

An interior point technique for solving bilevel programming problems

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Abstract This paper deals with bilevel programs with strictly convex lower level problems. We present the theoretical basis of a kind of necessary and sufficient optimality conditions that involve a single-level mathematical program satisfying the linear independence constraint qualification. These conditions are obtained by replacing the inner problem by their optimality conditions and relaxing their inequality constraints. An algorithm for the bilevel program, based on a well known technique for classical smooth constrained optimization, is also studied. The algorithm obtains a solution of this problem with an effort similar to that required by a classical well-behaved nonlinear constrained optimization problem. Several illustrative problems which include linear, quadratic and general nonlinear functions and constraints are solved, and very good results are obtained for all cases.

Keywords Bilevel programming · Nonlinear optimization · Feasible direction algorithms

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1 Introduction

We consider the following bilevel programming problem (BLPP),

$$(1.1) \quad \left\{ \begin{array}{l} \min_{x,y} F(x, y) \\ \text{subject to } G(x, y) \leq 0, \\ \text{where } y \text{ solves} \\ \left\{ \begin{array}{l} \min_y f(x, y) \\ \text{subject to } g(x, y) \leq 0, \end{array} \right. \end{array} \right. \quad (1)$$

also called *leader's problem*, where $x \in \mathbb{R}^m$; $y \in \mathbb{R}^n$; $F, f : \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}$; $G : \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^l$ and $g : \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^p$ are continuous, twice differentiable functions. We also assume that $G(x, y) \leq 0$ defines a compact set. The problem (1.1) will be referred to as the *lower level* or the *follower's problem*. The function F is called the *upper level objective* and the function G is the *upper level constraint*. Let the pair (x^*, y^*) be a solution of the BLPP. We suppose that at least one such pair exists.

The particular structure of bilevel programs facilitates the formulation of several practical problems that involve an hierarchical decision making process like engineering design, analysis of competitive economies, transport system planning, signal optimization, network design, strategic offensive and defensive force problems, government regulation, management, etc., see for example Ferris and Pang (1997), Shimizu et al. (1997), Vicente and Calamai (1994) and Dempe (2002).

We assume that the lower level problem is strictly convex in y for x fixed. Thus, it has a unique solution. Even if this assumption is strong, it is true for several engineering applications, where the lower level problem represents the mathematical model of a physical system. For instance, equilibrium of elastic structures can be obtained as the minimum of an energy functional under suitable constraints. Normally, designers seek for stable systems. Thus, the minimizer of the energy functional must be unique.

Convex bilevel programming presents difficulties normally not encountered in single-level mathematical programming. Since the implicit function $y(x) : \mathbb{R}^m \rightarrow \mathbb{R}^n$ is not always a differentiable mapping, the objective function F and all the constraints are generally nonsmooth. If we substitute the lower level problem by its Karush-Kuhn-Tucker (KKT) conditions, see Bazaraa et al. (1993), we obtain a particular case of the so-called Mathematical Program with Equilibrium Constraints (MPEC), Luo et al. (1996), Anitescu et al. (2007). This is a problem that does not satisfy linear independence (LICQ) or Mangasarian Fromovitz (MFCQ) constraints qualifications, generally required for smooth nonlinear programs. In consequence, optimization methods based on KKT conditions cannot be directly applied to MPEC problem. However, several algorithms for MPEC and Mathematical Programs with Complementarity Constraints (MPCC) have been proposed (Leyffer et al. 2006; Fletcher and Leyffer 2004; Raghunathan and Biegler 2005). Anitescu (2000, 2005) studied the conditions for existence of multipliers for MPCC. Fletcher et al. (2006) proved local convergence of a SQP algorithm when applied to MPCC.

This paper presents the theoretical analysis of a bilevel programming technique introduced in Herskovits et al. (1997, 2000), applied to shape optimization of solids in contact, and in a Stackelberg-Cournot-Nash equilibrium problem, Leontiev and Herskovits (1997).

Various approaches have been developed in different branches of numerical optimization to solve bilevel programs, see Dempe (2003). We mention extreme point, branch and bound algorithms, cutting plane techniques, mixed integer programming, grid search, parametric complementary pivoting, and penalty function methods. A detailed list of references can be found in Shimizu et al. (1997) and Vicente and Calamai (1994).

