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Generalized analysis of methods for solving systems of nonlinear equations with point and interval coefficients

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Abstract
In this article we consider problems of interval estimation of a set of solutions to point and interval (partially interval) systems of nonlinear equations. Most developed interval methods are intended only for estimating solutions of point nonlinear systems in some given interval box. And methods for estimating solution sets of nonlinear interval systems are not yet very developed, since the solution sets of such systems geometrically represent a rather complex structure. Here we conducted a general analysis on existing classical interval methods to test their applicability for interval systems. In this case, we chose the methods of Newton and Krawczyk. The results of the analysis show that these and similar other iterative methods are generally not applicable for interval systems due to the limited admissible area. Based on the results of the analysis, a new combined vertex method for outer estimation of solution sets of interval nonlinear systems is proposed, which includes these classical interval methods. Numerical experiments have shown that the proposed method is more efficient and gives more accurate estimates in feasible regions than the direct application of Newton, Krawczyk or Hansen-Sengupta interval methods for interval systems.

Keywords Systems of nonlinear equations · Outer estimation of solution set · Interval box · Interval methods

1 Introduction

Let a system of nonlinear point (with real coefficients) equations be given

$$F(x) = 0, \tag{1}$$

where $F(x) = \big(F_1(x), F_2(x), \ldots, F_n(x)\big)^\top$, $x = (x_1, x_2, \ldots, x_n)^\top$.

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To apply interval methods, the formulation of the problem of solving systems of nonlinear point (1) is given in the following form:

$$\left[\begin{array}{c} \textit{For each solution of the system of equations} \\ F(x) = 0 \\ \textit{on a given set } D \subseteq \mathbb{R}^n \textit{ find guaranteed two-sided} \\ \textit{boundaries (if possible, the mast accurate)} \end{array}\right] \tag{2}$$

And the problem of outer estimation for interval systems is formulated as follows:

for systems of equations of the form

$$F(\boldsymbol{a}, x) = 0, \tag{3}$$

$F(\boldsymbol{a}, x) = \big(F_1(\boldsymbol{a}, x), F_2(\boldsymbol{a}, x), \ldots, F_n(\boldsymbol{a}, x)\big)^\top$, $\boldsymbol{a} \in \mathbb{IR}^l$, *find the outer interval estimate for the united set of solutions*

$$\Xi_{uni}(F, \boldsymbol{a}) = \{x \in \mathbb{R}^n \mid (\exists a \in \boldsymbol{a})\big(F(a, x) = 0\big)\}, \tag{4}$$

within the interval box $\boldsymbol{X} \in \mathbb{IR}^n$.

Nonlinear systems of equations and interval methods for solving them are described in detail in monographs [1–7]. Currently, there are many interval iterative methods for solving nonlinear point systems of equations, the most popular of which are Newton's method, Krawczyk's method [8] and the Hansen-Sengupta method [9]. These methods are considered classical and fundamental because many recently developed methods use them as a component or are based on their ideas (we refer to these methods as classical interval methods below). Other interval methods have also been developed, for example, T. Eftekhari [10] proposes an interval analogue of the classical Ostrovsky method. In [11], S. Singh et al. develop higher order interval multi-step methods. B. Kubica in [12] presents a multi-threaded interval solver for subdetermined and well-defined nonlinear systems, where some options are proposed for ensuring consistency of boxes and eliminating initial boxes. E. Hansen et al. [13] offer a test for the existence of solutions to a nonlinear point system, where sufficient conditions for the existence of a solution for the Krawczyk method are checked. L. Kolev in [14] proposes a linearization method using modified affine arithmetic, which provides a more optimal shell compared to the well-known linear interval form [15].

It should be noted that most of the developed interval methods are intended for solving point nonlinear systems (1), and methods for estimating sets of solutions to interval systems of the form (3) are not yet sufficiently developed.

In this work, we limit ourselves to considering some special version of the equations, which is a special case of system (3):

$$F(a, x) = \boldsymbol{b}, \tag{5}$$

where $F(a,x) = \left(F_1(a,x), F_2(a,x), \ldots, F_n(a,x)\right)^\top, x = (x_1, x_1, \ldots, x_n)^\top \in \boldsymbol{X} \in \mathbb{IR}^n, a \in \mathbb{R}, \boldsymbol{b} \in \mathbb{IR}$, that the coefficients on the left side are point, and the coefficients on the right side are interval and the solutions are obtained in interval form.

Systems of the form (5) often arise in tolerance problems, optimization, and also in practice, for example, in calculating the parameters of electrical networks, when the parameters of the right side of the system are specified with limited errors in the form of intervals.

The set of solutions (4) of interval systems of equations in the general case geometrically form heterogeneous, complexly structured objects. Accurately describing such objects is an NP-hard problem [16, 17]. Therefore, such problems are reduced to interval estimation problems, i.e. tasks with less complexity of description. For clarity, we illustrate the solution areas and their approximate interval estimates (Fig. 1) for problems (2) and (5) using the following point and partially interval systems as examples, respectively:

$$\begin{cases} \cos(x_1) + x_2 = 1, \\ \frac{x_1^2}{3} - \frac{x_1^2}{3} = 1, \end{cases} \text{ and } \begin{cases} \cos(x_1) + x_2 = [1-r, 1+r], \\ \frac{x_1^2}{3} - \frac{x_1^2}{3} = [1-r, 1+r], \end{cases} \tag{6}$$

where r is the radius of the interval.

To solve problem (2), you can use classical interval methods, but with sufficiently small sizes of the initial box. To find global solutions, it becomes necessary to use constraint propagation methods [12]. In classical (real) numerical analysis there are almost no developed methods for solving this problem. Commonly used approaches with limited success include analytical research, multistart, and continuation methods [18].

The fundamental difference between interval systems of equations and point systems is that in the interval case, outer estimates of solution sets obtained using classical interval methods can have noticeable or even large errors, differing significantly from optimal (exact) outer estimates. This coarsening is naturally related to the width of the interval parameters. In addition, due to the increase in the size of decision sets due

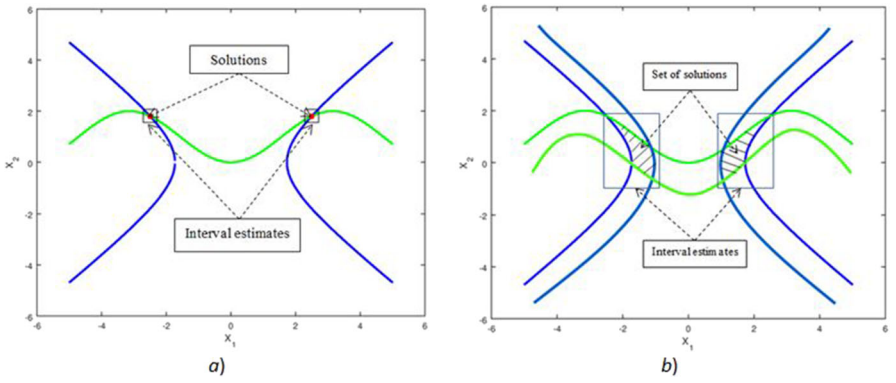

Fig. 1 Solutions and their approximate interval estimates for systems (6): a) point case; b) partial-interval case

to interval coefficients, the feasible region for classical interval methods will become incompatible with the region of decision sets. Therefore, these methods lose their applicability to systems (3) or (5).

Based on these considerations, in this paper we propose a combined vertex method that works regardless of the size of the solution sets of interval systems.

The structure of the article is organized in the following order. The basic facts and notations necessary for the article are given in Section 2. Section 3 provides an analysis of the theoretical and practical aspects of the Newton and Krawczyk methods. The proposed method and its algorithm are presented in Section 4. The results of a numerical experiment and a comparative analysis of the methods under consideration are given in Section 5. In the final Section 6, conclusions on this study are drawn.

