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Modified projection method and strong convergence theorem for solving variational inequality problems with non-Lipschitz operators

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Abstract
In this paper, we introduce a modified projection method and give a strong convergence theorem for solving variational inequality problems in real Hilbert spaces. Under mild assumptions, there exists a novel line-search rule that makes the proposed algorithm suitable for non-Lipschitz continuous and pseudo-monotone operators. Compared with other known algorithms in numerical experiments, it is shown that our algorithm has better numerical performance.

Keywords Hilbert space · Line-search rule · Projection method · Pseudo-monotone operator · Strong convergence

Mathematics Subject Classification (2010) 47H05 · 47H09 · 47J25 · 65K15

1 Introduction

We research the classical variational inequality problem, which is defined as follows: find $x^* \in \mathscr{C}$ such that

$$\langle \mathscr{A} x^*, x - x^* \rangle \geq 0, \ \forall x \in \mathscr{C}, \tag{1}$$

where $\mathscr{C}$ is a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$, $\mathscr{A} : \mathcal{H} \to \mathcal{H}$ is a nonlinear operator. For simplicity, we denote the solution set of variational inequality problem (1) by $Sol(\mathscr{C}, \mathscr{A})$.

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The variational inequality problem plays an important role in nonlinear analysis research, which not only promotes the development of optimization problems, mathematical statistics, fixed point theory and other mathematical disciplines, but also has a very wide range of applications in engineering, mechanics, physics and economics. So far, many methods for solving variational inequalities have been produced. In this paper, we mainly study a class of projection methods.

It is well known that one of the most classical projection methods is the gradient projection method involving the strongly monotone and Lipschitz continuous operator. Due to its mandatory conditions, it was later improved by Korpelevich [1] into the following algorithm, which is often referred to as the extragradient algorithm:

$$\begin{cases} y_n = P_{\mathscr{C}}(x_n - \tau \mathscr{A} x_n), \\ x_{n+1} = P_{\mathscr{C}}(x_n - \tau \mathscr{A} y_n), \end{cases}$$

where $\tau \in (0, \frac{1}{L})$ and the operator $\mathscr{A}$ is monotone and L-Lipschitz continuous. The extragradient algorithm was first used to solve the saddle point problem, and was later applied to the variational inequality problem by scholars. Since then, many meaningful results have been produced. For example, Yao et al. [2, 3] constructed extragradient algorithms involving Lipschitz continuous and monotone operators for solving variational inequalities and fixed point problems. Vuong [4] proved the convergence of the extragradient algorithm under the pseudo-monotone condition, and the operator is Lipschitz continuous and sequentially weakly continuous.

However, it is worth noting that when the operator $\mathscr{A}$ is non-Lipschitz continuous or the Lipschitz constant L is not easy to calculate, the above extragradient algorithm will fail due to the difficulty in determining the value of τ. To overcome this shortcoming, the authors apply different techniques. Thong [5] used the line-search process so that the involved operators only need to satisfy uniform continuity. Tan et al. [6, 7] and Duvocelle et al. [8] introduced different self adaptive rules so that the algorithms involving Lipschitz continuous and pseudo-monotone operators do not need to calculate the Lipschitz constant. In addition, Iusem [9] combined the line-search rule with the extragradient algorithm and obtained the following algorithm:

$$\begin{cases} y_n = P_{\mathscr{C}}(x_n - \tau_n \mathscr{A} x_n), \\ x_{n+1} = P_{\mathscr{C}}(x_n - \lambda_n \mathscr{A} y_n), \end{cases}$$

where $\tau_n := \gamma l^{m_n}$, $m_n := \min\{m \in \mathbb{N} : \gamma l^m \|\mathscr{A} x_n - \mathscr{A} y_n\| \le \mu \|x_n - y_n\|\}$, $\mu, l \in (0, 1)$ and $\lambda_n := \frac{\langle \mathscr{A} y_n, x_n - y_n \rangle}{\|\mathscr{A} y_n\|^2}$. Under suitable assumptions, Iusem established a weak convergence theorem without the need for the Lipschitz continuity.