Our technique is based on necessary and sufficient optimality conditions that involve an auxiliary single-level mathematical program. Under appropriate assumptions, the problem satisfies linear independence constraint qualifications. The auxiliary single level program is obtained by substituting the lower level problem by its KKT conditions and relaxing primal and dual feasibility constraints.

We prove that the set of nondegenerate solutions of the bilevel program is the same as the set of solutions of the auxiliary problem that are primal and dual feasible for the lower level problem.

The algorithm described here requires an initial point at the interior of the constraints of the first and second level problems. At each iteration, it computes a descent direction of the problem that is feasible with respect to both levels of constraints, by employing a formulation based on Herskovits's interior point method for nonlinear optimization, Herskovits (1998). A line search procedure ensures that the new point is feasible and the upper objective function is lower. In this way, it is obtained an interior point sequence converging to a local solution of the bilevel program.

We give numerical results for a set of test problems having linear, quadratic and nonlinear objective functions and constraints. Also an example of Stackelberg-Cournot-Nash equilibrium problem is included. In all numerical experiments and applications the problems were solved in a very efficient way, without any change in the internal parameters of the algorithm.

We present and prove optimality conditions in the next section. In Sect. 3 we describe FDIPA, the Feasible Direction Interior Point Algorithm for nonlinear optimization. In the section that follows, we propose an algorithm for BLPP and analyze its convergence. Finally, a set of numerical tests is described and our conclusions are presented.

2 Optimality conditions for bilevel programming

Let $\tilde{y}(x)$ be the global minimizer of the lower level problem (1.1) for a given x . The set

$$\Omega := \{(x, y) \in \mathbb{R}^m \times \mathbb{R}^n \mid G(x, y) \leq 0 \text{ and } g(x, y) \leq 0\}$$

is the so called *BLPP Constraint Region* and $\Omega(x) := \{y \mid g(x, y) \leq 0\}$, the *Lower's Feasible Region* for x fixed. Finally, the *BLPP Inducible Region*:

$$IR := \{(x, y) \mid G(x, y) \leq 0; y = \tilde{y}(x)\}$$

represents the set of feasible points for the upper level objective function. We deduce that $\tilde{y}(x)$ is a continuous closed point-to-point mapping. In consequence $F(x, \tilde{y}(x))$ is continuous and, since $G(x, y) \leq 0$ defines a compact set, the inducible region IR is also compact. Thus, the leader minimizes a continuous function over a compact set, Edmunds and Bard (1991).

We assume that, for x fixed, a solution of the lower level problem satisfies linear independence regularity conditions, see Luenberger (1984). Replacing the lower level problem (1.1) by Karush-Kuhn-Tucker conditions, the following nonlinear single-level program is obtained:

$$\left\{ \begin{array}{l} \min_{x,y,\xi} F(x, y) \\ \text{subject to } G(x, y) \leq 0 \\ \qquad \qquad \nabla_y f(x, y) + \xi \nabla_y g(x, y) = 0 \\ \qquad \qquad \xi_i g_i(x, y) = 0, \quad i = 1, \dots, p \\ \qquad \qquad g(x, y) \leq 0 \\ \qquad \qquad \xi \geq 0 \end{array} \right. \tag{2}$$

where $\xi \in \mathbb{R}^p$ is a vector of Kuhn-Tucker multipliers for the follower’s subproblem for x fixed. Since this equivalent problem is nonconvex, local minimum may exist. Note that ∇ represents the derivative with respect to (x, y, ξ) and that the derivative with respect a particular component is represented as ∇_x, ∇_y or ∇_{ξ} .

Theorem 1 *If the follower’s objective function f is strictly convex and $\Omega(x)$ is a compact convex set for all allowable choices of the leader variables, then (x^*, y^*) is a solution of (1) if and only if (x^*, y^*, ξ^*) is a solution of the problem (2). See Edmunds and Bard (1991).*

A deep discussion of the previous theorem and of the equivalence of problems (1) and (2) can be found in a recent paper, Dempe and Dutta (2010).