2 Preliminaries

An informal international standard has been adopted for notation in interval analysis; the latest modified version of this text is available on the Internet and published in the article [19]. According to this standard, all interval quantities in the text of the article are highlighted in italic bold mathematical font, and point (real, non-interval) quantities are not specially highlighted in any way.

The interval number $\boldsymbol{a} = [\underline{a}, \overline{a}] \subset \mathbb{R}$ is defined as the set of all numbers of the real axis located between the given boundaries $\underline{a}$ and $\overline{a}$ including them, i.e.

$$\boldsymbol{a} = [\underline{a}, \overline{a}] := \{x \in \mathbb{R} \mid \underline{a} \le x \le \overline{a}\}.$$

Next we use some interval characteristic functions: *width*: $\mathrm{wid}(\boldsymbol{a}) = \overline{a} - \underline{a}$; *radius*: $\mathrm{rad}(\boldsymbol{a}) = \frac{1}{2}(\overline{a} - \underline{a}) = \frac{1}{2}\mathrm{wid}(\boldsymbol{a})$; *Middle* (*center*): $\mathrm{mid}(\boldsymbol{a}) = \frac{1}{2}(\underline{a} + \overline{a})$; *Absolute value* (*modulus* or *magnitude*): $|\boldsymbol{a}| = \max\{|a| \mid a \in \boldsymbol{a}\} = \max\{|\underline{a}|, |\overline{a}|\}$.

Definition 1 Let $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{IR}$. Then the distance between intervals $\boldsymbol{a}$ and $\boldsymbol{b}$ is defined as:

$$\mathtt{dist}(\boldsymbol{a}, \boldsymbol{b}) := \max\{|\underline{a} - \underline{b}|, |\overline{a} - \overline{b}|\}.$$

Definition 2 Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{IR}^n$. Then the metric on a multidimensional interval space for vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined as:

$$\mathtt{dist}(\boldsymbol{x}, \boldsymbol{y}) := \max\{\|\underline{x} - \underline{y}\|, \|\overline{x} - \overline{y}\|\},$$

where $\|\cdot\|$ is the absolute vector norm in $\mathbb{R}^n$.

3 Analysis of interval iterative methods for solving point systems

In this section we discuss the application of Newton and Krawczyk methods for solving point systems of nonlinear equations according to problem (2).

3.1 Newton's interval method

Suppose that system (1) with n equations and n unknowns is given on the box $\boldsymbol{X}$. Let $\boldsymbol{L}$ be the interval Lipschitz matrix of the mapping F onto $\boldsymbol{X}$. In particular, $\boldsymbol{L}$ can be an interval extension of the Jacobian of the mapping F onto $\boldsymbol{X}$. Then for any points of $x, \tilde{x} \in \boldsymbol{X}$ the following representation holds:

$$F(x) \in F(\tilde{x}) + \boldsymbol{L}(x - \tilde{x}).$$

In particular, if $x = x^*$ is a solution to the system of equations (1), i.e. $F(x^*) = 0$, then

$$0 \in F(\tilde{x}) + \boldsymbol{L}(x^* - \tilde{x}. \tag{7}$$

We present Beck's criterion for the united set of solutions of interval systems of linear algebraic equations (ISLAE):

Theorem 1 (Beeck [20]) *Let $\boldsymbol{A} \in \mathbb{IR}^{m\times n}$ and $\boldsymbol{b} \in \mathbb{IR}^m$ be given. The vector $\tilde{x} \in \mathbb{R}^n$ belongs to the united set of solutions $\Xi(\boldsymbol{A}, \boldsymbol{b})$ of the interval system of linear algebraic equations $\boldsymbol{A}x = \boldsymbol{b}$ if and only if $\boldsymbol{A}\tilde{x} \cap \boldsymbol{b} \neq \varnothing \;\Leftrightarrow\; 0 \in \boldsymbol{A}\tilde{x} - \boldsymbol{b}$.*

It follows that the point x^* satisfies inclusion (3) if and only if it belongs to the united set of solutions of the interval linear system

$$\boldsymbol{L}(x - \tilde{x}) = -F(\tilde{x}).$$

Let *Encl* [6] be a procedure for outer estimation of the set of solutions of the interval linear system $\boldsymbol{A}x = \boldsymbol{b}$, it is easy to show that the following relation $\Xi(\boldsymbol{A}, \boldsymbol{b}) \subseteq \mathit{Encl}(\boldsymbol{A}, \boldsymbol{b})$ holds. Then the inclusion

$$x^* - \tilde{x} \in \mathit{Encl}\big(\boldsymbol{L}, -F(\tilde{x})\big) \quad \Rightarrow \quad x^* \in \tilde{x} - \mathit{Encl}\big(\boldsymbol{L}, F(\tilde{x})\big),$$

because $\mathit{Encl}\big(\boldsymbol{L}, -F(\tilde{x})\big) = -\mathit{Encl}\big(\boldsymbol{L}, -F(\tilde{x})\big)$.

Here, it is preferable to use the Gauss-Seidel method as the *Encl* procedure, since this method does not require the regularity of the Jacobi matrix. Even if the Jacobian matrix is singular and the solution set is unlimited, the algorithm gives the result on a finite box.

Let the interval Lipschitz matrix $\boldsymbol{L} \in \mathbb{IR}^{n\times n}$ be known for the mapping $F : \mathbb{R} \supseteq D \to \mathbb{IR}^n$. Then the mapping $\mathcal{N} : \mathbb{ID} \times \mathbb{R} \to \mathbb{IR}^n$, given by rule

$$\mathcal{N}\big(\boldsymbol{X}, \tilde{x}\big) := \tilde{x} - \mathit{Encl}\big(\boldsymbol{L}, F(\tilde{x})\big), \tag{8}$$

defines the interval Newton operator on $\mathbb{ID}$ relative to the point $\tilde{x}$. In this case, the iterative scheme of Newton's interval method is determined by the formula:

$$\begin{cases} \boldsymbol{X}^{(0)} \leftarrow \boldsymbol{X} \\ \boldsymbol{X}^{(k+1)} \leftarrow \boldsymbol{X}^{(k)} \cap \mathcal{N}\big(\boldsymbol{X}^{(k)}, \tilde{x}^{(k)}\big), \quad \tilde{x}^{(k)} \in \boldsymbol{X}^{(k)}, \quad k = 0, 1, 2, \ldots \end{cases} \tag{9}$$

As a consequence, if $\boldsymbol{X} \cap \mathcal{N}\left(\boldsymbol{X}, \tilde{x}\right) = \varnothing$, then the box $\boldsymbol{X}$ contains no solutions to the system $F(x) = 0$. If for some $\tilde{x}$ the inclusion $\mathcal{N}\left(\boldsymbol{X}, \tilde{x}\right) \subseteq \boldsymbol{X}$ holds, then according to Brouwer's theorem there is exactly one solution to the system $F(x) = 0$ in the box $\boldsymbol{X}$. The most unfavorable situation when operating Newton's interval method is the appearance of an inclusion

$$\mathcal{N}\left(\boldsymbol{X}, \tilde{x}\right) \supseteq \boldsymbol{X}. \tag{10}$$

In case (10), all subsequent steps are looped on the box $\boldsymbol{X}$ and do not provide any additional information about the required solutions of the system.

3.1.1 Discussion on the application of Newton's method

Here we discuss some issues on the application of Newton's interval method.

1. *Why does Newton's interval method not fully correspond to the point version?*

In the point (real) case, the calculation formula of Newton's method for solving a system of equations is usually written in the form

$$X^{(k+1)} \leftarrow X^{(k)} - \left(J(X^{(k)})\right)^{-1} \cdot F(X^{(k)}),$$

where the matrix appears, inverse to the Jacobian $J(X^{(k)})$ of the mapping F at point $X^{(k)}$. But we do not need this inverse matrix itself "*in its pure form*", but only its product by the vector $F(X^{(k)})$, which, when implemented, is more convenient to represent as a solution to a system of linear algebraic equations with the matrix $J(X^{(k)})$. This results in a gain in labor intensity and better accuracy of the result. These arguments remain fully valid in the interval case and, in addition, another good reason is added to them. Writing an interval estimate of the values of the product $\left(J(X^{(k)})\right)^{-1} \cdot F(X^{(k)})$ through outer evaluation of the set of ISLAE solutions reduces the coarsening of the result. That is why the calculation formulas of Newton's multidimensional interval method are defined in the form (8), which does not fully correspond to the point case.

