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RESEARCH

Communication in multiplex transportation networks

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Abstract
Complex networks are made up of vertices and edges. The edges, which may be directed or undirected, are equipped with positive weights. Modeling complex systems that consist of different types of objects leads to multilayer networks, in which vertices in distinct layers represent different kinds of objects. Multiplex networks are special vertex-aligned multilayer networks, in which vertices in distinct layers are identified with each other and inter-layer edges connect each vertex with its copy in other layers and have a fixed weight $\gamma > 0$ associated with the ease of communication between layers. This paper discusses two different approaches to analyze communication in a multiplex. One approach focuses on the multiplex global efficiency by using the multiplex path length matrix, the other approach considers the multiplex total communicability. The sensitivity of both the multiplex global efficiency and the multiplex total communicability to structural perturbations in the network is investigated to help to identify intra-layer edges that should be strengthened to enhance communicability.

Keywords Multiplex network · Network analysis · Total communicability · Global efficiency · Sensitivity analysis · Multiplex path length matrix

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1 Introduction

Multilayer networks arise when one seeks to model a complex system that contains connections and objects with distinct properties; see, e.g., [17, 24]. Multiplex networks,

Dedicated to our friend Michela Redivo-Zaglia on the occasion of her retirement.

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or briefly *multiplexes*, are special multilayer networks in which vertices in distinct layers are identified with each other, i.e., every vertex in some layer has a copy in all other layers and is connected to them. Connections between vertices in distinct layers are furnished by *inter-layer edges* that connect instances of the same vertex in different layers; connections between vertices in the same layer are represented by *intra-layer* edges.

Let the multiplex have L layers and let the graph for layer ℓ have N vertices. This graph is represented by an adjacency matrix $A^{(\ell)} = [a_{ij}^{(\ell)}]_{i,j=1}^{N}$, whose entry $a_{ij}^{(\ell)}$ is positive if there is an edge from vertex v_i to vertex v_j in layer ℓ; if there is no such edge, then $a_{ij}^{(\ell)} = 0$. The graph is said to be *undirected* if $a_{ij}^{(\ell)} = a_{ji}^{(\ell)}$ for all $1 \leq i, j \leq N$; otherwise the graph is *directed*. When $a_{ij}^{(\ell)} > 0$, this quantity is the *weight* of the edge from vertex v_i to vertex v_j in layer ℓ; we denote this intra-layer edge by $e(v_i^\ell \rightarrow v_j^\ell)$. A graph is said to be *unweighted* if all nonvanishing edge-weights equal one; otherwise the the graph is weighted. All matrices $A^{(\ell)}$, $1 \leq \ell \leq L$, that make up a multiplex are of the same size and all inter-layer edges are undirected and have the same weight $\gamma > 0$.

Applications of multiplexes include modeling transportation networks that are made up of train and bus routes, where the train routes and bus routes define intra-layer edges in different layers, and the train stations and bus stops define vertices with diverse properties. The weight of an intra-layer edge may account for the time needed to travel along the road or rail represented by the edge, while the weight γ of the inter-layer edges may model the average time spent transferring between a train station and an adjacent bus stop; see, e.g., [5].

A multiplex with N vertices $\{v_1, v_2, \ldots, v_N\}$ and L layers may be represented by a third-order adjacency tensor $\mathcal{A} \in \mathbb{R}^{N \times N \times L}$ and a parameter γ. The horizontal slices of the tensor are the adjacency matrices $A^{(\ell)}$, i.e., $\mathcal{A} = [a_{ij}^{(\ell)}]_{i,j=1,2,\ldots,N,\, \ell=1,2,\ldots,L}$. This multiplex also may be represented by a supra-adjacency matrix $B \in \mathbb{R}^{NL \times NL}$ with $N \times N$ blocks, where the adjacency matrix $A^{(\ell)}$ is the ℓth diagonal block, for $\ell = 1, 2, \ldots, L$, and every off-diagonal block, in position (ℓ_1, ℓ_2) for $1 \leq \ell_1, \ell_2 \leq L$ and $\ell_1 \neq \ell_2$, equals γI_N with the same $\gamma > 0$. The off-diagonal block in position (ℓ_1, ℓ_2) represents the inter-layer connection between the layers ℓ_1 and ℓ_2. Thus,

$$B = B(\gamma) = \text{blkdiag}[A^{(1)}, A^{(2)}, \ldots, A^{(L)}] + \gamma(\mathbf{1}_L \mathbf{1}_L^T \otimes I_N - I_{NL}), \qquad (1)$$

where $\otimes$ denotes the Kronecker product. Here $I_N \in \mathbb{R}^{N \times N}$ is the identity matrix, and $\mathbf{1}_L = [1, 1, \ldots, 1]^T \in \mathbb{R}^L$; see [17].

In a weighted multiplex, the edge-weights depend on the network model. In this paper, the weight associated with each intra-layer edge accounts for its “importance”, e.g., the number of flights along the route modeled by the edge, or the width of a highway segment modeled by the edge. Then the larger the edge-weight, the easier is the communication between the vertices that are connected by the edge. As an example, let there be an edge $e(v_i^\ell \rightarrow v_j^\ell)$ with weight $a_{ij}^{(\ell)}$. If this weight equals the number of flights from vertex v_i to vertex v_j offered by airline ℓ, then the reciprocal weight, $1/a_{ij}^{(\ell)}$, may be considered the average wait time between flights with this airline along this route. Then doubling the number of flights along a route corresponds to halving the wait time between flights along the edge.

We will need the notions of *path* and *walk* in a multiplex. A walk with $k+1$ vertices is a sequence of vertices $v_{i_1}, v_{i_2}, \ldots, v_{i_{k+1}}$ and an associated sequence of k intra-layer edges $e(v_{i_1}^{\ell_1} \to v_{i_2}^{\ell_1}), \ldots, e(v_{i_k}^{\ell_h} \to v_{i_{k+1}}^{\ell_h})$ connected by h inter-layer edges, with $1 \leq h \leq k$. The length of the walk defined by these vertices and edges is the sum of the *reciprocal weights* of the edges that make up the walk. Vertices and edges of a walk may be repeated. A path is a walk in which no vertex is repeated. Let there be a path from vertex v_i to vertex v_j. Then the distance $d(v_i, v_j)$ from vertex v_i to vertex v_j is the length of the shortest path from vertex v_i to vertex v_j measured by the sum of the reciprocal weights of the edges of the path. If the multiplex is unweighted and $\gamma = 1$, then $d(v_i, v_j)$ is the number of edges in a shortest path from v_i to v_j. Note that $d(v_i, v_j)$ may differ from $d(v_j, v_i)$; in fact, some distances might not be defined.

