Optim Eng (2011) 12:535–557
DOI 10.1007/s11081-010-9118-y

Sequential Approximate Optimization using Radial Basis Function network for engineering optimization

Satoshi Kitayama · Masao Arakawa · Koetsu Yamazaki

Received: 28 July 2009 / Accepted: 8 August 2010 / Published online: 19 August 2010

Abstract This paper presents a Sequential Approximate Optimization (SAO) procedure that uses the Radial Basis Function (RBF) network. If the objective and constraints are not known explicitly but can be evaluated through a computationally intensive numerical simulation, the response surface, which is often called metamodeling, is an attractive method for finding an approximate global minimum with a small number of function evaluations. An RBF network is used to construct the response surface. The Gaussian function is employed as the basis function in this paper. In order to obtain the response surface with good approximation, the width of this Gaussian function should be adjusted. Therefore, we first examine the width. Through this examination, some sufficient conditions are introduced. Then, a simple method to determine the width of the Gaussian function is proposed. In addition, a new technique called the adaptive scaling technique is also proposed. The sufficient conditions for the width are satisfied by introducing this scaling technique. Second, the SAO algorithm is developed. The optimum of the response surface is taken as a new sampling point for local approximation. In addition, it is necessary to add new sampling points in the sparse region for global approximation. Thus, an important issue for SAO is to determine the sparse region among the sampling points. To achieve this, a new function called the density function is constructed using the RBF network. The global minimum of the density function is taken as the new sampling point. Through the sampling strategy proposed in this paper, the approximate global

S. Kitayama (✉) · K. Yamazaki
Kanazawa University, Kakuma-machi, Kanazawa 920-1192, Japan
e-mail: kitagon@t.kanazawa-u.ac.jp

K. Yamazaki
e-mail: yamazaki@t.kanazawa-u.ac.jp

M. Arakawa
Kagawa University, Hayashi-chi, Takamatsu, Kagawa 761-0396, Japan
e-mail: arakawa@eng.kagawa-u.ac.jp

minimum can be found with a small number of function evaluations. Through numerical examples, the validities of the width and sampling strategy are examined in this paper.

Keywords Response surface · Sequential approximate optimization · RBF network · Density function · Engineering optimization

1 Introduction

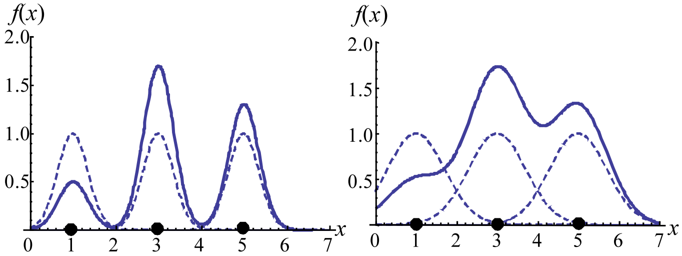
In recent years, many commercial software programs for design optimization have been widely utilized in a variety of industries. Recently, some population-based global optimization techniques, such as the Genetic Algorithm (GA), the Particle Swarm Optimization (PSO), and the Differential Evolution (DE), have been developed in comparison with classical mathematical programming. These global optimization techniques have been applied to practical design optimization. In addition, these methods are applicable to the multi-objective optimization problems. In general, the population-based optimization techniques require a large number of function evaluations to find the global minimum and a set of Pareto-optimal solutions. This makes the direct application of these optimization techniques to practical design optimization problems difficult in some cases due to the time-consuming. Since classical mathematical programming requires the sensitivity of the objective and constraints, it is not applicable to non-differentiable problems. In addition, function evaluations for calculating the sensitivity and determining the step-size are required. Nowadays, the time made available to develop new products is continuously being shortened, making it preferable to reduce the computing-time required for optimization. This implies that one of the most important aspects is reducing the function evaluations in practical design optimization. It is important to find the global minimum with high accuracy using global optimization techniques, and these global optimization techniques generally require a large number of function evaluations. However, it is also important to find an approximate global minimum for a design problem with a small number of function evaluations even when the objective and constraints are not known explicitly.

If the objective and constraints are not known explicitly but can be evaluated through computationally intensive numerical simulation, the response surface, which is called meta-modeling, is an attractive method for finding an approximate global minimum with a small number of function evaluations (Wang and Shan 2007). The Design of Experiment (DOE) is one of the most popular response surface methods (Myers and Montgomery 1995). The general and classical response surface procedure is briefly summarized as follows:

Step 1 First, numerous sampling points in the design variable space are set. The orthogonal array or Latin Hypercube Design (LHD) is often used to determine these sampling points.

Step 2 The objective and constraints are evaluated at these sampling points. Thus, the number of sampling points is equal to the function evaluations.

Step 3 Then, a response surface which approximates the objective and constraints is constructed. Quadratic polynomials, the Kriging, and the Radial Basis Function (RBF) network are used to construct the response surface.



(a) Small value of the width
width=0.5

(b) Large value of the width
width=1.0

Fig. 1 Effect of width in the Gaussian function

Step 4 Finally, the approximate optimum can be obtained by optimizing the response surface. The optimum of the response surface is taken as the approximate optimum of the original design optimization problem.

It is understood that the response surface is one of the approximation techniques. It is clear from the general flow described above that the number of function evaluations is drastically reduced by using the response surface approach. Using the quadratic polynomials as the response surface, it is possible to approximate the original function globally. It is possible to use the Kriging (Donald et al. 1998; Simpson et al. 2001a; Martin and Simpson 2005) and the RBF Network (Muller and Messac 2005; Hussain et al. 2002; Jin et al. 2001; Nakayama et al. 2002; Fang and Horstemeyer 2006; McDonald et al. 2007) to approximate the original function locally and globally, because these two methods utilize the Gaussian function as the basis function. Thus, the response surface using the Kriging and RBF Network is expressed by the linear combination of the weight and Gaussian function. In addition, the global and local approximations by the Kriging and the RBF Network imply that the response surface by these methods will be a multimodal function. However, an appropriate parameter should be adjusted in order to approximate the original function locally and globally. Therefore, one key for a good approximation is to determine the parameter appropriately. Adjusting this parameter appropriately will allow the global minimum to be found with high accuracy. This parameter is the width of the Gaussian function (Nakayama et al. 2002; Haykin 1994). If this width is small, the response surface will become peaky. Otherwise, the response surface will become smooth. The effect of the width is shown in Figs. 1(a) and (b). In Fig. 1, the black dots represent the sampling points, the dashed line represents the Gaussian function, and the bold line denotes the response surface. The following weights are assigned to the sampling points: $w_1 = 0.5$ at $x = 1$, $w_2 = 1.7$ at $x = 3$, and $w_3 = 1.3$ at $x = 5$. The difference between Fig. 1(a) and Fig. 1(b) is the value of the basis function width. The widths in Figs. 1(a) and (b) are set to 0.5 and 1.0, respectively. It is clear from Fig. 1 that the determination of the width plays an important role.