On the basis of the algorithm proposed by Iusem [9], Iusem et al. [10], Thong et al. [11, 12] and Xie et al. [13] have more or less improved the line-search rule and obtained the convergence of the corresponding algorithms. Specifically, the improved algorithm in [12] is as follows:

$$\begin{cases} y_n = P_{\mathscr{C}}(x_n - \tau_n \mathscr{A} x_n), \\ z_n = P_{\mathscr{C}}(x_n - \lambda_n \mathscr{A} y_n), \\ x_{n+1} = \beta_n x_0 + (1 - \beta_n) z_n, \end{cases}$$

where $\tau_n := \gamma l^{m_n}$, $m_n := \min\{m \in \mathbb{N} : \gamma l^m \langle \mathscr{A} x_n - \mathscr{A} y_n, x_n - y_n \rangle \leq \mu \|x_n - y_n\|^2\}$, $\mu, l \in (0, 1)$ and $\lambda_n := \frac{1-\mu}{\gamma} \frac{\|x_n - y_n\|^2}{\|\mathscr{A} y_n\|^2}$. They effectively modified the line-search rule, thereby increasing the range of choices for the sequence $\{\tau_n\}$. Under suitable assumptions, a strong convergence theorem is obtained by introducing Halpern-type iteration.

In addition, it is noted that during the operation of the extragradient algorithm, each iteration needs to calculate the projection on the feasible set $\mathscr{C}$ twice, but the generality of $\mathscr{C}$ will increase the computational complexity. To this end, the authors have proposed various methods to improve this situation. Vuong et al. [14] and Reich et al. [15] proposed different projection algorithms by using different line-search rules, and obtained the corresponding strong convergence theorems by combining Halpern iteration and viscosity iteration respectively. Censor et al. [16, 17] constructed a subgradient half-space $\{x \in \mathcal{H} : \langle x_n - \tau \mathscr{A} x_n - y_n, x - y_n \rangle \leq 0\}$ to replace the feasible set $\mathscr{C}$ in the second projection process in the extragradient algorithm, which accelerated the iteration rate of the algorithm. Tseng [18] directly omitted the second projection and used $y_n - \tau(\mathscr{A} y_n - \mathscr{A} x_n)$ to replace the computation of $P_{\mathscr{C}}(x_n - \tau \mathscr{A} y_n)$. Based on the algorithm proposed by Tseng, Reich et al. [19] and Yao et al. [20] add a single inertial term and a double inertial term respectively, and the convergence results of the related algorithms are obtained by combining self adaptive rule. In this paper, we mainly research such a projection-type algorithm modified by Vuong and Shehu [14]. The detailed algorithm is as follows:

Vuong and Shehu [14] introduced a half-space C_n in Algorithm 1 to replace the second-step projection in the iterative process, which is an interesting improvement and reduces the difficulty of projection computation. At the same time, they obtained the strong convergence theorem involving uniformly continuous, pseudo-monotone and sequentially weakly continuous operators by combining the line-search rule and Halpern-type iteration method. Based on the result of Algorithm 1, Reich et al. [15] made some improvements while speeding up the convergence of the following algorithm:

Reich et al. [15] proposed a different line-search rule than in Algorithm 1, thereby adjusting the half-space C_n. The operator $\mathscr{A}$ in Algorithm 2 is pseudo-monotone, uniformly continuous and satisfies $\|\mathscr{A} q\| \leq \liminf_{n\to\infty} \|\mathscr{A} x_n\|$ whenever $\{x_n\} \subset \mathscr{C}$ and $x_n \rightharpoonup q$. On the other hand, the introduced viscosity iteration method further accelerates the convergence process of Algorithm 2. Under the imposition of appropriate assumptions on the parameters, they obtained a strong convergence theorem.