It is of interest to determine the ease of communication between vertices in a network in a well-defined sense. We consider two approaches in our analysis of the communication in a multiplex:

1. Construct the multiplex path length matrix $P = P(\gamma) = [p_{ij}]_{i,j=1}^N$ (to be defined in Section 3) and consider the multiplex average inverse geodesic length,

$$e_{\mathcal{A}}(\gamma) = \frac{1}{N(N-1)} \sum_{i, j \neq i} \frac{1}{p_{ij}},$$

   which we will refer to as the *multiplex global efficiency*; see [1, 29]. In this approach an edge $e(v_i^\ell \to v_j^\ell)$ is considered important when it is *efficient* to transmit information along the edge, e.g., if several paths of short length end at vertex v_i and/or several paths of short length start at vertex v_j. We will refer to this technique as the *efficiency approach*.
2. Consider the *multiplex total communicability*, which is defined by

$$tc_B(\gamma) = \mathbf{1}_{NL}^T \exp_0(B) \mathbf{1}_{NL},$$

   where $\exp_0(B)$ is the modified exponential matrix, with $\exp_0(t) = \exp(t) - 1$; see [2, 4, 20]. In this approach an edge $e(v_i^\ell \to v_j^\ell)$ is considered important when it is *popular* in transmitting information, i.e., when vertex v_i has several in-edges with large weight in layer ℓ and/or vertex v_j has several out-edges with large weight in layer ℓ. This technique will be referred to as the *popularity approach*.

To assess the sensitivity of a measure of communication between the vertices to perturbations in intra-layer edge-weights, we analyze the "structured" sensitivity to changes of the positive entries of the tensor $\mathcal{A}$ to determine which intra-layer edges should be strengthened to enhance the global efficiency or the total communicability the most.

It is the purpose of the present paper to discuss and compare two approaches to analyze communication in a multiplex. The *efficiency approach* focuses on the multiplex global efficiency by using the multiplex path length matrix. This way to analyze multiplex networks was introduced in Noschese and Reichel [29]. We remark that in Noschese and Reichel [29] the weight associated with each intra-layer edge accounts for some kind of “distance”, e.g., the time required to travel from one location to another, the geographic distance between the locations associated with the vertices that are connected by the edge, or the cost of traversing along the edge. Consequently, in Noschese and Reichel [29] the length of a walk is defined as the sum of the weights of the edges that make up the walk. In the present paper, the length of a walk instead is defined as the sum of the reciprocal weights associated with the edges of the walk. This results in a novel derivation of the multiplex path length matrix, which is used in the computation of the multiplex global efficiency. New bounds for the latter are derived.

We also consider the multiplex total communicability and approximate this quantity by the multiplex Perron communicability, which was defined in El-Halouy et al. [20]. The application of this measure is referred to as the *popularity approach.* The multiplex global efficiency and the Perron communicability help us identify intra-layer edges that should be strengthened to enhance communicability in the network.

We are interested in comparing the efficiency and popularity approaches. In particular, we would like to study how sensitive the multiplex global efficiency and the Perron communicability are to perturbations of the multiplex network. Related investigations of single-layer networks are presented in [18, 30].

This paper is organized as follows. In Section 2 we discuss how the sparsity structure of the multiplex network can be exploited for sensitivity analysis. Sections 3 and 4 focus on the efficiency and popularity approaches, respectively. Numerical examples that compare the efficiency and popularity approaches are reported in Section 5. Section 6 contains concluding remarks.

It is a pleasure to dedicate this paper to Michela Redivo-Zaglia. She has made important contributions in many areas of computational mathematics including the solution of linear discrete ill-posed problems, handling breakdown in the Lanczos method, tensor and network computations, extrapolation, sequence transformation, and also written papers and books on the history of mathematics; see, e.g., [7–14].

2 Structured multiplex Perron sensitivity analysis

The following notions form the basis for our sensitivity analysis of multiplex networks. Let the matrix $A \in \mathbb{R}^{N\times n}$ be nonnegative and irreducible. Then it follows from the Perron-Frobenius theory that A has a unique eigenvalue $\rho > 0$ of largest magnitude (the Perron root) and that the associated right and left eigenvectors, $\mathbf{x}$ and $\mathbf{y}$, respectively,

$$A\mathbf{x} = \rho\mathbf{x}, \qquad \mathbf{y}^T A = \rho\mathbf{y}^T,$$

can be normalized to be of unit Euclidean norm with all components positive. They are referred to as Perron vectors. Let $E \in \mathbb{R}^{N\times N}$ be a nonnegative matrix of unit spectral norm, $\|E\|_2 = 1$. Introduce a small positive parameter ε and denote the Perron root of $A + \varepsilon E$ by $\rho + \delta\rho$. Then

$$\delta\rho = \varepsilon \frac{\mathbf{y}^T E\mathbf{x}}{\mathbf{y}^T\mathbf{x}} + \mathcal{O}(\varepsilon^2) \quad \text{as} \quad \varepsilon \searrow 0$$

and

$$\frac{\mathbf{y}^T E\mathbf{x}}{\mathbf{y}^T\mathbf{x}} = \frac{|\mathbf{y}^T E\mathbf{x}|}{\mathbf{y}^T\mathbf{x}} \leq \frac{\|\mathbf{y}\|_2\|E\|_2\|\mathbf{x}\|_2}{\mathbf{y}^T\mathbf{x}} = \frac{1}{\mathbf{y}^T\mathbf{x}},$$

with equality attained when E is the *Wilkinson perturbation* $W_N = \mathbf{y}\mathbf{x}^T$ associated with ρ; see [32]. The quantity $\frac{1}{\mathbf{y}^T\mathbf{x}}$ is referred to as the *condition number* of ρ and denoted by $\kappa(\rho)$. We note that the spectral norm may be replaced by the Frobenius norm.

Consider the cone $\mathcal{S}$ of all nonnegative matrices in $\mathbb{R}^{N\times N}$ with a given sparsity structure and let $M|_{\mathcal{S}}$ denote a matrix in $\mathcal{S}$ that is closest to a given nonnegative matrix M with respect to the Frobenius norm, i.e., $M|_{\mathcal{S}}$ is the projection of M onto the cone $\mathcal{S}$. Then $M|_{\mathcal{S}}$ is obtained by setting all the positive entries of M outside the sparsity structure $\mathcal{S}$ to zero.