In recent years, the Sequential Approximate Optimization (SAO) has been widely studied (Simpson et al. 2001b; Wang 2003; Rodriguez et al. 2001; Perez et al. 2002; Sobester et al. 2005; Huang et al. 2006; Simpson and Mistree 2001), compared with

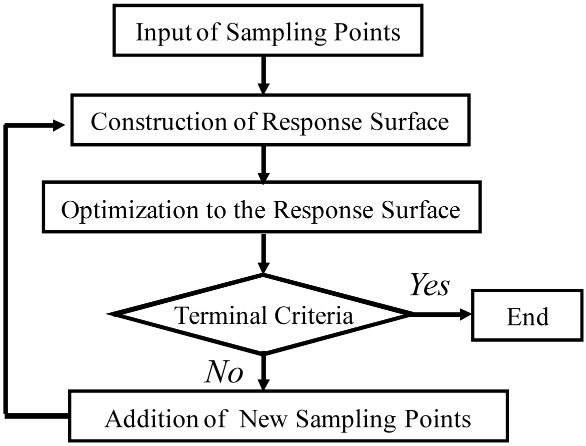

Input of Sampling Points
Construction of Response Surface
Optimization to the Response Surface
Terminal Criteria
Yes
End
No
Addition of New Sampling Points

Fig. 2 General procedure of SAO

the classical response surface approach described above. The general procedure for SAO is shown in Fig. 2.

In SAO, the response surface is constructed repeatedly by adding new sampling points, until the terminal criterion determined by the decision-maker is satisfied. In comparison with the classical response surface approach described above, it is expected that an approximate global minimum with high accuracy can be obtained through the addition of the new sampling points. In order to obtain an approximate global minimum with high accuracy, it has been reported that the most important requirement is simultaneously adding the new sampling points around (1) the optimum of the response surface and (2) the sparse region in the design variable space (Donald et al. 1998; Nakayama et al. 2002; Simpson et al. 2001b; Wang 2003; Rodriguez et al. 2001; Perez et al. 2002; Sobester et al. 2005; Huang et al. 2006; Simpson and Mistree 2001; Sasena 2002a, 2002b; Sharif et al. 2008). Let us consider the first objective, which is to add the optimum of the response surface as the new sampling point. This will lead to a local approximation with high accuracy. The zooming method belongs to this category (Kurtaran et al. 2002). However, only the successive additions of the optimum of the response surface may result in finding the local minimum. Then, the second objective, which is to add a sampling point in the sparse region, plays an important role. The addition of a new sampling point in the sparse region will lead to the global approximation. By this addition, it is possible to avoid falling into the local minimum. Thus, global and local approximations will be achieved simultaneously through the above sequential sampling strategy. In this sequential sampling strategy, it is important to find the sparse region in the design variable space. This paper will roughly belong to Donald et al. (1998); Sasena (2002a, 2002b). In these references, the expected improvement (EI) algorithm is employed to find the sparse region. In the EI algorithm, the region with high uncertainty corresponds to the sparse region. By adding the new sampling points to the regions with high uncertainty, a global approximation can be achieved. However, the Gaussian function is also employed in the EI algorithm. In order to find the sparse region with the EI algorithm, the parameter in the Gaussian function should be adjusted. Therefore, the common subject is the determination of the width in the Gaussian function with a simple manner.

In this paper, we use the RBF network to construct the response surface, in which the Gaussian function is employed as the basis function. In particular, we will consider the determination of the width and the exploration of the sparse region in the design variable space. First, we discuss the width of the Gaussian function, which affects the accuracy of the response surface. Two equations for determining the width have been proposed (Nakayama et al. 2002; Haykin 1994). Among these, the equation proposed by Nakayama et al. (2002) is effective through the author's numerical experiences in the case of one or two design variables. Thus, a good approximation can be achieved by using the equation proposed by Nakayama in the case of one or two design variables. However, it may be impossible to find an approximate global minimum with high accuracy in a case of involving more than three design variables. Therefore, a new equation for determining the width is necessary. By examining the equation proposed by Nakayama, some sufficient conditions are introduced. Then, a new equation for determining the width is proposed. In addition, a new technique called the adaptive scaling technique is also proposed in this paper. Second, an effective method for determining the sparse region in the design variable space is considered. In Nakayama et al. (2002), a simple method for determining the sparse region was proposed. However, this method basically depends on the randomness. Thus, a deterministic method for finding the sparse region is preferable from the viewpoint of efficiency.

The remainder of this paper is organized as follows: In Sect. 2, the RBF network is described briefly. In this section, the width proposed by Nakayama is also analyzed, and some sufficient conditions for a good approximation are determined. Then, a new equation for the width is proposed. In addition, the adaptive scaling technique is also described. In Sect. 3, the new function to find the sparse region in the design variable space, which is called the density function, is introduced, and the details of an SAO algorithm that uses the density function are shown in Sect. 4. The density function utilizes the RBF network, making its construction easy. In Sect. 5, some benchmark problems are discussed in order to examine the proposed SAO algorithm.

2 Radial Basis Function network

2.1 Learning of RBF network

An RBF network is a three-layer feed-forward network. The output of the network $f_a(\boldsymbol{x})$, which corresponds to the response surface, is given by

$$f_a(\boldsymbol{x}) = \sum_{i=1}^{m} w_i h_i(\boldsymbol{x}) \tag{1}$$

where m represents the number of sampling points, $h_i(\boldsymbol{x})$ is the ith basis function, and w_i denotes the weight of the ith basis function. In this paper, the following Gaussian function is used as the basis function.

$$h_i(\boldsymbol{x}) = \exp\left(-\frac{(\boldsymbol{x}-\boldsymbol{x}_i)^T(\boldsymbol{x}-\boldsymbol{x}_i)}{r_i^2}\right) \tag{2}$$

In (2), $\boldsymbol{x}_i$ represents the ith sampling point, and r_i is the width of the ith basis function. The response y_i is calculated at sampling point $\boldsymbol{x}_i$. The learning of the RBF network is usually accomplished by solving

$$E = \sum_{i=1}^{m}(y_i - f_a(\boldsymbol{x}_i))^2 + \sum_{i=1}^{m}\lambda_i w_i^2 \to \min \tag{3}$$

where the second term is introduced for the purpose of regularization. It is recommended that λ_i in (3) have a sufficiently small value (e.g. $\lambda_i = 1.0 \times 10^{-3}$). Thus, the learning of the RBF network is equivalent to finding the weight vector $\boldsymbol{w}$ (Orr 1996). The necessary condition of (3) leads to the following equation:

$$\boldsymbol{w} = (\boldsymbol{H}^T\boldsymbol{H} + \boldsymbol{\Lambda})^{-1}\boldsymbol{H}^T\boldsymbol{y} \tag{4}$$

where $\boldsymbol{H}$, $\boldsymbol{\Lambda}$, and $\boldsymbol{y}$ are given as follows:

$$\boldsymbol{H} = \begin{bmatrix} h_1(\boldsymbol{x}_1) & h_2(\boldsymbol{x}_1) & \cdots & h_m(\boldsymbol{x}_1) \\ h_1(\boldsymbol{x}_2) & h_2(\boldsymbol{x}_2) & \cdots & h_m(\boldsymbol{x}_2) \\ \vdots & \vdots & \ddots & \vdots \\ h_1(\boldsymbol{x}_m) & h_2(\boldsymbol{x}_m) & \cdots & h_m(\boldsymbol{x}_m) \end{bmatrix} \tag{5}$$