Motivated and inspired by the above results, we propose a new algorithm for solving

Algorithm 1 Halpern-type projection method

Initialization: Given $\{\beta_n\} \subset (0, 1),\ l \in (0, 1),\ \mu \in (0, 1)$. Let $x_1 \in \mathscr{C}$ be arbitrary.
Iterative Steps: Given the current iterate x_n $(n \geq 1)$, calculate x_{n+1} as follows:
Step 1. Compute

$$z_n = P_{\mathscr{C}}(x_n - \mathscr{A}x_n)$$

and $r(x_n) := x_n - z_n$. If $r(x_n) = 0$, then stop; x_n belongs to $Sol(\mathscr{C}, \mathscr{A})$. Otherwise,
Step 2. Compute

$$y_n = x_n - \tau_n r(x_n),$$

where $\tau_n := l^{m_n}$ and m_n is the smallest non-negative integer m satisfying

$$\langle \mathscr{A}(x_n - l^m r(x_n)), r(x_n)\rangle \geq \frac{\mu}{2}\|r(x_n)\|^2.$$

Step 3. Compute

$$x_{n+1} = \beta_n x_1 + (1 - \beta_n)P_{C_n}(x_n),$$

where

$$C_n := \{x \in \mathscr{C} : h_n(x_n) \leq 0\}$$

and

$$h_n(x) := \langle \mathscr{A}y_n, x - y_n\rangle.$$

Set $n := n + 1$ and go to **Step 1**.

Algorithm 2 Viscosity projection method

Initialization: Given $\{\beta_n\} \subset (0, 1),\ \mu > 0,\ l \in (0, 1),\ \lambda \in (0, \frac{1}{\mu})$. Let $x_1 \in \mathscr{C}$ be arbitrary.
Iterative Steps: Given the current iterate x_n $(n \geq 1)$, calculate x_{n+1} as follows:
Step 1. Compute

$$z_n = P_{\mathscr{C}}(x_n - \lambda\mathscr{A}x_n)$$

and $r(x_n) := x_n - z_n$. If $r(x_n) = 0$, then stop; x_n belongs to $Sol(\mathscr{C}, \mathscr{A})$. Otherwise
Step 2. Compute

$$y_n = x_n - \tau_n r(x_n),$$

where $\tau_n := l^{m_n}$ and m_n is the smallest non-negative integer m satisfying

$$\langle \mathscr{A}x_n - \mathscr{A}(x_n - l^m r(x_n)), r(x_n)\rangle \leq \frac{\mu}{2}\|r(x_n)\|^2.$$

Step 3. Compute

$$x_{n+1} = \beta_n f(x_n) + (1 - \beta_n)P_{C_n}(x_n),$$

where

$$C_n := \{x \in \mathscr{C} : h_n(x_n) \leq 0\}$$

and

$$h_n(x) := \langle \mathscr{A}y_n, x - x_n\rangle + \frac{\tau_n}{2\lambda}\|r(x_n)\|^2.$$

Set $n := n + 1$ and go to **Step 1**.

variational inequalities with uniformly continuous pseudo-monotone operators. Precisely, we create a novel line-search rule to determine the value of the key sequence and make appropriate adjustments to C_n. In addition, we introduced inertial technology to speed up iteration efficiency. Finally, by comparing with known results in numerical experiments, it is confirmed that our proposed algorithm indeed has better behavior.

2 Preliminaries

In this section, we recall some basic concepts and facts.

Let $\mathcal{H}$ be a real Hilbert space. Then

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle \tag{2}$$

and

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2, \tag{3}$$

for every $x, y \in \mathcal{H}$ and $\alpha \in \mathbb{R}$.

Let $\mathscr{C}$ be a nonempty subset of $\mathcal{H}$. Then an operator $\mathscr{A}: \mathscr{C} \to \mathcal{H}$ is called

(a) L-Lipschitz continuous with $L > 0$ if

$$\|\mathscr{A}x - \mathscr{A}y\| \leq L\|x-y\|, \ \forall x, y \in \mathscr{C}. \tag{4}$$

If $L \in (0, 1)$, $\mathscr{A}$ is called a contraction.

(b) monotone if

$$\langle \mathscr{A}x - \mathscr{A}y, x-y\rangle \geq 0, \ \forall x, y \in \mathscr{C}. \tag{5}$$

(c) pseudo-monotone if

$$\langle \mathscr{A}x, y-x\rangle \geq 0 \Rightarrow \langle \mathscr{A}y, y-x\rangle \geq 0, \ \forall x, y \in \mathscr{C}. \tag{6}$$

(d) sequentially weakly continuous if for each sequence $\{x_n\} \subset \mathscr{C}$ we have: $x_n \rightharpoonup x$ implies $\mathscr{A}x_n \rightharpoonup \mathscr{A}x$ as $n \to \infty$.