Since the set of constraints

$$g_i(x, y) \leq 0, \quad \xi_i \geq 0, \quad \text{and} \quad \xi_i g_i(x, y) = 0, \quad i = 1, \dots, p.$$

do not satisfy the linear independence constraint qualification, Bazaraa et al. (1993), problem (2) cannot be directly solved with classical algorithms exploiting KKT conditions. In fact, let (x, y, ξ) be a feasible point of problem (2). We have that

$$\begin{aligned} \nabla g_i(x, y) &= (\nabla_x g_i(x, y), \nabla_y g_i(x, y), 0)^t, \\ \nabla \xi_i &= (0, 0, \nabla_{\xi} \xi_i)^t, \\ \nabla(\xi_i g_i(x, y)) &= (\xi_i \nabla_x g_i(x, y), \xi_i \nabla_y g_i(x, y), g_i(x, y) \nabla_{\xi} \xi_i)^t, \end{aligned}$$

and denote $I = \{i : g_i(x, y) = 0\}$. If $i \in I$, then $\nabla g_i(x, y)$ and $\nabla(\xi_i g_i(x, y))$ are linearly dependent. If $i \notin I$, then $\nabla \xi_i$ and $\nabla(\xi_i g_i(x, y))$ are linearly dependent. Thus,

the set of constraints of problem (2) do not satisfy the linear independence constraint qualification regardless of what function $g(x, y)$ is taken and what assumptions about the properties of problem (1) are made, Scheel and Scholtes (2000). In Leyffer et al. (2006), an active set SQP method was proposed to overcome this problem.

We are going to show that the solutions of problem (1) can be found from the set of solutions to the problem

$$\left\{ \begin{array}{l} \min_{x, y, \xi} F(x, y) \\ \text{subject to } G(x, y) \leq 0, \\ \qquad \qquad \qquad \nabla_y L_f(x, y, \xi) = 0, \\ \qquad \qquad \qquad \xi_i g_i(x, y) = 0, i = 1, \dots, p \end{array} \right. \tag{3}$$

where $L_f(x, y, \xi)$ represents the Lagrangian function associated with the problem of the second level: $L_f(x, y, \xi) = f(x, y) + \xi^t g(x, y)$.

We say that the solution \bar{y} of the problem (1.1) is *nondegenerate* if the strict complementary condition holds: $\bar{\xi}_i > 0$ when $g_i(\bar{x}, \bar{y}) = 0$ for all $i = 1, \dots, p$. A solution of the bilevel problem (1) is said to be nondegenerate if this is the case for the lower level problem. We will be looking only for nondegenerate solutions, even if degenerate solutions can exist.

Theorem 2 (Necessary optimality condition) *Let f and g be differentiable at (x^*, y^*) with respect to y , g is continuous at (x^*, y^*) with respect to x and a constraint qualification for problem (1.1) holds. If (x^*, y^*) is a nondegenerate solution of the problem (1), there exists $\xi^* \in \mathbb{R}^p, \xi^* \geq 0$ such that (x^*, y^*, ξ^*) is a solution of (3).*

Proof Since (x^*, y^*) is a solution of (1), from Theorem 1 there exists ξ^* such that the triple (x^*, y^*, ξ^*) solves (2). Since (x^*, y^*) is nondegenerate and the function g is continuous with respect to x and y at (x^*, y^*) , there exists a neighborhood $\mathcal{O}^* \subset \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^p$ of the point (x^*, y^*, ξ^*) such that $\forall (x, y, \xi) \in \mathcal{O}^*$ the conditions $\xi_i g_i(x, y) = 0, i = 1, \dots, p$ imply that $g_i(x, y) < 0$ and $\xi_i = 0$ or that $\xi_i > 0$ and $g_i(x, y) = 0$. This means that the set of constraints $\{\xi_i g_i(x, y) = 0, i = 1, \dots, p\}$ is equivalent to $\{g(x, y) \leq 0, \xi \geq 0, \xi_i g_i(x, y) = 0, i = 1, \dots, p.\}$ on \mathcal{O}^* . Then, the feasible sets of problem (2) and (3) on \mathcal{O}^* are identical. Thus, (x^*, y^*, ξ^*) gives a solution of the problem (3). Moreover we have $g(x^*, y^*) \leq 0$ and $\xi^* \geq 0$. The proof is complete. □

Theorem 3 (Sufficient optimality condition) *If (x^*, y^*, ξ^*) is a solution of the problem (3) such that*

$$g(x^*, y^*) \leq 0 \quad \text{and} \quad \xi^* \geq 0. \tag{4}$$

Then (x^, y^*) is a solution of the BLPP (1).*

The proof is trivial. Thus, (3)–(4) can be considered as an optimality condition for a nondegenerate solution of the BLPP (1). Now, it is possible to assume that the

problem (3) verifies linear independence constraint qualification. This assumption is generically satisfied, even if counterexamples can be constructed. For example, the assumption is not true in the uncommon case when the set of derivatives of both levels constraints of problem (1) are not linearly independent. An interesting discussion about this assumption can be found in Scholtes and Stöhr (2001).