Let $E \in \mathcal{S}$ be a nonnegative matrix of unit Frobenius norm, $\|E\|_F = 1$. Then

$$\frac{\mathbf{y}^T E\mathbf{x}}{\mathbf{y}^T\mathbf{x}} = \frac{|\mathbf{y}^T E\mathbf{x}|}{\mathbf{y}^T\mathbf{x}} \leq \frac{\|\mathbf{y}\|_2\|\,\|W_N|_{\mathcal{S}}\|_F\|\mathbf{x}\|_2}{\mathbf{y}^T\mathbf{x}} = \frac{\|W_N|_{\mathcal{S}}\|_F}{\mathbf{y}^T\mathbf{x}},$$

with equality for the *structured analogue of the Wilkinson perturbation* W_N,

$$E = \frac{W_N|_{\mathcal{S}}}{\|W_N|_{\mathcal{S}}\|_F}.$$

This is the worst-case perturbation for the Perron root ρ induced by a unit norm matrix $E \in \mathcal{S}$; see [27]. The quantity

$$\kappa^{\text{struct}}(\rho) = \frac{\|W_N|_{\mathcal{S}}\|_F}{\mathbf{y}^T\mathbf{x}} = \kappa(\rho)\|W_N|_{\mathcal{S}}\|_F \tag{2}$$

is referred to as the *structured condition number* of ρ. It satisfies $\kappa^{\text{struct}}(\rho) \leq \kappa(\rho)$.

Assume that the matrix $A^+ := \sum_{\ell=1}^{L} A^{(\ell)}$ is irreducible. Then also the supra-adjacency matrix B is irreducible. To determine which edge-weight(s) should be increased to enhance communicability the most, we apply the Perron-Frobenius theory.

1. As for the efficiency approach, we analyze the Wilkinson perturbation W_N associated with the Perron root of an $N \times N$ efficiency matrix (defined in Section 3.3), projected onto the cone $\mathcal{S}_{A^+} \subseteq \mathbb{R}^{N\times N}$ of all nonnegative matrices with the same sparsity structure as A^+.

2. As for the popularity approach, we analyze the Wilkinson perturbation W_{NL} associated with the Perron root of the supra-adjacency matrix, projected onto the cone $\mathcal{S}_{B_d} \subseteq \mathbb{R}^{NL\times NL}$ of all nonnegative matrices with the same sparsity structure as $B_d := \mathrm{blkdiag}[A^{(1)}, \ldots, A^{(L)}]$.

3 The efficiency approach

This section introduces the path length matrix for multiplexes and describes how it can be applied to determine the importance of an edge. The sensitivity of the edge importance to perturbations of the weights is investigated.

3.1 The multiplex 1-path length matrix

To construct the multiplex path length matrix $P = P(\gamma)$ associated with the given multiplex network, we first introduce the third-order tensor $\mathcal{P} = [p_{ij}^{(\ell)}]_{i,j=1,2,\ldots,N,\ \ell=1,2,\ldots,L} \in \mathbb{R}^{N\times N\times L}$ with entries

$$p_{ij}^{(\ell)} = \begin{cases} 0, & \text{if } i = j, \\ 1/a_{ij}^{(\ell)}, & \text{if } a_{ij}^{(\ell)} > 0, \\ \infty, & \text{otherwise.} \end{cases}$$

Then define the multiplex 1-path length matrix

$$P^1 = [p_{ij}^1]_{i,j=1}^N, \quad \text{with } p_{ij}^1 = \min_{\ell=1,2,\ldots,L} p_{ij}^{(\ell)}.$$

The entry p_{ij}^1 with $i \neq j$ represents the length of the shortest path from vertex v_i to vertex v_j made up of a single intra-layer edge, or equals infinity if there is no edge in any layer from vertex v_i to vertex v_j. For example, let $a_{ij}^{(\ell)}$ be the number of direct flights from vertex v_i to vertex v_j offered by airline ℓ. Its reciprocal $1/a_{ij}^{(\ell)}$ can be interpreted as the average wait time between these flights. The extra-diagonal entry p_{ij}^1 then either represents the average wait time for any direct flight or equals infinity if no airline offers a direct flight.

3.2 Constructing the multiplex *K*-path length matrix

We discuss how to construct the multiplex K-path length matrix $P^K = P^K(\gamma) = [p_{ij}^K]_{i,j=1}^N$, whose entry p_{ij}^K with $i \neq j$ is the length of the shortest path from vertex v_i to vertex v_j made up of at most K intra-layer edges. An analogous path length matrix has previously been introduced in Noschese and Reichel [28] to investigate single-layer networks. The diagonal entries of P^K are zero by definition. We note that the multiplex path length matrix $P = [p_{ij}]_{i,j=1}^N$ satisfies $P \equiv P^{N-1}(\gamma)$, because in a multiplex path, the number of intra-layer edges is at most $N-1$.

The construction of the multiplex K-path length matrix uses *min-plus matrix multiplication*, i.e., we carry out matrix multiplication in the tropical algebra; see [25]:

$$C = A \star B: \qquad c_{ij} = \min_{h=1,2,\ldots,N} \{a_{ih} + b_{hj}\}, \qquad 1 \leq i, j \leq N,$$

where $A = [a_{ij}]_{i,j=1}^{N}$, $B = [b_{ij}]_{i,j=1}^{N}$, and $C = [c_{ij}]_{i,j=1}^{N}$ are real $N \times N$ matrices.

The multiplex K-path length matrix $P^K = [p_{ij}^K]_{i,j=1}^{N}$ can be constructed by means of min-plus powers of P^1. For single-layer networks, i.e., when $L = 1$, the matrix $P^K = [p_{ij}^K]_{i,j=1}^{N}$ is for $1 < K \leq N-1$ given by

$$p_{ij}^K = \min_{h=1,2,\ldots,N} \{p_{ih}^{K-1} + p_{hj}^1\}, \text{ if } i \neq j,$$

and $p_{ij}^K = 0$ otherwise; see [28]. When determining the entry p_{ij}^K for a multiplex, one has to include the cost $1/\gamma$ for each layer switch in the sum of the reciprocal weights of intra-layer edges of a path, because all intra-layer edges of a shortest path of a multiplex do not necessarily belong to the same layer. In order to see if such switching cost is relevant, one takes into account the layer of the last edge (i.e., the intra-layer edge from the penultimate vertex to the last vertex) of all shortest paths from vertex v_i to vertex v_h made up of at most $K-1$ edges (in case $0 < p_{ih}^{K-1} < \infty$). Only if there exists an edge from vertex v_h to vertex v_j in layer ℓ (i.e., if $0 < p_{hj}^{(\ell)} < \infty$) and such layer is different from all the above mentioned layers, then the cost $1/\gamma$ is included in the computation of the length of the relevant path made up of at most K intra-layer edges, because there is a layer-switch preceding the last intra-layer edge.

