Opt Quant Electron (2016) 48:116
DOI 10.1007/s11082-016-0371-y

Dispersive dark optical soliton with Tzitzéica type nonlinear evolution equations arising in nonlinear optics

Dispersive dark optical soliton with Tzitzéica ...

Jalil Manafian[1] · Mehrdad Lakestani[1]

Received: 7 November 2015 / Accepted: 2 January 2016 / Published online: 20 January 2016

Abstract A improvement of the expansion methods namely the improved $\tan(\Phi(\xi)/2)$-expansion method for solving the Tzitzéica type nonlinear evolution equations is proposed. In this work, the dispersive optical solitons that are governed by the Tzitzéica type nonlinear evolution equations. As a result, many new and more general exact travelling wave solutions are obtained including periodic function solutions, soliton-like solutions and trigonometric function solutions. The exact particular solutions containing four types hyperbolic function solution, trigonometric function solution, exponential solution and rational solution. We obtained the further solutions comparing with other methods. Recently this method is developed for searching exact travelling wave solutions of nonlinear partial differential equations. Abundant exact travelling wave solutions including solitons, kink, periodic and rational solutions have been found. These solutions might play important role in engineering fields. It is shown that this method, with the help of symbolic computation, provides a straightforward and powerful mathematical tool for solving the nonlinear problems.

Keywords Improved $\tan(\Phi(\xi)/2)$-expansion method · Tzitzéica type nonlinear equation · Travelling wave · Periodic function solutions · Soliton-like solutions and trigonometric function solutions

✉ Jalil Manafian
j_manafianheris@tabrizu.ac.ir

Mehrdad Lakestani
lakestani@tabrizu.ac.ir

[1] Department of Applied Mathematics, Faculty of Mathematics Science, University of Tabriz, Tabriz, Islamic Republic of Iran

1 Introduction

Nonlinear evolution equations (NLEEs) are very important equations in mathematical physics, engineering and applied mathematics for describing diverse types of physical mechanisms of natural phenomena in the field of applied sciences, biochemical and engineering. Nonlinear wave equations play a substantial role in various fields such as plasma physics, fluid mechanics, optical fibers, solid state physics, chemical kinetics, geochemistry, nonlinear optics and so on (Gray and Scott 1990; Kabir and Khajeh 2009). Much work has been down over the years on the subject of obtaining the analytical solutions to the nonlinear partial differential equations (NPDEs). One of the most exciting advances of nonlinear science and theoretical physics has been a development of methods to look for exact solutions for NPDEs. With the rapid development of nonlinear sciences based on computer algebraic system, many effective methods have been presented, such as, the homotopy analysis method (Dehghan et al. 2010), the variational iteration method (He 1999; Mohyud-Din et al. 2009; Dehghan et al. 2010; Jafari et al. 2014), the homotopy perturbation method (Dehghan and Manafian 2009; Mohyud-Din et al. 2011; Zhang et al. 2014), the sine-cosine method (Wazwaz 2006), the tanh-coth method (Manafian Heris and Lakestani 2013), the modified extended tanh-function method (Abdou and Soliman 2006; El-Wakil and Abdou 2007), the Exp-function method (Dehghan et al. 2011; Ebrahimi Ghogdia et al. 2015; Yildirim and Pinar 2010; Abdou et al. 2007; Noor et al. 2008, 2010; Mohyud-Din et al. 2010, 2012; Manafian and Lakestani 2015b), Taylor expansion method (Ma and You 2004), the exp(-$\Phi(\xi)$)-expansion method (Roshid and Rahman 2014; Hafez et al. 2015), the $(\frac{G'}{G})$-expansion method (Naher et al. 2013; Saba et al. 2015; Manafian and Lakestani 2015a), the modified simple equation method (Jawad et al. 2010), the novel $(\frac{G'}{G})$-expansion method (Alam et al. 2014), the new approach of the generalized $(\frac{G'}{G})$-expansion method (Naher and Abdullah 2013), the Jacobi elliptic function method (Chen and Wang 2005), the homogeneous balance method (Zhao et al. 2006) and so on. Here, we use of an effective method namely the improved $\tan(\Phi(\xi)/2)$-expansion method (Manafian et al. 2015; Manafian and Lakestani 2016) for constructing a range of exact solutions for the following ordinary partial differential equations that in this paper we developed solutions as well. In this paper, we put forth the new approach of improved $\tan(\Phi(\xi)/2)$-expansion method to construct exact travelling wave solutions including solitons, kink, periodic and rational solutions to the Tzitzéica type nonlinear evolution equations. Wazwaz (Wazwaz 2005) applied the tanh method to the Dodd–Bullough–Mikhailov and the Tzitzéica–Dodd–Bullough equations. Huber (2008) applied a class of solitary-like solutions of the Tzitzéica equation where generated by a similarity reduction. Borhanifar and Moghanlu (2011) obtained the exact traveling wave solutions of the DBM equation and TDB equation are using the (G’/G)-expansion method. The Tzitzéica equation (Tzitzéica 1910)

$$u_{tt} - u_{xx} - e^{u} + e^{-2u} = 0, \tag{1}$$

has been investigated by Tzitzéica in the study of integrable surfaces. The Dodd–Bullough–Mikhailov (DBM) equation is given as

$$u_{xt} + e^{u} + e^{-2u} = 0, \tag{2}$$

and consider the Tzitzéica–Dodd–Bullough (TDB) equation as follows

$$u_{xt} - e^{-u} - e^{-2u} = 0, \tag{3}$$

where these equations play a significant role in many scientific applications such as solid state physics, nonlinear optics and the quantum field theory (Abazari 2010). Also, Mikhailov (1910) deducted the N-soliton solutions for Tzitzéica equation by the inverse scattering method. The purpose of this paper is to obtain exact solutions of the Tzitzéica type nonlinear evolution equations and to determine the accuracy of the improved $\tan(\Phi(\xi)/2)$-expansion method in solving these kind of problems. The paper is organized as follows: In Sect. 2, we describe the improved $\tan(\Phi(\xi)/2)$-expansion method. In Sect. 3, we examine the Tzitzéica equation, DBM equation and TDB equation respectively with method introduced in Sect. 2. In Sect. 4, we examine the physical interpretations of the solutions of the Tzitzéica type nonlinear evolution equations. Also conclusion is given in Sect. 5. Finally some references are given at the end of this paper.

2 Description of improved $\tan(\Phi(\xi)/2)$-expansion technique

The $\tan(\Phi(\xi)/2)$-expansion method is well-known analytical method which first presented and developed in Manafian et al. (2015). Also Manafian and Lakestani (2015c) applied the stated method to solving the generalized Fitzhugh–Nagumo equation with time-dependent coefficients.

Step 1. We suppose that the given nonlinear partial differential equation for $u(x, t)$ to be in the form

$$\mathcal{N}(u, u_x, u_t, u_{xx}, u_{tt}, \ldots) = 0, \tag{4}$$

which can be converted to an ODE

$$\mathcal{Q}(u, u', -\mu u', u'', \mu^2 u'', \ldots) = 0, \tag{5}$$

by the transformation $\xi = x - \mu t$, is wave variable. Also, μ is constant to be determined later.

Step 2. Suppose the traveling wave solution of Eq. (5) can be expressed as follows

$$u(\xi) = S(\Phi) = \sum_{k=0}^{m} A_k \left[p + \tan\left(\frac{\Phi(\xi)}{2}\right)\right]^k + \sum_{k=1}^{m} B_k \left[p + \tan\left(\frac{\Phi(\xi)}{2}\right)\right]^{-k}, \tag{6}$$

where $A_k (0 \le k \le m)$ and $B_k (1 \le k \le m)$ are constants to be determined, such that $A_m \neq 0, B_m \neq 0$ and $\Phi = \Phi(\xi)$ satisfies the following ordinary differential equation:

$$\Phi'(\xi) = a\sin(\Phi(\xi)) + b\cos(\Phi(\xi)) + c. \tag{7}$$

We will consider the following special solutions of equation (7):

Family 1: **When** $a^2 + b^2 - c^2 < 0$ **and** $b - c \neq 0$, then $\Phi(\xi) = 2\arctan\left[\frac{a}{b-c} - \frac{\sqrt{c^2-b^2-a^2}}{b-c}\tan\left(\frac{\sqrt{c^2-b^2-a^2}}{2}(\xi + C)\right)\right]$.

Family 2: **When** $a^2 + b^2 - c^2 > 0$ **and** $b - c \neq 0$, then $\Phi(\xi) = 2\arctan\left[\frac{a}{b-c} + \frac{\sqrt{b^2+a^2-c^2}}{b-c}\tanh\left(\frac{\sqrt{b^2+a^2-c^2}}{2}(\xi + C)\right)\right]$.

Family 3: **When** $a^2 + b^2 - c^2 > 0$, $b \neq 0$ **and** $c = 0$, then $\Phi(\xi) = 2\arctan\left[\frac{a}{b} + \frac{\sqrt{b^2+a^2}}{b}\tanh\left(\frac{\sqrt{b^2+a^2}}{2}(\xi + C)\right)\right]$.

Family 4: **When** $a^2 + b^2 - c^2 < 0$, $c \neq 0$ **and** $b = 0$, then $\Phi(\xi) = 2\arctan$

$\left[-\frac{a}{c}+\frac{\sqrt{c^2-a^2}}{c}\tan\left(\frac{\sqrt{c^2-a^2}}{2}(\xi+C)\right)\right]$.

Family 5: **When** $a^2+b^2-c^2>0,\ b-c\neq 0$ **and** $a=0$, then $\Phi(\xi)=2\arctan\left[\sqrt{\frac{b+c}{b-c}}\tanh\left(\frac{\sqrt{b^2-c^2}}{2}(\xi+C)\right)\right]$.

Family 6: **When** $a=0$ **and** $c=0$, then $\Phi(\xi)=\arctan\left[\frac{e^{2b(\xi+C)}-1}{e^{2b(\xi+C)}+1},\ \frac{2e^{b(\xi+C)}}{e^{2b(\xi+C)}+1}\right]$.

Family 7: **When** $b=0$ **and** $c=0$, then $\Phi(\xi)=\arctan\left[\frac{2e^{a(\xi+C)}}{e^{2a(\xi+C)}+1},\ \frac{e^{2a(\xi+C)}-1}{e^{2a(\xi+C)}+1}\right]$.

Family 8: **When** $a^2+b^2=c^2$, then $\Phi(\xi)=-2\arctan\left[\frac{(b+c)(a(\xi+C)+2)}{a^2(\xi+C)}\right]$.

Family 9: **When** $a=b=c=ka$, then $\Phi(\xi)=2\arctan\left[e^{ka(\xi+C)}-1\right]$.

Family 10: **When** $a=c=ka$ **and** $b=-ka$, then $\Phi(\xi)=2\arctan\left[\frac{e^{ka(\xi+C)}}{1-e^{ka(\xi+C)}}\right]$.

Family 11: **When** $c=a$, then $\Phi(\xi)=-2\arctan\left[\frac{(a+b)e^{b(\xi+C)}-1}{(a-b)e^{b(\xi+C)}-1}\right]$.

Family 12: **When** $a=c$, then $\Phi(\xi)=2\arctan\left[\frac{(b+c)e^{b(\xi+C)}+1}{(b-c)e^{b(\xi+C)}-1}\right]$.

Family 13: **When** $c=-a$, then $\Phi(\xi)=2\arctan\left[\frac{e^{b(\xi+C)}+b-a}{e^{b(\xi+C)}-b-a}\right]$.

Family 14: **When** $b=-c$, then $\Phi(\xi)=-2\arctan\left[\frac{ae^{a(\xi+C)}}{ce^{a(\xi+C)}-1}\right]$.

Family 15: **When** b $=0$ **and** $a=c$, then $\Phi(\xi)=-2\arctan\left[\frac{c(\xi+C)+2}{c(\xi+C)}\right]$.

Family 16: **When** $a=0$ **and** $b=c$, then $\Phi(\xi)=2\arctan[c(\xi+C)]$.

Family 17: **When** $a=0$ **and** $b=-c$, then $\Phi(\xi)=-2\arctan\left[\frac{1}{c(\xi+C)}\right]$.

Family 18: **When** $a=b=0$ then $\Phi(\xi)=c\xi+C$.

Family 19: **When** $b=c$ then $\Phi(\xi)=2\arctan\left[\frac{e^{a(\xi+C)}-b}{a}\right]$,

where $A_k, B_k(k=1,2,\ldots,m), a, b$ and c are constants to be determined later. But, the positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in Eq. (7). If m is not an integer, then a transformation formula should be used to overcome this difficulty.

Step 3. Substituting (6) into Eq. (5) with the value of m obtained in Step 2. Collecting the coefficients of $\tan^k\left(\frac{\Phi(\xi)}{2}\right)$, $\cot^k\left(\frac{\Phi(\xi)}{2}\right)(k=0,1,2,\ldots)$, then setting each coefficient to zero, we can get a set of over-determined equations for $A_0, A_k, B_k(k=1,2,\ldots,m)$ a, b, c and p with the aid of symbolic computation Maple.

Step 4. Solving the algebraic equations in Step 3, then substituting $A_0, A_1, B_1, \ldots, A_m, B_m, \mu, p$ in (6).

3 The Tzitzéica type nonlinear evolution equations

In this section, we will exert the improved $\tan(\Phi(\xi)/2)$-expansion technique to obtain new and more general exact solutions of the Tzitzéica type nonlinear evolution equations.

3.1 The Tzitzéica equation

We consider the Tzitzéica equation as follows

$$u_{tt}-u_{xx}-e^{u}+e^{-2u}=0, \tag{8}$$

by using the transformation $v(x,t) = e^{-u(x,t)}$, Eq. (8) transforms into the following partial differential equation,

$$vv_{tt} - v_t^2 - vv_{xx} + v_x^2 - v^3 + 1 = 0. \tag{9}$$

The travelling wave transformation $\xi = kx + wt$ reduces Eq. (9) to the following ODE

$$(w^2 - k^2)(vv'' - (v')^2) - v^3 + 1 = 0, \tag{10}$$

where $v' = \frac{dv}{d\xi}$ and $v'' = \frac{d^2v}{d\xi^2}$. Balancing the terms vv'' and v^3 by using homogenous principle the following result could be obtained

$$2M + 2 = 3M, \qquad M = 2. \tag{11}$$

Here, for simplicity, we set $p = 0$. Then (6) reduces to

$$v(\xi) = A_0 + A_1 \tan\left(\frac{\Phi(\xi)}{2}\right) + A_2 \tan^2\left(\frac{\Phi(\xi)}{2}\right) + B_1 \cot\left(\frac{\Phi(\xi)}{2}\right) + B_2 \cot^2\left(\frac{\Phi(\xi)}{2}\right). \tag{12}$$

Substituting (12) and (7) into Eq. (9) and by using the well-known Maple software, we obtain the following sets of non-trivial solutions

Set I: We have the following:

$$\begin{aligned} &a = \sqrt{c^2 - b^2 - \frac{3}{\Delta}}, \quad \Delta = k^2 - w^2, \quad b = b, \quad c = c, \quad A_0 = \frac{1}{2}\Delta(b^2 - c^2) + 1, \\ &A_1 = 0, \quad A_2 = 0, \quad k = k, \quad w = w, \end{aligned} \tag{13}$$

$$\begin{aligned} B_1 = (b + c)\sqrt{-\Delta^2(b^2 - c^2) - 3\Delta}, \quad B_2 = -\frac{1}{2}\Delta(b + c)^2, \\ v(\xi) = A_0 + B_1 \cot\left(\frac{\Phi(\xi)}{2}\right) + B_2 \cot^2\left(\frac{\Phi(\xi)}{2}\right), \end{aligned} \tag{14}$$

where a, b and c are arbitrary constants. Using the transformations $u = \ln v$ and according to (12), we obtain several types of travelling wave solutions of Eq. (9) as follows:

By using of the **Families 1, 2, 3, 4, 11** and **19** can be written, respectively, as

$$\begin{aligned} u_1(\xi) = \ln\Bigg\{ & 1 + \frac{1}{2}\Delta(b^2 - c^2) + (b^2 - c^2)\Delta\left[1 - \sqrt{-\frac{3}{3 + \Delta(b^2 - c^2)}}\tan\left(\frac{1}{2}\sqrt{\frac{3}{\Delta}}(\xi + C)\right)\right]^{-1} \\ & - \frac{1}{2}\Delta^2(b^2 - c^2)^2\left[\sqrt{-\Delta(b^2 - c^2) - 3} - \sqrt{3}\tan\left(\frac{1}{2}\sqrt{\frac{3}{\Delta}}(\xi + C)\right)\right]^{-2}\Bigg\}, \end{aligned} \tag{15}$$

$$\begin{aligned} u_2(\xi) = \ln\Bigg\{ & 1 + \frac{1}{2}\Delta(b^2 - c^2) + (b^2 - c^2)\Delta\left[1 + \sqrt{\frac{3}{3 + \Delta(b^2 - c^2)}}\tanh\left(\frac{1}{2}\sqrt{-\frac{3}{\Delta}}(\xi + C)\right)\right]^{-1} \\ & - \frac{1}{2}\Delta^2(b^2 - c^2)^2\left[\sqrt{-\Delta(b^2 - c^2) - 3} + \sqrt{-3}\tanh\left(\frac{1}{2}\sqrt{-\frac{3}{\Delta}}(\xi + C)\right)\right]^{-2}\Bigg\}, \end{aligned} \tag{16}$$

$$u_3(\xi) = \ln\left\{1 + \frac{1}{2}\Delta b^2 + b^2\Delta\left[1 + \sqrt{\frac{3}{3+\Delta b^2}}\tanh\left(\frac{1}{2}\sqrt{-\frac{3}{\Delta}}(\xi + C)\right)\right]^{-1}\right.$$
$$\left. - \frac{1}{2}\Delta^2 b^4\left[\sqrt{-\Delta b^2 - 3} + \sqrt{-3}\tanh\left(\frac{1}{2}\sqrt{-\frac{3}{\Delta}}(\xi + C)\right)\right]^{-2}\right\}, \tag{17}$$

$$u_4(\xi) = \ln\left\{1 - \frac{1}{2}\Delta c^2 - c^2\Delta\left[1 - \sqrt{\frac{3}{\Delta c^2 - 3}}\tan\left(\frac{1}{2}\sqrt{\frac{3}{\Delta}}(\xi + C)\right)\right]^{-1}\right.$$
$$\left. - \frac{1}{2}\Delta^2 c^4\left[\sqrt{\Delta c^2 - 3} - \sqrt{3}\tan\left(\frac{1}{2}\sqrt{\frac{3}{\Delta}}(\xi + C)\right)\right]^{-2}\right\}, \tag{18}$$

$$u_5(\xi) = \ln\left\{1 + \frac{1}{2}(3 - a^2\Delta) - \left(a + \sqrt{\frac{3}{\Delta}}\right)\sqrt{-\Delta(6 - a^2\Delta)}\left[\frac{\left(a + \sqrt{\frac{3}{\Delta}}\right)e^{\sqrt{\frac{3}{\Delta}}(\xi+C)} - 1}{\left(a - \sqrt{\frac{3}{\Delta}}\right)e^{\sqrt{\frac{3}{\Delta}}(\xi+C)} - 1}\right]^{-1}\right.$$
$$\left. - \frac{1}{2}\Delta\left(a + \sqrt{\frac{3}{\Delta}}\right)^2\left[\frac{\left(a + \sqrt{\frac{3}{\Delta}}\right)e^{\sqrt{\frac{3}{\Delta}}(\xi+C)} - 1}{\left(a - \sqrt{\frac{3}{\Delta}}\right)e^{\sqrt{\frac{3}{\Delta}}(\xi+C)} - 1}\right]^{-2}\right\}, \tag{19}$$

$$u_6(\xi) = \ln\left\{1 + 6b\left[e^{\sqrt{-\frac{3}{\Delta}}(\xi+C)} - b\right]^{-1} - 6b^2\left[e^{\sqrt{-\frac{3}{\Delta}}(\xi+C)} - b\right]^{-2}\right\}, \tag{20}$$

where $\xi = kx + wt$.

Set II: We have the following:

$$a = \sqrt{c^2 - b^2 - \frac{3}{\Delta}}, \quad \Delta = k^2 - w^2, \quad b = b, \quad c = c, \quad A_0 = 1 + \frac{1}{2}\Delta(b^2 - c^2),$$
$$B_1 = 0, \quad B_2 = 0, \quad k = k, \quad w = w, \tag{21}$$

$$A_1 = (b - c)\sqrt{-\Delta^2(b^2 - c^2) - 3\Delta}, \quad A_2 = -\frac{1}{2}\Delta(b - c)^2,$$
$$v(\xi) = A_0 + A_1\tan\left(\frac{\Phi(\xi)}{2}\right) + A_2\tan^2\left(\frac{\Phi(\xi)}{2}\right). \tag{22}$$

By using of the **Families 1, 2, 3, 4, 11** and **14** can be written, respectively, as

$$u_7(\xi) = \ln\left\{-2 - \frac{1}{2}\Delta(b^2 - c^2) - \sqrt{-3\Delta(b^2 - c^2) - 9}\tan\left(\frac{1}{2}\sqrt{\frac{3}{\Delta}}(\xi + C)\right)\right.$$
$$\left. - \frac{1}{2}\left[\sqrt{-\Delta(b^2 - c^2) - 3} - \sqrt{3}\tan\left(\frac{1}{2}\sqrt{\frac{3}{\Delta}}(\xi + C)\right)\right]^2\right\}, \tag{23}$$

$$u_8(\xi) = \ln\left\{-2 - \frac{1}{2}\Delta(b^2 - c^2) + \sqrt{3\Delta(b^2 - c^2) + 9}\tanh\left(\frac{1}{2}\sqrt{-\frac{3}{\Delta}}(\xi + C)\right) - \frac{1}{2}\left[\sqrt{-\Delta(b^2 - c^2) - 3} - \sqrt{-3}\tanh\left(\frac{1}{2}\sqrt{-\frac{3}{\Delta}}(\xi + C)\right)\right]^2\right\}, \tag{24}$$

$$u_9(\xi) = \ln\left\{-2 - \frac{1}{2}\Delta b^2 - \sqrt{-3\Delta b^2 - 9}\tan\left(\frac{1}{2}\sqrt{\frac{3}{\Delta}}(\xi + C)\right) - \frac{1}{2}\left[\sqrt{-\Delta b^2 - 3} - \sqrt{3}\tan\left(\frac{1}{2}\sqrt{\frac{3}{\Delta}}(\xi + C)\right)\right]^2\right\}, \tag{25}$$

$$u_{10}(\xi) = \ln\left\{-2 + \frac{1}{2}\Delta c^2 + \sqrt{-3\Delta c^2 + 9}\tanh\left(\frac{1}{2}\sqrt{-\frac{3}{\Delta}}(\xi + C)\right) - \frac{1}{2}\left[\sqrt{\Delta c^2 - 3} - \sqrt{-3}\tanh\left(\frac{1}{2}\sqrt{-\frac{3}{\Delta}}(\xi + C)\right)\right]^2\right\}, \tag{26}$$

$$u_{11}(\xi) = \ln\left\{1 + \frac{1}{2}(3 - a^2\Delta) - \left(\sqrt{\frac{3}{\Delta}} - a\right)\sqrt{-\Delta(6 - a^2\Delta)}\left[\frac{\left(a + \sqrt{\frac{3}{\Delta}}\right)e^{\sqrt{\frac{3}{\Delta}}(\xi+C)} - 1}{\left(a - \sqrt{\frac{3}{\Delta}}\right)e^{\sqrt{\frac{3}{\Delta}}(\xi+C)} - 1}\right] - \frac{1}{2}\Delta\left(\sqrt{\frac{3}{\Delta}} - a\right)^2\left[\frac{\left(a + \sqrt{\frac{3}{\Delta}}\right)e^{\sqrt{\frac{3}{\Delta}}(\xi+C)} - 1}{\left(a - \sqrt{\frac{3}{\Delta}}\right)e^{\sqrt{\frac{3}{\Delta}}(\xi+C)} - 1}\right]^2\right\}, \tag{27}$$

$$u_{12}(\xi) = \ln\left\{1 - 2b\sqrt{-3\Delta}\left[\frac{\sqrt{-\frac{3}{\Delta}}e^{\sqrt{-\frac{3}{\Delta}}(\xi+C)}}{be^{\sqrt{-\frac{3}{\Delta}}(\xi+C)} - 1}\right] - 2b^2\Delta\left[\frac{\sqrt{-\frac{3}{\Delta}}e^{\sqrt{-\frac{3}{\Delta}}(\xi+C)}}{be^{\sqrt{-\frac{3}{\Delta}}(\xi+C)} - 1}\right]^2\right\}, \tag{28}$$

where $\xi = kx + wt$.

Set III: We have the following:

$$a = \frac{1}{\sqrt{-\Delta}}, \quad \Delta = k^2 - w^2, \quad b = \sqrt{c^2 - \frac{2}{\Delta}}, \quad c = c, \quad A_0 = 0, \quad B_1 = 0, \quad B_2 = 0, \quad k = k, \quad w = w, \tag{29}$$

$$A_1 = (c - b)\sqrt{-\Delta}, \quad A_2 = 1 - c^2\Delta + c\sqrt{2\Delta - c^2\Delta^2}, \quad v(\xi) = A_1\tan\left(\frac{\Phi(\xi)}{2}\right) + A_2\tan^2\left(\frac{\Phi(\xi)}{2}\right). \tag{30}$$

By using of the **Families 1, 2, 11** and **13** can be written, respectively, as

$$u_{13}(\xi)=\ln\left\{-1+\sqrt{-3}\tan\left(\frac{1}{2}\sqrt{\frac{3}{\Delta}}(\xi+C)\right)-\frac{1}{2}\left[1-\sqrt{-3}\tan\left(\frac{1}{2}\sqrt{\frac{3}{\Delta}}(\xi+C)\right)\right]^2\right\}, \tag{31}$$

$$u_{14}(\xi)=\ln\left\{-1-\sqrt{3}\tanh\left(\frac{1}{2}\sqrt{-\frac{3}{\Delta}}(\xi+C)\right)-\frac{1}{2}\left[1+\sqrt{3}\tanh\left(\frac{1}{2}\sqrt{-\frac{3}{\Delta}}(\xi+C)\right)\right]^2\right\}, \tag{32}$$

$$u_{15}(\xi)=\ln\left\{(1-\sqrt{3})\left[\frac{(1+\sqrt{3})e^{\sqrt{-\frac{3}{\Delta}}(\xi+C)}-\sqrt{-\Delta}}{(1-\sqrt{3})e^{\sqrt{-\frac{3}{\Delta}}(\xi+C)}-\sqrt{-\Delta}}\right]+(2+\sqrt{3})\left[\frac{(1+\sqrt{3})e^{\sqrt{-\frac{3}{\Delta}}(\xi+C)}-\sqrt{-\Delta}}{(1-\sqrt{3})e^{\sqrt{-\frac{3}{\Delta}}(\xi+C)}-\sqrt{-\Delta}}\right]^2\right\}, \tag{33}$$

$$u_{16}(\xi)=\ln\left\{-(1+\sqrt{3})\left[\frac{\sqrt{-\Delta}e^{\sqrt{-\frac{3}{\Delta}}(\xi+C)}-(\sqrt{3}-1)}{\sqrt{-\Delta}e^{\sqrt{-\frac{3}{\Delta}}(\xi+C)}-(\sqrt{3}+1)}\right]+(2+\sqrt{3})\left[\frac{\sqrt{-\Delta}e^{\sqrt{-\frac{3}{\Delta}}(\xi+C)}-(\sqrt{3}-1)}{\sqrt{-\Delta}e^{\sqrt{-\frac{3}{\Delta}}(\xi+C)}-(\sqrt{3}+1)}\right]^2\right\}, \tag{34}$$

where $\xi=kx+wt$.

Set IV: We have the following:

$$a=\sqrt{\frac{1-\sqrt{3}i}{2\Delta}},\quad \Delta=k^2-w^2,\quad b=\sqrt{c^2+\frac{1-\sqrt{3}i}{\Delta}},\quad c=c,\quad A_0=0,\quad B_1=0,$$
$$B_2=0,\quad k=k,\quad w=w, \tag{35}$$

$$\begin{gathered}A_1=(b-c)\sqrt{\frac{1-\sqrt{3}i}{2}\Delta},\quad A_2=-\frac{1}{2}\Delta(b-c)^2,\\ v(\xi)=A_1\tan\left(\frac{\Phi(\xi)}{2}\right)+A_2\tan^2\left(\frac{\Phi(\xi)}{2}\right).\end{gathered} \tag{36}$$

By using of the **Families 1, 2** and **11** can be written, respectively, as

$$u_{17}(x,t)=\ln\left\{\frac{1-\sqrt{3}i}{4}\left(2\left[1-\sqrt{-3}\tan\left(\sqrt{\frac{3\sqrt{3}i-3}{8\Delta}}(kx+wt)\right)\right]-\left[1-\sqrt{-3}\tan\left(\sqrt{\frac{3\sqrt{3}i-3}{8\Delta}}(kx+wt)\right)\right]^2\right)\right\}, \tag{37}$$

$$u_{18}(x,t)=\ln\left\{\frac{1-\sqrt{3}i}{4}\left(2\left[1+\sqrt{3}\tanh\left(\sqrt{\frac{3-3\sqrt{3}i}{8\Delta}}(kx+wt)\right)\right]-\left[1+\sqrt{3}\tanh\left(\sqrt{\frac{3-3\sqrt{3}i}{8\Delta}}(kx+wt)\right)\right]^2\right)\right\}, \tag{38}$$

$$u_{19}(x,t) = \ln\left\{\frac{1}{2}(\sqrt{3}-1)(1-\sqrt{3}i)\left[\frac{(1+\sqrt{3})\sqrt{1-\sqrt{3}i}e^{\sqrt{-\frac{3}{\Delta}}(kx+wt)}-\sqrt{2\Delta}}{(1-\sqrt{3})\sqrt{1-\sqrt{3}i}e^{\sqrt{-\frac{3}{\Delta}}(kx+wt)}-\sqrt{2\Delta}}\right]\right.$$
$$\left.-\frac{1}{4}(\sqrt{3}-1)^2(1-\sqrt{3}i)\left[\frac{(1+\sqrt{3})\sqrt{1-\sqrt{3}i}e^{\sqrt{-\frac{3}{\Delta}}(kx+wt)}-\sqrt{2\Delta}}{(1-\sqrt{3})\sqrt{1-\sqrt{3}i}e^{\sqrt{-\frac{3}{\Delta}}(kx+wt)}-\sqrt{2\Delta}}\right]^2\right\}. \tag{39}$$

Set V: We have the following:

$$\begin{aligned} a &= \sqrt{(b^2-c^2)i - \frac{3\sqrt{3}-3i}{2\Delta}}, \quad \Delta = k^2-w^2, \quad b=b, \quad c=c, \\ A_0 &= \frac{1-2i}{6}\Delta(b^2-c^2) - \frac{\sqrt{3}-i}{2}, \quad A_1 = 0, \end{aligned} \tag{40}$$

$$A_2 = 0, \quad B_1 = -(b+c)\Delta\sqrt{(b^2-c^2)i - \frac{3\sqrt{3}-3i}{2\Delta}}, B_2 = -\frac{1}{2}\Delta(b+c)^2, k=k, \quad w=w,$$
$$v(\xi) = A_0 + B_1\cot\left(\frac{\Phi(\xi)}{2}\right) + B_2\cot^2\left(\frac{\Phi(\xi)}{2}\right). \tag{41}$$