For any point $x \in \mathcal{H}$, it is obvious that there exists a unique nearest point in $\mathscr{C}$, denoted by $P_{\mathscr{C}}x$ satisfying $\|x - P_{\mathscr{C}}x\| \leq \|x-y\|$, $\forall y \in \mathscr{C}$. $P_{\mathscr{C}}$ is called the metric projection of $\mathcal{H}$ onto $\mathscr{C}$. The projection formula will be applied to numerical experiments (Sect. 4). The projection of x onto a half-space $\mathscr{C}_{u,v} = \{x : \langle u, x\rangle \leq v\}$ is computed by

$$P_{\mathscr{C}_{u,v}} = x - \max\{[\langle u, x\rangle - v]/\|u\|^2, 0\}u.$$

Throughout this paper, let $\mathscr{C}$ be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. Next, list some lemmas that will be needed later.

Lemma 1 ([21]) *Given $x \in \mathcal{H}$ and $z \in \mathscr{C}$. Then*

$$z = P_{\mathscr{C}}x \Leftrightarrow \langle x-z, z-y\rangle \geq 0, \ \forall y \in \mathscr{C}.$$

Lemma 2 ([21]) *Given $x \in \mathcal{H}$. Then*

(1) $\|P_{\mathscr{C}}x - P_{\mathscr{C}}y\|^2 \leq \langle P_{\mathscr{C}}x - P_{\mathscr{C}}y, x-y\rangle, \ \forall y \in \mathcal{H}$;

(2) $\|P_{\mathscr{C}}x - y\|^2 \leq \|x-y\|^2 - \|x - P_{\mathscr{C}}x\|^2, \ \forall y \in \mathscr{C}$.

Lemma 3 ([22]) *Given $x \in \mathcal{H}$ and $\alpha \geq \beta > 0$. Then*

(1) $\|x - P_{\mathscr{C}}(x - \beta\mathscr{A}x)\| \leq \|x - P_{\mathscr{C}}(x - \alpha\mathscr{A}x)\|$;

(2) $\dfrac{\|x - P_{\mathscr{C}}(x - \alpha\mathscr{A}x)\|}{\alpha} \leq \dfrac{\|x - P_{\mathscr{C}}(x - \beta\mathscr{A}x)\|}{\beta}$.

Lemma 4 ([23]) *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces. Suppose $\mathscr{A} : \mathcal{H}_1 \to \mathcal{H}_2$ is uniformly continuous on bounded subsets of $\mathcal{H}_1$ and M is a bounded subset of $\mathcal{H}_1$. Then $\mathscr{A}(M)$ (the image of M under $\mathscr{A}$) is bounded.*

Lemma 5 ([24]) *Let $\mathscr{A} : \mathscr{C} \to \mathcal{H}$ be pseudo-monotone and continuous. Then, x^* belongs to $Sol(\mathscr{C}, \mathscr{A})$ if and only if*

$$\langle \mathscr{A}x, x - x^* \rangle \geq 0, \ \forall x \in \mathscr{C}.$$

Lemma 6 ([25]) *Let h be a real-valued function on $\mathcal{H}$ and defined $K := \{x \in \mathscr{C} : h(x) \leq 0\}$. If K is nonempty and h is Lipschitz continuous on $\mathscr{C}$ with modulus $\theta > 0$, then*

$$dist(x, K) \geq \theta^{-1} \max\{h(x), 0\}, \ \forall x \in \mathscr{C},$$

where dist(x,K) denotes the distance of x to K.

Lemma 7 ([26]) *Let $\{d_n\}$ be a sequence of non-negative real number such that there exists a subsequence $\{d_{n_j}\} \subset \{d_n\}$ such that $d_{n_j} < d_{n_j+1}$ for all $j \in \mathbb{N}$. Then there exists a non-decreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $\lim_{k\to\infty} m_k = \infty$ and the following properties are satisfied by all (sufficiently large) number $k \in \mathbb{N}$:*

$$d_{m_k} \leq d_{m_k+1}, \quad d_k \leq d_{m_k+1}.$$

In fact, $m_k = \max\{n \in \mathbb{N} : d_n < d_{n+1}, n \leq k\}$.