2. *What is the best way to choose the center of the $\tilde{x}$ expansion when using Newton's interval method?*

It makes sense to do this so that the value of $\|F(\tilde{x})\|$ is as small as possible. The smaller the norm of the vector function $F(\tilde{x})$, the smaller will be the norm of the vectors forming the set of solutions of the interval linear system

$$\boldsymbol{L}(x - \tilde{x}) = -F(\tilde{x}),$$

which we must intersect with the original box, the smaller its dimensions will be. In this case, we will most likely obtain a narrower outer estimate of the set of solutions of the original nonlinear system and more accurately determine the status of the box under study. Numerical experiments conducted by E. Hansen and R. Greenberg [21] confirm this conclusion. The book [13] discusses a simple iterative procedure called "*inner iteration*" which is designed to minimize $\|F(\tilde{x})\|$ within a given box.

3. *When does inclusion (10) appear and what to do in such situations?*

If the width of the box $\boldsymbol{X}$ is large, then the applied solution methods may not be successful. In such situations, we have inclusion (10), which does not give us any

information about the existence of a solution or its absence on a given box $\boldsymbol{X}$. One of the simple and universal techniques that allows you to move the refinement of solutions from the "dead point" is to split the original box into smaller sub-boxes. We consider this technique in more detail in the next Section 3.1.2

3.1.2 Forced crushing of interval box

The most common technique in such situations is *bisection* - dividing the box $\boldsymbol{X}$ into two equal or unequal parts along some edge, for example, into halves (Fig. 2).

$$\boldsymbol{X} = \boldsymbol{X}' \cup \boldsymbol{X}'',$$

where

$$\boldsymbol{X}' = \big(\boldsymbol{X}_1, \ldots, [\underline{X}_i, \operatorname{mid}(\boldsymbol{X}_i)], \ldots, \boldsymbol{X}_n\big),$$
$$\boldsymbol{X}'' = \big(\boldsymbol{X}_1, \ldots, [\operatorname{mid}(\boldsymbol{X}_i), \overline{X}_i], \ldots, \boldsymbol{X}_n\big),$$

for some $i \in \{1, 2, \ldots, n\}$. In this case, we select the component for crushing according to its maximum width, i.e. $\boldsymbol{X}_i = \max\{\operatorname{wid}(\boldsymbol{X})\}$.

Then we can apply Newton's interval method to $\boldsymbol{X}'$ and $\boldsymbol{X}''$ separately and combine the results. In this case, the sub-boxes $\boldsymbol{X}'$ and $\boldsymbol{X}''$ have a smaller width than the original $\boldsymbol{X}$, then the application of Newton's interval method to them can be more successful. Next, these descendants can be divided again, as many as necessary to achieve the desired smallness of their sizes, then we can successfully perform tests of the existence of solutions on these boxes. To organize such crushing, the book [6] suggests creating a list $\mathcal{L}$ of all descendants of the initial box $\boldsymbol{X}$ that are suspicious for the content of solutions. At the same time, in order to ensure that the operating time of the algorithm is limited, it is necessary to set a certain size δ for the boxes; when this size is reached, it no longer makes sense to further crush the box.

Algorithms for crushing boxes are usually called *branch-and-cut methods*, where branching is the splitting of the original search area for solutions into subdomains, and cutting is the discarding of unpromising parts of the original area that certainly do not contain solutions.

Applying Algorithm 1, we get three lists of boxes:

$\mathcal{L}_1$ is a list consisting of boxes of size no more than δ, which are guaranteed to contain solutions, systems of equations in $\boldsymbol{X}$;

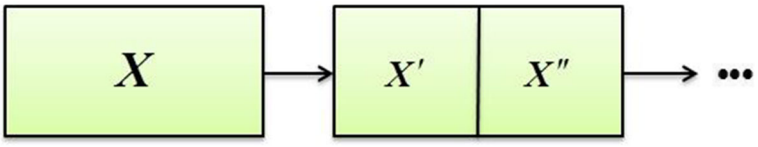

Fig. 2 Forced crushing of boxes into descendants

Algorithm 1 Algorithm of the branching and cutting method for the global probative solution of equations [6].

Require: System of equations $F(x) = 0$, Box $\boldsymbol{X} \in \mathbb{IR}^n$; Interval extension $\boldsymbol{F} : \mathbb{IX} \to \mathbb{IR}^n$ of function F; The given accuracy $\delta > 0$ of localization of system solutions
Ensure: Lists $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$
1: initialize the worklist $\mathcal{L}$ with the initial box $\boldsymbol{X}$;
2: **while** $\mathcal{L} \neq \varnothing$ **do**
3: extract box $\boldsymbol{Y}$ from the list $\mathcal{L}$;
4: apply existence tests to $\boldsymbol{Y}$ for a solution, their result is also denoted by $\boldsymbol{Y}$;
5: **if** (the absence of solutions is proven in $\boldsymbol{Y}$) **then**
6: remove box $\boldsymbol{Y}$ from consideration
7: **else**
8: **if** $\mathrm{wid}(\boldsymbol{Y}) < \delta$ **then**
9: enter $\boldsymbol{Y}$ into the corresponding list $\mathcal{L}_1$ or $\mathcal{L}_2$
10: **else**
11: cut $\boldsymbol{Y}$ into descendant boxes $\boldsymbol{Y}'$ and $\boldsymbol{Y}''$ and add them to the worklist $\mathcal{L}$
12: **end if**
13: **end if**
14: **end while**
15: all boxes from $\mathcal{L}$ are moved to list $\mathcal{L}_3$;

$\mathcal{L}_2$ is a list consisting of boxes of size no more than δ that are suspected of containing a solution to a system of equations in $\boldsymbol{X}$.

$\mathcal{L}_3$ - a list consisting of "*under-processed*" boxes that have a size greater than δ.

All solutions of the system of equations under consideration that do not belong to the boxes from the list $\mathcal{L}_1$ are contained in the boxes from the lists $\mathcal{L}_2$ and $\mathcal{L}_3$. This technique can be applied not only to Newton's interval method itself, but also to its other modifications, for example, the Krawczyk [8] and Hansen-Sengupta [9] methods.

3.2 Krawczyk method

The Krawczyk operator for the mapping $\mathcal{K} : \mathbb{ID} \times \mathbb{R} \to \mathbb{IR}^n$ is given by the expression

$$\mathcal{K}(\boldsymbol{X}, \tilde{x}) := \tilde{x} - \Lambda F(\tilde{x}) + (I - \Lambda \boldsymbol{L})(\boldsymbol{X} - \tilde{x}), \tag{11}$$

where $\Lambda \in \mathbb{R}^{n \times n}$ is a real matrix.

If the spectral radius is $\rho\big(|I - \Lambda \boldsymbol{L}|\big) < 1$, then, according to Schröder's theorem [6], the mapping has a unique fixed point, which is a solution to the system of equations under consideration. Thus, the box $\boldsymbol{X}$ contains exactly one solution to the system $F(x) = 0$.

The iterative scheme of the Krawczyk method has the following form:

$$\begin{cases} \boldsymbol{X}^{(0)} \leftarrow \boldsymbol{X} \\ \boldsymbol{X}^{(k+1)} \leftarrow \boldsymbol{X}^{(k)} \cap \mathcal{K}\big(\boldsymbol{X}^{(k)}, \tilde{x}^{(k)}\big), \quad \tilde{x}^{(k)} \in \boldsymbol{X}^{(k)}, \quad k = 0, 1, 2, \ldots \end{cases} \tag{12}$$

The iterative scheme (12) is called the Krawczyk method, and its special case is also used for outer estimation of solution sets of ISLAE.