$$\boldsymbol{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \lambda_m \end{bmatrix} \tag{6}$$

$$\boldsymbol{y} = (y_1, y_2, \ldots, y_m)^T \tag{7}$$

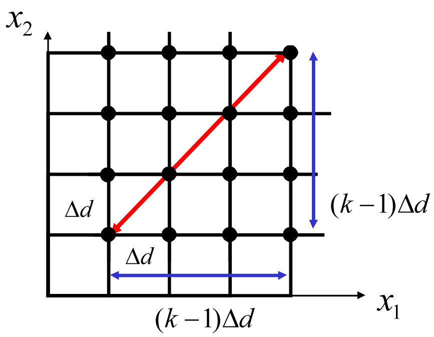
It is clear from (6) that the learning of the RBF network is equivalent to the matrix inversion $(\boldsymbol{H}^T\boldsymbol{H} + \boldsymbol{\Lambda})^{-1}$. In the SAO, the new sampling points are added. Using the RBF network, it is easy to calculate the weight vector $\boldsymbol{w}$, because the additional learning is reduced to the incremental calculation of the matrix inversion. The detailed procedure is found in Orr (1996).

2.2 Width of basis function

Determining the width of the basis function is the key factor for good approximation. The optimization with respect to width may be valid. However, the increment of the sampling points will cause some difficulties in optimizing the width, such as the local minimum. Thus, it is preferable to determine the width with a simple method. To determine the width easily, the following equation was proposed by Nakayama et al. (2002):

$$r = \frac{d_{\max}}{\sqrt[n]{nm}} \tag{8}$$

Fig. 3 Sampling points with two design variables



where $d_{\max}$ denotes the maximum distance among the sampling points. n denotes the number of design variables, and m is the number of sampling points. Equation (8) is applied to all basis functions. Thus, $r_1 = r_2 = \cdots = r_m = r$. Equation (8) is considered to be the generalization of the equation proposed by Simpson et al. (2001b).

Suppose that all of the design variables are equally scaled. This scaling technique, which is called the adaptive scaling technique, will be described in Sect. 2.3. Let us consider the K-level full factorial design, in which the regular interval is given by Δd. In this case, $d_{\max}$ is given by

$$d_{\max} = \sqrt{n}(K-1)\Delta d \tag{9}$$

The black dots in Fig. 3 show the sampling points with two design variables.

In the case of n design variables, the number of sampling points, m, is simply calculated as follows:

$$m = K^n \tag{10}$$

Equations (9) and (10) are substituted into (8). We solve (8) with respect to $r/\Delta d$, and then we can finally obtain the following equation:

$$\frac{r}{\Delta d} = n^{\frac{n-2}{2n}}\left(1 - \frac{1}{K}\right) \tag{11}$$

In (11), $K \to \infty$ is considered. This implies an ideal distribution of the sampling points in the design variable space. Table 1 shows the convergence at $K \to \infty$.

It is clear from Table 1 that the uniform convergence of $r/\Delta d$ can be achieved in the cases of $n = 1$ and $n = 2$. However, $r/\Delta d$ does not converge uniformly in the case of $n \geq 3$. Therefore, $r/\Delta d \to 1$ cannot be achieved at $K \to \infty$. It is assumed that the key factor for a good approximation is the uniform convergence, which is $r/\Delta d \to 1$ at $K \to \infty$. Then, on the basis of (8), some sufficient conditions for the width for a good approximation by the RBF network are summarized as follows:

(W1) It is preferable to consider the number of design variables, n.
(W2) It is also preferable to consider the number of sampling points, m.
(W3) It is preferable to consider the maximum distance among the sampling points, $d_{\max}$.

Table 1 Convergence of $r/\Delta d$ at $K \to \infty$

The number of design n variables	$r/\Delta d$
1	1.000
2	1.000
3	1.201
4	1.414
5	1.621
6	1.817
7	2.003
8	2.181

(W4) It is preferable to consider the uniform convergence of $r/\Delta d$ through the increment of the number of design variables ($r/\Delta d \to 1$ at $K \to \infty$).

In order to satisfy the above sufficient conditions, the following equation for the width may be valid:

$$r_1 = r_2 = \cdots = r_m = \frac{d_{\max}}{\sqrt{n}\sqrt[n]{m}} \tag{12}$$

Since (12) satisfies the above sufficient conditions at $K \to \infty$, a good approximation can be expected. However, (12) does not consider the sparseness and density of the sampling points. In addition, it is clear from (10) that numerous sampling points are required for a good approximation, using (12). Then, the following equation considering the sparseness and density of the sampling points is proposed in this paper.

$$r_i = \frac{d_{i,\max}}{\sqrt{n}\sqrt[n]{m-1}} \quad i = 1, 2, \ldots, m \tag{13}$$

where $d_{i,\max}$ denotes the maximum distance from the ith sampling point. Equation (13) is applied to each basis function individually, unlike (8) and (12).

2.3 Adaptive scaling technique

As already described, all of the design variables should be scaled equally in the development of (13). A simple scaling technique, called the adaptive scaling technique, is introduced in this section. The following equation is used to scale all of the design variables:

$$X_I = \frac{x_I - x_I^L}{x_I^U - x_I^L} \times s \quad I = 1, 2, \ldots, n \tag{14}$$

where x_I is the Ith design variable. x_I^U and x_I^L denote the upper and lower bounds of the Ith design variable, respectively. s (> 0) in (14) denotes the scaling coefficient. Using (14), all of the design variables are scaled between 0 and s. The scaling coefficient s plays an important. If the scaling coefficient s is fixed, (W4) described above may not be satisfied. Thus, scaling coefficient s should be adjusted adaptively. Then, we develop the adaptive scaling technique to satisfy (W4). The algorithm for this technique is summarized as follows:

Step 1 Initial scaling coefficient s (> 0) is set up.
Step 2 All of the design variables are scaled by (14).
Step 3 The width given by (13) is calculated in the scaled space.
Step 4 The minimum width $r_{\min}$ is found.

$$r_{\min} = \min_{1 \leq i \leq m} \{r_i\} \tag{15}$$

Step 5 If $r_{\min} \leq 1$, then scaling coefficient s is updated as follows:

$$s = \alpha \times s \quad (\alpha > 1) \tag{16}$$

Otherwise, the adaptive scaling algorithm will be terminated. On the basis of the author's numerical experiences, $\alpha = 1.2$ is recommended.

The characteristics of this scaling technique are as follows: (1) it can be used to calculate the width in the scaled space and (2) it can be used to examine (W4), which is one of the sufficient conditions. Therefore, the sufficient conditions for a good approximation are always verified.

3 Density function using RBF network

The objective of the density function is to discover a sparse region in the design variable space. It is expected that the addition of new sampling points in the sparse region will lead to the global approximation. An important issue is the construction of a density function using the RBF network. In the SAO, the approximate global minimum can be found through the addition of new sampling points. From the point of view of computer-programming code, it is not preferable to construct numerous subroutines. For simpler computer-programming code, it is preferable to use one or two subroutines multiple times. Thus, the density function using the RBF network is developed.