By using of the **Families 1, 2, 14** and **18** can be written, respectively, as

$$u_{20}(x,t) = \ln\left\{\frac{1-2i}{6}\Delta(b^2-c^2) - \frac{\sqrt{3}-i}{2} + (b^2-c^2)\left[1-\sqrt{\frac{2\Delta(c^2-b^2)(1+i)+3\sqrt{3}-3i}{2\Delta(b^2-c^2)i-3\sqrt{3}+3i}}\right.\right.$$
$$\left.\times\tan\left(\frac{\sqrt{\frac{3\sqrt{3}-3i}{2\Delta}-(b^2-c^2)(1+i)}}{2}(kx+wt)\right)\right]^{-1} - \frac{\Delta^2(b^2-c^2)^2}{2\Delta(b^2-c^2)i-3\sqrt{3}+3i}$$
$$\left.\times\left[1-\sqrt{\frac{2\Delta(c^2-b^2)(1+i)+3\sqrt{3}-3i}{2\Delta(b^2-c^2)i-3\sqrt{3}+3i}}\tan\left(\frac{\sqrt{\frac{3\sqrt{3}-3i}{2\Delta}-(b^2-c^2)(1+i)}}{2}(kx+wt)\right)\right]^{-2}\right\}, \tag{42}$$

$$u_{21}(x,t) = \ln\left\{\frac{1-2i}{6}\Delta(b^2-c^2) - \frac{\sqrt{3}-i}{2} + (b^2-c^2)\left[1+\sqrt{\frac{2\Delta(b^2-c^2)(1+i)-3\sqrt{3}+3i}{2\Delta(b^2-c^2)i-3\sqrt{3}+3i}}\right.\right.$$
$$\left.\times\tanh\left(\frac{\sqrt{\frac{-3\sqrt{3}+3i}{2\Delta}+(b^2-c^2)(1+i)}}{2}(kx+wt)\right)\right]^{-1} - \frac{\Delta^2(b^2-c^2)^2}{2\Delta(b^2-c^2)i-3\sqrt{3}+3i}$$
$$\left.\times\left[1+\sqrt{\frac{2\Delta(b^2-c^2)(1+i)-3\sqrt{3}+3i}{2\Delta(b^2-c^2)i-3\sqrt{3}+3i}}\tanh\left(\frac{\sqrt{\frac{-3\sqrt{3}+3i}{2\Delta}+(b^2-c^2)(1+i)}}{2}(kx+wt)\right)\right]^{-2}\right\}, \tag{43}$$

$$u_{22}(x,t)=\ln\left\{-\frac{\sqrt{3}-i}{2}-2c\sqrt{\Delta\left(\frac{3\sqrt{3}-3i}{2}\right)}\left[\frac{\sqrt{-\frac{3\sqrt{3}-3i}{2\Delta}}e^{\sqrt{-\frac{3\sqrt{3}-3i}{2\Delta}}(\xi+C)}}{ce^{\sqrt{-\frac{3\sqrt{3}-3i}{2\Delta}}(\xi+C)}-1}\right]^{-1}\right.$$
$$\left.-2\Delta c^2\left[\frac{\sqrt{-\frac{3\sqrt{3}-3i}{2\Delta}}e^{\sqrt{-\frac{3\sqrt{3}-3i}{2\Delta}}(\xi+C)}}{ce^{\sqrt{-\frac{3\sqrt{3}-3i}{2\Delta}}(\xi+C)}-1}\right]^{-2}\right\}, \tag{44}$$

$$u_{23}(x,t)=\ln\left\{\frac{(4-\sqrt{3})i-1-4\sqrt{3}}{4}+(\sqrt{3}+i)\cot\left(\sqrt{\frac{3+3\sqrt{3}i}{8\Delta}}(kx+wt)\right)\right.$$
$$\left.+\frac{3+3\sqrt{3}i}{4}\cot^2\left(\sqrt{\frac{3+3\sqrt{3}i}{8\Delta}}(kx+wt)\right)\right\}. \tag{45}$$

Set VI: We have the following:

$$a=0,\quad b=\sqrt{c^2-\frac{3}{\Delta}},\quad c=c,\quad A_1=0,\quad A_2=\frac{3}{2}-c^2\Delta+c\Delta\sqrt{c^2-\frac{3}{\Delta}},\quad A_0=-\frac{1}{2}, \tag{46}$$

$$B_1=0,\quad B_2=0,\quad k=k,\quad w=w,\quad v(\xi)=A_0+A_2\tan^2\left(\frac{\Phi(\xi)}{2}\right). \tag{47}$$

By using of the **Families 5, 6** and **18** can be written, respectively, as

$$u_{24}(x,t)=\ln\left\{-\frac{1}{2}+\left(\frac{3}{2}+c\Delta\sqrt{c^2-\frac{3}{\Delta}}-c^2\Delta\right)\frac{\sqrt{c^2-\frac{3}{\Delta}}+c}{\sqrt{c^2-\frac{3}{\Delta}}-c}\tanh^2\left(\frac{1}{2}\sqrt{-\frac{3}{\Delta}}(kx+wt)\right)\right\}, \tag{48}$$

$$u_{25}(x,t)=\ln\left\{-\frac{1}{2}+\frac{3}{2}\tan^2\left(\frac{1}{2}\arctan\left[\frac{e^{\sqrt{-\frac{12}{\Delta}}(kx+wt)}-1}{e^{\sqrt{-\frac{12}{\Delta}}(kx+wt)}+1},\ \frac{2e^{\sqrt{-\frac{3}{\Delta}}(kx+wt)}}{e^{\sqrt{-\frac{12}{\Delta}}(kx+wt)}+1}\right]\right)\right\}, \tag{49}$$

$$u_{26}(x,t)=\ln\left\{-\frac{1}{2}-\frac{3}{2}\tan^2\left(\sqrt{\frac{3}{4\Delta}}(kx+wt)\right)\right\}. \tag{50}$$

Set VII: We have the following:

$$a=0,\quad b=\sqrt{ic^2-\frac{3i+3\sqrt{3}}{2\Delta}},\quad c=c,\quad A_1=0,\quad A_2=-\frac{1}{2}\Delta(b-c)^2, \tag{51}$$

$$A_0=\frac{c^2(i-1)\Delta}{6}+\frac{i+\sqrt{3}}{4},\quad B_1=0,\quad B_2=0,\quad k=k,\quad w=w,$$
$$v(\xi)=A_0+A_2\tan^2\left(\frac{\Phi(\xi)}{2}\right). \tag{52}$$

By using of the **Families 5, 6** and **18** can be written, respectively, as

$$u_{27}(x,t)=\ln\left\{\frac{c^2(i-1)\Delta}{6}+\frac{i+\sqrt{3}}{4}-\frac{1}{2}\Delta\left((i-1)c^2-\frac{3i+3\sqrt{3}}{2\Delta}\right)\tanh^2\left(\frac{1}{2}\sqrt{(i-1)c^2-\frac{3i+3\sqrt{3}}{2\Delta}}(kx+wt)\right)\right\},\tag{53}$$

$$u_{28}(x,t)=\ln\left\{\frac{i+\sqrt{3}}{4}+\frac{3i+3\sqrt{3}}{4}\tan^2\left(\frac{1}{2}\arctan\left[\frac{e^{\sqrt{-\frac{6i+6\sqrt{3}}{\Delta}}(kx+wt)}-1}{e^{\sqrt{-\frac{6i+6\sqrt{3}}{\Delta}}(kx+wt)}+1},\ \frac{2e^{\sqrt{-\frac{3i+3\sqrt{3}}{2\Delta}}(2kx+2wt)}}{e^{\sqrt{-\frac{6i+6\sqrt{3}}{\Delta}}(kx+wt)}+1}\right]\right)\right\},\tag{54}$$

$$u_{29}(x,t)=\ln\left\{\frac{(2\sqrt{3}-1)+(2+\sqrt{3})i}{4}-\frac{3-3\sqrt{3}i}{4}\tan^2\left(\sqrt{\frac{3-3\sqrt{3}i}{8\Delta}}(kx+wt)\right)\right\}.\tag{55}$$

Set VIII: We have the following:

$$a=0,\quad b=\sqrt{c^2-\frac{3}{4\Delta}},\quad c=c,\quad A_0=\frac{1}{4},\quad A_1=0,\quad A_2=\frac{3}{8}-\Delta c^2+c\Delta\sqrt{c^2-\frac{3}{4\Delta}},\quad B_1=0,\tag{56}$$

$$B_2=\frac{3}{8}-\Delta c^2-c\Delta\sqrt{c^2-\frac{3}{4\Delta}},\quad k=k,\quad w=w,$$
$$v(\xi)=A_0+A_2\tan^2\left(\frac{\Phi(\xi)}{2}\right)+B_2\cot^2\left(\frac{\Phi(\xi)}{2}\right),\tag{57}$$

where $\Delta=k^2-w^2$. By using of the **Families 5, 6** and **18** can be written, respectively, as

$$u_{30}(x,t)=\ln\left\{\frac{1}{4}+\left(\frac{3}{8}-\Delta c^2+c\Delta\sqrt{c^2-\frac{3}{4\Delta}}\right)\frac{\sqrt{c^2-\frac{3}{4\Delta}}+c}{\sqrt{c^2-\frac{3}{4\Delta}}-c}\tanh^2\left(\frac{1}{4}\sqrt{-\frac{3}{\Delta}}(kx+wt)\right)\right.$$
$$\left.+\left(\frac{3}{8}-\Delta c^2-c\Delta\sqrt{c^2-\frac{3}{4\Delta}}\right)\frac{\sqrt{c^2-\frac{3}{4\Delta}}-c}{\sqrt{c^2-\frac{3}{4\Delta}}+c}\coth^2\left(\frac{1}{4}\sqrt{-\frac{3}{\Delta}}(kx+wt)\right)\right\},\tag{58}$$

$$u_{31}(x,t)=\ln\left\{\frac{1}{4}+\frac{3}{8}\tan^2\left(\frac{1}{2}\arctan\left[\frac{e^{\sqrt{-\frac{3}{\Delta}}(kx+wt)}-1}{e^{\sqrt{-\frac{3}{\Delta}}(kx+wt)}+1},\frac{2e^{\frac{1}{2}\sqrt{-\frac{3}{\Delta}}(kx+wt)}}{e^{\sqrt{-\frac{3}{\Delta}}(kx+wt)}+1}\right]\right)\right.$$
$$\left.+\frac{3}{8}\cot^2\left(\frac{1}{2}\arctan\left[\frac{e^{\sqrt{-\frac{3}{\Delta}}(kx+wt)}-1}{e^{\sqrt{-\frac{3}{\Delta}}(kx+wt)}+1},\frac{2e^{\frac{1}{2}\sqrt{-\frac{3}{\Delta}}(kx+wt)}}{e^{\sqrt{-\frac{3}{\Delta}}(kx+wt)}+1}\right]\right)\right\},\tag{59}$$

$$u_{32}(x,t)=\ln\left\{\frac{1}{4}-\frac{3}{8}\tan^2\left(\frac{1}{2}\sqrt{\frac{3}{\Delta}}(kx+wt)\right)-\frac{3}{8}\cot^2\left(\frac{1}{2}\sqrt{\frac{3}{\Delta}}(kx+wt)\right)\right\}.\tag{60}$$

Set IX: We have the following:

$$a=0, b=\sqrt{c^2+\frac{3-3\sqrt{3}i}{8\Delta}}, \quad c=c, \quad A_1=0, \quad A_2=-\frac{1}{2}\Delta(b-c)^2,$$
$$A_0=\frac{1}{3}(c^2-b^2)\Delta, \quad B_1=0, \tag{61}$$

$$B_2=-\frac{1}{2}\Delta(b+c)^2, \quad k=k, \quad w=w, \quad v(\xi)=A_0+A_2\tan^2\left(\frac{\Phi(\xi)}{2}\right)+B_2\cot^2\left(\frac{\Phi(\xi)}{2}\right), \tag{62}$$

where $\Delta=k^2-w^2$. By using of the **Families 5, 6** and **18** can be written, respectively, as

$$u_{33}(x,t)=\ln\left\{\frac{3\sqrt{3}i-3}{24}+\frac{3\sqrt{3}i-3}{16}\left[\tanh^2\left(\frac{1}{4}\sqrt{\frac{3-3\sqrt{3}i}{2\Delta}}(kx+wt)\right)+\coth^2\left(\frac{1}{4}\sqrt{\frac{3-3\sqrt{3}i}{2\Delta}}(kx+wt)\right)\right]\right\}, \tag{63}$$

$$u_{34}(x,t)=\ln\left\{\frac{3\sqrt{3}i-3}{24}+\frac{3\sqrt{3}i-3}{16}\tan^2\left(\frac{1}{2}\arctan\left[\frac{e^{\sqrt{\frac{3-3\sqrt{3}i}{2\Delta}}(kx+wt)}-1}{e^{\sqrt{\frac{3-3\sqrt{3}i}{2\Delta}}(kx+wt)}+1},\frac{2e^{\frac{1}{2}\sqrt{\frac{3-3\sqrt{3}i}{2\Delta}}(kx+wt)}}{e^{\sqrt{\frac{3-3\sqrt{3}i}{2\Delta}}(kx+wt)}+1}\right]\right)\right.$$
$$\left.+\frac{3\sqrt{3}i-3}{16}\cot^2\left(\frac{1}{2}\arctan\left[\frac{e^{\sqrt{\frac{3-3\sqrt{3}i}{2\Delta}}(kx+wt)}-1}{e^{\sqrt{\frac{3-3\sqrt{3}i}{2\Delta}}(kx+wt)}+1},\frac{2e^{\frac{1}{2}\sqrt{\frac{3-3\sqrt{3}i}{2\Delta}}(kx+wt)}}{e^{\sqrt{\frac{3-3\sqrt{3}i}{2\Delta}}(kx+wt)}+1}\right]\right)\right\}, \tag{64}$$

$$u_{35}(x,t)=\ln\left\{\frac{3\sqrt{3}i-3}{24}-\frac{3\sqrt{3}i-3}{16}\tan^2\left(\frac{1}{4}\sqrt{\frac{3\sqrt{3}i-3}{2\Delta}}(kx+wt)\right)-\frac{3\sqrt{3}i-3}{16}\cot^2\left(\frac{1}{4}\sqrt{\frac{3\sqrt{3}i-3}{2\Delta}}\xi+C\right)\right\}. \tag{65}$$

3.2 Dodd–Bullough–Mikhailov (DBM) equation

Now we consider the Dodd–Bullough–Mikhailov (DBM) equation as follows

$$u_{xt}+e^u+e^{-2u}=0, \tag{66}$$

by using the transformation $v(x,t)=e^{-u(x,t)}$, Eq. (66) transforms into the following partial differential equation,

$$vv_{xt}-v_xv_t+v^3+1=0. \tag{67}$$

The travelling wave transformation $\xi=kx+wt$ reduces Eq. (67) to the following ODE

$$kw(vv''-(v')^2)+v^3+1=0, \tag{68}$$

where $v'=\frac{dv}{d\xi}$ and $v''=\frac{d^2v}{d\xi^2}$. Balancing the terms vv'' and v^3 by using homogenous principle one can be found

$$2M+2=3M, \quad M=2. \tag{69}$$

Here, for simplicity, we set $p=0$. Then (6) reduces to

$$v(\xi) = A_0 + A_1 \tan\left(\frac{\Phi(\xi)}{2}\right) + A_2 \tan^2\left(\frac{\Phi(\xi)}{2}\right) + B_1 \cot\left(\frac{\Phi(\xi)}{2}\right) + B_2 \cot^2\left(\frac{\Phi(\xi)}{2}\right). \quad (70)$$

Substituting (70) and (7) into Eq. (68), the following solutions are obtained.

