_2\left(c\left(x - \frac{kt^\alpha}{\alpha}\right)\right)^{-2},
\end{aligned} \tag{5.63}
$$

$$
\begin{aligned}
v_8(x,t) = & \frac{k^3}{\mu} - kB_1\left(c\left(x - \frac{kt^\alpha}{\alpha}\right)\right) - kB_2\left(c\left(x - \frac{kt^\alpha}{\alpha}\right)\right)^2 + kB_1\left(c\left(x - \frac{kt^\alpha}{\alpha}\right)\right)^{-1} \\
& + kB_2\left(c\left(x - \frac{kt^\alpha}{\alpha}\right)\right)^{-2},
\end{aligned} \tag{5.64}
$$

when $a = 0$ and $b = c$ (*Family 13*).

6 Conclusion

In this study, by utilizing three integration methods, namely, the extended trial equation method, the $\exp(-\Omega(\eta))$-expansion method and the $\tan(\phi(\eta)/2)$-expansion method, with the help of Maple software, we investigated the solutions of the nonlinear conformable time-fractional Boussinesq equations. We obtained some new dark solitons, bright solitons, solitary wave, periodic solitary wave, rational function, and elliptic function solutions. All the obtained solutions in this study verified the conformable time-fractional Boussinesq equations, we checked this using the same program in Maple 13. The aforementioned methods are powerful and efficient mathematical tool that can be used with the aid of symbolic software such as Maple or Mathematica in exploring search for the solutions of the various nonlinear fractional equations arising in the various field of nonlinear sciences.

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