Our approach to find points that verify the present optimality conditions, consists in solving iteratively the nonlinear program (3) so that all the iterates verify (4).

3 The feasible direction interior point algorithm

This algorithm consists in Newton like iterations to solve the system of nonlinear equations, in the primal and the dual variables, given by the equalities included in KKT optimality conditions. To describe the basic ideas we consider the following optimization problem,

$$\begin{cases} \min F(z) \\ \text{subject to } G(z) \leq 0, \\ H(z) = 0 \end{cases} \tag{5}$$

where $z \in \mathbb{R}^{nz}$, $G : \mathbb{R}^{nz} \rightarrow \mathbb{R}^{mz}$ and $H : \mathbb{R}^{nz} \rightarrow \mathbb{R}^{pz}$.

Let $\nabla G \in \mathbb{R}^{nz \times mz}$ and $\nabla H \in \mathbb{R}^{nz \times pz}$ be the matrices of derivatives of G and H . We call $\lambda \in \mathbb{R}^{mz}$ and $\mu \in \mathbb{R}^{pz}$ the vectors of dual variables corresponding to the inequality and equality constraints and define $\mathcal{D}[v] = \text{diag}[v]$, a diagonal matrix such that $\mathcal{D}[v]_{i,i} = v_i$, for a vector v in a finite dimensional space.

The FDIPA requires the following assumptions about Problem (5):

Assumption 1 *The functions $F(z)$, $G(z)$ and $H(z)$ are continuously differentiable in $\Omega' \equiv \{z \in \mathbb{R}^{nz} \mid G(z) \leq 0 \text{ and } H(z) \leq 0\}$ and their derivatives satisfy a Lipschitz condition.*

Assumption 2 *Each $z \in \text{int}(\Omega')$ satisfies $G(z) < 0$, $H(z) < 0$.*

Assumption 3 (Regularity condition) *For all $z \in \Omega'$, the vectors $\nabla G_i(z)$, for i such that $G_i(z) = 0$, and $\nabla H_j(z)$, for $j = 1, \dots, pz$, are linearly independent.*

Let z^* be a regular point of the problem (5), Herskovits (1998), then the corresponding Karush Kuhn Tucker first order necessary optimality conditions are expressed as: If z^* is a local minimum of (5) then there exists $\lambda^* \in \mathbb{R}^{mz}$ and $\mu^* \in \mathbb{R}^{pz}$ such that

$$\nabla F(z^*) + \nabla G(z^*)\lambda^* + \nabla H(z^*)\mu^* = 0, \tag{6}$$

$$\mathcal{D}[G(z^*)]\lambda^* = 0, \tag{7}$$

$$H(z^*) = 0, \tag{8}$$

$$G(z^*) \leq 0,$$

$$\lambda^* \geq 0.$$

A quasi-Newton iteration for the solution of (6) to (8) in (z, λ, μ) can be stated as

$$\begin{aligned} & \begin{bmatrix} B & \nabla G(z) & \nabla H(z) \\ \mathcal{D}[\lambda]G(z)^t & \mathcal{D}[G(z)] & 0 \\ \nabla H(z)^t & 0 & 0 \end{bmatrix} \begin{bmatrix} z_\alpha - z \\ \lambda_\alpha - \lambda \\ \mu_\alpha - \mu \end{bmatrix} \\ & = - \begin{bmatrix} \nabla F(z) & +\nabla G(z)\lambda + \nabla H(z)\mu \\ \mathcal{D}[G(z)]\lambda \\ H(z) \end{bmatrix} \end{aligned} \tag{9}$$

where B represents a quasi-Newton approximation of the Hessian of the Lagrangian

$$\nabla^2 F(z) + \sum_{i=1}^{mz} \lambda_i \nabla^2 G_i(z) + \sum_{i=1}^{pz} \mu_i \nabla^2 H_i(z),$$

(z, λ, μ) is the current point and $(z_\alpha, \lambda_\alpha, \mu_\alpha)$ is a new estimate.