The basic concept of the density function is very simple. The local maxima are generated at the sampling points. To achieve this objective, every output $\boldsymbol{y}$ of the RBF network is replaced with $+1$. Suppose that the number of sampling points is m, and $D(\boldsymbol{x})$ denotes the density function. Equation (13) with the adaptive scaling technique is also used for the density function. The detailed procedure to construct the density function is summarized as follows:

D-Step 1 The following vector $\boldsymbol{y}^D$ is prepared at the sampling points.

$$\boldsymbol{y}^D = (1, 1, \ldots, 1)^T_{m \times 1} \tag{17}$$

D-Step 2 The weight vector $\boldsymbol{w}^D$ of the density function $D(\boldsymbol{x})$ is calculated as follows:

$$\boldsymbol{w}^D = (\boldsymbol{H}^T \boldsymbol{H} + \boldsymbol{\Lambda}) \boldsymbol{H}^T \boldsymbol{y}^D \tag{18}$$

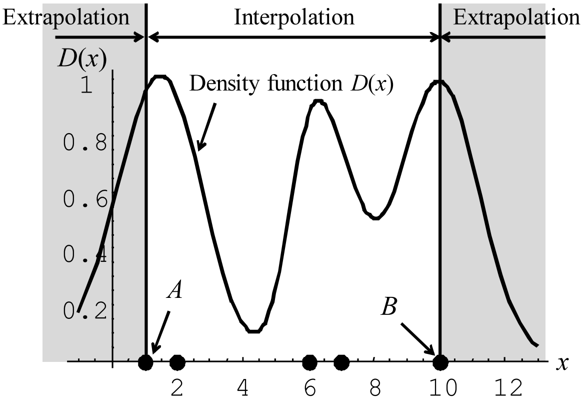


Fig. 4 Illustrative example of density function in one dimension

D-Step 3 The density function $D(\boldsymbol{x})$ is minimized to determine the sparse region in the design variable space.

$$D(\boldsymbol{x}) = \sum_{i=1}^{m} w_i^D h_i(\boldsymbol{x}) \to \min \tag{19}$$

D-Step 4 The point at which the density function $D(\boldsymbol{x})$ is minimized is taken as the new sampling point.

Figure 4 shows an illustrative example in one dimension. The black dots denote the sampling points.

It is clear from Fig. 4 that local minima are generated in the sparse region of the sampling points and that local maxima are also generated at the sampling points. The RBF network is basically the interpolation between sampling points: therefore, points A and B in Fig. 4 are the lower and upper bounds of the density function.

4 Algorithm for SAO using RBF network

Figure 5 shows the detailed algorithm for SAO using the RBF network. In this paper, the terminal criterion of SAO is determined by the maximum number of sampling points, $m_{\max}$.

The proposed SAO algorithm is roughly divided into two phases. The first phase is used to construct the response surface and add the optimum of response surface as a new sampling point. Thus, in the first phase, the number of new sampling points is one that is the optimum of the response surface. The second phase is used to construct the density function and add the optimum of the density function as a new sampling point. It should be noted that the density function is constructed until the terminal criterion, which is described later, is satisfied. As a result, many new sampling points will be added, according to the number of design variables, n.

Let us consider the first phase. First, the initial sampling points are determined by using the orthogonal array, the LHD, and so on. The number of sampling points is m. The initial scaling coefficient is also set up. The objective and constraints are

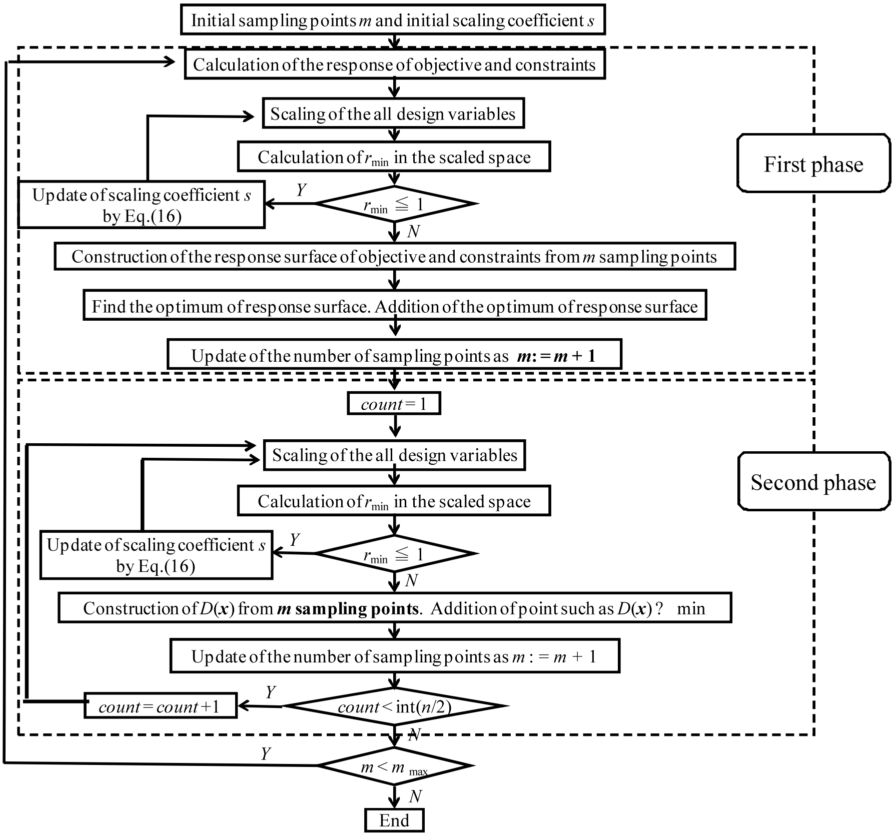

Initial sampling points m and initial scaling coefficient s
Calculation of the response of objective and constraints
Scaling of the all design variables
Calculation of r_min in the scaled space
First phase
Update of scaling coefficient s by Eq.(16)
Y
r_min ≦ 1
N
Construction of the response surface of objective and constraints from m sampling points
Find the optimum of response surface. Addition of the optimum of response surface
Update of the number of sampling points as m: = m + 1
count = 1
Scaling of the all design variables
Second phase
Calculation of r_min in the scaled space
Update of scaling coefficient s by Eq.(16)
Y
r_min ≦ 1
N
Construction of D(x) from m sampling points. Addition of point such as D(x) ? min
Update of the number of sampling points as m : = m + 1
Y
count = count + 1
count < int(n/2)
N
Y
m < m max
N
End

Fig. 5 Proposed SAO algorithm

calculated at the sampling points. Then, the adaptive scaling technique is applied. After the scaling coefficient is determined with the adaptive scaling technique, the response surface of the objective and constraints is constructed from m sampling points, by RBF network. The global optimization technique is applied to the response surface, in order to find the optimum of the response surface. The optimum of the response surface is directly taken as the new sampling point. In this phase, the number of sampling points is updated as $m = m + 1$.