Each solution of the system $F(x) = 0$ on the box $\boldsymbol{X}$ also lies in $\mathcal{K}\left(\boldsymbol{X}, \tilde{x}\right)$, therefore, to determine more accurate boundaries, we find $\boldsymbol{X} \cap \mathcal{K}\left(\boldsymbol{X}, \tilde{x}\right)$. If $\boldsymbol{X} \cap \mathcal{K}\left(\boldsymbol{X}, \tilde{x}\right) = \varnothing$, then the box $\boldsymbol{X}$ contains no solutions to the system $F(x) = 0$.

Let us present two main facts about the existence of a solution from [6]:

- if $\mathcal{K}\left(\boldsymbol{X}, \tilde{x}\right) \subseteq \boldsymbol{X}$ and the matrix Λ in the Krawczyk operator is regular, then the box $\boldsymbol{X}$ contains at least one solution to the system $F(x) = 0$;
- if $\mathcal{K}_i\left(\boldsymbol{X}, \tilde{x}\right) \subset \boldsymbol{X}_i, \quad i = 1, 2, \ldots, n$, then the interval Lipschitz matrix $\boldsymbol{L}$ is strongly regular and the box $\boldsymbol{X}$ contains exactly one solution to the system $F(x) = 0$.

The Krawczyk operator and the iterative scheme of the Krawczyk method work for fairly arbitrary $\tilde{x}$ and Λ, but to obtain the best results it is advisable to choose them so that the result of the Krawczyk operator is, if possible, the narrowest box possible. It can be noted that the smallest width of values of the Krawczyk operator is achieved when choosing $\tilde{x} = \text{mid}(\boldsymbol{X})$, and for the matrix Λ the most favorable value is $\left(\text{mid}(\boldsymbol{L})\right)^{-1}$, since at $\tilde{x} = \text{mid}(\boldsymbol{X})$ the width of values $\mathcal{K}\left(\boldsymbol{X}, \tilde{x}\right)$ is minimized.

Another improvement of the Krawczyk operator is associated with a strengthened version of Brouwer's fixed point theorem, in which it is necessary to check the nesting conditions for the image not for the entire box, but only for its boundary [22].

4 The problem of estimating a set of solutions for partially interval nonlinear systems

Classical interval methods were considered in the previous section for finding solutions to point systems of nonlinear equations in a given interval box. For outer estimation of solution sets of interval systems of the form (5), they are not always applicable, i.e. can sometimes be used when the interval coefficients have fairly small widths. This circumstance limits the applicability of these methods. Below we propose a new method that works in combination with some well-known algorithms.

4.1 The idea of the proposed method

W. Dong and H. Shah [23] proposed the so-called "*Vertex Method*" for calculating the boundaries of the value domains of functions of fuzzy variables. Z.Qiu and his co-authors in [24, 25] use the vertex method to determine the dynamic response boundary of mechanical systems with interval parameters.

Here we use the idea of this method to solve systems of nonlinear equations of the form (5). To do this, we write system (5) in a more understandable form:

$$\begin{cases} F_1\ (x_1, x_2, \ldots, x_n) = [\underline{b}_1, \overline{b}_1], \\ F_2\ (x_1, x_2, \ldots, x_n) = [\underline{b}_2, \overline{b}_2], \\ \vdots \qquad\quad \ddots \qquad\quad \vdots \quad\ \vdots \\ F_n\ (x_1, x_2, \ldots, x_n) = [\underline{b}_n, \overline{b}_n]. \end{cases} \tag{13}$$

Next, with all possible combinations of left and right boundaries of intervals $\boldsymbol{b}_i = [\underline{b}_i, \overline{b}_i]$ $(i = 1, 2, \ldots, n)$, we will create a sequence of point systems of nonlinear equations. Then for n intervals $\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_n \in \mathbb{IR}$ - this will be 2n combinations. For example, for $n = 2$, we have:

$$\begin{cases} F_1(x_1, x_2) = [\underline{b}_1, \overline{b}_1], \\ F_2(x_1, x_2) = [\underline{b}_2, \overline{b}_2]. \end{cases} \tag{14}$$

Thus, we get $2^2 = 4$ point equations:

$$\begin{cases} F_1(x_1, x_2) = \underline{b}_1 \\ F_2(x_1, x_2) = \underline{b}_2 \end{cases} \begin{cases} F_1(x_1, x_2) = \underline{b}_1 \\ F_2(x_1, x_2) = \overline{b}_2 \end{cases} \begin{cases} F_1(x_1, x_2) = \overline{b}_1 \\ F_2(x_1, x_2) = \underline{b}_2 \end{cases} \begin{cases} F_1(x_1, x_2) = \overline{b}_1 \\ F_2(x_1, x_2) = \overline{b}_2 \end{cases}. \tag{15}$$

Having solved each system of nonlinear equations (15) on a given box $\boldsymbol{X}$ using the Newton or Krawczyk interval method, we obtain four pairs of interval solutions $(\boldsymbol{x}_{1j}, \boldsymbol{x}_{2j})$ $(j = 1, 2, 3, 4)$, which can be written in more detail in the following form:

$$\begin{array}{ll} \boldsymbol{x}_{11} = [\underline{x}_{11}, \overline{x}_{11}], & \boldsymbol{x}_{21} = [\underline{x}_{21}, \overline{x}_{21}], \\ \boldsymbol{x}_{12} = [\underline{x}_{12}, \overline{x}_{12}], & \boldsymbol{x}_{22} = [\underline{x}_{22}, \overline{x}_{22}], \\ \boldsymbol{x}_{13} = [\underline{x}_{13}, \overline{x}_{13}], & \boldsymbol{x}_{23} = [\underline{x}_{23}, \overline{x}_{23}], \\ \boldsymbol{x}_{14} = [\underline{x}_{14}, \overline{x}_{14}], & \boldsymbol{x}_{24} = [\underline{x}_{24}, \overline{x}_{24}]. \end{array} \tag{16}$$

From the calculated intervals we find the components of the solution to (15), calculating the minimum and maximum of the corresponding boundaries of the intervals from (16), as a result we obtain the final outer interval estimates:

$$\boldsymbol{x}_1 = [\underline{x}_1, \overline{x}_1] = \left[\min_{j=1}^{4}\{\underline{x}_{1j}\}, \max_{j=1}^{4}\{\overline{x}_{1j}\}\right],$$

$$\boldsymbol{x}_2 = [\underline{x}_2, \overline{x}_2] = \left[\min_{j=1}^{4}\{\underline{x}_{2j}\}, \max_{j=1}^{4}\{\overline{x}_{2j}\}\right].$$

This can be generalized to n equations:

$$\boldsymbol{x}_i = [\underline{x}_i, \overline{x}_i] = \left[\min_{j=1}^{n}\{\underline{x}_{ij}\}, \max_{j=1}^{n}\{\overline{x}_{ij}\}\right], \quad i = 1, 2, \ldots, n. \tag{17}$$

Here, the found components $\boldsymbol{x}_i$ $(i = 1, 2, \ldots, n)$ determine the interval estimate of the set of solutions of system (13) in the form of an interval vector-bar $\boldsymbol{X} = (\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n)$.

Thus, by reducing the interval system to point systems, we will simplify a complex problem into a simpler problem. At the same time, we can prevent overestimation of interval estimates, the so-called "*Wrapping effect*", which is often found in interval calculations [26].

4.2 Algorithm

The main algorithm of this method includes two auxiliary algorithms.