By denoting $d_0 = z_\alpha - z$, $\lambda_0 = \lambda_\alpha$ and $\mu_0 = \mu_\alpha$, then (9) becomes

$$Bd_0 + \nabla G(z)\lambda_0 + \nabla H(z)\mu_0 = -\nabla F(z) \tag{10}$$

$$\mathcal{D}[\lambda]\nabla G(z)^t d_0 + \mathcal{D}[G(z)]\lambda_0 = 0 \tag{11}$$

$$\nabla H(z)^t d_0 = -H(z) \tag{12}$$

which is independent of the current value of μ . In Herskovits (1998), it is proved that d_0 is a descent direction of the objective function $F(z)$. However, d_0 cannot be employed as a search direction, since it is not necessarily a feasible direction.

An appropriate potential function is needed for the line search. We consider

$$\phi_c(z) = F(z) + \sum_{i=1}^{pz} c_i |H_i(z)|, \tag{13}$$

where c_i are positive constants. It can be shown that, if c_i are large enough, then $\phi_c(z)$ is an Exact Penalty Function of the equality constraints Luenberger (1984). On the other hand, ϕ_c has no derivatives at points where there are active equality constraints. FDIPA uses ϕ_c in the line search, but avoids the points where this function is nonsmooth. It generates a sequence with decreasing values of ϕ_c at the interior of the inequality constraints and such that the equalities are negative. With this purpose, the system (10)–(12) is modified in a way to obtain a feasible direction of the inequalities that points to the negative side of the equalities and, at the same time, is a descent direction of ϕ_c .

We define the linear system

$$Bd_1 + \nabla G(z)\lambda_1 + \nabla H(z)\mu_1 = 0$$

$$\mathcal{D}[\lambda]\nabla G(z)^t d_1 + \mathcal{D}[G(z)]\lambda_1 = -\lambda$$

$$\nabla H(z)^t d_1 = -e$$

where $e = [1, 1, \dots, 1]^t \in \mathbb{R}^{pz}$.

Now, the search direction can be calculated by $d = d_0 + \rho d_1$, where we consider ρ as follows: If $d_1^t \nabla \phi_c(z) > 0$, set

$$\rho = \min \left\{ \varphi \|d_0\|_2^2; (\gamma - 1) \frac{d_0^t \nabla \phi_c(z)}{d_1^t \nabla \phi_c(z)} \right\}, \quad \text{where } \gamma \in (0, 1) \text{ and } \varphi > 0.$$

Otherwise, set $\rho = \varphi \|d_0\|_2^2$.

The following assumptions about the algorithm are required:

Assumption 4 *There exists positive numbers λ^I, λ^S and \bar{G} such that $0 < \lambda_i \leq \lambda^S$, $i = 1, \dots, mz$ and $\lambda_i \geq \lambda^I$ for any i such that $G_i(z) \geq -\bar{G}$, at every iterate of the algorithm.*

Assumption 5 *There exist positive numbers σ_1 and σ_2 such that*

$$\sigma_1 \|d\|_2^2 \leq d^T B d \leq \sigma_2 \|d\|_2^2 \quad \text{for any } d \in \mathbb{R}^{nz}.$$

Assuming that the iterates given by the present algorithm are in a compact set, it was proved in Herskovits (1998) that the FDIPA is globally convergent to a Karush Kuhn Tucker point for any initial point at the interior of Ω' and for any way of updating B and λ according to both the previous assumptions.

4 A numerical algorithm for bilevel programming

We present an algorithm that computes a solution of the problem

$$\begin{cases} \min F(x, y, \xi) \\ \text{subject to } G(x, y, \xi) \leq 0, \\ H(x, y, \xi) = 0, \end{cases} \tag{14}$$

where $H(x, y, \xi) := (\nabla_y L_f(x, y, \xi), \xi_1 g_1(x, y), \dots, \xi_p g_p(x, y))$. It follows from Theorem 3 that these are also solutions of the bilevel program. We assume that the computed solutions of problem (1) are nondegenerate.

We suppose that problem (14) verifies the assumptions required in Herskovits (1998) and satisfy the conditions in (4). Thus, we need to define the following sets:

$$\Omega_1 := \{(x, y, \xi) \mid G(x, y, \xi) \leq 0 \text{ and } H(x, y, \xi) \leq 0\},$$

and

$$\Omega_2 := \{(x, y, \xi) \mid g(x, y) \leq 0 \text{ and } \xi \geq 0\}.$$

In order to find solutions of (14) that verify (4), we take the initial point at the interior of $\Omega^+ \equiv \Omega_1 \cap \Omega_2$.