In this way the off-diagonal entries of the multiplex K-path length matrix $P^K = [p_{ij}^K]_{i,j=1}^{N}$, for $1 < K \leq N-1$, are computed according to

$$p_{ij}^K = p_{i\bar{h}}^{K-1} + p_{\bar{h}j}^{(\bar{\ell})} + \frac{1}{\gamma}\delta_{\bar{h}}^{(\bar{\ell})} \text{ where } (\bar{h}, \bar{\ell}) = \arg\min_{h,\ell} \{p_{ih}^{K-1} + p_{hj}^{(\ell)} + \frac{1}{\gamma}\delta_h^{(\ell)}\},$$

with $\delta_h^{(\ell)} = 0$, if one of the following conditions holds:

- $p_{ih}^{K-1} = 0$, i.e., $v_i = v_h$;
- $p_{ih}^{K-1} = \infty$, i.e., there is no path from vertex v_i to vertex v_h made up of at most $K-1$ edges;
- $p_{hj}^{(\ell)} = 0$, for all $\ell = 1, 2, \ldots, L$, i.e., $v_h = v_j$;
- $p_{hj}^{(\ell)} = \infty$, for all $\ell = 1, 2, \ldots, L$, i.e., there are no intra-layer edges from vertex v_h to vertex v_j;
- the intra-layer edge from vertex v_h to vertex v_j with weight $p_{hj}^{(\ell)}$ belongs to the same layer ℓ as the last edge of a shortest path made up of at most $K-1$ edges from vertex v_i to vertex v_h of length p_{ih}^{K-1},

and $\delta_h^{(\ell)} = 1$ otherwise; see [29].

3.2.1 Min-plus powers versus powers

Consider for ease of discussion an undirected and unweighted single-layer network, i.e., a simple graph. Let $A = [a_{ij}]_{i,j=1}^N$ be the adjacency matrix for the graph and define its hth power $A^h = [a_{ij}^{(h)}]_{i,j=1}^N$. The entry $a_{ij}^{(h)}$ counts the number of walks of length h between the vertices v_i and v_j. Estrada and Rodriguez-Velazquez [22] defined the communicability between the vertices v_i and v_j for $i \neq j$ as the (i, j)th entry of the matrix

$$\exp_0(A) = \sum_{h=1}^{\infty} \frac{A^h}{h!}. \tag{3}$$

The rapid growth of the denominator with h ensures that the expansion converges and that terms $\frac{A^h}{h!}$ with h large contribute only little to $\exp_0(A)$. This is in agreement with the intuition that messages propagate better along short walks than along long ones. The (i, i)th entry of the sum (3) is commonly referred to as the subgraph centrality of vertex v_i; see [22][1].

We turn to the min-plus power P^K. The (i, j)th entry of P^K gives the length of the shortest path between the vertices v_i and v_j made up of at most K edges, i.e., it counts the number of edges of such shortest paths (since the graph is unweighted). Compare with the Kth partial sum $\sum_{h=1}^K A^h/h!$, whose (i, j)th entry is related to the number of walks between v_i and v_j made up of at most K edges. Thus, when considering the path length matrix P instead of $\exp_0(A)$, one emphasizes the availability of a short shortest path more than the availability of several paths that can be used for communication, thus focusing more on efficiency than popularity.

3.3 Estimating the multiplex global efficiency

For $1 \leq K \leq N - 1$, we introduce the *multiplex K-efficiency matrix* $P_{-1}^K = [p_{ij}^{K,-1}]_{i,j=1}^N$. It is obtained by replacing the off-diagonal entries of the K-path length matrix P^K by their reciprocals, i.e.,

$$p_{ij}^{K,-1} = 1/p_{ij}^K, \quad 1 \leq i, j \leq N, \quad i \neq j.$$

Moreover, we define the *multiplex global K-efficiency*

$$e_{\mathcal{A}}^K(\gamma) = \frac{1}{N(N-1)} \sum_{i,j \neq i} \frac{1}{p_{ij}^K} = \frac{1}{N(N-1)} \mathbf{1}_N^T P_{-1}^K \mathbf{1}_N,$$

with $1/\infty$ identified with 0; see [29].

[1] In [22] $\exp(A)$ is used instead of $\exp_0(A)$, but the term I_N has no natural interpretation in network modeling.

For the multiplex global efficiency $e_{\mathcal{A}}(\gamma)$, one has

$$e_{\mathcal{A}}(\gamma) = \frac{1}{N(N-1)} \sum_{i,j \neq i} \frac{1}{p_{ij}} = \frac{1}{N(N-1)} \mathbf{1}_N^T P_{-1} \mathbf{1}_N,$$

where P_{-1} will be referred to as the *multiplex efficiency matrix*. The following result shows how we can estimate the multiplex global efficiency.

Proposition 1 *The multiplex global K-efficiency, for $1 \leq K \leq N-1$, satisfies the inequality*

$$e_{\mathcal{A}}^1(\gamma) \leq e_{\mathcal{A}}^2(\gamma) \leq \cdots \leq e_{\mathcal{A}}^{N-1} = e_{\mathcal{A}}(\gamma).$$

Proof The proof follows by observing that by construction $P_{-1}^K \leq P_{-1}^{K+1}$, for $1 \leq K < N-1$, and that $P_{-1} \equiv P_{-1}^{N-1}$. □

3.4 Increasing the multiplex efficiency

We would like to identify the intra-layer edges that should be strengthened to enhance the multiplex global efficiency the most and will refer to these edges as "efficient". Consider the nonnegative vectors $\mathbf{h}_{\text{in}} \in \mathbb{R}^N$ and $\mathbf{h}_{\text{out}} \in \mathbb{R}^N$ defined as follows: the ith entry of the former is the harmonic in-centrality $\sum_{h \neq i} 1/p_{hi}$ of vertex v_i and the ith entry of the latter is its harmonic out-centrality $\sum_{h \neq i} 1/p_{ih}$; see [1]. The following results hold.

Proposition 2 *For the multiplex global efficiency and the harmonic in- and out-centralities, we have*

$$e_{\mathcal{A}}(\gamma) = \frac{1}{N(N-1)} \|\mathbf{h}_{\text{in}}\|_1 = \frac{1}{N(N-1)} \|\mathbf{h}_{\text{out}}\|_1.$$

Proof Since all the entries of the multiplex efficiency matrix P_{-1} are nonnegative quantities, one has that every entry of both harmonic in- and out-centrality vectors equals its modulus. Thus, the equalities $\|\mathbf{h}_{\text{in}}\|_1 = \|\mathbf{h}_{\text{out}}\|_1 = \mathbf{1}_N^T P_{-1} \mathbf{1}_N$ give the desired equality. □

Proposition 3 *For the Perron root of the multiplex efficiency matrix and the harmonic in- and out-centrality vectors, one has*

$$\rho \leq \min\{\|\mathbf{h}_{\text{in}}\|_\infty, \|\mathbf{h}_{\text{out}}\|_\infty\}.$$

Proof One can see that $\|P_{-1}\|_1 = \|\mathbf{h}_{\text{in}}\|_\infty$ and $\|P_{-1}\|_\infty = \|\mathbf{h}_{\text{out}}\|_\infty$. Since the spectral radius is less than or equal to any natural matrix norm, it follows that $\rho \leq \|P_{-1}\|_1$ and $\rho \leq \|P_{-1}\|_\infty$. This yields the desired inequality. □

Proposition 4 *For the Perron root of the multiplex efficiency matrix and the multiplex global efficiency, one has*

$$\rho \leq N(N-1) e_{\mathcal{A}}(\gamma).$$

Proof By Proposition 3, both $\rho \leq \|\mathbf{h}_{\text{in}}\|_1$ and $\rho \leq \|\mathbf{h}_{\text{out}}\|_1$ hold, having observed that for any vector $\mathbf{z}$ one has $\|\mathbf{z}\|_\infty \leq \|\mathbf{z}\|_1$. The proof now follows by Proposition 2. $\square$

The above results lead us to expect that the multiplex global efficiency increases the most by increasing the edge-weights that make ρ increase the most. We therefore should strengthen the existing edges $e(v_i^\ell \to v_j^\ell)$, for $\ell = 1, 2 \ldots, L$, determined by the (i, j)th entry of $A^+ = \sum_{\ell=1}^L A^{(\ell)}$ that corresponds to a largest entry of the Wilkinson perturbation W_N associated with ρ, projected onto the cone $\mathcal{S}_{A^+}$.