Lemma 8 ([27]) *Let $\{d_n\}$ be a sequence of non-negative real numbers such that*

$$d_{n+1} \leq (1 - a_n)d_n + a_n b_n, \ \forall n \geq 0,$$

where $\{a_n\} \subset (0, 1)$ and $\{b_n\}$ is a real sequence such that
(a) $\sum_{n=1}^{\infty} a_n = \infty$;
(b) $\limsup_{n\to\infty} b_n \leq 0$.
Then $\lim_{n\to\infty} d_n = 0$.

Lemma 9 ([12]) *Let $\mathscr{C}$ be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$ and $\mathscr{A} : \mathscr{C} \to \mathcal{H}$ be pseudo-monotone, sequentially weakly continuous and uniformly continuous on the bounded subsets of $\mathscr{C}$. There exists a subsequence $\{w_{n_k}\} \subset \{w_n\}$ such that $w_{n_k} \rightharpoonup q \in \mathscr{C}$ and $\lim_{k\to\infty} \|w_{n_k} - P_{\mathscr{C}}(w_{n_k} - \tau_{n_k}\mathscr{A}w_{n_k})\| = 0$, where τ_n is a positive sequence. If*

$$\liminf_{k\to\infty} \langle \mathscr{A}w_{n_k}, x - w_{n_k} \rangle \geq 0, \ \forall x \in \mathscr{C}.$$

Then $q \in Sol(\mathscr{C}, \mathscr{A})$.

3 Main results

3.1 Strong convergence

In this section, we propose an improved projection-type method for solving variational inequality problems in Hilbert spaces. First, we give the following conditions:

(C1) The feasible set $\mathscr{C}$ is a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. The solution set $Sol(\mathscr{C}, \mathscr{A})$ is nonempty.

(C2) The operator $\mathscr{A} : \mathscr{C} \to \mathcal{H}$ is uniformly continuous, pseudo-monotone and sequentially weakly continuous on $\mathscr{C}$.

(C3) Let $f : \mathscr{C} \to \mathscr{C}$ be a contraction mapping with a coefficient $\rho \in [0, 1)$ and β_n be a sequence in $(0, 1)$ such that

$$\lim_{n\to\infty} \beta_n = 0, \ \sum_{n=1}^{\infty} \beta_n = \infty.$$

The sequence $\{\alpha_n\}$ satisfies $\lim_{n\to\infty} \frac{\alpha_n}{\beta_n} = 0$.

Next, we introduce our new algorithm.

Algorithm 3 Modified viscosity projection method

Initialization: Let $\mu \in (0, 1], \ l \in (0, 1), \ \gamma > 0$ and $x_1 \in \mathscr{C}$ be arbitrary.
Iterative steps: Given the current iterate $x_n \ (n \geq 1)$.
Step 1. Evaluate

$$w_n = (1 - \alpha_n)x_n + \alpha_n x_1.$$

Step 2. Compute

$$z_n = P_{\mathscr{C}}(w_n - \tau_n \mathscr{A} w_n),$$
$$y_n = w_n - \tau_n r(w_n),$$

where $\tau_n := \gamma l^{m_n}$ and m_n is the smallest non-negative integer m satisfying

$$\gamma l^m \langle \mathscr{A} w_n - \mathscr{A}(w_n - \gamma l^m r(w_n)), r(w_n)\rangle \leq \frac{\mu}{2}\|r(w_n)\|^2, \tag{7}$$

and $r(w_n) := w_n - z_n$. If $r(w_n) = 0$: Stop. Otherwise, go to **Step 3**.
Step 3. Calculate

$$x_{n+1} = \beta_n f(w_n) + (1 - \beta_n) P_{C_n}(w_n),$$

where

$$C_n := \{x \in \mathscr{C} : h_n(w_n) \leq 0\}$$

and

$$h_n(x) = \langle \mathscr{A} y_n, x - w_n\rangle + \frac{\mu}{2}\|r(w_n)\|^2. \tag{8}$$

Set $n := n + 1$ and return to **Step 1**.