1) To determine the tuple of coefficients (vector components) of the right-hand sides of point systems of equations having the form (15), we propose the following simple and effective auxiliary algorithm (Algorithm 2), which can be tested directly on the Python interpreter:

Algorithm 2 Selecting vertices c_i $(i = 1, 2, \ldots, 2^n)$, for point equations.

```
import numpy as np
n=int(input('Enter n= '))#Enter n
b=[0]*2*n
for i in range(2*n):
    i = str(i + 1)
    print("Enter array element " + i, end = " ")
    i = int(i)
    i = i - 1
    b[i] = int(input())
print("")
for i in range(len(b)):
    print(b[i], end = " ")
print("")
b = np.array(b).reshape(n, 2)#Reduction to interval form
index = []
for i in [bin(b)[2:].zfill(n)
          for b in range(2 ** n)]:
    index.append(list(map(int, list(i))))

for j in index:
    total = []
    for k in range(n):
        total.append(b[k][j[k]])
    print(total)
```

Here, to implement Algorithm 2, the boundaries of the intervals on the right side are entered sequentially as elements of a one-dimensional array.

Example 1 Let's show the work of Algorithm 2 for $n = 3$, $\boldsymbol{b}_1 = [13, 15]$, $\boldsymbol{b}_2 = [7, 8]$, $\boldsymbol{b}_3 = [11, 12]$:

$$\begin{array}{ll} c_1 = [13, 7, 11], & c_5 = [15, 7, 11], \\ c_2 = [13, 7, 12], & c_6 = [15, 7, 12], \\ c_3 = [13, 8, 11], & c_7 = [15, 8, 11], \\ c_4 = [13, 8, 12], & c_8 = [15, 8, 12]. \end{array} \tag{18}$$

Here, each tuple c_i defines the vector of the right side of a point systems of the form $F(x) = c_i$ $(i = 1, 2, \ldots, 8)$, for example, the first system for the tuple $c_1 = [13, 7, 11]$ is defined as

$$\begin{cases} F_1(x_1, x_2, x_3) = 13, \\ F_2(x_1, x_2, x_3) = 7, \\ F_3(x_1, x_2, x_3) = 11. \end{cases}$$

2) To localize solutions, we connect Algorithm 1. This algorithm creates lists of boxes that are guaranteed to contain solutions to the system, removing boxes that do not contain solutions. In this case, Algorithm 1 is run for each point system separately.

Finally, we present the general algorithm design for the proposed combined vertex method (Algorithm 3).

Algorithm 3 Combined vertex method for solving partially interval systems.

Require: Interval system of equations $F(x) = \boldsymbol{b}_i,\ i = 1, 2, \ldots, n$, Box $\boldsymbol{X} \in \mathbb{IR}^n$; The specified accuracy is $\varepsilon > 0$

Ensure: Outer interval estimate (vector-box) $\boldsymbol{X^*} \in \mathbb{IR}^n$ of the interval system of equations $F(x) = \boldsymbol{b}_i,\ i = 1, 2, \ldots, n$.

1: Create point systems $F(x) = c_i$ using Algorithm 2;
2: Determine the feasible region $\boldsymbol{X}^{(0)} \in \mathbb{IR}^n$ using Algorithm 1 for each point system;
3: Using one of the interval algorithms (for example, Newton, Krawczyk, Hansen-Sengupta, or others), find interval estimates of all point systems created in step `step` 1;
4: Find the minimums and maximums from formula (17), from the estimates obtained in `step` 3, and output the final results.

When applying iterative algorithms to solve point systems of equations (`step` 3), the criterion for stopping the iterative process is the condition $\texttt{dist}\left(\boldsymbol{x}^{(k)}, \boldsymbol{x}^{(k+1)}\right) < \varepsilon$, where ε is the required accuracy.

5 Numerical experiments

In this section we provide several examples showing the process of solving point and partially interval systems. First, Example 2 analyzes the operation of the interval methods of Newton and Krawczyk for point systems, comparing their results in terms of the permissible area of application and the number of iterations. Example 3 demonstrates the direct application of these methods to partially interval systems. Next, in Example 4, the performance of the proposed method is tested, comparing the numerical results on the quality of the found interval estimates, relative to the Krawczyk method. The last example 5 considers the application of the proposed method to one problem of electrical circuit analysis, where classical interval methods are not applicable.

All numerical experiments were carried out on an Intel® CoreTM i3-3217U laptop, `CPU 1.83 GHz, 4 GB DDR3 Memory`, the program code was written in GNU Octave v.4.2.1, using the "octave-interval v.3.2.0-3_amd64" software package [27]. This package complies with `IEEE Std 1788-2015` for interval arithmetic [28].

Example 2 Let us be given a system of nonlinear equation

$$\begin{cases} x_1^2 + x_2^2 = 1, \\ x_1 + x_2^2 = 0. \end{cases} \tag{19}$$

For this system we define the corresponding functions

$$\begin{cases} F_1(x_1, x_2) = x_1^2 + x_2^2 - 1, \\ F_2(x_1, x_2) = x_1 + x_2^2. \end{cases}$$

Table 1 Results for various initial data for system (19)

$\boldsymbol{X}^{(0)}$	Newton method			Krawchyk method		
	$k_{\mathcal{N}}$	$\boldsymbol{X}_1$	$\boldsymbol{X}_2$	$k_{\mathcal{K}}$	$\boldsymbol{X}_1$	$\boldsymbol{X}_2$
$[0, 1] \times [0, 1]$	-	NaN	NaN	9	[0.61803, 0.61804]	[0.78615, 0.78616]
$[0.1, 1] \times [0.1, 1]$	-	NaN	NaN	8	[0.61803, 0.61804]	[0.78615, 0.78616]
$[0.2, 1] \times [0.2, 1]$	-	NaN	NaN	7	[0.61803, 0.61804]	[0.78615, 0.78616]
$[0.3, 1] \times [0.3, 1]$	∞	[0.3, 1]	[0.3, 1]	6	[0.61803, 0.61804]	[0.78615, 0.78616]
$[0.4, 1] \times [0.4, 1]$	3	[0.61803, 0.61804]	[0.78615, 0.78616]	5	[0.61803, 0.61804]	[0.78615, 0.78616]
$[0.5, 1] \times [0.5, 1]$	3	[0.61803, 0.61804]	[0.78615, 0.78616]	5	[0.61803, 0.61804]	[0.78615, 0.78616]

The results of applying the interval methods of Newton (9) and Krawczyk (12) for system (19) are given in Table 1 (where $\boldsymbol{X}^{(0)}$ is the initial approximation, $k_{\mathcal{N}}$, $k_{\mathcal{K}}$ is the iteration number of the interval methods of Newton and Krawczyk, respectively).

From Table 1 it is clear that in the first three cases, Newton's interval method does not provide a solution, this is due to the fact that the interval Lipschitz matrix $\boldsymbol{L}$ contains singular point matrices and therefore the set of solutions is unlimited. Krawczyk's method converges more slowly than Newton's interval method, but provides a solution in areas where Newton's method is not applicable. Newton's method works with quadratic convergence when the width of the interval vector of the initial approximation is small enough. Both methods produce the same and more accurate interval estimates.

If we are only interested in the point solution, then we can take the midpoint of these interval estimates as

$$x_1 = \mathrm{mid}\boldsymbol{X}_1 = \frac{\underline{X}_1 + \overline{X}_1}{2} = 0.618035, \quad x_2 = \mathrm{mid}\boldsymbol{X}_2 = \frac{\underline{X}_2 + \overline{X}_2}{2} = 0.786155.$$

For graphical illustration, let us consider in detail the case with the initial approximation $\boldsymbol{X}^{(0)} = [0.5, 1] \times [0.5, 1]$, in which the results of the iterative calculation are given in Table 2.