Since multiplexes typically have a large number of vertices, we focus on techniques that are well suited for large-scale networks. In case the computation of the path length matrix $P \equiv P^{N-1}$ is too expensive to be attractive, one may instead consider the Perron root ρ_K of the multiplex K-efficiency matrix P_{-1}^K and the associated Perron vectors $\mathbf{x}_K$ and $\mathbf{y}_K$, and determine the Wilkinson matrix $W_N^K = \mathbf{y}_K \mathbf{x}_K^T$, for $1 \leq K \leq N-1$ large enough, to identify edges whose strengthening may be advantageous; see [30]. This leads us to propose to strengthen existing edges $e(v_i^\ell \to v_j^\ell)$, for $\ell = 1, 2, \ldots, L$, determined by the (i, j)th entry of A^+ that correspond to a largest entry of the Wilkinson perturbation W_N^K projected onto the cone $\mathcal{S}_{A^+}$. The following result motivates this approach.

Proposition 5 *One has $\rho_K \leq \rho_{K+1} \leq \rho$, for $1 \leq K < N-1$.*

Proof The proof follows by observing that $P_{-1}^K \leq P_{-1}^{K+1}$, for $1 \leq K < N-1$, and that $P_{-1} \equiv P_{-1}^{N-1}$. $\square$

Moreover, consider the vector $\mathbf{h}_{\text{in}}^K \in \mathbb{R}^N$ whose ith entry is the harmonic K_{in}-centrality $\sum_{h \neq i} 1/p_{hi}^K$ of v_i and the vector $\mathbf{h}_{\text{out}}^K \in \mathbb{R}^N$ whose ith entry is its harmonic K_{out}-centrality $\sum_{h \neq i} 1/p_{ih}^K$; see [30]. The following propositions can be shown similarly as Propositions 2, 3, and 4.

Proposition 6 *For the multiplex global K-efficiency and the harmonic K_{in}- and K_{out}-centralities, the following equalities hold*

$$e_{\mathcal{A}}^K(\gamma) = \frac{1}{N(N-1)} \|\mathbf{h}_{\text{in}}^K\|_1 = \frac{1}{N(N-1)} \|\mathbf{h}_{\text{out}}^K\|_1, \quad 1 \leq K \leq N-1.$$

Proposition 7 *For the Perron root of the multiplex K-efficiency matrix and the harmonic K_{in}- and K_{out}-centralities one has, for $1 \leq K \leq N-1$,*

$$\rho_K \leq \min\{\|\mathbf{h}_{\text{in}}^K\|_\infty, \|\mathbf{h}_{\text{out}}^K\|_\infty\}.$$

Proposition 8 *For the Perron root of the multiplex K-efficiency matrix and the multiplex global K-efficiency one has, for $1 \leq K \leq N-1$,*

$$\rho_K \leq N(N-1) e_{\mathcal{A}}^K(\gamma).$$

The multiplex global K-efficiency is expected to increase the most by increasing the edge-weights that make the Perron root ρ_K increase the most. In fact, when analyzing

P_{-1}^{K}, for $1 \le K \le N-1$, we prefer to consider the eigenvector centrality over the degree, that is to say, the ith entry of $\mathbf{y}_K$ instead of the in-degree of vertex v_i (its harmonic K_{in}-centrality) and the jth entry of $\mathbf{x}_K$ instead of the out-degree of vertex v_j (its harmonic K_{out}-centrality); see, e.g., [6, 21, 26] for discussions on eigenvector centrality.

4 The popularity approach

This section considers the multiplex Perron communicability and discusses how it can be used to determine the importance of an edge. The sensitivity of this measure to perturbations of the weights is studied. The multiplex Perron communicability has prviously been described in El-Halouy et al. [20]. It is the aim of the present paper to compare the performance of the techniques of this section and Section 3.

The evaluation of the matrix $\exp_0(B)$ is very time-consuming when the supra-adjacency matrix B is large. We therefore are interested in estimating the multiplex total communicability without calculating $\exp_0(B)$ by using the Perron root ρ of $B = B(\gamma)$, the associated Perron vectors $\mathbf{x}$ and $\mathbf{y}$, and the Wilkinson matrix $W_{NL} = \mathbf{y}\mathbf{x}^T$. We propose to approximate $tc_B(\gamma) = \mathbf{1}_{NL}^T \exp_0(B)\mathbf{1}_{NL}$ by means of the *multiplex Perron communicability*

$$Pc_B(\gamma) = \exp_0(\rho)\, \mathbf{1}_{NL}^T W_{NL} \mathbf{1}_{NL},$$

which is much easier to compute; see [18, 20] for related discussions. The following result holds.

Proposition 9 [18] *If the Perron root ρ of B is significantly larger than the magnitude of the other eigenvalues of B, then*

$$tc_B(\gamma) \approx \kappa(\rho) Pc_B(\gamma).$$

Thus, $tc_B(\gamma)$ depends on $Pc_B(\gamma)$ and the conditioning of the Perron root ρ. In the special case of an undirected network, the Perron vectors $\mathbf{x}$ and $\mathbf{y}$ coincide and, therefore, $\kappa(\rho) = 1/\mathbf{y}^T\mathbf{x} = 1$. Under the assumption of Proposition 9, we then obtain that

$$tc_B(\gamma) \approx Pc_B(\gamma).$$

Additionally, since $\mathbf{1}_{NL}^T W_{NL} \mathbf{1}_{NL} = \|\mathbf{x}\|_1 \|\mathbf{y}\|_1$ and, $\forall \mathbf{z} \in \mathbb{C}^n$, one has $\|\mathbf{z}\|_2 \le \|\mathbf{z}\|_1 \le \sqrt{n}\|\mathbf{z}\|_2$, the following bounds for the multiplex Perron communicability hold.