Then, the second phase, in which the density function is constructed, is considered. In order to construct the density function by the RBF network, the adaptive scaling technique is also employed. Thus, the scaling coefficient and width are determined and these values are used to construct the density function. The point at which the density function is minimized is then found. The optimum of the density function is taken as the new sampling point, and the number of sampling points is updated as shown in Fig. 5. In Fig. 5, the parameter *count* is introduced. This parameter controls the number of sampling points that can be obtained by the density function. Thus, in the proposed algorithm, the number of sampling points by the density function varies according to the number of design variables. If the parameter *count* is less than *int*$(n/2)$, this parameter is increased as $count = count + 1$, and the adaptive scaling techniques is also employed as shown in Fig. 5. The terminal criterion in the second phase is given by *int*$(n/2)$, where *int*() represents the rounding-off. If the terminal criterion is satisfied, the number of sampling point m is compared with $m_{\max}$. If the m is less than $m_{\max}$, the objective function and constraints are calculated as shown in Fig. 5. Otherwise, the algorithm is terminated. In the SAO, several optima can be obtained because the response surface is constructed repeatedly through the addition of the new sampling points. In this paper, the optimum of the response surface at $m_{\max}$ is taken as the final optimum.

5 Numerical examples

The validity of the proposed SAO algorithm will now be examined through some typical numerical examples. The objective and constraints are approximated separately by the RBF network. These response surfaces and the density function become a multi-modal function: Therefore, the global optimization technique is required to find the global minimum of the response surface. Then, the Particle Swarm Optimization (PSO) is used as the global optimizer. In all of the numerical examples, the following PSO parameters are used: (1) the number of particles is 30 and (2) the maximum search iteration is set to 500. The initial scaling coefficient is set to 1 in all of the numerical examples.

5.1 Illustrative example

Let us consider the following optimization problem.

$$f(\boldsymbol{x}) = -(x_1 - 1)^2 - (x_2 - 0.5)^2 \rightarrow \min \tag{20}$$

$$g_1(\boldsymbol{x}) = \frac{[(x_1 - 3)^2 + (x_2 + 2)^2]\exp(-x_2^7)}{12} - 1 \leq 0 \tag{21}$$

$$g_2(\boldsymbol{x}) = (10x_1 + x_2)/7 - 1 \leq 0 \tag{22}$$

$$g_3(\boldsymbol{x}) = \frac{(x_1 - 0.5)^2 + (x_2 - 0.5)^2}{0.2} - 1 \leq 0 \tag{23}$$

$$0 \leq \boldsymbol{x} \leq 1 \tag{24}$$

In Fig. 6, the local minimum $\boldsymbol{x}^L$ and global minimum $\boldsymbol{x}^G$ are shown by the squares. The objective functions at $\boldsymbol{x}^L$ and $\boldsymbol{x}^G$ are given as follows:

$$f(\boldsymbol{x}_L) = -0.6867 \quad \text{at } \boldsymbol{x}_L = (0.2623, 0.1223)^T \tag{25}$$

$$f(\boldsymbol{x}_G) = -0.7484 \quad \text{at } \boldsymbol{x}_G = (0.2016, 0.8332)^T \tag{26}$$

Two constraints $g_1(\boldsymbol{x})$ and $g_3(\boldsymbol{x})$ are active at the local and global minima. It is clear from Fig. 6 that there are two separate feasible regions in this problem and that the response surface approach is valid.

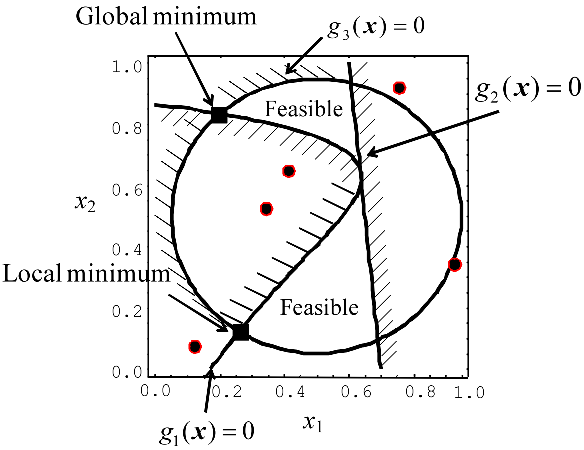


Fig. 6 Feasible region, and local and global minima

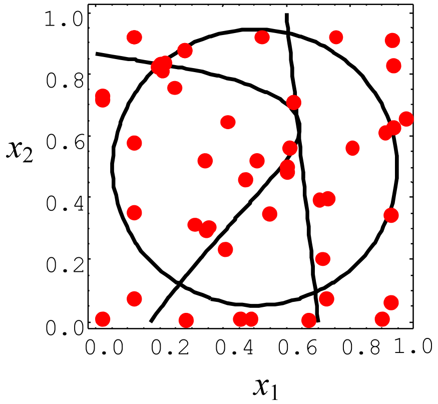


Fig. 7 Distribution of sampling points at $m_{\max}$

The five initial sampling points represented by the dots • in Fig. 6 are determined by the LHD, and the maximum number of sampling points $m_{\max}$ is set to 50. The objective and constraints are approximated separately, and the presented SAO algorithm is applied. The distribution of the sampling points at $m_{\max}$ is shown in Fig. 7, and the objective and constraints at the optimum of the response surface through the successive addition of the sampling points are shown in Table 2.

The sampling points are distributed around the global minimum, and are also distributed in the design variable space. The approximate global minimum is $\tilde{\boldsymbol{x}}_G = (0.2024, 0.8327)^T$, and the objective at this point is $f(\tilde{\boldsymbol{x}}_G) = -0.7468$.

5.2 Application to several benchmark problems

The validity of the proposed SAO algorithm is examined through five typical benchmark problems. These problems are listed in Table 3. The initial sampling points are determined by the LHD. In each of these problems, the initial number of sampling points is set to five. Twenty trials are performed with different random seeds. In the

Table 2 Objective and constraints at the optimum of the response surface

Number of sampling points	$f(\tilde{\boldsymbol{x}}_G)$	$g_1(\tilde{\boldsymbol{x}}_G)$	$g_2(\tilde{\boldsymbol{x}}_G)$	$g_3(\tilde{\boldsymbol{x}}_G)$
5	−0.7818	0.2189	−0.7468	−0.2441
11	−0.6321	0.0966	−0.5380	−0.3510
17	−0.7245	0.0414	−0.5886	−0.0920
23	−0.7257	−0.0159	−0.5686	−0.0309
29	−0.7510	0.0158	−0.6006	−0.0100
35	−0.7468	0.0000	−0.5919	0.0000
41	−0.7468	0.0000	−0.5919	0.0000
47	−0.7468	0.0000	−0.5919	0.0000