Set I: We have the following:

$$\begin{aligned} &a = \sqrt{c^2 - b^2 + \frac{3}{kw}}, \quad b = b, \quad c = c, A_0 = -1 + \frac{1}{2}kw(b^2 - c^2), \\ &A_1 = 0, \quad A_2 = 0, \quad k = k, \quad w = w, \end{aligned} \quad (71)$$

$$\begin{aligned} B_1 &= -(b + c)kw\sqrt{c^2 - b^2 + \frac{3}{kw}}, \quad B_2 = -\frac{1}{2}kw(b + c)^2, \\ v(\xi) &= A_0 + B_1 \cot\left(\frac{\Phi(\xi)}{2}\right) + B_2 \cot^2\left(\frac{\Phi(\xi)}{2}\right), \end{aligned} \quad (72)$$

By using of the **Families 1, 2, 12** and **19** can be written, respectively, as

$$\begin{aligned} u_1(x,t) = \ln\Bigg\{ &-1 + \frac{1}{2}kw(b^2 - c^2) - (b^2 - c^2)kw\left[1 - \sqrt{-\frac{3}{3 - kw(b^2 - c^2)}}\tan\left(\frac{1}{2}\sqrt{-\frac{3}{kw}}(kx + wt)\right)\right]^{-1} \\ &- \frac{1}{2}\frac{k^2w^2(b^2 - c^2)^2}{3 - (b^2 - c^2)kw}\left[1 - \sqrt{-\frac{3}{3 - kw(b^2 - c^2)}}\tan\left(\frac{1}{2}\sqrt{-\frac{3}{kw}}(kx + wt)\right)\right]^{-2}\Bigg\}, \end{aligned} \quad (73)$$

$$\begin{aligned} u_2(x,t) = \ln\Bigg\{ &-1 + \frac{1}{2}kw(b^2 - c^2) - (b^2 - c^2)kw\left[1 + \sqrt{\frac{3}{3 - kw(b^2 - c^2)}}\tanh\left(\frac{1}{2}\sqrt{\frac{3}{kw}}(kx + wt)\right)\right]^{-1} \\ &- \frac{1}{2}\frac{k^2w^2(b^2 - c^2)^2}{3 - (b^2 - c^2)kw}\left[1 + \sqrt{\frac{3}{3 - kw(b^2 - c^2)}}\tanh\left(\frac{1}{2}\sqrt{\frac{3}{kw}}(kx + wt)\right)\right]^{-2}\Bigg\}, \end{aligned} \quad (74)$$

$$\begin{aligned} u_3(x,t) = \ln\Bigg\{ &-1 + \frac{1}{2}(3 - kwc^2) - \left(\sqrt{\frac{3}{kw}} + c\right)kwc\left[\frac{\left(\sqrt{\frac{3}{kw}} + c\right)e^{\sqrt{\frac{3}{kw}}(kx+wt)} + 1}{\left(\sqrt{\frac{3}{kw}} - c\right)e^{\sqrt{\frac{3}{kw}}(kx+wt)} - 1}\right]^{-1} \\ &- \frac{1}{2}kw\left(\sqrt{\frac{3}{kw}} + c\right)^2\left[\frac{\left(\sqrt{\frac{3}{kw}} + c\right)e^{\sqrt{\frac{3}{kw}}(kx+wt)} + 1}{\left(\sqrt{\frac{3}{kw}} - c\right)e^{\sqrt{\frac{3}{kw}}(kx+wt)} - 1}\right]^{-2}\Bigg\}, \end{aligned} \quad (75)$$

$$u_4(x,t) = \ln\left\{-1 - 6c\left[\frac{1}{e^{\sqrt{\frac{3}{kw}}(kx+wt)} - c}\right] - 6c^2\left[\frac{1}{e^{\sqrt{\frac{3}{kw}}(kx+wt)} - c}\right]^2\right\}. \quad (76)$$

Set II: We have the following:

$$a=\sqrt{c^2-b^2-\frac{3+3\sqrt{3}i}{2kw}},\quad b=b,\quad c=c, A_0=\frac{1+\sqrt{3}i}{2}+\frac{1}{2}kw(b^2-c^2),\quad A_1=0,$$
$$A_2=0,\quad k=k,\quad w=w, \tag{77}$$

$$B_1=-(b+c)kw\sqrt{c^2-b^2-\frac{3+3\sqrt{3}i}{2kw}},\quad B_2=-\frac{1}{2}kw(b+c)^2,$$
$$v(\xi)=A_0+B_1\cot\left(\frac{\Phi(\xi)}{2}\right)+B_2\cot^2\left(\frac{\Phi(\xi)}{2}\right). \tag{78}$$

By using of the **Families 1, 2, 12** and **19** can be written, respectively, as

$$u_5(x,t)=\ln\left\{\frac{1+\sqrt{3}i+kw(b^2-c^2)}{2}-(b^2-c^2)kw\left[1-\sqrt{-\frac{3+3\sqrt{3}i}{3+3\sqrt{3}i+2kw(b^2-c^2)}}\tan\left(\sqrt{\frac{3+3\sqrt{3}i}{8kw}}(kx+wt)\right)\right]^{-1}\right.$$
$$\left.+\frac{1}{2}\frac{k^2w^2(b^2-c^2)^2}{(b^2-c^2)kw+3+3\sqrt{3}i}\left[1-\sqrt{-\frac{3+3\sqrt{3}i}{3+3\sqrt{3}i+2kw(b^2-c^2)}}\tan\left(\sqrt{\frac{3+3\sqrt{3}i}{8kw}}(kx+wt)\right)\right]^{-2}\right\}, \tag{79}$$

$$u_6(x,t)=\ln\left\{\frac{1+\sqrt{3}i+kw(b^2-c^2)}{2}-(b^2-c^2)kw\left[1+\sqrt{\frac{3+3\sqrt{3}i}{3+3\sqrt{3}i+2kw(b^2-c^2)}}\tanh\left(\sqrt{-\frac{3+3\sqrt{3}i}{8kw}}(kx+wt)\right)\right]^{-1}\right.$$
$$\left.+\frac{1}{2}\frac{k^2w^2(b^2-c^2)^2}{(b^2-c^2)kw+3+3\sqrt{3}i}\left[1+\sqrt{\frac{3+3\sqrt{3}i}{3+3\sqrt{3}i+2kw(b^2-c^2)}}\tanh\left(\sqrt{-\frac{3+3\sqrt{3}i}{8kw}}(kx+wt)\right)\right]^{-2}\right\}, \tag{80}$$

$$u_7(x,t)=\ln\left\{-\frac{1+\sqrt{3}i+2kwc^2}{4}-\left(\sqrt{-\frac{3+3\sqrt{3}i}{2kw}}+c\right)kwc\left[\frac{\left(\sqrt{-\frac{3+3\sqrt{3}i}{2kw}}+c\right)e^{\sqrt{-\frac{3+3\sqrt{3}i}{2kw}}(kx+wt)}+1}{\left(\sqrt{-\frac{3+3\sqrt{3}i}{2kw}}-c\right)e^{\sqrt{-\frac{3+3\sqrt{3}i}{2kw}}(kx+wt)}-1}\right]^{-1}\right.$$
$$\left.-\frac{1}{2}kw\left(\sqrt{-\frac{3+3\sqrt{3}i}{2kw}}+c\right)^2\left[\frac{\left(\sqrt{-\frac{3+3\sqrt{3}i}{2kw}}+c\right)e^{\sqrt{-\frac{3+3\sqrt{3}i}{2kw}}(kx+wt)}+1}{\left(\sqrt{-\frac{3+3\sqrt{3}i}{2kw}}-c\right)e^{\sqrt{-\frac{3+3\sqrt{3}i}{2kw}}(kx+wt)}-1}\right]^{-2}\right\}, \tag{81}$$

$$u_8(x,t)=\ln\left\{\frac{1+\sqrt{3}i}{2}+\left[\frac{c(3+3\sqrt{3}i)}{e^{\sqrt{-\frac{3+3\sqrt{3}i}{2kw}}(kx+wt)}-c}\right]+c^2(3+3\sqrt{3}i)\left[\frac{1}{e^{\sqrt{-\frac{3+3\sqrt{3}i}{2kw}}(kx+wt)}-c}\right]^2\right\}. \tag{82}$$

Set III: We have the following:

$$a=a,\quad b=b,\quad c=c, A_0=-\frac{2a^2+c^2-b^2}{2(a^2+b^2-c^2)},$$
$$A_1=\frac{3a(b-c)}{a^2+b^2-c^2}, A_2=-\frac{3(b-c)^2}{2(a^2+b^2-c^2)}, \tag{83}$$

$$B_1 = 0, \quad B_2 = 0, \quad k = \frac{3}{w(a^2 + b^2 - c^2)}, \\ w = w, v(\xi) = A_0 + A_1 \tan\left(\frac{\Phi(\xi)}{2}\right) + A_2 \tan^2\left(\frac{\Phi(\xi)}{2}\right). \tag{84}$$

By using of the **Families 1, 2, 6, 10, 12** and **18** can be written, respectively, as

$$u_9(x,t) = \ln\left\{\frac{6a^2 + b^2 - c^2}{2(a^2 + b^2 - c^2)} + \frac{3a}{\sqrt{c^2 - b^2 - a^2}} \tan\left(\frac{\sqrt{c^2 - b^2 - a^2}}{2}\left(\frac{3}{w(a^2 + b^2 - c^2)}x + wt\right)\right)\right. \\ \left. - \frac{3}{2(a^2 + b^2 - c^2)}\left[a - \sqrt{c^2 - b^2 - a^2} \tan\left(\frac{\sqrt{c^2 - b^2 - a^2}}{2}\left(\frac{3}{w(a^2 + b^2 - c^2)}x + wt\right)\right)\right]^2\right\}, \tag{85}$$

$$u_{10}(x,t) = \ln\left\{\frac{6a^2 + b^2 - c^2}{2(a^2 + b^2 - c^2)} - \frac{3a}{\sqrt{a^2 + b^2 - c^2}} \tanh\left(\frac{\sqrt{a^2 + b^2 - c^2}}{2}\left(\frac{3}{w(a^2 + b^2 - c^2)}x + wt\right)\right)\right. \\ \left. - \frac{3}{2(a^2 + b^2 - c^2)}\left[a + \sqrt{a^2 + b^2 - c^2} \tanh\left(\frac{\sqrt{a^2 + b^2 - c^2}}{2}\left(\frac{3}{w(a^2 + b^2 - c^2)}x + wt\right)\right)\right]^2\right\}, \tag{86}$$

$$u_{11}(x,t) = \ln\left\{\frac{1}{2} - \frac{3}{2} tan^2\left(\frac{1}{2}\arctan\left[\frac{e^{2b\left(\frac{3}{wb^2}x + wt\right)} - 1}{e^{2b\left(\frac{3}{wb^2}x + wt\right)} + 1}, \frac{2e^{b\left(\frac{3}{wb^2}x + wt\right)}}{e^{2b\left(\frac{3}{wb^2}x + wt\right)} + 1}\right]\right)\right\}, \tag{87}$$

$$u_{12}(x,t) = \ln\left\{-1 - 6\left[\frac{e^{ka\left(\frac{3}{wk^2a^2}x + wt\right)}}{1 - e^{ka\left(\frac{3}{wk^2a^2}x + wt\right)}}\right] - 6\left[\frac{e^{ka\left(\frac{3}{wk^2a^2}x + wt\right)}}{1 - e^{ka\left(\frac{3}{wk^2a^2}x + wt\right)}}\right]^2\right\}, \tag{88}$$

$$u_{13}(x,t) = \ln\left\{\frac{b^2 - 3c^2}{2b^2} + \frac{3c(b-c)}{b^2}\left[\frac{(b+c)e^{b\left(\frac{3}{wb^2}x + wt\right)} + 1}{(b-c)e^{b\left(\frac{3}{wb^2}x + wt\right)} - 1}\right] - \frac{3(b-c)^2}{2b^2}\left[\frac{(b+c)e^{b\left(\frac{3}{wb^2}x + wt\right)} + 1}{(b-c)e^{b\left(\frac{3}{wb^2}x + wt\right)} - 1}\right]^2\right\}, \tag{89}$$

$$u_{14}(x,t) = \ln\left\{-\frac{1}{2} - \frac{3}{2}\tan^2\left[\frac{1}{2}\left(-\frac{3}{cw}x + cwt + C\right)\right]\right\}. \tag{90}$$

Set IV: We have the following:

$$a = \sqrt{c^2 - b^2 - \frac{3 - 3\sqrt{3}i}{2kw}}, \quad b = b, \quad c = c, A_0 = \frac{1}{2}kw(b^2 - c^2) + \frac{3 - 3\sqrt{3}i}{6}, \quad B_1 = 0, \\ B_2 = 0, \quad k = k, \tag{91}$$

$$w = w, \quad A_1 = (b - c)kwa, \quad A_2 = -\frac{1}{2}kw(b - c)^2, \\ v(\xi) = A_0 + A_1 \tan\left(\frac{\Phi(\xi)}{2}\right) + A_2 \tan^2\left(\frac{\Phi(\xi)}{2}\right). \tag{92}$$

By using of the **Families 1, 2, 14** and **18** can be written, respectively, as

$$u_{15}(x,t)=\ln\left\{\frac{1}{2}kw(c^2-b^2)-\frac{3-3\sqrt{3}i}{3}-\frac{1}{2}\sqrt{18(1+\sqrt{3}i)-6kw(1-\sqrt{3}i)(b^2-c^2)}\tan\left(\sqrt{\frac{3-3\sqrt{3}i}{8kw}}(kx+wt)\right)\right.$$
$$\left.-\frac{1}{2}kw\left[\sqrt{c^2-b^2-\frac{3-3\sqrt{3}i}{2kw}}-\sqrt{\frac{3-3\sqrt{3}i}{2kw}}\tan\left(\sqrt{\frac{3-3\sqrt{3}i}{8kw}}(kx+wt)\right)\right]^2\right\}, \tag{93}$$

$$u_{16}(x,t)=\ln\left\{\frac{1}{2}kw(c^2-b^2)-\frac{3-3\sqrt{3}i}{3}+\frac{1}{2}\sqrt{6kw(1-\sqrt{3}i)(b^2-c^2)-18(1+\sqrt{3}i)}\tanh\left(\sqrt{\frac{3\sqrt{3}i-3}{8kw}}(kx+wt)\right)\right.$$
$$\left.-\frac{1}{2}kw\left[\sqrt{c^2-b^2-\frac{3-3\sqrt{3}i}{2kw}}+\sqrt{\frac{3\sqrt{3}i-3}{2kw}}\tanh\left(\sqrt{\frac{3\sqrt{3}i-3}{8kw}}(kx+wt)\right)\right]^2\right\}, \tag{94}$$

$$u_{17}(x,t)=\ln\left\{\frac{3-3\sqrt{3}i}{6}+2ckw\sqrt{-\frac{3-3\sqrt{3}i}{2kw}}\left[\frac{\sqrt{-\frac{3-3\sqrt{3}i}{2kw}}e^{\sqrt{-\frac{3-3\sqrt{3}i}{2kw}}(kx+wt)}}{ce^{\sqrt{-\frac{3-3\sqrt{3}i}{2kw}}(kx+wt)}-1}\right]\right.$$
$$\left.-2kwc^2\left[\frac{\sqrt{-\frac{3-3\sqrt{3}i}{2kw}}e^{\sqrt{-\frac{3-3\sqrt{3}i}{2kw}}(kx+wt)}}{ce^{\sqrt{-\frac{3-3\sqrt{3}i}{2kw}}(kx+wt)}-1}\right]^2\right\}, \tag{95}$$

$$u_{18}(x,t)=\ln\left\{\frac{1-\sqrt{3}i-kwc^2}{2}-\frac{1}{2}kwc^2\tan^2\left(\sqrt{\frac{3-3\sqrt{3}i}{8kw}}(kx+wt)+C\right)\right\}. \tag{96}$$