Remark 1 1. Since $\mathscr{C}$ is a convex set, it can be deduced that all iterates $\{x_n\}$, $\{w_n\}$, $\{y_n\}$ and $\{z_n\}$ generated by Algorithm 3 belong to $\mathscr{C}$. Therefore, the operator $\mathscr{A}$ needs to be defined only on $\mathscr{C}$, rather than necessarily on the entire space $\mathcal{H}$.

2. The operator $\mathscr{A}$ is pseudo-monotone and uniformly continuous, which allows us to prove the strong convergence theorem without Lipschitz prior knowledge.
3. The sequence $\{\alpha_n\}$ could be chosen such that

$$\alpha_n = \begin{cases} \min\{\frac{\theta_n}{\|x_1 - x_n\|}, \frac{\alpha}{2}\}, & \text{if } x_n \neq x_1, \\ \frac{\alpha}{2}, & \text{otherwise }, \end{cases}$$

   where α is a constant such that $0 \leq \alpha < 1$ and $\{\theta_n\}$ is a positive sequence such that $\lim_{n\to\infty} \frac{\theta_n}{\beta_n} = 0$. At this time, it is easy to see that $\lim_{n\to\infty} \frac{\alpha_n}{\beta_n} = 0$. In addition, if $\alpha = 0$, then $w_n = x_n$, this is a trivial case. In the following text we will mainly study non-trivial situations.
4. Compared to Algorithms 1 and 2, Algorithm 3 introduces a step size rule with parameter $\gamma > 0$ in (7), which allows the trial step τ_n to start from a value other than 1 in each outer iteration. This modification makes sense because there are cases where a larger step size $\tau_n > 1$ may be acceptable (resulting in faster convergence), and cases where $\tau_n < 1$ may be smaller (in which case it is beneficial to choose $\gamma < 1$ to avoid unnecessary evaluations of $\mathscr{A}$ in the inner loop). In addition, during the inner loop, we initially set $m = 0$, which gives $\tau_n = \gamma$. We then substitute this value into (7) to see if it satisfies the condition. If it does, we output the result, otherwise we set $m = 1$ and continue iterating until the condition in (7) is satisfied. Once satisfied, we output the corresponding y_n value and continue with further iterations.
5. In Algorithm 2, $h_n(x)$ is defined as $h_n(x) = \langle \mathscr{A} y_n, x - x_n\rangle + \dfrac{\tau_n}{2\lambda}\|r(x_n)\|^2$, where $\tau_n\|r(x_n)\|^2 \to 0$ is a crucial assumption for proving $r(x_n) \to 0$. However, in the original proof, it was necessary to consider two separate cases: $\liminf_{n\to\infty} \tau_n > 0$ and $\liminf_{n\to\infty} \tau_n = 0$, as detailed in [15, Lemma 3.5]. To streamline the proof, we have made an improvement to $h_n(x)$ in (8). This adjustment eliminates the need to consider separate cases and leads directly to the conclusion, thus simplifying the overall proof.

Lemma 10 *Assume that conditions (C1-C3) hold. Then the line-search rule* (7) *is well defined.*

Proof If $w_n \in Sol(\mathscr{C}, \mathscr{A})$, then $w_n = P_{\mathscr{C}}(w_n - \gamma\mathscr{A} w_n)$, thus (7) holds with $m_n = 0$. Next, we suppose that $w_n \notin Sol(\mathscr{C}, \mathscr{A})$ and assume the contrary. That is, for all m we have

$$\gamma l^m \langle \mathscr{A} w_n - \mathscr{A}(w_n - \gamma l^m r(w_n)), r(w_n)\rangle > \frac{\mu}{2}\|r(w_n)\|^2. \tag{9}$$

By the Cauchy-Schwarz inequality, then

$$\begin{aligned} &\gamma l^m \langle \mathscr{A} w_n - \mathscr{A}(w_n - \gamma l^m r(w_n)), r(w_n)\rangle \\ &\leq \gamma l^m \|\mathscr{A} w_n - \mathscr{A}(w_n - \gamma l^m r(w_n))\|\|r(w_n)\|. \end{aligned} \tag{10}$$

Combining (9) and (10), we have

$$\gamma l^m \|\mathscr{A} w_n - \mathscr{A}(w_n - \gamma l^m r(w_n))\| > \frac{\mu}{2}\|w_n - P_{\mathscr{C}}(w_n - \gamma l^m \mathscr{A} w_n)\|.$$