Table 2 Numerical results with the initial approximation $\boldsymbol{X}^{(0)} = [0.5, 1] \times [0.5, 1]$ of system (19)

k	Newton's method		Krawchyk method	
	$\boldsymbol{X}_1^{(k)}$	$\boldsymbol{X}_2^{(k)}$	$\boldsymbol{X}_1^{(k)}$	$\boldsymbol{X}_2^{(k)}$
0	[0.5, 1]	[0.5, 1]	[0.5, 1]	[0.5, 1]
1	[0.5, 0.7643]	[0.68441, 0.89893]	[0.5, 0.77501]	[0.67499, 0.90834]
2	[0.61363, 0.62262]	[0.78319, 0.78926]	[0.57764, 0.65876]	[0.75858, 0.81397]
3	[0.61803, 0.61804]	[0.78615, 0.78616]	[0.61519, 0.62088]	[0.78424, 0.78807]
4			[0.61802, 0.61805]	[0.78614, 0.78617]
5			[0.61803, 0.61804]	[0.78615, 0.78616]

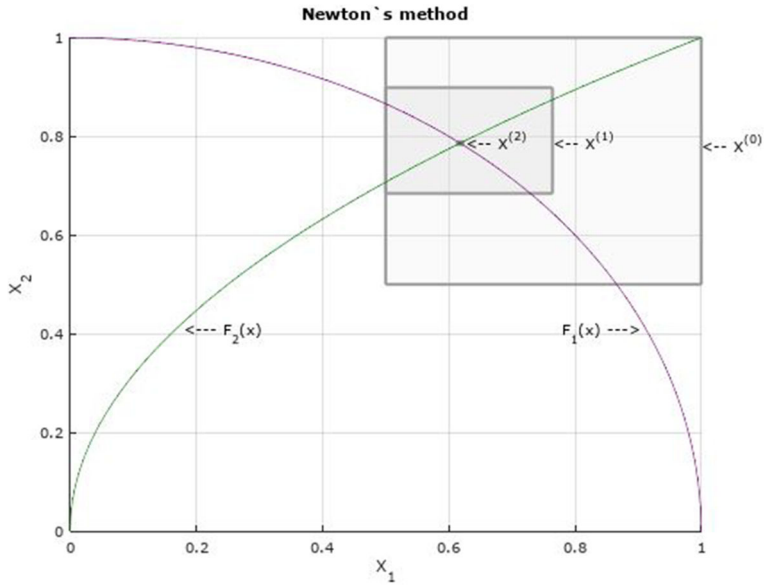

Fig. 3 Iterative process of Newton's method for system (19)

A graphical representation of the iterative process of interval methods is illustrated in Figs. 3 and 4, where nested sequences of rectangles are interval estimates in the corresponding iterations.

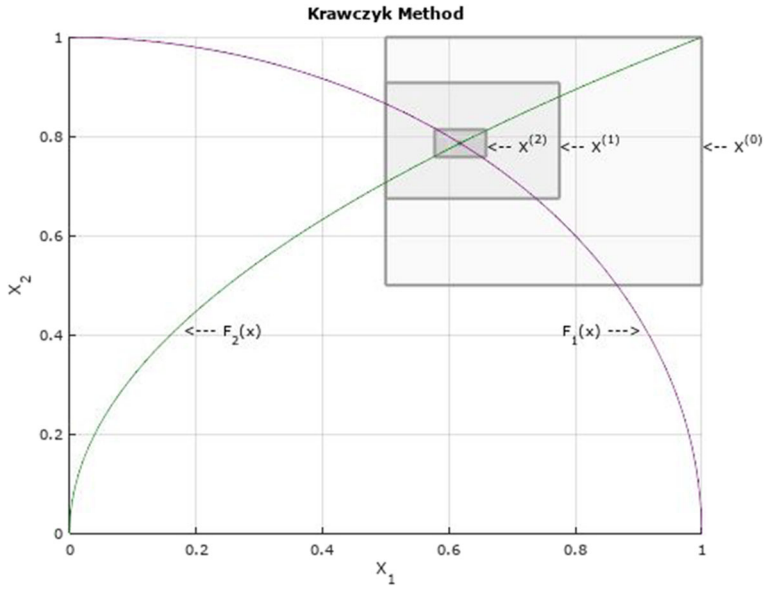

Fig. 4 Iterative process of the Krawczyk method for system (19)

Table 3 Numerical results with the initial approximation $\boldsymbol{X}^{(0)} = [0.5, 1] \times [0.5, 1]$ of system (20)

	Newton's method		Krawchyk method	
k	$\boldsymbol{X}_1^{(k)}$	$\boldsymbol{X}_2^{(k)}$	$\boldsymbol{X}_1^{(k)}$	$\boldsymbol{X}_2^{(k)}$
0	[0.5, 1]	[0.5, 1]	[0.5, 1]	[0.5, 1]
1	[0.5, 0.7664]	[0.6288, 0.9012]	[0.5, 0.8150]	[0.6483, 0.9350]
2	[0.53, 0.691]	[0.7407, 0.8356]	[0.5186, 0.7188]	[0.7198, 0.8534]
3	[0.5671, 0.6683]	[0.7536, 0.8182]	[0.5564, 0.6797]	[0.7464, 0.8259]
4	[0.5693, 0.6668]	[0.7550, 0.8173]	[0.5671, 0.6690]	[0.7535, 0.8188]
5	[0.5695, 0.6666]	[0.7551, 0.8172]	[0.5691, 0.6670]	[0.7549, 0.8174]
6			[0.5694, 0.6666]	[0.7551, 0.8172]
7			[0.5695, 0.6666]	[0.7551, 0.8172]
t (sec)	0.50		0.43	

The dimensions of the boxes $\boldsymbol{X}^{(3)}$, $\boldsymbol{X}^{(4)}$, $\boldsymbol{X}^{(5)}$ (see Table 2) are quite small, so they are almost invisible in the graphs (Figs. 3 and 4).

Example 3 Consider the system of (19), in partially interval form

$$\begin{cases} x_1^2 + x_2^2 = [1 - r, 1 + r], \\ x_1 + x_2^2 = 0, \end{cases} \tag{20}$$

where r is the radius of the created interval. Let, for this example, $r = 0.1$.

Using this fairly simple example, we discuss the work of Newton's and Krawczyk's interval methods in the process of solving interval systems.

Defining the relevant functions

$$\begin{cases} \boldsymbol{F}_1(x_1, x_2) = x_1^2 + x_2^2 - [0.9, 1.1], \\ F_2(x_1, x_2) = x_1 + x_2^2, \end{cases}$$

where $\boldsymbol{F}_1(x_1, x_2)$ is an interval-valued function, and $F_2(x_1, x_2)$ is a point (real) function. The results of the numerical experiment are presented in Table 3.

As the results from Table 3 show, when applying the Newton and Krawczyk interval methods for interval systems (20), the number of iterations increases compared to the application for point systems (19), but both methods always give the same final results. The process of approaching the interval solution for system (20) is presented graphically in Figs. 5 and 6.

From Figs. 5 and 6 it is clear that the interval-valued function $\boldsymbol{F}_1(x)$ forms families of point functions. The sequence of rectangles signifies approximations of the outer interval estimates of the united solution set, and the resulting interval estimate is highlighted in red.