Proposition 10 [18]

$$\exp_0(\rho) \le Pc_B(\gamma) \le NL \exp_0(\rho).$$

Typically, $\exp_0(\rho) \gg NL$. Therefore, it suffices to consider $\exp_0(\rho)$ to determine whether the multiplex Perron communicability is large or small.

4.1 Sensitivity of multiplex total communicability

We would like to identify the intra-layer edges that should be strengthened to enhance the multiplex total communicability the most. The canonical way to identify the entries of the block-diagonal portion of the supra-adjacency matrix (1), whose weights should be increased, is to evaluate the Fréchet derivative $L_{\exp_0}(B, E) \in \mathbb{R}^{NL \times NL}$ at $B = B(\gamma)$ in the direction $E = \mathbf{e}_i \mathbf{e}_j^T \in \mathbb{R}^{NL \times NL}$ for $1 \leq i, j \leq NL$. The Fréchet derivative $L_f(B, E)$ of a function f at the matrix B in the direction E is defined as

$$f(B + E) = f(B) + L_f(B, E) + o(\|E\|_2) \quad \text{as} \quad \|E\|_2 \to 0;$$

see, e.g., [23, 30, 31]. We are interested in determining intra-layer edges that have large weights, whose modification results in a relatively large change in the total communicability. Note that the sensitivity in the direction $\mathbf{e}_i \mathbf{e}_j^T$, i.e., $\mathbf{1}_{NL}^T L_{\exp_0}(B, \mathbf{e}_i \mathbf{e}_j^T) \mathbf{1}_{NL}$, is $\mathbf{e}_i^T L_{\exp_0}(B^T, \mathbf{1}_{NL} \mathbf{1}_{NL}^T) \mathbf{e}_j$; see [31]. However, the evaluation of $L_{\exp_0}(B^T, \mathbf{1}_{NL} \mathbf{1}_{NL}^T)$ is very demanding (about 8 times more arithmetic floating point operations than the evaluation of $\exp_0(B)$).

We remark that one could approximate the gradient of $tc_B(\gamma)$ by using Arnoldi or Lanczos decompositions, as proposed for large-scale single-layer networks in [30, 31]. In the following subsection, we focus on another approach, that takes into account the multiplex Perron communicability. The computations required are quite straightforward and not very demanding also for large-scale problems.

4.2 Increasing the multiplex communicability

We propose to determine the intra-layer edges with large weights whose modification yields a relatively large change in the Perron root ρ of the supra-adjacency matrix B. These intra-layer edges should be strengthened to enhance the multiplex Perron communicability the most. We will illustrate that modifications of the weights of the intra-layer edges identified by this technique give a relatively large change in the multiplex total communicability.

Following [30], we construct an "importance vector" by multiplying the positive entries of

$$B_d = \text{blkdiag}[A^{(1)}, \ldots, A^{(L)}]$$

element by element by the corresponding entries of $W_{NL}|_{\mathcal{S}_{B_d}}$, where $\mathcal{S}_{B_d} \subseteq \mathbb{R}^{NL \times NL}$ is the cone of all nonnegative matrices with the same sparsity structure as B_d, and then choose the weights $a_{ij}^{(\ell)} \in \mathcal{A}$ that correspond to the largest entries of this vector. Since supra-adjacency matrices typically are quite large, one generally computes their right and left Perron vectors by an iterative method that only requires the evaluation of matrix-vector products with the matrix B_d and its transpose. Clearly, one does not have to store B_d, but only $\mathcal{A}$, to evaluate matrix-vector products with the matrix B_d and its transpose.

To estimate the potential for increase in communicability, we propose to also evaluate an approximation of the *structured multiplex Perron communicability*, which is defined by

$$Pc_B^{\text{struct}}(\gamma) = \exp_0(\rho)\, \mathbf{1}_{NL}^T W_{NL}|_{\mathcal{S}_{B_d}} \mathbf{1}_{NL}.$$

The following result holds.

Proposition 11

$$Pc_B^{\text{struct}}(\gamma) \leq Pc_B(\gamma).$$

Proof The proof follows from the inequality $W_{NL}|_{\mathcal{S}_{B_d}} \leq W_{NL}$. □

Due to Proposition 10, we have $Pc_B^{\text{struct}}(\gamma) \leq NL \exp_0(\rho)$. This bound can be refined as the following result shows.

Proposition 12

$$Pc_B^{\text{struct}}(\gamma) \leq NL \exp_0(\rho) \frac{\kappa^{\text{struct}}(\rho)}{\kappa(\rho)}.$$

Proof Since $\mathbf{1}_{NL}^T W_{NL}|_{\mathcal{S}_{B_d}} \mathbf{1}_{NL} = \|\text{vec}(W_{NL}|_{\mathcal{S}_{B_d}})\|_1 \leq NL \|W_{NL}|_{\mathcal{S}_{B_d}}\|_F$, we have the upper bound $Pc_B^{\text{struct}}(\gamma) \leq NL \exp_0(\rho) \|W_{NL}|_{\mathcal{S}_{B_d}}\|_F$. The proof follows by observing that $\kappa^{\text{struct}}(\rho) = \kappa(\rho) \|W_{NL}|_{\mathcal{S}_{B_d}}\|_F$; cf. (2). □

5 Numerical tests

The numerical tests reported in this section have been carried out using MATLAB R2024a on a 3.2 GHz Intel Core i7 6 core iMac. The Perron root, and the left and right Perron vectors for small to moderately sized networks can easily be evaluated by using the MATLAB function eig. For large-scale multiplexes, these quantities can be computed by the MATLAB function eigs or by an Arnoldi algorithm (one-sided or two sided).

5.1 Single-layers networks

In the simple case when $L = 1$, the supra-adjacency matrix B, the third-order tensor $\mathcal{A}$, as well as B_d and A^+, reduce to the adjacency matrix $A \in \mathbb{R}^{N \times N}$ for the given single-layer network. Also, $\mathcal{S}_{A^+} \equiv \mathcal{S}_{B_d} \equiv \mathcal{S}_A$. Nevertheless, a few comments about the well-known *Air500* and *Autobahn* data sets may be of interest to a reader since these networks allow simple illustrations of the concepts of efficiency and popularity.

Example 5.1 (Air500 data set) Consider the adjacency matrix $A \in \mathbb{R}^{500 \times 500}$ for the network *Air500* in [19]. This data set describes flight connections for the top 500 airports worldwide based on total passenger volume. The flight connections between airports are for the year from 1 July 2007 to 30 June 2008. The network is represented by a directed unweighted connected graph $\mathcal{G}$ with $N = 500$ vertices and 24009 directed edges. The vertices of the network are the airports and the edges represent direct flight routes between two airports.