Set V: We have the following:

$$a=\sqrt{\frac{b^2-c^2}{2}},\quad b=b,\quad c=c,\quad A_0=0,\quad B_1=0,\quad B_2=0,\quad A_1=\frac{2}{b+c}\sqrt{\frac{b^2-c^2}{2}},$$
$$w=w, \tag{97}$$

$$A_2=-\frac{b-c}{b+c},\quad k=\frac{2}{w(b^2-c^2)}\left(\frac{-1+\sqrt{3}i}{2}\right),$$
$$v(\xi)=A_1\tan\left(\frac{\Phi(\xi)}{2}\right)+A_2\tan^2\left(\frac{\Phi(\xi)}{2}\right). \tag{98}$$

By using of the **Families 1** and **2** can be written, respectively, as

$$u_{19}(x,t)=\ln\left\{1-\sqrt{-3}\tan\left(\sqrt{-\frac{3}{8}(b^2-c^2)}\left(\frac{2}{w(b^2-c^2)}\left(\frac{-1+\sqrt{3}i}{2}\right)x+wt\right)\right)\right.$$
$$\left.-\frac{1}{2}\left[1-\sqrt{-3}\tan\left(\sqrt{-\frac{3}{8}(b^2-c^2)}\left(\frac{2}{w(b^2-c^2)}\left(\frac{-1+\sqrt{3}i}{2}\right)x+wt\right)\right)\right]^2\right\}, \tag{99}$$

$$u_{20}(x,t) = \ln\left\{1+\sqrt{3}\tanh\left(\sqrt{\frac{3}{8}(b^2-c^2)}\left(\frac{2}{w(b^2-c^2)}\left(\frac{-1+\sqrt{3}i}{2}\right)x+wt\right)\right) - \frac{1}{2}\left[1+\sqrt{3}\tanh\left(\sqrt{\frac{3}{8}(b^2-c^2)}\left(\frac{2}{w(b^2-c^2)}\left(\frac{-1+\sqrt{3}i}{2}\right)x+wt\right)\right)\right]^2\right\}. \tag{100}$$

Set VI: We have the following:

$$a = \sqrt{\frac{b^2-c^2}{2}}, \quad b=b, \quad c=c, A_0=0, \quad B_1=0, \\ B_2=0, A_1=\frac{2}{b+c}\sqrt{\frac{b^2-c^2}{2}}\left(\frac{-1+\sqrt{3}i}{2}\right), w=w, \tag{101}$$

$$A_2 = -\frac{b-c}{b+c}\left(\frac{-1+\sqrt{3}i}{2}\right), \quad k=\frac{2}{w(b^2-c^2)}\left(\frac{-1+\sqrt{3}i}{2}\right), \\ v(\xi) = A_1\tan\left(\frac{\Phi(\xi)}{2}\right)+A_2\tan^2\left(\frac{\Phi(\xi)}{2}\right). \tag{102}$$

By using of the **Families 1** and **2** can be written, respectively, as

$$u_{21}(x,t) = \ln\left\{\frac{-1+\sqrt{3}i}{2}\left[1-\sqrt{-3}\tan\left(\sqrt{-\frac{3}{8}(b^2-c^2)}\left(\frac{2}{w(b^2-c^2)}\left(\frac{-1+\sqrt{3}i}{2}\right)x+wt\right)\right)\right] - \frac{-1+\sqrt{3}i}{4}\left[1-\sqrt{-3}\tan\left(\sqrt{-\frac{3}{8}(b^2-c^2)}\left(\frac{2}{w(b^2-c^2)}\left(\frac{-1+\sqrt{3}i}{2}\right)x+wt\right)\right)\right]^2\right\}, \tag{103}$$

$$u_{22}(x,t) = \ln\left\{\frac{-1+\sqrt{3}i}{2}\left[1+\sqrt{3}\tanh\left(\sqrt{\frac{3}{8}(b^2-c^2)}\left(\frac{2}{w(b^2-c^2)}\left(\frac{-1+\sqrt{3}i}{2}\right)x+wt\right)\right)\right] - \frac{-1+\sqrt{3}i}{4}\left[1+\sqrt{3}\tanh\left(\sqrt{\frac{3}{8}(b^2-c^2)}\left(\frac{2}{w(b^2-c^2)}\left(\frac{-1+\sqrt{3}i}{2}\right)x+wt\right)\right)\right]^2\right\}. \tag{104}$$

Set VII: We have the following:

$$a=0, \quad b=b, \quad c=c, \quad A_0=\frac{1}{2}, \quad A_1=0, \quad A_2=-\frac{3}{2}\frac{b-c}{b+c}, \quad B_1=0, \quad B_2=0, \tag{105}$$

$$k=\frac{3}{w(b^2-c^2)}, \quad w=w, \quad v(\xi)=A_0+A_2\tan^2\left(\frac{\Phi(\xi)}{2}\right). \tag{106}$$

By using of the **Families 5** and **6** can be written, respectively, as

$$u_{23}(x,t) = \ln\left\{\frac{1}{2}-\frac{3}{2}\tan^2\left(\frac{\sqrt{b^2-c^2}}{2}\left(\frac{3}{w(b^2-c^2)}x+wt+C\right)\right)\right\}, \tag{107}$$

$$u_{24}(x,t) = \ln\left\{\frac{1}{2} - \frac{3}{2}\tan^2\left(\frac{1}{2}\arctan\left[\frac{e^{2b\left(\frac{3}{wb^2}x+wt+C\right)}-1}{e^{2b\left(\frac{3}{wb^2}x+wt+C\right)}+1}, \frac{2e^{b\left(\frac{3}{wb^2}x+wt+C\right)}}{e^{2b\left(\frac{3}{wb^2}x+wt+C\right)}+1}\right]\right)\right\}. \quad (108)$$

Set VIII: We have the following:

$$\begin{aligned} &a = 0, \quad b = b, \quad c = c, \quad A_0 = \left(\frac{-1+\sqrt{3}i}{4}\right), \quad A_1 = 0, \\ &A_2 = -\frac{3}{2}\frac{b-c}{b+c}\left(\frac{-1+\sqrt{3}i}{2}\right), \quad B_1 = 0, \end{aligned} \quad (109)$$

$$B_2 = 0, \quad k = \frac{3}{w(b^2-c^2)}\left(\frac{-1+\sqrt{3}i}{2}\right), \quad w = w, \quad v(\xi) = A_0 + A_2\tan^2\left(\frac{\Phi(\xi)}{2}\right). \quad (110)$$

By using of the **Families 5** and **6** can be written, respectively, as

$$u_{25}(x,t) = \ln\left\{\frac{-1+\sqrt{3}i}{4} - \frac{3}{2}\left(\frac{-1+\sqrt{3}i}{4}\right)\tan^2\left(\frac{\sqrt{b^2-c^2}}{2}\left(\frac{3}{w(b^2-c^2)}\left(\frac{-1+\sqrt{3}i}{4}\right)x + wt + C\right)\right)\right\}, \quad (111)$$

$$u_{26}(x,t) = \ln\left\{\frac{-1+\sqrt{3}i}{4}\left[1 - \frac{3}{2}\tan^2\left(\frac{1}{2}\arctan\left[\frac{e^{2b\left(\frac{3}{wb^2}\left(\frac{-1+\sqrt{3}i}{2}\right)x+wt\right)}-1}{e^{2b\left(\frac{3}{wb^2}\left(\frac{-1+\sqrt{3}i}{2}\right)x+wt\right)}+1}, \frac{2e^{b\left(\frac{3}{wb^2}\left(\frac{-1+\sqrt{3}i}{2}\right)x+wt\right)}}{e^{2b\left(\frac{3}{wb^2}\left(\frac{-1+\sqrt{3}i}{2}\right)x+wt\right)}+1}\right]\right)\right]\right\}. \quad (112)$$

Set IX: We have the following:

$$\begin{aligned} &a = 0, \quad b = b, \quad c = c, \quad A_0 = -\frac{1}{4}, \quad A_1 = 0, \quad A_2 = -\frac{3}{8}\frac{b-c}{b+c}, \\ &B_1 = 0, B_2 = -\frac{3}{8}\frac{b+c}{b-c}, \end{aligned} \quad (113)$$

$$k = \frac{3}{4w(b^2-c^2)}, \quad w = w, \quad v(\xi) = A_0 + A_2\tan^2\left(\frac{\Phi(\xi)}{2}\right) + B_2\cot^2\left(\frac{\Phi(\xi)}{2}\right). \quad (114)$$

By using of the **Families 5** and **6** can be written, respectively, as

$$u_{27}(x,t) = \ln\left\{-\frac{1}{4} - \frac{3}{8}\left[\tan^2\left(\frac{\sqrt{b^2-c^2}}{2}\left(\frac{3}{4w(b^2-c^2)}x + wt\right)\right) + \cot^2\left(\frac{\sqrt{b^2-c^2}}{2}\left(\frac{3}{4w(b^2-c^2)}x + wt\right)\right)\right]\right\}, \quad (115)$$

$$\begin{aligned} u_{28}(x,t) = \ln\Bigg\{ &-\frac{1}{4} - \frac{3}{8}\tan^2\left(\frac{1}{2}\arctan\left[\frac{e^{2b\left(\frac{3}{4wb^2}x+wt\right)}-1}{e^{2b\left(\frac{3}{4wb^2}x+wt\right)}+1}, \frac{2e^{b\left(\frac{3}{4wb^2}x+wt\right)}}{e^{2b\left(\frac{3}{4wb^2}x+wt\right)}+1}\right]\right) \\ &- \frac{3}{8}\cot^2\left(\frac{1}{2}\arctan\left[\frac{e^{2b\left(\frac{3}{4wb^2}x+wt\right)}-1}{e^{2b\left(\frac{3}{4wb^2}x+wt\right)}+1}, \frac{2e^{b\left(\frac{3}{4wb^2}x+wt\right)}}{e^{2b\left(\frac{3}{4wb^2}x+wt\right)}+1}\right]\right)\Bigg\}. \end{aligned} \quad (116)$$

Set X: We have the following:

$$a=0,\quad b=b,\quad c=c,\quad A_1=0,\quad A_0=-\frac{1}{4}\left(\frac{-1+\sqrt{3}i}{2}\right),\\ A_2=B_2=-\frac{3}{8}\frac{b-c}{b+c}\left(\frac{-1+\sqrt{3}i}{2}\right),\tag{117}$$

$$B_1=0,k=\frac{3}{4w(b^2-c^2)}\left(\frac{-1+\sqrt{3}i}{2}\right),\quad w=w,\\ v(\xi)=A_0+A_2\tan^2\left(\frac{\Phi(\xi)}{2}\right)+B_2\cot^2\left(\frac{\Phi(\xi)}{2}\right).\tag{118}$$

By using of the **Families 5** and **6** can be written, respectively, as

$$u_{29}(x,t)=\ln\left\{-\frac{1}{4}\left(\frac{-1+\sqrt{3}i}{2}\right)-\frac{3}{8}\left(\frac{-1+\sqrt{3}i}{2}\right)\left[\tan^2\left(\frac{\sqrt{b^2-c^2}}{2}\left(\frac{3}{4w(b^2-c^2)}\left(\frac{-1+\sqrt{3}i}{2}\right)x+wt+C\right)\right)+\right.\right.\\ \left.\left.\times\cot^2\left(\frac{\sqrt{b^2-c^2}}{2}\left(\frac{3}{4w(b^2-c^2)}\left(\frac{-1+\sqrt{3}i}{2}\right)x+wt+C\right)\right)\right]\right\},\tag{119}$$

$$u_{30}(x,t)=\ln\left\{-\frac{1}{4}-\frac{3}{8}\left(\frac{-1+\sqrt{3}i}{2}\right)\tan^2\left(\frac{1}{2}\arctan\left[\frac{e^{2b\left(\frac{3}{4wb^2}\left(\frac{-1+\sqrt{3}i}{2}\right)x+wt\right)}-1}{e^{2b\left(\frac{3}{4wb^2}\left(\frac{-1+\sqrt{3}i}{2}\right)x+wt\right)}+1},\frac{2e^{b\left(\frac{3}{4wb^2}\left(\frac{-1+\sqrt{3}i}{2}\right)x+wt\right)}}{e^{2b\left(\frac{3}{4wb^2}\left(\frac{-1+\sqrt{3}i}{2}\right)x+wt\right)}+1}\right]\right)\right.\\ \left.-\frac{3}{8}\left(\frac{-1+\sqrt{3}i}{2}\right)\cot^2\left(\frac{1}{2}\arctan\left[\frac{e^{2b\left(\frac{3}{4wb^2}\left(\frac{-1+\sqrt{3}i}{2}\right)x+wt\right)}-1}{e^{2b\left(\frac{3}{4wb^2}\left(\frac{-1+\sqrt{3}i}{2}\right)x+wt\right)}+1},\frac{2e^{b\left(\frac{3}{4wb^2}\left(\frac{-1+\sqrt{3}i}{2}\right)x+wt\right)}}{e^{2b\left(\frac{3}{4wb^2}\left(\frac{-1+\sqrt{3}i}{2}\right)x+wt\right)}+1}\right]\right)\right\}.\tag{120}$$

3.3 Tzitzéica–Dodd–Bullough (TDB) equation

As last example, we consider the Tzitzéica–Dodd–Bullough equation as follows:

$$u_{xt}-e^{-u}-e^{-2u}=0,\tag{121}$$

by using the transformation $v(x,t)=e^{-u(x,t)}$, Eq. (121) transforms into the following partial differential equation,

$$-vv_{xt}+v_xv_t-v^3-v^4=0.\tag{122}$$

The travelling wave transformation $\xi=kx+wt$ reduces Eq. (122) to the following ODE

$$kw((v')^2-vv'')-v^3-v^4=0.\tag{123}$$

Balancing the terms vv'' and v^4 by using homogenous principle one can be found $M=1$. Then (6) reduces to

$$v(\xi)=A_0+A_1\left[p+\tan\left(\frac{\Phi(\xi)}{2}\right)\right]+B_1\left[p+\tan\left(\frac{\Phi(\xi)}{2}\right)\right]^{-1}.\tag{124}$$

Substituting (124) and (7) into Eq. (122), the following solutions are obtained.

Set I: We have the following:

$$k = k, \quad \Lambda = a^2 + b^2 - c^2 p = p, \quad A_0 = \frac{p(c-b) - a}{2\sqrt{\Lambda}}, \quad A_1 = \frac{b-c}{2\sqrt{\Lambda}}, \quad B_1 = 0, \quad (125)$$

$$w = -\frac{1}{k\Lambda}, \quad v(\xi) = A_0 + A_1\left[p + \tan\left(\frac{\Phi(\xi)}{2}\right)\right]. \quad (126)$$

Using the transformation $u = \ln v$ and according to (126), the following solutions are obtained.