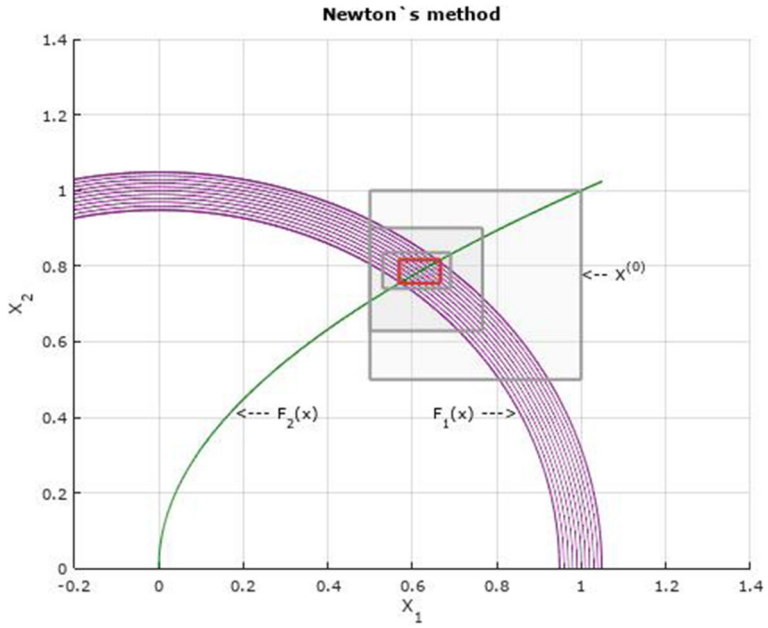

Fig. 5 Iterative process of Newton's method for system (20)

Fig. 6 Iterative process of the Krawczyk method for system (20)

Example 4 Now consider an interval system of third-order nonlinear algebraic equations

$$\begin{cases} x_1 + x_2^2 + x_3^2 = [11.9, 12.1], \\ x_1^2 + x_2 + x_3^2 = [13.8, 14.2], \\ x_1^2 + x_2^2 + x_3 = [7.8, 8.1]. \end{cases} \tag{21}$$

To create an iterative process, we select the initial approximation $\boldsymbol{X}^{(0)} = [1.5, 2.5] \times [0.5, 1.5] \times [2.5, 3.5]$, since the point (2; 1; 3) is satisfied by all equations in the system (21). In this case, the spectral radius is $\rho\big(|I - \Lambda \boldsymbol{L}|\big) \approx 1.2896 > 1$, which means convergence is not guaranteed.

Based on this example, we will conduct a comparative analysis of the results on the quality of interval estimates between the direct application of the Krawczyk method to solve system (21) and the proposed approach, which uses the Krawczyk method to solve the generated systems of point equations. To do this, we will consider two approaches.

First approach (direct application of the Krawczyk method) First, let us consider the case of direct application of the Krawczyk method for the interval system (21). We create the corresponding interval-valued functions $\boldsymbol{F} : \mathbb{IR}^3 \to \mathbb{IR}$:

$$\boldsymbol{F}(x) = \begin{pmatrix} \boldsymbol{F}_1(x_1, x_2, x_3) = x_1 + x_2^2 + x_3^2 - [11.9, 12.1], \\ \boldsymbol{F}_2(x_1, x_2, x_3) = x_1^2 + x_2 + x_3^2 - [13.8, 14.2], \\ \boldsymbol{F}_3(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3 - [7.8, 8.1]. \end{pmatrix}.$$

The interval Lipschitz matrix of the system is equal to:

$$\boldsymbol{L}(\boldsymbol{X}) = \begin{pmatrix} [1, 1] & 2 \cdot \boldsymbol{X}_2 & 2 \cdot \boldsymbol{X}_3 \\ 2 \cdot \boldsymbol{X}_1 & [1, 1] & 2 \cdot \boldsymbol{X}_3 \\ 2 \cdot \boldsymbol{X}_1 & 2 \cdot \boldsymbol{X}_2 & [1, 1] \end{pmatrix}.$$

We find the inverse matrix $\Lambda = \Big(\boldsymbol{L}\big(\mathrm{mid}(\boldsymbol{X})\big)\Big)^{-1}$. The first step of the process is decisive, if the box $\big(I - \Lambda \boldsymbol{L}\big)\big(\boldsymbol{X} - \mathrm{mid}(\boldsymbol{X})\big)$ from expression (11) decreases in size, then the process is successful. The results of the computational experiment are shown in Table 4.

From Table 4 you will notice that the values of $\boldsymbol{X}_2$ do not change, since $\rho\big(|I - \Lambda \boldsymbol{L}|\big) > 1$ is the same in all iterations.

So, as a result of direct application of Krawzcyk's method, we obtained the outer interval estimate $\boldsymbol{X}_{\mathcal{K}}$ ($\mathcal{K}$ - Krawzcyk):

$$\boldsymbol{X}_{\mathcal{K}} = \begin{pmatrix} \boldsymbol{X}_1 \\ \boldsymbol{X}_2 \\ \boldsymbol{X}_3 \end{pmatrix} = \begin{pmatrix} [1.6334, 2.3553] \\ [0.5, 1.5] \\ [2.7680, 3.2452] \end{pmatrix}. \tag{22}$$

Table 4 Results of direct application of the Krawczyk method for system (21)

k	$\boldsymbol{X}_1^{(k)}$	$\boldsymbol{X}_2^{(k)}$	$\boldsymbol{X}_3^{(k)}$
0	[1.5, 2.5]	[0.5, 1.5]	[2.5, 3.5]
1	[1.5, 2.5]	[0.5, 1.5]	[2.6358, 3.3774]
2	[1.5, 2.4901]	[0.5, 1.5]	[2.6791, 3.3341]
⋮	⋮	⋮	⋮
23	[1.6334, 2.3553]	[0.5, 1.5]	[2.7680, 3.2452]
t (sec)		3.31	

Second approach (application of the proposed method) To apply the proposed method, we divide system (21) into eight ($2^3 = 8$) point systems using Algorithm 2, as shown in Example 1. Therefore, to solve these point systems we use the Krawczyk method, since this method is more applicable than Newton's interval method.

Next, we present the interval estimates of the corresponding point systems of equations obtained by the Krawczyk method with the initial approximation $\boldsymbol{X}^{(0)} = [1.5, 2.5] \times [0.5, 1.5] \times [2.5, 3.5]$ (Table 5), where k is the total number of iterations, and $c = (c_1, c_2, c_3)^\top$ are the coefficients of the right side obtained using Algorithm 2, as shown in (18).

From the obtained local interval estimates for point systems, we find global interval estimates for the interval system (21):

$$\boldsymbol{X}_1 = [\underline{x}_1, \overline{x}_1] = \left[\min_{j=1}^{8}\{\underline{x}_{1j}\}, \max_{j=1}^{8}\{\overline{x}_{1j}\}\right] = [1.9170, 2.0693],$$

$$\boldsymbol{X}_2 = [\underline{x}_2, \overline{x}_2] = \left[\min_{j=1}^{8}\{\underline{x}_{2j}\}, \max_{j=1}^{8}\{\overline{x}_{2j}\}\right] = [0.7698, 1.1439],$$

$$\boldsymbol{X}_3 = [\underline{x}_3, \overline{x}_3] = \left[\min_{j=1}^{8}\{\underline{x}_{3j}\}, \max_{j=1}^{8}\{\overline{x}_{3j}\}\right] = [2.9565, 3.0570].$$

Table 5 Interval estimates of point systems of equations, compiled from the interval system (21)

k	$c = (c_1, c_2, c_3)^\top$	$\boldsymbol{X}_1$	$\boldsymbol{X}_2$	$\boldsymbol{X}_3$	t (sec.)
7	$(11.9, 13.8, 7.8)^\top$	[1.9599, 1.9600]	[0.9810, 0.9810]	[2.9962, 2.9963]	1.22
7	$(11.9, 13.8, 8.1)^\top$	[1.9170, 1.9171]	[1.0550, 1.0551]	[3.0115, 3.0116]	1.05
7	$(11.9, 14.2, 7.8)^\top$	[1.9949, 1.9950]	[1.0786, 1.0787]	[2.9565, 2.9566]	1.04
7	$(11.9, 14.2, 8.1)^\top$	[1.9541, 1.9542]	[1.1438, 1.1439]	[2.9727, 2.9728]	1.09
7	$(12.1, 13.8, 7.8)^\top$	[2.0403, 2.0404]	[0.7698, 0.7698]	[3.0441, 3.0442]	1.04
7	$(12.1, 13.8, 8.1)^\top$	[1.9959, 1.9960]	[0.8713, 0.8713]	[3.0569, 3.0570]	0.94
7	$(12.1, 14.2, 7.8)^\top$	[2.0692, 2.0693]	[0.9032, 0.9032]	[3.0025, 3.0026]	1.09
7	$(12.1, 14.2, 8.1)^\top$	[2.0280, 2.0281]	[0.9848, 0.9848]	[3.0169, 3.0170]	1.01