Assumption 6 *The set Ω^+ has an nonempty interior, $\text{int}(\Omega^+)$.*

Assumption 7 Each $(x, y, \xi) \in \text{int}(\Omega^+)$ satisfies $G(x, y, \xi) < 0, H(x, y, \xi) < 0$.

Assumption 8 The functions $F(x, y, \xi), G(x, y, \xi)$ and $H(x, y, \xi)$ are continuously differentiable in Ω^+ and their derivatives satisfy a Lipschitz condition.

Assumption 9 (Regularity condition) For all $(x, y, \xi) \in \Omega^+$ the vectors $\nabla G_i(x, y, \xi),$ for i such that $G_i(x, y, \xi) = 0,$ and $\nabla H_j(x, y, \xi),$ for $j = 1, \dots, n + p,$ are linearly independent.

Now, consider the function

$$\Phi_c(x, y, \xi) = F(x, y, \xi) + \sum_{i=1}^{n+p} c_i |H_i(x, y, \xi)|,$$

where $H : \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}^{n+p}$ is defined by

$$H_i(x, y, \xi) := \delta_{in} \frac{\partial L_f}{\partial y_i}(x, y, \xi) + (1 - \delta_{in}) \xi_{i-n} g_{i-n}(x, y),$$

such that,

$$\delta_{in} := \begin{cases} 1 & \text{if } i \leq n \\ 0 & \text{otherwise,} \end{cases}$$

and c_i are positive constants determined by the algorithm.

The present algorithm starts with an initial point $(x_0, y_0, \xi_0) \in \text{int}(\Omega^+)$. At each iteration we have that $G(x, y, \xi) < 0$ and the equality constraints hold as negative inequalities, $H(x, y, \xi) < 0,$ being active only at the limit. For a given point $(x, y, \xi),$ the algorithm finds a search direction $d \equiv (d_x, d_y, d_\xi)$ that is a descent direction of $\Phi_c(x, y, \xi)$ and points to the negative side of the equality and inequality constraints. A line search is then performed looking for a step length t such that $(x, y, \xi) + td \in \text{int}(\Omega^+)$ and a decrease of $F(x, y, \xi)$ is obtained.

Then, the algorithm is stated as follows:

Parameters. Let be $\varepsilon > 0, \varphi > 0, \gamma \in (0, 1)$ and $\nu \in (0, 1).$

Data. Define initial values for $(x, y, \xi) \in \text{int}(\Omega^+), B \in \mathbb{R}^{(m+n+p) \times (m+n+p)}$ symmetric and positive definite, $\lambda \in \mathbb{R}^l, \lambda > 0$ and $0 < c \in \mathbb{R}^{n+p}.$

Step 1. Computation of a search direction

(i) Compute (d_0, λ_0, μ_0) by solving the linear system

$$Bd_0 + \nabla G(x, y, \xi)\lambda_0 + \nabla H(x, y, \xi)\mu_0 = -\nabla F(x, y, \xi) \tag{15}$$

$$\mathcal{D}[\lambda]\nabla G(x, y, \xi)^t d_0 + \mathcal{D}[G(x, y, \xi)]\lambda_0 = 0 \tag{16}$$

$$\nabla H(x, y, \xi)^t d_0 = -H(x, y, \xi). \tag{17}$$

If $d_0 \leq \varepsilon,$ stop.

(ii) Compute (d_1, λ_1, μ_1) by solving the linear system

$$Bd_1 + \nabla G(x, y, \xi)\lambda_1 + \nabla H(x, y, \xi)\mu_1 = 0 \tag{18}$$

$$\mathcal{D}[\lambda]\nabla G(x, y, \xi)^t d_1 + \mathcal{D}[G(x, y, \xi)]\lambda_1 = -\lambda \tag{19}$$

$$\nabla H(x, y, \xi)^t d_1 = -\mathbf{e} \tag{20}$$

where $\mathbf{e} = [1, \dots, 1]^t, \mathbf{e} \in \mathbb{R}^{n+p}$.

(iii) If $c_i < -1.2\mu_{0i}$, then set $c_i = -2\mu_{0i}$, for $i = 1, \dots, n + p$.