By using of the **Families 1, 2, 6, 10, 11** and **18** can be written, respectively, as

$$u_1(x,t) = \ln\left\{-\frac{\sqrt{-1}}{2}\tan\left(\frac{\sqrt{c^2 - b^2 - a^2}}{2}\left(kx - \frac{1}{k(a^2 + b^2 - c^2)}t\right)\right)\right\}, \quad (127)$$

$$u_2(x,t) = \ln\left\{\frac{1}{2}\tanh\left(\frac{\sqrt{a^2 + b^2 - c^2}}{2}\left(kx - \frac{1}{k(a^2 + b^2 - c^2)}t\right)\right)\right\}, \quad (128)$$

$$u_3(x,t) = \ln\left\{\frac{1}{2}\tan\left(\frac{1}{2}\arctan\left[\frac{e^{2b\left(kx - \frac{1}{kb^2}t\right)} - 1}{e^{2b\left(kx - \frac{1}{kb^2}t\right)} + 1}, \frac{2e^{b\left(kx - \frac{1}{kb^2}t\right)}}{e^{2b\left(kx - \frac{1}{kb^2}t\right)} + 1}\right]\right)\right\}, \quad (129)$$

$$u_4(x,t) = \ln\left\{-\frac{1}{2} + \frac{e^{a\left(kx - \frac{1}{ka^2}t\right)}}{-1 + e^{a\left(kx - \frac{1}{ka^2}t\right)}}\right\},$$
$$u_5(\xi) = \ln\left\{\frac{-a}{2b} - \frac{b-a}{2b}\,\frac{(a+b)e^{b\left(kx - \frac{1}{kb^2}t\right)} - 1}{(a-b)e^{b\left(kx - \frac{1}{kb^2}t\right)} - 1}\right\}, \quad (130)$$

$$u_6(x,t) = \ln\left\{\frac{1}{2}i\tan\left[\frac{1}{2}\left(kx + \frac{1}{kc^2}t + C\right)\right]\right\}. \quad (131)$$

Set II: We have the following:

$$w = w, \quad p = -\frac{a}{b-c}, \quad A_0 = \frac{-1}{2}, \quad A_1 = 0, \quad B_1 = \frac{\sqrt{\Delta}}{2(b-c)}, \quad k = -\frac{1}{w\Delta},$$
$$v(\xi) = A_0 + B_1\left[p + \tan\left(\frac{\Phi(\xi)}{2}\right)\right]^{-1}, \quad (132)$$

where $\Delta = a^2 + b^2 - c^2$. By using of the **Families 1, 2, 6, 10, 11** and **18** can be written, respectively, as

$$u_7(x,t) = \ln\left\{\frac{-1}{2} - \frac{1}{2}i\cot\left(\frac{\sqrt{c^2 - b^2 - a^2}}{2}\left(-\frac{1}{w(a^2 + b^2 - c^2)}x + wt\right)\right)\right\}, \quad (133)$$

$$u_8(x,t) = \ln\left\{\frac{-1}{2} + \frac{1}{2}\coth\left(\frac{\sqrt{a^2 + b^2 - c^2}}{2}\left(-\frac{1}{w(a^2 + b^2 - c^2)}x + wt\right)\right)\right\}, \quad (134)$$

$$u_9(x,t) = \ln\left\{-\frac{1}{2}+\frac{1}{2}\cot\left(\frac{1}{2}\arctan\left[\frac{e^{2b\left(-\frac{1}{wb^2}x+wt\right)}-1}{e^{2b\left(-\frac{1}{wb^2}x+wt\right)}+1}, \frac{2e^{b\left(-\frac{1}{wb^2}x+wt\right)}}{e^{2b\left(-\frac{1}{wb^2}x+wt\right)}+1}\right]\right)\right\}, \quad (135)$$

$$u_{10}(x,t) = \ln\left\{-\frac{1}{2}-\frac{1}{4}\left[\frac{1}{2}-\frac{e^{a\left(-\frac{1}{wa^2}x+wt\right)}}{e^{a\left(-\frac{1}{wa^2}x+wt\right)-1}}\right]^{-1}\right\}, \quad (136)$$

$$u_{11}(x,t) = \ln\left\{-\frac{1}{2}+\frac{b}{2(b-a)}\left[-\frac{a}{b-a}-\frac{(a+b)e^{b\left(-\frac{1}{wb^2}x+wt\right)}-1}{(a-b)e^{b\left(-\frac{1}{wb^2}x+wt\right)}-1}\right]^{-1}\right\},$$
$$u_{12}(x,t) = \ln\left\{-\frac{1}{2}-\frac{1}{2}i\cot\left[\frac{1}{2}\left(\frac{1}{wc^2}x+wt+C\right)\right]\right\}. \quad (137)$$

Set III: We have the following:

$$p = \frac{-a+3\sqrt{a^2+b^2-c^2}}{b-c}, \quad A_0 = -2, \quad A_1 = 0, \quad B_1 = \frac{4}{b-c}\sqrt{a^2+b^2-c^2}, \quad (138)$$

$$w = w, k = -\frac{1}{w(a^2+b^2-c^2)}, \quad v(\xi) = A_0 + B_1\left[p+\tan\left(\frac{\Phi(\xi)}{2}\right)\right]^{-1}. \quad (139)$$

By using of the **Families 1, 2, 6, 10, 11** and **18** can be written, respectively, as

$$u_{13}(x,t) = \ln\left\{-2+4\left[3-i\tan\left(\frac{\sqrt{c^2-b^2-a^2}}{2}\left(-\frac{1}{w(a^2+b^2-c^2)}x+wt\right)\right)\right]^{-1}\right\}, \quad (140)$$

$$u_{14}(x,t) = \ln\left\{-2+4\left[3+\tanh\left(\frac{\sqrt{a^2+b^2-c^2}}{2}\left(-\frac{1}{w(a^2+b^2-c^2)}x+wt\right)\right)\right]^{-1}\right\}, \quad (141)$$

$$u_{15}(x,t) = \ln\left\{-2+4\left[3+\tan\left(\frac{1}{2}\arctan\left[\frac{e^{2b\left(-\frac{1}{wb^2}x+wt\right)}-1}{e^{2b\left(-\frac{1}{wb^2}x+wt\right)}+1}, \frac{2e^{b\left(-\frac{1}{wb^2}x+wt\right)}}{e^{2b\left(-\frac{1}{wb^2}x+wt\right)}+1}\right]\right)\right]^{-1}\right\}. \quad (142)$$

$$u_{16}(x,t) = \ln\left\{-2+2\left[1+\frac{e^{a\left(-\frac{1}{wa^2}x+wt\right)}}{1-e^{a\left(-\frac{1}{wa^2}x+wt\right)}}\right]^{-1}\right\},$$
$$u_{17}(x,t) = \ln\left\{-2+\frac{4b}{b-a}\left[\frac{3b-a}{b-a}-\frac{(a+b)e^{b\left(-\frac{1}{wb^2}x+wt\right)}-1}{(a-b)e^{b\left(-\frac{1}{wb^2}x+wt\right)}-1}\right]^{-1}\right\}, \quad (143)$$

$$u_{18}(x,t) = \ln\left\{-2-4i\left[-3i+\tan\left(\frac{1}{2}\left(\frac{1}{wc}x+cwt\right)\right)\right]^{-1}\right\}. \quad (144)$$

Set IV: We have the following:

$$p=-\frac{2(2A_0+1)(a^2+b^2-c^2)+2a\sqrt{a^2+b^2-c^2}}{2(b-c)\sqrt{a^2+b^2-c^2}},\quad A_1=0,$$
$$B_1=\frac{2A_0(A_0+1)}{b-c}\sqrt{a^2+b^2-c^2}, \tag{145}$$

$$A_0=A_0,\quad w=w,\quad k=-\frac{1}{w(a^2+b^2-c^2)},\quad v(\xi)=A_0+B_1\left[p+\tan\left(\frac{\Phi(\xi)}{2}\right)\right]^{-1}. \tag{146}$$

By using of the **Families 1, 2, 6, 10, 11** and **18** can be written, respectively, as

$$u_{19}(x,t)=\ln\left\{A_0-2A_0(A_0+1)\left[2A_0+1+i\tan\left(\frac{\sqrt{c^2-b^2-a^2}}{2}\left(-\frac{1}{w(a^2+b^2-c^2)}x+wt\right)\right)\right]^{-1}\right\}, \tag{147}$$

$$u_{20}(x,t)=\ln\left\{A_0-2A_0(A_0+1)\left[2A_0+1-\tanh\left(\frac{\sqrt{a^2+b^2-c^2}}{2}\left(-\frac{1}{w(a^2+b^2-c^2)}x+wt\right)\right)\right]^{-1}\right\}, \tag{148}$$

$$u_{21}(x,t)=\ln\left\{A_0-2A_0(A_0+1)\left[2A_0+1-\tan\left(\frac{1}{2}\arctan\left[\frac{e^{2b\left(-\frac{1}{wb^2}x+wt\right)}-1}{e^{2b\left(-\frac{1}{wb^2}x+wt\right)}+1},\frac{2e^{b\left(-\frac{1}{wb^2}x+wt\right)}}{e^{2b\left(-\frac{1}{wb^2}x+wt\right)}+1}\right]\right)\right]^{-1}\right\}, \tag{149}$$

$$u_{22}(x,t)=\ln\left\{A_0-A_0(A_0+1)\left[2A_0+2-\frac{e^{a\left(-\frac{1}{wa^2}x+wt\right)}}{e^{a\left(-\frac{1}{wa^2}x+wt\right)}-1}\right]^{-1}\right\},$$
$$u_{23}(x,t)=\ln\left\{A_0+\frac{2bA_0(A_0+1)}{b-a}\left[\frac{(2A_0+1)b+a}{a-b}-\frac{(a+b)e^{b\left(-\frac{1}{wb^2}x+wt\right)}-1}{(a-b)e^{b\left(-\frac{1}{wb^2}x+wt\right)}-1}\right]^{-1}\right\}, \tag{150}$$

$$u_{24}(x,t)=\ln\left\{A_0-2A_0(A_0+1)i\left[(2A_0+1)i+\tan\left(\frac{1}{2wc}x+\frac{cw}{2}t+C\right)\right]^{-1}\right\}. \tag{151}$$

Set V: We have the following:

$$p=\frac{b-c-2aA_1}{2(b-c)A_1},\quad A_0=-1,\quad A_1=A_1,\quad B_1=\frac{(b-c)^2-4A_1^2(a^2+b^2-c^2)}{4A_1(b-c)^2}, \tag{152}$$

$$w = -\frac{4A_1^2}{k(b-c)^2}, \quad k = k,$$
$$v(\xi) = A_0 + A_1\left[p + \tan\left(\frac{\Phi(\xi)}{2}\right)\right] + B_1\left[p + \tan\left(\frac{\Phi(\xi)}{2}\right)\right]^{-1}. \tag{153}$$

By using of the **Families 1, 2, 6, 10, 11** and **18** can be written, respectively, as

$$u_{25}(x,t) = \ln\left\{-\frac{1}{2} - A_1\frac{\sqrt{c^2-b^2-a^2}}{b-c}\tan\left(\frac{\sqrt{c^2-b^2-a^2}}{2}\left(kx - \frac{4A_1^2}{k(b-c)^2}t\right)\right)\right.$$
$$\left. + \frac{(b-c)^2 - 4A_1^2(a^2+b^2-c^2)}{4A_1(b-c)^2}\left[\frac{1}{2A_1} - \frac{\sqrt{c^2-b^2-a^2}}{b-c}\tan\left(\frac{\sqrt{c^2-b^2-a^2}}{2}\left(kx - \frac{4A_1^2}{k(b-c)^2}t\right)\right)\right]^{-1}\right\}, \tag{154}$$

$$u_{26}(x,t) = \ln\left\{-\frac{1}{2} + A_1\frac{\sqrt{a^2+b^2-c^2}}{b-c}\tanh\left(\frac{\sqrt{a^2+b^2-c^2}}{2}\left(kx - \frac{4A_1^2}{k(b-c)^2}t\right)\right)\right.$$
$$\left. + \frac{(b-c)^2 - 4A_1^2(a^2+b^2-c^2)}{4A_1(b-c)^2}\left[\frac{1}{2A_1} + \frac{\sqrt{a^2+b^2-c^2}}{b-c}\tanh\left(\frac{\sqrt{a^2+b^2-c^2}}{2}\left(kx - \frac{4A_1^2}{k(b-c)^2}t\right)\right)\right]^{-1}\right\}, \tag{155}$$

$$u_{27}(x,t) = \ln\left\{-\frac{1}{2} + A_1\tan\left(\frac{1}{2}\arctan\left[\frac{e^{2b\left(kx-\frac{4A_1^2}{kb^2}t\right)} - 1}{e^{2b\left(kx-\frac{4A_1^2}{kb^2}t\right)} + 1}, \frac{2e^{b\left(kx-\frac{4A_1^2}{kb^2}t\right)}}{e^{2b\left(kx-\frac{4A_1^2}{kb^2}t\right)} + 1}\right]\right)\right.$$
$$\left. + \frac{1-4A_1^2}{4A_1}\left[\frac{1}{2A_1} + \tan\left(\frac{1}{2}\arctan\left[\frac{e^{2b\left(kx-\frac{4A_1^2}{kb^2}t\right)} - 1}{e^{2b\left(kx-\frac{4A_1^2}{kb^2}t\right)} + 1}, \frac{2e^{b\left(kx-\frac{4A_1^2}{kb^2}t\right)}}{e^{2b\left(kx-\frac{4A_1^2}{kb^2}t\right)} + 1}\right]\right)\right]^{-1}\right\}, \tag{156}$$

$$u_{28}(x,t) = \ln\left\{-\frac{1}{2} + \frac{A_1}{2} + \frac{A_1 e^{a\left(kx-\frac{A_1^2}{ka^2}t\right)}}{1 - e^{a\left(kx-\frac{A_1^2}{ka^2}t\right)}} + \frac{1-A_1^2}{A_1}\left[\frac{1+A_1}{2A_1} + \frac{e^{a\left(kx-\frac{A_1^2}{ka^2}t\right)}}{1 - e^{a\left(kx-\frac{A_1^2}{ka^2}t\right)}}\right]^{-1}\right\}, \tag{157}$$

$$u_{29}(x,t) = \ln\left\{-\frac{1}{2} - \frac{aA_1}{b-a} - \frac{(a+b)e^{b\left(kx-\frac{4A_1^2}{k(b-a)^2}t\right)} - 1}{(a-b)e^{b\left(kx-\frac{4A_1^2}{k(b-a)^2}t\right)} - 1}\right.$$
$$\left. + \frac{(b-a)^2 - 4A_1^2b^2}{4A_1(b-a)^2}\left[\frac{b-a-2aA_1}{2(b-a)A_1} - \frac{(a+b)e^{b\left(kx-\frac{4A_1^2}{k(b-a)^2}t\right)} - 1}{(a-b)e^{b\left(kx-\frac{4A_1^2}{k(b-a)^2}t\right)} - 1}\right]^{-1}\right\}, \tag{158}$$

$$u_{30}(x,t) = \ln\left\{-\frac{1}{2} + A_1 \tan\left(\frac{1}{2}kcx - \frac{2A_1^2}{kc}t\right) + \frac{1+4A_1^2}{4A_1}\left[\frac{1}{2A_1} + \tan\left(\frac{1}{2}kcx - \frac{2A_1^2}{kc}t\right)\right]^{-1}\right\}. \tag{159}$$