The global efficiency is $e_A = e_A^{499} \equiv e_A^5 = 4.8392 \cdot 10^{-1}$, see [29, Example 5], and the total communicability is $tc_A = 1.9164 \cdot 10^{38}$. The Perron communicability is $Pc_A = 1.9132 \cdot 10^{38}$. The flight connection from the Frankfurt FRA Airport (vertex v_{161}) to the New York JFK Airport (vertex v_{224}) is more efficient than the flight connection from New York JFK Airport (vertex v_{224}) to the Atlanta ATL Airport (vertex v_{24}), i.e., strengthening the former edge has a larger impact on the global efficiency than strengthening the latter edge. The latter edge is the most popular edge, i.e., increasing the weight for this edge increases the total communicability the most. The former edge appears in several shortest paths that connect airports, while the latter picks up travelers from several major airports and transmits them to several major airports. Note that the information provided by the efficiency matrix is the same as the one given by P_{-1}^2, so that the perturbation that increases the global 2-efficiency the most also increases the global efficiency the most; cf. Table 1.

Example 5.2 (Autobahn data set) Consider the undirected unweighted graph $\mathcal{G}$ that represents the German highway system network *Autobahn*. The graph, which is available at [19], has $N = 1168$ vertices representing German locations and 1243 edges representing highway segments that connect them. Therefore, the adjacency matrix $A \in \mathbb{R}^{1168\times 1168}$ for this network has 2486 nonvanishing entries.

The global efficiency is $e_A = e_A^{1167} \equiv e_A^{62} = 6.7175 \cdot 10^{-2}$; see [29, Example 6]. The total communicability is $tc_A = 1.2563 \cdot 10^4$ and the Perron communicability is $Pc_A = 2.2448 \cdot 10^3$. The highway segment that connects Duisburg (vertex v_{219}) and Krefeld (vertex v_{565}) turns out to be more efficient than the highway segment that connects Duisburg (vertex v_{219}) and Düsseldorf (vertex v_{217}), which instead turned out to be the most popular edge. Note that the information provided by the efficiency matrix is the same as the one given by P_{-1}^4, so that the perturbation that increases the global 4-efficiency the most also increases the global efficiency the most; cf. Table 2.

5.2 Multiplex networks

The computations in this subsection use the two-sided Arnoldi method described in Zwaan and Hochstenbach [33].

Table 1 Air500 data set

K	(h, k)	$e_A^K(1)$
5	(161, 224)	$4.8392 \cdot 10^{-1}$
4	(161, 224)	$4.8387 \cdot 10^{-1}$
3	(161, 224)	$4.7909 \cdot 10^{-1}$
2	(161, 224)	$3.6044 \cdot 10^{-1}$
1	(224, 24)	$9.6228 \cdot 10^{-2}$

Indices chosen by the procedure are shown in the second column and the global K-efficiency is displayed in the third column for $K = 1, 2, \ldots, 5$

Table 2 Autobahn data set

K	(h, k)	$e_A^K(1)$
62	$(565, 219)$	$6.7175 \cdot 10^{-2}$
$\vdots$	$\vdots$	$\vdots$
5	$(565, 219)$	$7.9991 \cdot 10^{-3}$
4	$(565, 219)$	$6.1823 \cdot 10^{-3}$
3	$(219, 217)$	$4.6017 \cdot 10^{-3}$
2	$(693, 543)$	$3.2082 \cdot 10^{-3}$
1	$(219, 217)$	$1.8238 \cdot 10^{-3}$

The second column shows indices chosen by the procedure for $K = 1, 2, \ldots, 5$ and $K = 62$, and the third column displays the global K-efficiency

Example 5.3 (European airlines data set) We consider the undirected, unweighted, and connected network consisting of $N = 417$ vertices that represent European airports and $L = 37$ layers that represent different airlines operating in Europe. Each edge represents a flight between airports. The network can be downloaded from [3]. In this multiplex all the maximal shortest paths are made up of 7 intra-layer edges and 2 layer switches, with $\gamma = 1$ that reflects the effort required to change airlines for connecting flights. Thus, for any $K \geq 7$, one has $e_{\mathcal{A}}(1) = e_{\mathcal{A}}^K(1)$ with

$$e_{\mathcal{A}}(1) = \frac{1}{N(N-1)} \sum_{i,j \neq i} \frac{1}{p_{ij}} = 0.3477.$$

For all $K \geq 3$, the largest entries of $W_N^K|_{\mathcal{S}_{A^+}}$ correspond to the route between vertices v_{40} and v_{15}. Thus, the information provided by the multiplex efficiency matrix P_{-1} is the same as the information given by P_{-1}^3, so that the perturbation that increases the multiplex global 3-efficiency the most is the same that increases the multiplex global efficiency the most; cf. Table 3. This indicates that the Perron root ρ of P_{-1} may be increased the most by doubling the number of flights between the Barcelona

Table 3 European airlines data set

K	(h, k)	$e_{\mathcal{A}}^K(1)$
7	(40, 15)	$3.4766 \cdot 10^{-1}$
6	(40, 15)	$3.4766 \cdot 10^{-1}$
5	(40, 15)	$3.4746 \cdot 10^{-1}$
4	(40, 15)	$3.4416 \cdot 10^{-1}$
3	(40, 15)	$3.1949 \cdot 10^{-1}$
2	(38, 15)	$1.8393 \cdot 10^{-1}$
1	(38, 15)	$3.4046 \cdot 10^{-2}$

Indices chosen by the procedure in the second column and multiplex global K-efficiency in the third column for $K = 1, 2, \ldots, 7$

(vertex v_{40}) and Amsterdam (vertex v_{15}) airports. The airlines that operate this route are EasyJet (layer 3), KLM (layer 9), Vueling (layer 21), and Transavia Holland (layer 27), all with the same number of flights.

Increasing the number of flights of each airline by 25 percent, so as to increase by one the total number of flights on the route, yields the perturbed third-order adjacency tensor $\widetilde{\mathcal{A}}$ and the global efficiency

$$e_{\widetilde{\mathcal{A}}}(1) = \frac{1}{N(N-1)} \sum_{i,j\neq i} \frac{1}{\tilde{p}_{ij}} = 0.3480.$$

As for the popularity approach, one has

$$tc_B(1) = \mathbf{1}_{NL}^T \exp_0(B)\mathbf{1}_{NL} = 2.4930 \cdot 10^{20}.$$

The Perron root $\rho = 38.3714$ of B (with $\exp_0(\rho) = 4.6183 \cdot 10^{16}$, $\kappa(\rho) = 1$ and $\kappa^{\text{struct}}(\rho) = 5.3310 \cdot 10^{-2}$) is significantly larger than the other eigenvalues. We obtain

$$Pc_B(1) = \exp_0(\rho)\, \mathbf{1}_{NL}^T W_{NL} \mathbf{1}_{NL} = 1.9637 \cdot 10^{20},$$
$$Pc_B^{\text{struct}}(1) = \exp_0(\rho)\, \mathbf{1}_{NL}^T W_{NL}|_{\mathcal{S}_{B_d}} \mathbf{1}_{NL} = 1.4733 \cdot 10^{17}.$$