Thus, we obtained an interval estimate of $\boldsymbol{X}_{\mathcal{V}}$ ($\mathcal{V}$ - vertex) of system (21):

$$\boldsymbol{X}_{\mathcal{V}} = \begin{pmatrix} \boldsymbol{X}_1 \\ \boldsymbol{X}_2 \\ \boldsymbol{X}_3 \end{pmatrix} = \begin{pmatrix} [1.9170, 2.0693] \\ [0.7698, 1.1439] \\ [2.9565, 3.0570] \end{pmatrix}. \tag{23}$$

From the obtained numerical results (22) and (23), we note that the proposed method (second approach) gives better interval estimates than the direct application of interval iterative methods (first approach) for solving interval systems of nonlinear algebraic equations:

$$\boldsymbol{X}_{\mathcal{V}} = \begin{pmatrix} [1.9170, 2.0693] \\ [0.7698, 1.1439] \\ [2.9565, 3.0570] \end{pmatrix} \subset \begin{pmatrix} [1.6334, 2.3553] \\ [0.5, 1.5] \\ [2.7680, 3.2452] \end{pmatrix} = \boldsymbol{X}_{\mathcal{K}}.$$

Example 5 (Application) In this example, consider the problem of analyzing a known electrical circuit [29, 30], where the voltage transfer function (Butterworth) is defined as

$$T(s) = \frac{1/R_i}{2 + s(C_1 + C_3 + L_2/R_i) \cdot s^2 \cdot (L_2C_3 + C_1L_2/R_i)} + \frac{1/R_i}{C_1L_2C_3s^3} =$$

$$= \frac{1/R_i}{2 + a_1s + a_2s^2 + a_3s^3}.$$

By introducing the designations for the circuit parameters as $x_1 = C_1$, $x_2 = L_2$, $x_3 = C_3$, the coefficients a_1, a_2, a_3 are expressed by the equations:

$$\begin{aligned} a_1 &= x_1 + x_3 + x_2/R_i, \\ a_2 &= x_2x_3 + x_1x_2/R_i, \\ a_3 &= x_1x_2x_3. \end{aligned} \tag{24}$$

As shown in [30], at $R = 0.5$ we obtain a specific system of equations for calculating the circuit parameters

$$\begin{cases} x_1 + 2x_2 + x_3 = 6, \\ 2x_1x_2 + x_2x_3 = 6, \\ x_1x_2x_3 = 3. \end{cases} \tag{25}$$

System (25) has two point solutions $x_1 = (1, 1.5, 2)^\top$ and $x_2 = (3.261, 0.779, 1.181)^\top$.

Let us assume that the values of the coefficients a_1, a_2, a_3 in (24) vary within certain intervals

$$\begin{cases} x_1 + 2x_2 + x_3 = [5.8, 6.5], \\ 2x_1x_2 + x_2x_3 = [5.5, 6.4], \\ x_1x_2x_3 = [2.6, 3.5]. \end{cases} \tag{26}$$

For this example, the Newton and Krawczyk methods are not applicable, since these methods work with sufficiently small sizes of the initial box, and the region of the solution set of system (26) is larger than the size of the admissible region of these iterative methods, for which they can lead us to a convergent iterative process.

When applying the proposed method, step 2 of Algorithm 3 narrows the size of the initial box by discarding the part that does not contain a solution. As a result, we will obtain an admissible region in which Krawczyk's method will be applicable for each point system of type (15). For example, Krawczyk's method with initial box $\boldsymbol{X}^{(0)} = [3, 3.5] \times [0.5, 1] \times [1, 1.5]$ does not converge to a solution for the first point equation

$$\begin{cases} x_1 + 2x_2 + x_3 = 5.8, \\ 2x_1x_2 + x_2x_3 = 5.5, \\ x_1x_2x_3 = 2.6. \end{cases}$$

At the very first crushing (step 2 Algorithm 3), we get two sub-boxes $\boldsymbol{X}^{(0)} = \left(\boldsymbol{X}^{(0)}\right)' \bigcup \left(\boldsymbol{X}^{(0)}\right)'' = ([3, 3.25] \times [0.5, 1] \times [1, 1.5])' \cup ([3.25, 3.5] \times [0.5, 1] \times [1, 1.5])''$, which the first sub-box $([3, 3.25] \times [0.5, 1] \times [1, 1.5])'$ already advances the process towards a solution and in 7 iterations we get $\boldsymbol{X} = [3.2455, 3.2456] \times [0.7239, 0.7239] \times [1.1066, 1.1067]$, and discard the second half $([3.25, 3.5] \times [0.5, 1] \times [1, 1.5])''$, since there are no solutions in this box. Here we can choose a larger initial box $\boldsymbol{X}^{(0)}$, but the execution time of the algorithm increases significantly.

The numerical results when applying Algorithm 3 for system (26) are given in Table 6. Here we present the final result, omitting all intermediate data.

From Table 6 it is clear that we have found two partially intersecting parallelotopes that cover the set of solutions of the interval system (26), corresponding to point solutions of the system (25).

Partially interval systems often arise in the analysis of parameters of mathematical models, for example, if in system (25) the value of the parameter $\boldsymbol{x}_1 = [\underline{x}_1, \overline{x}_1]$ is given in intervals, and it is necessary to estimate the values of the remaining variables $\boldsymbol{x}_2$ and $\boldsymbol{x}_3$, then by transferring all $\boldsymbol{x}_1$ in the system equations to one side of the equality, we can create a new partially interval system. This transformation is useful when analyzing individual parameters of a mathematical model if they have an interval type of uncertainty.

Table 6 Numerical results for system (26)

$\boldsymbol{X}^{(0)}$	$\boldsymbol{X}_1$	$\boldsymbol{X}_2$	$\boldsymbol{X}_3$	t (sec.)
$[0, 5] \times [0, 2] \times [0, 4]$	[0.6404, 2.0117]	[1.0000, 1.8675]	[1.1446, 3.6109]	7.46
	[2.0117, 4.2991]	[0.5693, 1.1582]	[0.8987, 1.7475]	

6 Conclusion

The main objects considered in this article were the problems of estimating a set of solutions to point and interval systems of nonlinear equations and their solution methods. In order to study the advantages and disadvantages of the known classical interval methods for solving these problems, we first analyzed the Newton and Krawczyk methods from a theoretical and practical point of view. The results of the analysis showed that these methods work for fairly small sizes of the initial boxes, and in the general case are not suitable for solving systems with interval coefficients, even when used in combination with branch-and-prune type methods [12]. Based on the refined properties of these methods, a combined vertex method was proposed for estimating the united set of solutions of interval nonlinear systems. Due to the limited application of Newton's and Krawczyk's methods, we cannot directly apply them to solve systems with wider interval coefficients. Numerical experiments have shown that the proposed method is more efficient and gives more accurate estimates in feasible areas than the direct application of classical interval iterative methods. In the work, in particular, in the examples given, we limited ourselves to considering only partially interval systems with a small size in order to outline the essence of the idea of this approach. In future research, we intend to develop a theoretical rationale for this approach and consider larger-scale examples. The approaches considered and statements made in this work can be used by engineers when solving practical problems where systems of nonlinear equations arise. For example, when solving nodal nonlinear equations for calculating parameters in electrical circuits [31, 32], as well as when using the method of integrating differential equations [33] under conditions of interval data uncertainty.

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Code Availability Program codes of the proposed algorithms and other materialsrelated to this work can be obtained from the author uponreasonable request.

Declarations

Ethical approval The author declares that he has followed all current scientific practice.

Consent to participate The author approves of his participation in this work.

Consent for publication The author approves the publication of this research.

Competing interests The authors declare no competing interests.

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