Table 3 Benchmark problems considered in this paper

No.	Objective and constraints	$f(\boldsymbol{x}_G)$
1	$f(\boldsymbol{x} = \sum_{i=1}^{5} i\cos[(i+1)x_1 + i] \to \min$ $0 \le x_1 \le 7.5$	$f(\boldsymbol{x}_G) = -12.871$
2	$f(\boldsymbol{x}) = 2 + 0.01(x_2 - x_1^2)^2 + (1 - x_1)^2 + 2(2 - x_2)^2$ $+ 7\sin(0.5x_1)\sin(0.7x_1x_2) \to \min$ $0 \le \boldsymbol{x} \le 5$	$f(\boldsymbol{x}_G) = -1.4565$
3	$f(\boldsymbol{x}) = \sum_{i=1}^{2} \lvert x_i \sin x_i + 0.1x_i \rvert \to \min$ $-10 \le \boldsymbol{x} \le 10$	$f(\boldsymbol{x}_G) = 0$
4	$f(\boldsymbol{x}) = x_1^2 + x_2^2 \to \min$ $g_1(\boldsymbol{x}) = -(x_1 + 4)^2/3 - (x_2 - 0.1)^2 + 20 \le 10$ $-6 \le x_1 \le 4 \quad -4 \le x_2 \le 6$	$f(\boldsymbol{x}_G) = 11.4371$
5	$f(\boldsymbol{x}) = -(x_1 - 1)^2 - (x_2 - 0.5)^2 \to \min$ $g_1(\boldsymbol{x}) = [(x_1 - 3)^2 + (x_2 + 2)^2]\exp(-x_2^7) - 12 \le 0$ $g_2(\boldsymbol{x}) = 10x_1 + x_2 - 7 \le 0$ $g_3(\boldsymbol{x}) = (x_1 - 0.5)^2 + (x_2 - 0.5)^2 - 0.2 \le 0$ $0 \le \boldsymbol{x} \le 1$	$f(\boldsymbol{x}_G) = -0.7483$

constrained problems, the objective and constraints are approximated separately. The results are shown in Table 4.

It is clear from Table 4 that the proposed SAO algorithm is valid for the benchmark problems considered here.

5.3 Comparison with other SAO algorithms

It is difficult to examine and evaluate all of the SAO algorithms because they employ the specific parameters in the algorithm. In addition, various sequential sampling algorithms are developed. One of the important aspects in the SAO is to reduce the function evaluations. Thus, it may be possible to examine the validity of the proposed algorithm from the view point of function evaluations. In this section, the pro-

Table 4 Results of benchmark problems

	Test 1	Test 2	Test 3	Test 4	Tests
$m_{\max}$	15	50	50	50	50
Minimum of objective	−12.8708	−1.4557	4.3601E–04	11.4426	−0.7486
Maximum of objective	−12.3941	−1.3107	9.3849E–03	1 1 .9480	−0.7431
Average of objective	−12.7723	−1.4061	3.5725E–03	11.6164	−0.7467
Standard deviation of objective	1.4444E–01	4.8181E–02	3.1839E–03	1.7970E–01	1.7961E–03

posed SAO algorithm is compared with other SAO algorithms through benchmark problems. The benchmark problems are taken from Hussain et al. (2002) and Wang (2003), in which the function evaluations are clearly described. The proposed SAO algorithm is compared through six benchmark problems. In the proposed algorithm, initial sampling points are determined by the LHD, and initial number of sampling points is set to 10. The comparisons of function evaluations and the objective at the approximate global minimum are listed in Table 5.

5.4 Examination of width

The validity of the width proposed in this paper is examined through the following problem.

$$f(\boldsymbol{x}) = \frac{1}{2}\sum_{i=1}^{n}(x_i^4 - 16x_i^2 + 5x_i) \rightarrow \min \tag{27}$$

$$-5 \leq \boldsymbol{x} \leq 5 \tag{28}$$

The global minimum $\boldsymbol{x}_G$ is $\boldsymbol{x}_G = (-2.9035, -2.9035, \ldots, -2.9035)^T$. The number of design variables n is set to 10. In this case, the objective function at $\boldsymbol{x}_G$ is $f(\boldsymbol{x}_G) = -391.661$. The PSO is applied to this problem directly. The number of particles is set to 20, and the maximum search iteration is set to 500. Therefore, 10000 function evaluations are required to find the global minimum. The PSO results are shown in Table 6 through 10 trials.

To reduce the function evaluations, we try to find the global minimum with 500 function evaluations. Thus, the maximum number of sampling points is set to 500. First, 30 sampling points are distributed at random in the design variable space. Ten trials are performed using (8) and (13). The results are shown in Table 7 and Table 8.

By comparing Table 7 with Table 8, it is clear that better results can be obtained, by using (13). Through five trials (Trial No. 1–Trial No. 5), the approximate global minimum could be obtained by (13). In addition, the other trials (Trial No. 6—Trial No. 10) yield quasi-optimums. However, an approximate global minimum cannot be obtained with (8). These results imply that it is preferable to apply a different width to each basis function.

Table 5 Comparison of other method based on Hussain et al. (2002) and Wang (2003)

Reference	Objective and constraints	$f(\boldsymbol{x}_G)$	Objective at approximate global minimum	Function evaluations	Proposed algorithm	Function evaluations by the proposed algorithm
Wang (2003)	$f(\boldsymbol{x}) = 4x_1^2 - 2.1x_1^4 + \frac{x_1^6}{3} + x_1x_2 - 4x_2^2 + 4x_2^4 \to \min$ $2 \le \boldsymbol{x} \le 2$	−1.032	−1.029	44	−1.016	28
Wang (2003)	$f(\boldsymbol{x}) = x_1^2 + x_2^2 - \cos(18x_1) - \cos(18x_2) \to \min$ $1 \le \boldsymbol{x} \le 1$	−2.0	−1.854	60	−1.990	28
Wang (2003)	$f(\boldsymbol{x}) = (x_2 - \frac{5.1}{4\pi^2}x_1^2 + \frac{5}{\pi}x_1 - 6)^2$ $+ 10(1 - \frac{1}{8\pi})\cos x_1 + 10 \to \min$ $-5 \le x_1 \le 10 \quad 0 \le x_2 \le 15$	0.398	0.398	36	0.398	34
Wang (2003)	$f(\boldsymbol{x}) = [1 + (x_1 + x_2 + 1)^2 \times (19 - 14x_1 + 3x_1^2$ $- 14x_2 + 6x_1x_2 + 3x_2^2)][30 + (2x_1 - 3x_2)^2$ $\times (18 - 32x_1 + 12x_1^2 + 48x_2 - 36x_1x_2 + 27x_2^2]$ $\to \min$ $-2 \le \boldsymbol{x} \le 2$	3.000	3.000	77	3.050	60
Hussain et al. (2002)	$f(\boldsymbol{x}) = x_1 \sin(x_2) + x_2 \sin(x_1) \to \min$ $-2\pi \le \boldsymbol{x} \le 2\pi$	−9.629	N/A	80	−9.628	40
Hussain et al. (2002)	$f(\boldsymbol{x}) = x_1 \sin(x_1) + x_2 \sin(x_2) \to \min$ $-2\pi \le \boldsymbol{x} \le 2\pi$	−9.629	N/A	60	−9.623	30