Set VI: We have the following:

$$p = \frac{-a}{b-c}, \quad A_0 = -\frac{1}{2}, \quad A_1 = \frac{b-c}{4\sqrt{\Delta}}, \quad B_1 = \frac{\sqrt{\Delta}}{4(b-c)}, \tag{160}$$

$$w = -\frac{1}{4k\Delta}, \quad k = k, \quad v(\xi) = A_0 + A_1\left[p + \tan\left(\frac{\Phi(\xi)}{2}\right)\right] + B_1\left[p + \tan\left(\frac{\Phi(\xi)}{2}\right)\right]^{-1}, \tag{161}$$

where $\Delta = a^2 + b^2 - c^2$. By using of the **Families 1, 2, 6, 10, 11** and **18** can be written, respectively, as

$$u_{31}(x,t) = \ln\left\{-\frac{1}{2} - \frac{1}{4}i\left[\tan\left(\frac{\sqrt{-\Delta}}{2}\left(kx - \frac{1}{4k\Delta}t\right)\right) + \cot\left(\frac{\sqrt{-\Delta}}{2}\left(kx - \frac{1}{4k\Delta}t\right)\right)\right]\right\}, \tag{162}$$

$$u_{32}(x,t) = \ln\left\{-\frac{1}{2} + \frac{1}{4}\left[\tanh\left(\frac{\sqrt{\Delta}}{2}\left(kx - \frac{1}{4k\Delta}t\right)\right) + \coth\left(\frac{\sqrt{\Delta}}{2}\left(kx - \frac{1}{4k\Delta}t\right)\right)\right]\right\}, \tag{163}$$

$$\begin{aligned} u_{33}(x,t) &= \ln\left\{-\frac{1}{2} + \frac{1}{4}\tan\left(\frac{1}{2}\arctan\left[\frac{e^{2b\left(kx-\frac{1}{4kb^2}t\right)}-1}{e^{2b\left(kx-\frac{1}{4kb^2}t\right)}+1}, \frac{2e^{b\left(kx-\frac{1}{4kb^2}t\right)}}{e^{2b\left(kx-\frac{1}{4kb^2}t\right)}+1}\right]\right)\right. \\ &\quad \left.+ \frac{1}{4}\cot\left(\frac{1}{2}\arctan\left[\frac{e^{2b\left(kx-\frac{1}{4kb^2}t\right)}-1}{e^{2b\left(kx-\frac{1}{4kb^2}t\right)}+1}, \frac{2e^{b\left(kx-\frac{1}{4kb^2}t\right)}}{e^{2b\left(kx-\frac{1}{4kb^2}t\right)}+1}\right]\right)\right\}, \end{aligned} \tag{164}$$

$$u_{34}(x,t) = \ln\left\{-\frac{3}{4} - \frac{1}{2}\frac{e^{a\left(kx-\frac{1}{4ka^2}t\right)}}{e^{a\left(kx-\frac{1}{4ka^2}t\right)}-1} - \frac{1}{8}\left[\frac{1}{2} - \frac{e^{a\left(kx-\frac{1}{4ka^2}t\right)}}{e^{a\left(kx-\frac{1}{4ka^2}t\right)}-1}\right]^{-1}\right\}, \tag{165}$$

$$\begin{aligned} u_{35}(x,t) &= \ln\left\{-\frac{1}{2} - \frac{a}{4b} + \frac{a-b}{4b}\frac{(a+b)e^{b\left(kx-\frac{1}{4kb^2}t\right)}-1}{(a-b)e^{b\left(kx-\frac{1}{4kb^2}t\right)}-1}\right. \\ &\quad \left.+ \frac{b}{4(b-a)}\left[\frac{a}{a-b} - \frac{(a+b)e^{b\left(kx-\frac{1}{4kb^2}t\right)}-1}{(a-b)e^{b\left(kx-\frac{1}{4kb^2}t\right)}-1}\right]^{-1}\right\}, \end{aligned} \tag{166}$$

$$u_{36}(x,t) = \ln\left\{-\frac{1}{2} + \frac{1}{4}i\tan\left(\frac{kc}{2}x + \frac{1}{8kc}t\right) - \frac{1}{4}i\cot\left(\frac{kc}{2}x + \frac{1}{8kc}t\right)\right\}. \tag{167}$$

• **Note that**: All the obtained results have been checked with Maple 13 by putting them back into the original equation and found correct.

4 Physical interpretations of the solutions

In this section, we depict the graph and signify the obtained solutions to the Tzitzéica type nonlinear evolution equations. Now, we will discuss all possible physical significance for parameter.

Remark 1 In Figures 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 and 11, we plot three dimensional graphics of real values of (15), (16), (20), (54), (73), (74), (76), (108), (127), (128), (130) respectively, which denote the dynamics of solutions with appropriate parametric selections. We plot three dimensional graphics of Figs. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 and 11, when $-10<x<10, 0<t<10$. To the best of our knowledge, these optical soliton solutions have not been submitted to literature in advance. The analytical solutions and figures obtained in this paper give us a different physical interpretation for the Tzitzéica type nonlinear evolution equations.

Remark 2 Solutions $u_1, u_4, u_7, u_9, u_{13}, u_{17}, u_{20}, u_{23}, u_{26}, u_{29}, u_{32}, u_{35}$ of the Tzitzéica equation and $u_1, u_5, u_9, u_{14}, u_{15}$, $u_{18}, u_{19}, u_{21}, u_{23}, u_{25}, u_{29}$ of the DBM equation and $u_1, u_6, u_7, u_{12}, u_{13}, u_{18}, u_{19}, u_{24}, u_{25}, u_{30}, u_{31}, u_{36}$ of TDB equation represent the exact periodic traveling wave solutions. Periodic solutions are traveling wave solutions that are periodic, such as $sin(x+t)$. Figures 1, 5 and 9 show the shape of the periodic solutions of u_1 of (15), (73) and (127) respectively. The other figures are ignored for simplicity.

Remark 3 Solutions $u_2, u_3, u_8, u_{10}, u_{14}, u_{18}, u_{21}, u_{24}, u_{27}, u_{30}, u_{33}$ of the Tzitzéica equation and u_2, u_6, u_{10}, u_{16}, u_{20}, u_{22} of the DBM equation and $u_2, u_8, u_{14}, u_{20}, u_{26}, u_{32}$ of TDB

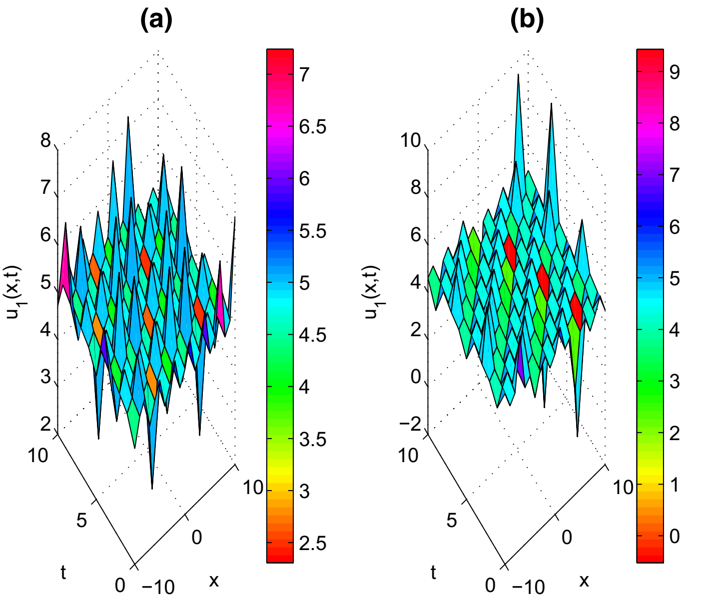

Fig. 1 Graphs of u_1 real values of (15) are demonstrated at **a** $k=5, w=3, b=2, c=3$, **b** $k=3, w=2, b=2, c=4$, when $-10<x<10$, $0<t<10$

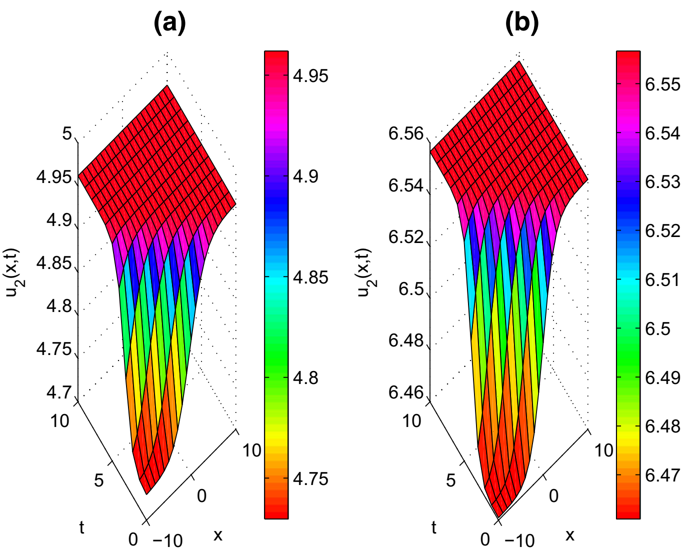

Fig. 2 Graphs of u_2 real values of (16) are demonstrated at **c** $k = 1, w = 3, b = 3, c = 1$, **d** $k = 2, w = 5, b = 5, c = 3$, when $-10 < x < 10$, $0 < t < 10$

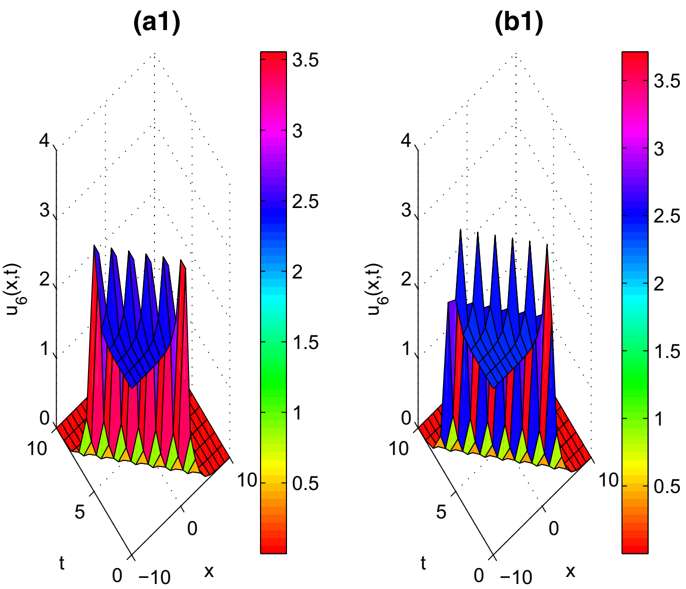

Fig. 3 Graphs of u_6 real values of (20) are demonstrated at **a1** $k = 1, w = 2, b = 2$, **b1** $k = 1, w = 2, b = 5$, when $-10 < x < 10$, $0 < t < 10$

equation are the soliton solutions. Solitons are special kinds of solitary waves. Solitons have a remarkable property that keeps its identity upon interacting with other solitons. Soliton solutions have particle-like structures, for example, magnetic monopoles, and

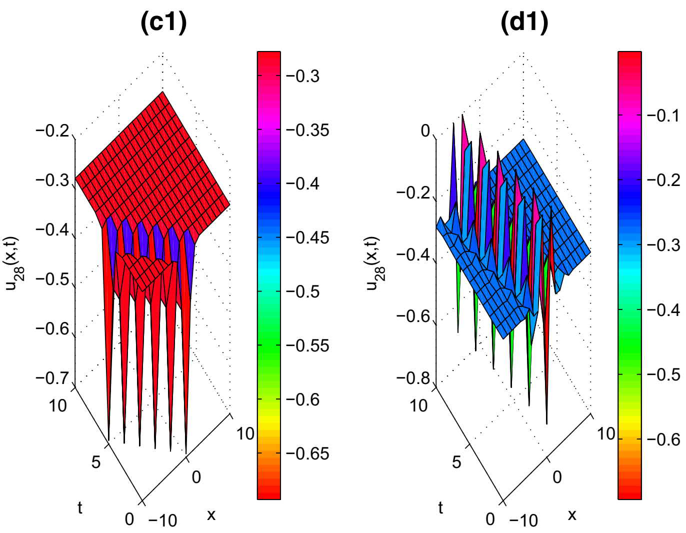

Fig. 4 Graphs of u_{28} real values of (54) are demonstrated at **c1** $k = 1, w = 2$ and **d1** $k = 2, w = 1$, when $-10 < x < 10$, $0 < t < 10$

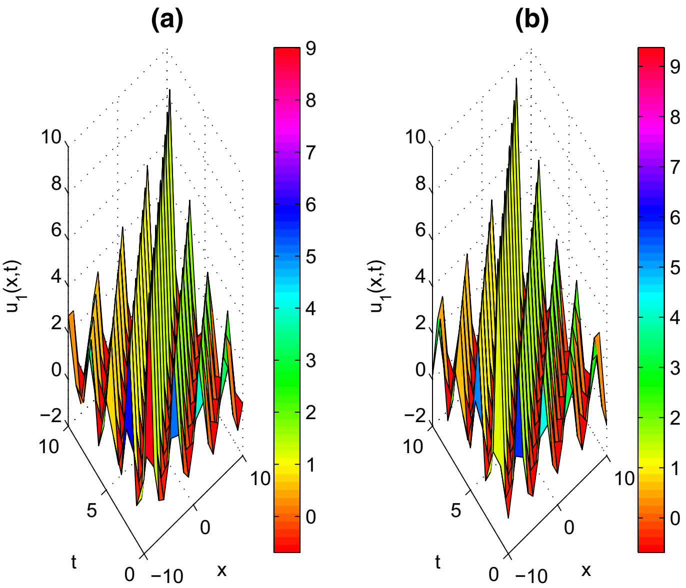

Fig. 5 Graphs of u_1 real values of (73) are demonstrated at **a** $k = 1, w = -2, b = 1, c = 2$ and **b** $k = 3, w = -6, b = 2, c = 4$, when $-10 < x < 10$, $0 < t < 10$

extended structures, like, domain walls and cosmic strings, that have implications in cosmology of the early universe. Figures 2, 6 and 10 show the shape of the exact soliton solutions of u_2 of (16), (74) and (128) respectively. The other figures are ignored for simplicity.

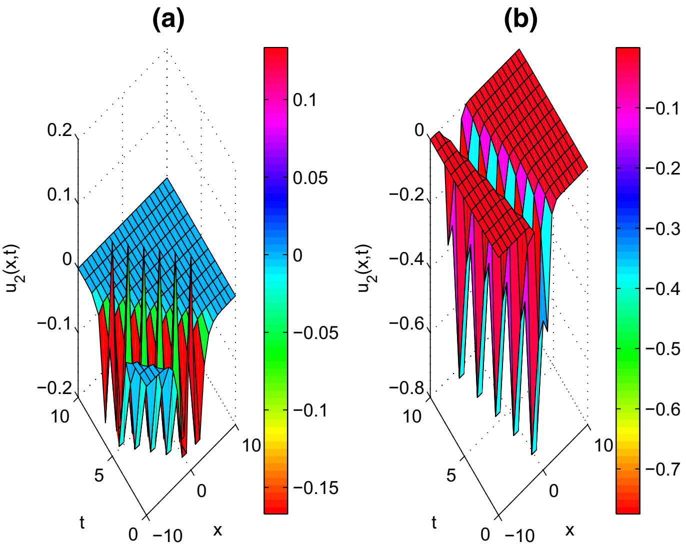

Fig. 6 Graphs of u_2 real values of (74) are demonstrated at **c** $k=1, w=2, b=3, c=2$ and **d** $k=2, w=1, b=4, c=2$, when $-10<x<10$, $0<t<10$

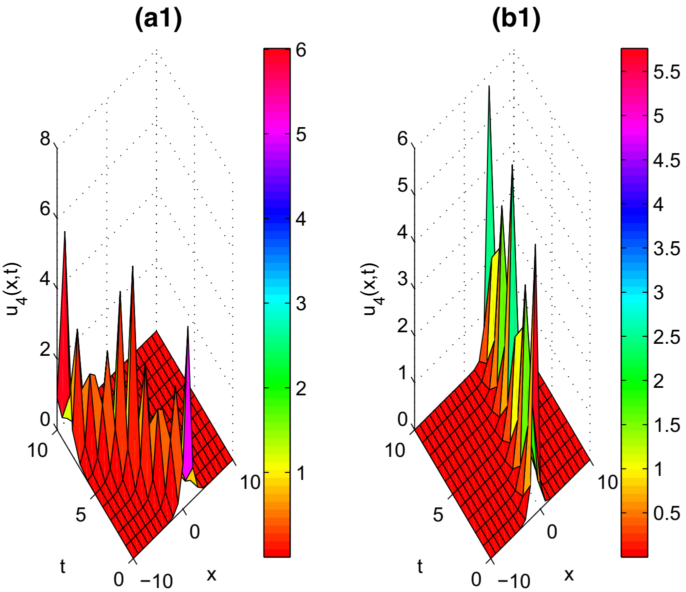

Fig. 7 Graphs of u_4 real values of (76) are demonstrated at **a1** $k=5, w=6, c=4$ and **b1** $k=-10, w=6, c=10$, when $-10<x<10$, $0<t<10$