The largest entries of $W_{NL}|_{\mathcal{S}_{B_d}}$ correspond to edge $e(v_{38}^1 \leftrightarrow v_2^1)$. This indicates that the Perron root may be increased the most by doubling the number of flights operated by Lufthansa airline between the Munich (vertex v_{38}) and Frankfurt (vertex v_2) airports. Note that Lufthansa (layer 1) is the only operating company for this route. For the perturbed supra-adjacency matrix $\widehat{B}$ one has the Perron root $\hat{\rho} = 38.3798$, with $\exp_0(\hat{\rho}) = 4.6572 \cdot 10^{16}$, and

$$tc_{\widehat{B}}(1) = \mathbf{1}_{NL}^T \exp_0(\widehat{B})\mathbf{1}_{NL} = 2.5056 \cdot 10^{20}.$$

Finally, we observe that

$$e_{\widehat{\mathcal{A}}}(1) = 0.3479 < 0.3480 = e_{\widetilde{\mathcal{A}}}(1); \quad tc_{\widetilde{B}}(1) = 2.4972 \cdot 10^{20} < 2.5056 \cdot 10^{20} = tc_{\widehat{B}}(1).$$

Example 5.4 (London transportation data set) Consider the undirected, weighted, and connected network consisting of $N = 369$ vertices that represent train stations in London and $L = 3$ layers that represent the networks of stations connected by

1. Tube - All underground lines (e.g., District, Circle, etc) aggregated;
2. Overground;
3. Docklands Light Railway (DLR).

Each intra-layer edge represents a route between stations. Data was collected in 2013. The network can be downloaded from Bergermann [15].

In this multiplex all the maximal shortest paths are made up of 40 intra-layer edges and 2 layer switches, with $\gamma = 1$.

Table 4 London transportation data set

K	(h, k)	$e_{\mathcal{A}}^{K}(1)$
40	(185, 182)	$1.1261 \cdot 10^{-1}$
$\vdots$	$\vdots$	$\vdots$
10	(185, 182)	$6.4761 \cdot 10^{-2}$
9	(182, 39)	$5.8334 \cdot 10^{-2}$
8	(182, 39)	$5.1699 \cdot 10^{-2}$
7	(182, 39)	$4.4932 \cdot 10^{-2}$
6	(182, 39)	$3.7978 \cdot 10^{-2}$
5	(182, 39)	$3.1098 \cdot 10^{-2}$
4	(185, 182)	$2.4577 \cdot 10^{-2}$
3	(185, 182)	$1.8541 \cdot 10^{-2}$
2	(182, 39)	$1.2894 \cdot 10^{-2}$
1	(182, 39)	$7.2464 \cdot 10^{-3}$

The second column shows indices chosen by the procedure for $K = 1, 2, \ldots, 10$ and $K = 40$, and the third column displays the multiplex global K-efficiency

Thus, for all $K \geq 40$, one has $e_{\mathcal{A}}^{K}(1) = e_{\mathcal{A}}(1)$, with

$$e_{\mathcal{A}}(1) = \frac{1}{N(N-1)} \mathbf{1}_N^T P_{-1} \mathbf{1}_N = 0.1126.$$

Also, for all $K \geq 10$, the largest entries of $W_N^K|_{\mathcal{S}_{A+}}$ correspond to the route between vertices v_{185} and v_{182}; cf. Table 4. This indicates that the Perron root of the multiplex efficiency matrix, $\rho(P_{-1}) = 46.5551$, may be increased the most by adding a new underground line to the 3 lines operating between the Euston Square (v_{185}) and King's Cross St. Pancras (v_{182}).

As for the popularity approach, one has

$$tc_B(1) = \mathbf{1}_{NL}^T \exp_0(B) \mathbf{1}_{NL} = 5.6238 \cdot 10^4.$$

The Perron root $\rho = 6.5138$ of B (with $\exp_0(\rho) = 6.7341 \cdot 10^2$, $\kappa(\rho) = 1$ and $\kappa^{\text{struct}}(\rho) = 5.0028 \cdot 10^{-1}$) is significantly larger than the other eigenvalues.

$$Pc_B(1) = \exp_0(\rho)\, \mathbf{1}_{NL}^T W_{NL} \mathbf{1}_{NL} = 2.0831 \cdot 10^4$$

$$Pc_B^{\text{struct}}(1) = \exp_0(\rho)\, \mathbf{1}_{NL}^T W_{NL}|_{\mathcal{S}_{B_d}} \mathbf{1}_{NL} = 1.6214 \cdot 10^3.$$

The largest entries of $W_{NL}|_{\mathcal{S}_{B_d}}$ correspond to the edge $e(v_{182}^1 \leftrightarrow v_{39}^1)$ in layer 1. This indicates that the Perron root may be increased the most by adding a new underground line to the 3 lines operating between King's Cross St. Pancras (v_{182}) and Farrington Station (v_{39}).

Let us take into account both the efficiency and the popularity approaches, by adding the underground line Euston Square - King's Cross St. Pancras - Farrington Station. As for the perturbed multiplex, one has the Perron root $\rho(\widehat{P}_{-1}) = 46.9491$ and

$$e_{\widehat{\mathcal{A}}}(1) = \frac{1}{N(N-1)}\mathbf{1}_N^T \widehat{P}_{-1}\mathbf{1}_N = 0.1132,$$

whereas, for the perturbed supra-adjacency matrix $\widehat{B}$, one has the Perron root $\hat{\rho} = 7.4155$, $\exp_0(\hat{\rho}) = 1.6606 \cdot 10^3$, and

$$tc_{\widehat{B}}(1) = \mathbf{1}_{NL}^T \exp_0(\widehat{B})\mathbf{1}_{NL} = 7.0644 \cdot 10^4.$$

In the above examples, we used the Wilkinson perturbation associated with the Perron root of B and projected onto $\mathcal{S}_{B_d}$ (or associated with the Perron root of the P_{-1}^K, for a suitable K, and projected onto $\mathcal{S}_{A+}$) to determine the edge, such that a change in its weight has the largest effect on the total communicability (or the global efficiency) of the multiplex. If we are interested in identifying more than one edge, whose weight should be changed to increase such measure of communication, then we can either repeat the computations with the network obtained by having modified one edge-weight or consider the edge that is identified by the second largest entry of the same matrix.

6 Concluding remarks

Two measures of communicability in multiplex networks are considered: multiplex global efficiency and multiplex total communicability. Their sensitivity to changes in edge-weights is investigated. Approximations of both measures can be evaluated also for large multiplex networks. They measure different aspects of communicability and shed light on which edges should be strengthened (e.g., which roads should be widened) to increase the communicability of the network the most. Application of these concept to single-layer and multiplex transportation networks are presented.

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Declarations

Conflict of Interest The authors declare that they have no conflict of interest.

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