Table 6 Results of direct search by the PSO

Trial	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	Obj.
1	−2.9035	−2.9035	−2.9035	−2.9035	−2.9035	−2.9035	−2.9035	−2.9035	−2.9035	−2.9035	−391.6600
2	−2.9036	−2.9036	−2.9036	−2.9035	−2.9035	−2.9035	−2.9035	2.7468	−2.9035	−2.9035	−377.5200
3	−2.9046	−2.9027	−2.9021	−2.9038	−2.9032	−2.9047	−2.9015	−2.9043	−2.9029	−2.9049	−391.6600
4	−2.9036	−2.9035	−2.9036	−2.9036	−2.9035	−2.9035	−2.9036	−2.9035	−2.9036	−2.9036	−391.6600
5	−2.9035	−2.9035	−2.9035	−2.9035	−2.9035	−2.9035	−2.9035	−2.9035	−2.9035	−2.9035	−391.6600
6	−2.9035	−2.9035	−2.9035	−2.9035	−2.9036	−2.9035	2.7468	−2.9036	−2.9035	−2.9035	−377.5200
7	−2.9035	−2.9035	−2.9035	−2.9035	−2.9035	−2.9035	−2.9035	−2.9035	−2.9035	−2.9035	−391.6600
8	−2.9035	−2.9035	−2.9035	−2.9035	−2.9035	−2.9035	−2.9035	−2.9035	−2.9035	−2.9035	−391.6600
9	−2.9036	−2.9034	−2.9036	−2.9036	−2.9034	−2.9036	−2.9035	−2.9035	−2.9036	−2.9036	−391.6600
10	−2.9035	−2.9035	−2.9035	−2.9035	−2.9035	−2.9035	−2.9035	−2.9035	−2.9035	−2.9035	−391.6600

Table 7 The result by using (8)

Trial	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	Obj.
1	−2.6590	−2.7695	−2.1054	3.0079	−3.2740	−2.7108	−2.6121	−2.3852	2.7003	−0.9410	−314.1416
2	−2.0759	−2.7692	2.2905	−1.9245	2.7989	−2.6619	1.9850	−2.0531	2.4101	−1.4912	−273.9235
3	−2.9410	−2.8063	−3.0479	−2.6390	−2.9142	−2.8250	−2.8031	0.9497	−2.7844	0.7700	−318.2305
4	−2.8926	−2.9712	−2.0732	−3.2507	−2.8878	−2.7514	−3.3558	−1.2210	−2.2069	−1.5235	−324.6039
5	−2.9462	2.5448	−2.9124	−2.7911	−3.0628	2.3251	0.6797	−2.9464	−2.9779	2.7018	−308.3509
6	2.4241	−0.3086	3.1126	−3.6446	−1.6238	−2.6086	−3.4843	1.1949	−0.0281	1.9153	−192.1104
7	−2.8263	−2.9896	2.7270	−3.1419	−2.6376	−2.6945	−2.4389	−3.2174	−1.6055	−3.3261	−347.9492
8	−2.9634	−2.7063	2.5490	−2.6656	−2.5071	−2.9957	−2.9850	2.8242	−3.0073	2.2608	−341.3516
9	−2.6664	2.4891	−2.7121	−2.2098	−2.7480	−0.1432	1.5402	−2.7654	−3.2011	−3.0773	−300.2067
10	−2.3563	−2.7148	−2.7854	−2.6837	2.3436	−1.2683	2.4122	−2.6995	−2.7142	−2.1761	−321.3327

Table 8 The result by using (13)

Trial	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	Obj.
1	−2.8096	−2.8428	−2.8138	−2.8923	−2.9249	−2.8322	−2.7792	−2.9189	−2.9251	−2.8174	−390.8271
2	−2.8796	−2.7992	−2.9779	−2.9068	−2.8315	−2.9377	−2.9344	−2.9574	−2.9395	−2.8551	−391.1336
3	−2.9828	−2.8559	−2.8442	−2.7923	−2.8739	−2.9228	−2.7560	−2.9362	−2.9070	−2.8233	−390.7393
4	−2.8815	−2.8551	−2.9466	−2.8876	−2.8993	−2.8246	−2.8708	−2.8602	−2.8473	−2.8967	−391.3663
5	−2.9453	−2.9166	−2.7878	−2.8491	−2.8713	−2.7786	−2.9730	−2.7636	−2.9755	−2.9006	−390.5783
6	−2.9192	−2.8345	−2.9351	−2.8462	−2.9019	−2.8998	−2.8975	−2.9197	2.6884	−2.8685	−377.2915
7	−2.9376	2.6801	−2.8858	−2.8898	−2.8615	−2.8626	−2.9049	−2.8773	−2.8294	−2.8921	−377.2669
8	2.6569	−2.9470	−2.9554	−2.8557	−2.8986	−2.9416	−2.8419	−2.8627	−2.9532	−2.8738	−377.1143
9	−2.9211	−2.8614	2.7425	−2.9349	−2.8388	−2.8856	−2.8809	−2.8981	−2.9182	−2.8922	−377.3801
10	−2.8901	−2.8468	−2.8828	−2.9049	−2.9252	2.6898	−2.8235	−2.9735	−2.8813	−2.8938	−377.2004

Table 9 Orthogonal array $L9$

	x_1	x_2	x_3
No. 1	0.05	0.25	2
No. 2	0.05	0.775	8.5
No. 3	0.05	1.3	15
No. 4	1.025	0.25	8.5
No. 5	1.025	0.075	15
No. 6	1.025	1.3	2
No. 7	2	0.25	15
No. 8	2	0.075	2
No. 9	2	1.3	8.5

5.5 Optimum design of tension/compression spring

One of the most popular test problems proposed by Arora can be considered (Arora 1989). Many researchers have used this as benchmark problem in the structural optimization (Coello Coello 2000; Ray and Saini 2001; Hu et al. 2003). The design variables are (1) the diameter d $(= x_1)$, (2) mean coil diameter D $(= x_2)$, and (3) number of active coils N $(= x_3)$. The problem can be formulated as follows:

$$f(\boldsymbol{x}) = (2 + x_3)x_1^2 x_2 \rightarrow \min \tag{29}$$

$$g_1(\boldsymbol{x}) = 1 - \frac{x_2^3 x_3}{71785 x_1^4} \leq 0 \tag{30}$$

$$g_2(\boldsymbol{x}) = \frac{4x_2^2 - x_1 x_2}{12566(x_2 x_1^3 - x_1^4)} + \frac{1}{5108 x_1^2} - 1 \leq 0 \tag{31}$$

$$g_3(\boldsymbol{x}) = 1 - \frac{140.45 x_1}{x_2^2 x_3} \leq 0 \tag{32}$$

$$g_4(\boldsymbol{x}) = \frac{x_1 + x_2}{1.5} - 1 \leq 0 \tag{33}$$

$$0.05 \leq x_1 \leq 2.00 \tag{34}$$

$$0.25 \leq x_2 \leq 1.30 \tag{35}$$

$$2.00 \leq x_3 \leq 15.0 \tag{36}$$

The orthogonal array $L9$, which is shown in Table 9, is used to determine the initial sampling points.

$m_{\max}$ is set to 150, and 11 trials are performed to compare the past researches. The results obtained by applying the proposed SAO algorithm are shown in Table 10. It is clear from Table 10 that the function evaluations are drastically reduced in comparison with those in the past researches.