(iv) If $d_1^t \nabla \Phi_c(x, y, \xi) > 0$, set

$$\rho = \min \left\{ \varphi \|d_0\|_2^2 ; (\gamma - 1) \frac{d_0^t \nabla \Phi_c(x, y, \xi)}{d_1^t \nabla \Phi_c(x, y, \xi)} \right\}.$$

Otherwise, set $\rho = \varphi \|d_0\|_2^2$.

(v) Compute the search direction $d = d_0 + \rho d_1$ and also $\bar{\lambda} = \lambda_0 + \rho \lambda_1$.

Step 2. Line search Compute t , the first number of the sequence $\{1, v, v^2, v^3, \dots\}$ satisfying

$$\Phi_c((x, y, \xi) + td) \leq \Phi_c(x, y, \xi) + t\eta d^t \nabla \Phi_c(x, y, \xi), \tag{21}$$

$$\nabla_y L_f((x, y, \xi) + td) < 0, \tag{22}$$

$$(\xi + td_\xi)_i > 0, \quad \text{for } i = 1, \dots, p \tag{23}$$

$$g_i(x + td_x, y + td_y) < 0, \quad \text{for } i = 1, \dots, p \tag{24}$$

$$G_j((x, y, \xi) + td) < 0, \quad \text{for } j = 1, \dots, l \text{ if } \bar{\lambda}_j \geq 0, \tag{25}$$

$$G_j((x, y, \xi) + td) \leq G_j(x, y, \xi), \quad \text{for } j = 1, \dots, l \text{ otherwise.} \tag{26}$$

Step 3. Updates

(i) Set $(x, y, \xi) := (x, y, \xi) + td$ and define new values for $\lambda > 0$ and B symmetric and positive definite.

(ii) Go to Step 1.

In Herskovits (1998) was proved global convergence of the FDIPA. However, in the previous algorithm, the line search includes the conditions (23) and (24) to have the new iterate at the interior of Ω^+ . With this additional condition the result of Lemma 4.4 in Herskovits (1998) remains valid. In effect, by Assumption 8, there exist $k_i > 0$ such that

$$(\xi_i + td_{\xi_i})g_i(x + td_x, y + td_y) \leq \xi_i g_i(x, y) + t d^t \nabla(\xi_i g_i(x, y)) + t^2 k_i \|d\|_2^2 \tag{27}$$

holds for any (x, y, ξ) and $(x, y, \xi) + td \in \text{int } \Omega^+$ such that $[(x, y, \xi), (x, y, \xi) + td] \subset \text{int } \Omega^+$.

By (17) and (20) we have that $d^t \nabla(\xi_i g_i(x, y)) = -\xi_i g_i(x, y) - \rho e$. Then,

$$\xi_i g_i(x, y) + t d^t \nabla(\xi_i g_i(x, y)) + t^2 k_i \|d\|_2^2 = (1 - t)\xi_i g_i(x, y) - \rho t e + t^2 k_i \|d\|_2^2,$$

and we deduce that

$$(\xi_i + td_{\xi_i})g_i(x + td_x, y + td_y) < 0, \quad (28)$$

if the following conditions are true:

$$(1 - t) > 0 \quad \text{and} \quad \rho - tk_i \|d\|_2^2 > 0.$$

Both inequality hold for any $t \in [0, \tau_i]$ such that

$$\tau_i < \min\{1; \rho/(k_i \|d\|_2^2)\}. \quad (29)$$

As in Herskovits (1998), we have that there exists $\bar{\rho}_i > 0$ such that $\bar{\rho}_i < \rho/(k_i \|d\|_2^2)$ for $(x, y, \xi) \in \Omega^+$. Then, $\tau_i < \min\{1; \bar{\rho}_i\}$. In consequence, (28) is true for any $t \in [0, \tau_i]$ and we deduce that $(\xi_i + td_{\xi_i}) > 0$ and $g_i(x + td_x, y + td_y) < 0$ for any $t \in [0, \tau_i]$.

5 Tests examples

We carried out several numerical experiments with the proposed algorithm applied to the solution of benchmark problems. These problems are listed in Table 1. We note that all these problems have only nondegenerate solutions.