Remark 4 Solutions $u_5, u_6, u_{11}, u_{12}, u_{15}, u_{16}, u_{19}, u_{22}, u_{25}, u_{28}, u_{31}, u_{34}$ of the Tzitzéica equation and u_3, u_4, u_7, u_8, $u_{11}, u_{12}, u_{13}, u_{17}, u_{24}, u_{26}, u_{28}, u_{30}$ of the DBM equation and $u_3, u_4, u_5, u_9, u_{10}, u_{11}$, $u_{15}, u_{16}, u_{17}, u_{21}$, u_{22}, u_{23}, u_{27}, u_{28}, u_{29}, u_{33}, u_{34}, u_{35} of TDB equation

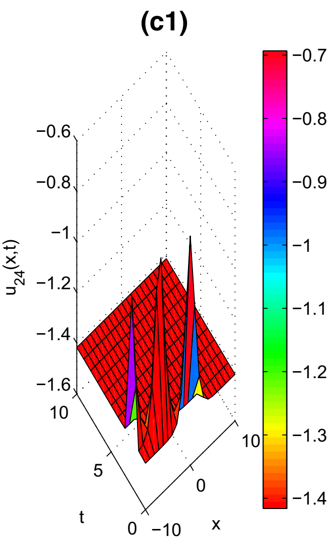

Fig. 8 Graphs of u_{24} real values of (108) are demonstrated at **c1** $w = 2, b = 2$ and **d1** $w = 10, b = -5$, when $-10 < x < 10$, $0 < t < 10$

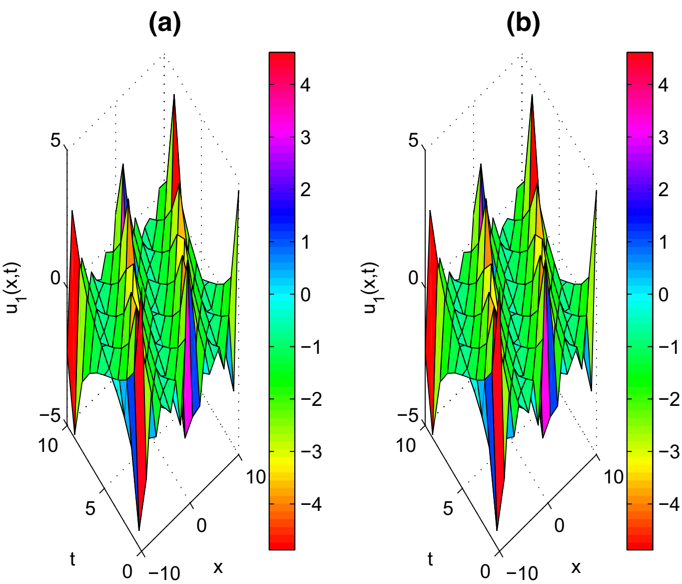

Fig. 9 Graphs of u_1 real values of (127) are demonstrated at **a** $k = 10, a = 3, b = 2, c = 5$ and **b** $k = -10, a = 4, b = 5, c = 7$, when $-10 < x < 10$, $0 < t < 10$

represent the singular kink-type traveling wave solutions. Figures 3, 4, 7, 8 and 11 show the shape of the exact singular kink-type solution of u_6, u_{28} of (20), (54), u_4, u_{24} of (76), (108) and u_4 of (130) respectively. We note that kink solution acquires non-vanishing values as $x \longrightarrow \infty$. For convenience the other figures are omitted.

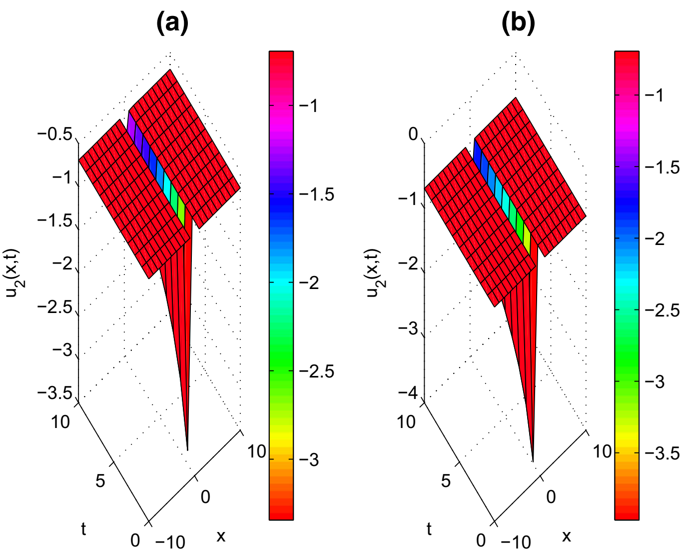

Fig. 10 Graphs of u_2 real values of (128) are demonstrated at **c** $k = 5, a = 3, b = 3, c = 4$ and **d** $k = 5, a = 4, b = 4, c = 5$, when $-10 < x < 10$, $0 < t < 10$

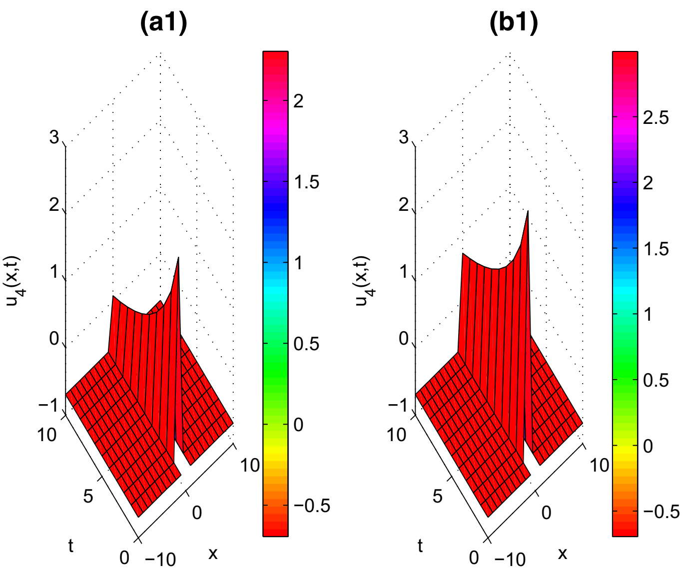

Fig. 11 Graphs of u_4 real values of (130) are demonstrated at **a1** $k = 5, a = 2$ and **b1** $k = 10, a = 2$, when $-10 < x < 10$, $0 < t < 10$

5 Conclusion

In this paper, the new approach of expansion method namely the improved $\tan(\Phi(\xi)/2)$-expansion method has successfully been implemented to investigate the Tzitzéica type nonlinear evolution equations. Abundant exact travelling wave solutions including solitons, kink, periodic and rational solutions are obtained. It is worth mentioning that some of newly obtained solutions are identical to already published results. It has been shown that the applied method is effective and more wide-ranging than the generalized and improved (G'/G)-expansion method because it gives many new solutions. Also figures analysis for Tzitzéica type nonlinear evolution equations have surveyed as well. Therefore, this method can be applied to study many other nonlinear partial differential equations which frequently arise in engineering, mathematical physics and nonlinear optics and the quantum field theory.

References

Abazari, R.: The (G'/G)-expansion method for Tzitzéica type nonlinear evolution equations. Math. Comput. Model. **52**, 1834–1845 (2010)

Abdou, M.A., Soliman, A.A., Basyony, S.T.: New application of exp-function method for improved Boussinesq equation. Phys. Lett. A **369**, 469–475 (2007)

Abdou, M.A., Soliman, A.A.: Modified extended tanh-function method and its application on nonlinear physical equations. Phys. Lett. A **353**, 487–492 (2006)

Alam, M.N., Akbar, M.A., Mohyud-Din, S.T.: A novel (G'/G)-expansion method and its application to the Boussinesq equation. Chin. Phys. B **23**, 020202 (2014)

Borhanifar, A., Moghanlu, A.Z.: Application of the (G'/G)-expansion method for the Zhiber–Shabat equation and other related equations. Math. Comput. Model. **54**, 2109–2116 (2011)

Chen, Y., Wang, Q.: Extended Jacobi elliptic function rational expansion method and abundant families of Jacobi elliptic functions solutions to (1+1)-dimensional dispersive long wave equation. Chaos Solitons Fractals **24**, 745–757 (2005)

Dehghan, M., Manafian, J., Saadatmandi, A.: Solving nonlinear fractional partial differential equations using the homotopy analysis method. Numer. Methods Partial Differ. Equ. J. **26**, 448–479 (2010)

Dehghan, M., Manafian, J., Saadatmandi, A.: Application of semi-analytic methods for the Fitzhugh–Nagumo equation, which models the transmission of nerve impulses. Math. Methods Appl. Sci. **33**, 1384–1398 (2010)

Dehghan, M., Manafian Heris, J., Saadatmandi, A.: Application of the Exp-function method for solving a partial differential equation arising in biology and population genetics. Int. J. Numer. Methods Heat Fluid Flow **21**, 736–753 (2011)

Dehghan, M., Manafian, J.: The solution of the variable coefficients fourth–order parabolic partial differential equations by homotopy perturbation method. Z. Naturfor. **64a**, 420–430 (2009)

Ebrahimi Ghogdia, S., Ghomanjani, F., Saberi-Nadjafi, J.: Expansion of the Exp-function method for solving systems of two-dimensional Navier–Stokes equations. J. Taibah Univ. Sci. **9**, 121–125 (2015)

El-Wakil, S.A., Abdou, M.A.: New exact travelling wave solutions using modified extended tanh-function method. Chaos Solitons Fractals **31**, 840–852 (2007)

Gray, P., Scott, S.K.: Chemical Oscillation and Instabilities-Nonlinear Chemical Kinetics. Oxford Science Publications, Clarendon, Oxford (1990)

Hafez, M.G., Alam, M.N., Akbar, M.A.: Traveling wave solutions for some important coupled nonlinear physical models via the coupled Higgs equation and the Maccari system. J. King Saud Univ.-Sci. **27**, 105–112 (2015)

He, J.H.: Variational iteration method a kind of non-linear analytical technique: some examples. Int. J. Nonlinear Mech. **34**, 699–708 (1999)

Huber, A.: A note on a class of solitary-like solutions of the Tzitzéica equation generated by a similarity reduction. Phys. D. **237**, 1079–1087 (2008)

Jafari, H., Kadem, A., Baleanu, D.: Variational iteration method for a fractional-order Brusselator system. Abstract Appl. Anal. **2014**, 1–6 (2014)

Jawad, A.J.M., Petkovic, M.D., Biswas, A.: Modified simple equation method for nonlinear evolution equations. Appl. Math. Comput. **217**, 869–877 (2010)

Kabir, M.M., Khajeh, A.: New explicit solutions for the Vakhnenko and a generalized form of the nonlinear heat conduction equations via exp-function method. Int. J. Nonlinear Sci. Numer. Simul. **10**, 1307–1318 (2009)

Ma, W.X., You, Y.: Rational solutions of the Toda lattice equation in Casoratian form. Chaos Solitons Fractals **22**, 395–406 (2004)
Manafian, J., Lakestani, M.: Solitary wave and periodic wave solutions for Burgers, Fisher, Huxley and combined forms of these equations by the (G'/G)-expansion method. Pramana J. Phys. **4**, 1–22 (2015a)
Manafian, J., Lakestani, M.: Optical solitons with Biswas–Milovic equation for Kerr law nonlinearity. Eur. Phys. J. Plus **130**, 1–12 (2015b)
Manafian, J., Lakestani, M.: New improvement of the expansion methods for solving the generalized Fitzhugh-Nagumo equation with time-dependent coefficients. Int. J. Eng. Math. **2015**, 107978 (2015c). doi:10.1155/2015/107978
Manafian, J., Lakestani, M.: Application of $tan(\phi/2)$-expansion method for solving the Biswas–Milovic equation for Kerr law nonlinearity. Optik-Int. J. Light Electron Opt. **127**, 2040–2054 (2016)
Manafian Heris, J., Lakestani, M.: Solitary wave and periodic wave solutions for variants of the KdV-Burger and the K(n, n)-Burger equations by the generalized tanh-coth method. Commun. Numer. Anal. **2013**, 1–18 (2013)
Manafian, J., Lakestani, M., Bekir, A.: Study of the analytical treatment of the (2+1)-dimensional zoomeron, the Duffing and the SRLW equations via a new analytical approach. Int. J. Appl. Comput. Math. **1**, 1–26 (2015). doi:10.1007/s40819-015-0058-2
Mikhailov, A.V.: The reduction problem and the inverse scattering method. Physica **3D**(2), 73–117 (1910)
Mohyud-Din, S.T., Noor, M.A., Noor, K.I.: Some relatively new techniques for nonlinear problems. Math. Problems Eng. **2009**, 1–26 (2009). doi:10.1155/2009/234849. Article ID 234849
Mohyud-Din, S.T., Noor, M.A., Asif, W.: Exp-function method for generalized traveling solutions of Calogero-Degasperis-Fokas equation. Z. Naturfor. A **65a**, 78–84 (2010)
Mohyud-Din, S.T., Yildirim, A., Sezer, S.A.: Numerical soliton solutions of the improved Boussinesq equation. Int. J. Numer. Methods Heat Fluid Flow **21**, 822–827 (2011)
Mohyud-Din, S.T., Yildirim, A., Sariaydin, S.: Numerical soliton solution of the Kaup-Kupershmidt equation. Int. J. Numer. Methods Heat Fluid Flow **21**, 272–281 (2011)
Mohyud-Din, S.T., Khan, Y., Naeem, F., Yildirim, A.: Exp-function method for solitary and periodic solutions of Fitzhugh Nagumo equations. Int. J. Numer. Methods Heat Fluid Flow **22**, 335–341 (2012)
Naher, H., Abdullah, F.A., Mohyud-Din, S.T.: Extended generalized Riccati equation mapping method for the fifth-order Sawada–Kotera equation. AIP Adv. **3**, 052104 (2013). doi:10.1063/1.4804433
Naher, H., Abdullah, F.A.: New approach of (G'/G)-expansion method and new approach of generalized (G'/G)-expansion method for nonlinear evolution equation. AIP Adv. **3**, 032116 (2013). doi:10.1063/1.4794947
Noor, M.A., Mohyud-Din, S.T., Asif, W.: Exp-function method for generalized traveling solutions of master partial differential equations. Acta Appl. Math. **104**, 131–137 (2008)
Noor, M.A., Mohyud-Din, S.T., Asif, W., Eisa, A.A.S.: Exp-function method for traveling wave solutions of nonlinear evolution equations. Appl. Math. Comput. **216**, 477–483 (2010)
Roshid, O.R., Rahman, M.A.: The exp(-$\Phi(\xi)$)-expansion method with application in the (1+1)-dimensional classical Boussinesq equations. Results Phys. **4**, 150–155 (2014)
Saba, F., Jabeen, S., Akbar, H., Tauseef Mohyud-Din, S.: Modified alternative (G'/G)-expansion method to general Sawada–Kotera equation of fifth-order. J. Egypt. Math. Soc. **23**, 416–423 (2015)
Tzitzéica, G.: Géometric infinitésimale-sur une nouvelle classe de surface. C. R. Math. Acad. Sci. Paris **150**, 227–232 (1910)
Wazwaz, A.M.: The tanh method: solitons and periodic solutions for the Dodd–Bullough–Mikhailov and the Tzitzéica–Dodd–Bullough equations. Chaos Solitons Fractals **25**, 55–63 (2005)
Wazwaz, A.M.: Travelling wave solutions for combined and double combined sine–cosine–Gordon equations by the variable separated ODE method. Appl. Math. Comput. **177**, 755–760 (2006)
Yildirim, A., Pinar, Z.: Application of the exp-function method for solving nonlinear reaction-diffusion equations arising in mathematical biology. Comput. Math. Appl. **60**, 1873–1880 (2010)
Zhang, X., Zhao, J., Liu, J., Tang, B.: Homotopy perturbation method for two dimensional time-fractional wave equation. Appl. Math. Model. **38**, 5545–5552 (2014)
Zhao, X., Wang, L., Sun, W.: The repeated homogeneous balance method and its applications to nonlinear partial differential equations. Chaos Solitons Fractals **28**, 448–453 (2006)