Table 10 Comparison of results on the optimum design of tension/compression spring

Design variables	Best solutions found				
	Arora (1989)	Coello Coello (2000)	Ray and Saini (2001)	Hu et al. (2003)	This research
x_1	0.053396	0.051480	0.050417	0.051466	0.050000
x_2	0.399180	0.351661	0.321532	0.351384	0.314777
x_3	9.185400	11.632201	13.979915	11.608659	14.650042
$g_1(\boldsymbol{x})$	0.000019	-0.002080	-0.001926	-0.003336	-0.018820
$g_2(\boldsymbol{x})$	-0.000018	-0.000110	-0.012944	-0.000110	-0.006566
$g_3(\boldsymbol{x})$	-4.123832	-4.026318	-3.899430	-4.026318	-3.837790
$g_4(\boldsymbol{x})$	-0.698283	-0.731239	-0.752034	-0.731324	-0.756815
$f(\boldsymbol{x})$	0.012730	0.012705	0.013060	0.012667	0.013103
Function call	N/A	900000	1291	N/A	66
Average of $f(\boldsymbol{x})$	N/A	0.012769	0.013436	0.012719	0.013273
Worst of $f(\boldsymbol{x})$	N/A	0.012822	0.013580	N/A	0.013643

6 Conclusions

In this paper, the Sequential Approximate Optimization (SAO) algorithm using the RBF network has been proposed. The Gaussian function is employed as the basis function. We have examined the width of the Gaussian function, which affects the accuracy of the response surface. By examining the width equation proposed by Nakayama, some sufficient conditions for a good approximation are introduced. Then, a new equation to determine the width has been proposed. In addition, a simple scaling technique called the adaptive scaling technique has also been proposed. In this technique, the sufficient conditions for a good approximation are always verified. Clearly, it is better to optimize the width in these kinds of meta-modeling techniques. However, optimization of the width is very is time-consuming task. Therefore, it is preferable to determine the width with a simple manner. Many meta-modeling techniques, such as Kriging, RBF network, and Support Vector Regression (SVR), have been proposed. The Gaussian function, which is sometimes called the Gaussian kernel, is commonly employed in all these methods. The equivalence between ordinary Kriging and SVR has been reported under the assumption that the covariance function is used as the kernel function (An and Sun 2006). The equivalence between SVM and the regularization neural network has been also reported (Andras 2002). This equivalence can be extended to RBF network, considering the suggestions of Poggio and Girosi (1990). Thus, it is considered that the equivalence between SVM and RBF network can be established. In the Gaussian kernel, the width plays an important role. Therefore, one of the important issues is the determination of the width with a simple manner. It is expected that the proposed width with the adaptive scaling technique is applicable to Kriging, SVR, and so on, in which the Gaussian function is employed.

Second, the sampling strategy has been examined. In the SAO, the optimum of the response surface is taken as the new sampling point in order to improve the local accuracy. In addition, new sampling points in the sparse region are required for a global approximation. To determine the sparse region, the density function constructed by the RBF network has been developed. This density function generates local minima in the sparse region, so that the minimum of this function can be taken as a new sampling point. In the proposed SAO algorithm, the density function is constructed repeatedly until the terminal criterion is satisfied. As the result, many new sampling points can be obtained. Through typical mathematical and engineering optimization problems, the validity of the proposed SAO algorithm has been examined.

Acknowledgements The authors have developed (13), through in-depth discussions with Prof. Nakayama, H. (Konan University). The reviewers also gave us the useful and constructive suggestions to improve the quality of this paper. We would like to thank to them for their useful and constructive comments.

References

An W, Sun Y (2006) An equivalence between SILF-SVR and ordinary Kriging. Neural Process Lett 23:133–141

Andras P (2002) The equivalence of support vector machine and regularization neural networks. Neural Process Lett 65:97–104

Arora JS (1989) Introduction to optimum design. McGraw-Hill, New York

Coello Coello CA (2000) Use of a self-adaptive penalty approach for engineering optimization problems. Comput Ind 41:113–127

Donald J, Schonlau M, Welch WJ (1998) Efficient global optimization of expensive black-box functions. J Glob Optim 13:455–492

Fang H, Horstemeyer MF (2006) Global response approximation with radial basis function. Eng Optim 38(4):407–424

Haykin S (1994) Neural networks: a comprehensive foundation. Macmillan Co, New York, pp 236–284

Hu XH, Eberhart RC, Shi YH (2003) Engineering optimization with particle swarm. In: IEEE swarm intelligence symposium, pp 53–57

Huang D, Allen TT, Notz WI, Zeng N (2006) Global optimization of stochastic black-box systems via sequential Kriging meta-models. J Glob Optim 34:441–466

Hussain MF, Barton RR, Joshi SB (2002) Metamodeling: radial basis functions, versus polynomials. Eur J Oper Res 138:142–154

Jin R, Chen W, Simpson TW (2001) Comparative studies of metamodelling techniques under multiple modelling criteria. Struct Multidiscip Optim 23:1–13

Kurtaran H, Eskandarian A, Marzougui D, Bedewi NE (2002) Crashworthiness design optimization using successive response surface approximations. Comput Mech 29:409–421

Martin JD, Simpson TW (2005) Use of Kriging models to approximate deterministic computer models. AIAA J 43(4):853–863

McDonald DB, Grantham WJ, Tabor WL, Murphy MJ (2007) Global and local optimization using radial basis function response surface models. Appl Math Model 31:2095–2110

Muller AA, Messac A (2005) Extended radial basis functions: more flexible and effective metamodeling. AIAA J 43(6):1306–1315

Myers RH, Montgomery DH (1995) Response surface methodology. Wiley, New York

Nakayama H, Arakawa M, Sasaki R (2002) Simulation-based optimization using computational intelligence. Optim Eng 3:201–214

Orr MJL (1996) Introduction to radial basis function networks. http://www.anc.ed.ac.uk/rbf/rbf.html (Online available)

Perez VM, Renaud JE, Watson LT (2002) Adaptive experimental design for construction of response surface approximations. AIAA J 40(12):2495–2503

Poggio T, Girosi F (1990) Networks for approximation and learning. Proc IEEE 78(9):1481–1497

Ray T, Saini P (2001) Engineering design optimization using swarm with an intelligent information sharing among individuals. Eng Optim 33:735–748

Rodriguez JF, Perez VM, Padmanabhan D, Renaud JE (2001) Sequential approximate optimization using variable fidelity response surface approximations. Struct Multidiscip Optim 22:24–34

Sasena MJ, Papalambros PY, Goovaerts P (2002a) Exploration of metamodeling sampling criteria for constrained global optimization. Eng Optim 34(3):263–278

Sasena MJ, Papalambros PY, Goovaerts P (2002b) Global optimization of problems with disconnected feasible regions via surrogate modeling. In: 9th/ISSMO symposium on multidisciplinary analysis and optimization, AIAA2002-5573

Sharif B, Wang GG, EiMekkawy TY (2008) Mode pursuing sampling method for discrete variable optimization on expensive black-box functions. J Mech Des 130:021402

Simpson TW, Mistree F (2001) Kriging models for global approximation in simulation-based multidisciplinary design optimization. AIAA J 30(12):2233–2241

Simpson TW, Mauery TM, Korte JJ, Mistree F (2001a) Kriging models for global approximation in simulation-based multidisciplinary design optimization. AIAA J 39(12):2233–2241

Simpson TW, Peplinski JD, Kock PN, Allen JK (2001b) Metamodels for computer-based engineering design: survey and recommendations. Eng Comput 17:129–150

Sobester A, Leary SJ, Keane AJ (2005) On the design of optimization strategies based on global response surface approximation models. J Glob Optim 33:31–59

Wang GG (2003) Adaptive response surface method using inherited Latin hypercube design points. J Mech Des 125:210–220

Wang GG, Shan S (2007) Review of metamodeling techniques in support of engineering design optimization. J Mech Des 129:370–380