The numerical results are presented in Table 2. The abbreviation attached to the problem (column PROB.) describes the original source and the ordinal number in which the problem is given there. The column CLASS gives the classification of the problem. The characters denote the type of upper level objective function, upper level constraints, lower level objective function, and lower level constraints in the prescribed order, and are the following: **L** for linear, **Q** for quadratic, **N** for nonlinear

Table 1 Test problems

Problem	Author
CF1	Chen and Florian (1991)
BCC3	Bi et al. (1989)
YZ1	Ye and Zhu (1993)
EB1	Edmunds and Bard (1991)
FTC5 and FTC4	Friesz et al. (1990)
VSJ1 and VSJ2	Vicente et al. (1994)
CV1 ^a	Calamai and Vicente (1994)
GS1	Savard and Gauvin (1994)
YZZ2 and YZZ1	Ye et al. (1993)
AS1	Aiyoshi and Shimizu (1984)
B5	Bard (1984)
T1 (Stackelberg-Cournot-Nash equilibrium problem)	Tobin (1992)

^aThe transformed problem described in pp. 116–117 in Calamai and Vicente (1994) was solved

Table 2 Numerical results

PROB.	CLASS	x^0	y^0	ITER	x^*	y^*	MIN
CF1	L_LL	10	9	10	19	14	-37
BCC3	L_LL	3	3	11	4	1	5
YZ1	LBQB	-0.5	0.5	10	-1	-1	-2
EB1	LBNB	0	(0.5, 0)	10	1	(0, 0)	-1
FTC5	QBQB	80	30	7	93.3333	26.6667	3266.67 (max)
VSJ1	QBQB	(0.1, 0.1)	0.1	12	(0.8, 0.2)	0	0.5
VSJ2	QBQB	(0.1, 0.3)	0.1	12	(1, 0.4)	0	0.32
CV1	Q_QL	(0,0,0,0)	(0.1, 0.1)	12	(**)	(0.025, 0.05)	0.3125
GS1	QBQL	8	11	10	2	14	20
YZZ2	QBQL	0	0.5	10	2	4	-4
AS1	QLQL	5	1	20	10	10	100
FTC4	QBQQ	(0.4, 0.4)	(1.5, 1.5)	12	(0.5, 0.5)	(0.5, 0.5)	-1
B5	QQQQ	(1.5, 0.5)	(1, 0)	13	(2, 0)	(1.6, 0.2)	-14.36
YZZ1	NBNB	-0.5	0	10	1	-1	0
T1	NLNB	10	(10, 5, 1)	10	23.8580	(5.2906, 0, 0)	406.629 (max)

(**) (0.1308, 0.0520, 0.1022, 0.0674)

in both x and y variables functions, and **B** for “box” constraints. We use the blank position “_” if constraints are absent. The following notation is also used: (x_0, y_0) for initial point of iterations; ITER for number of iterations; (x^*, y^*) for calculated solution; MIN for value of $F(x^*, y^*)$.

Each iteration includes one computation of the search direction and the line search. In all examples our results correspond to the solution indicated in the original sources. We stop the algorithm when the norm of the gradient of the Lagrangian and the absolute value of the equality constraints are all less than $\varepsilon = 10^{-6}$. All the iterates are strictly feasible with respect to the inequality constraints of both level problems.

The present technique was initially applied for shape optimization of nonlinear elastic solids in contact, Herskovits et al. (2000). This problem, that satisfies the assumptions required here, employs a finite element model and requires the solution of large bilevel programs. The results were obtained in about 10 iterations.

6 Conclusions

We have presented the theoretical basis of a class of optimality conditions for bilevel programming and of a numerical algorithm to solve them. This algorithm, that employs ideas of a feasible direction interior point technique for smooth optimization, solves two linear systems with the same matrix and performs an inexact line search at each iteration. We remark that the present approach is free of parameters to be adjusted and avoids bad conditioning produced by large penalty parameters or regularization techniques. The linear systems are in general well conditioned. If regularity

conditions are not true at the solution, the linear systems conditioning can become worst in the final iterations.

Iterates are at the interior of both levels inequality constraints, being the objective function reduced at each iteration. Our algorithm is very simple to code. Since it works with linear systems, several numerical methods or techniques that take advantage of the structure can be employed.

The numerical results shown here and the large scale applications described in Herskovits et al. (2000, 1997) were all solved with a small number of iterations, employing the same set on parameters for line search and stopping criteria, suggesting that our technique is robust. We believe that the approach can be extended to mathematical programs with equilibrium constraints (MPEC) and generalized bilevel programming problems.

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