It follows that

$$\|\mathscr{A} w_n - \mathscr{A}(w_n - \gamma l^m r(w_n))\| > \frac{\mu}{2} \frac{\|w_n - P_{\mathscr{C}}(w_n - \gamma l^m \mathscr{A} w_n)\|}{\gamma l^m}. \tag{11}$$

Since $w_n \in \mathscr{C}$, $P_{\mathscr{C}}$ is continuous and $\mathscr{A}$ is uniformly continuous, we obtain

$$\lim_{m\to\infty} \|w_n - P_{\mathscr{C}}(w_n - \gamma l^m \mathscr{A} w_n)\| = 0, \tag{12}$$

and

$$\lim_{m\to\infty} \|\mathscr{A} w_n - \mathscr{A}(w_n - \gamma l^m r(w_n))\| = 0.$$

Noticing (11), we get

$$\lim_{m\to\infty} \frac{\|w_n - P_{\mathscr{C}}(w_n - \gamma l^m \mathscr{A} w_n)\|}{\gamma l^m} = 0. \tag{13}$$

Let $z_m = P_{\mathscr{C}}(w_n - \gamma l^m \mathscr{A} w_n)$. By Lemma 1, we have

$$\langle z_m - w_n + \gamma l^m \mathscr{A} w_n, x - z_m\rangle \geq 0, \ \forall x \in \mathscr{C}.$$

That is

$$\langle \frac{z_m - w_n}{\gamma l^m}, x - z_m\rangle + \langle \mathscr{A} w_n, x - z_m\rangle \geq 0, \ \forall x \in \mathscr{C}.$$

Consequently

$$\langle \frac{z_m - w_n}{\gamma l^m}, x - z_m\rangle + \langle \mathscr{A} w_n, x - w_n\rangle + \langle \mathscr{A} w_n, w_n - z_m\rangle \geq 0, \ \forall x \in \mathscr{C}. \tag{14}$$

Taking $m \to \infty$ in (14) and using (12) and (13), we obtain

$$\langle \mathscr{A} w_n, x - w_n\rangle \geq 0, \ \forall x \in \mathscr{C},$$

it follows that $w_n \in Sol(\mathscr{C}, \mathscr{A})$. This contradiction implies that (7) is well defined. □

Lemma 11 *Assume that conditions (C1-C3) hold. Let the function h_n be defined by (8) and $p \in Sol(\mathscr{C}, \mathscr{A})$. Then $h_n(w_n) = \frac{\mu}{2}\|r(w_n)\|^2$ and $h_n(p) \leq 0$. Particularly, $h_n(w_n) > 0$ whenever $r(w_n) \neq 0$.*

Proof According to (8), $h_n(w_n) = \frac{\mu}{2}\|r(w_n)\|^2$ is obvious. On the other hand, from the pseudomonotonicity of $\mathscr{A}$ and $p \in Sol(\mathscr{C}, \mathscr{A})$, we have $\langle \mathscr{A} y_n, y_n - p\rangle \geq 0$ and

$$\begin{aligned} h_n(p) &= \langle \mathscr{A} y_n, p - w_n\rangle + \frac{\mu}{2}\|r(w_n)\|^2 \\ &= \langle \mathscr{A} y_n, p - y_n\rangle + \langle \mathscr{A} y_n, y_n - w_n\rangle + \frac{\mu}{2}\|r(w_n)\|^2 \\ &\leq -\tau_n \langle \mathscr{A} y_n, r(w_n)\rangle + \frac{\mu}{2}\|r(w_n)\|^2. \end{aligned} \tag{15}$$

From the rule (7) and the definition of $\{y_n\}$, we obtain

$$\tau_n \langle \mathscr{A} w_n - \mathscr{A} y_n, r(w_n)\rangle \leq \frac{\mu}{2}\|r(w_n)\|^2,$$

that is

$$\tau_n \langle \mathscr{A} y_n, r(w_n)\rangle \geq \tau_n \langle \mathscr{A} w_n, r(w_n)\rangle - \frac{\mu}{2}\|r(w_n)\|^2. \tag{16}$$

According to Lemma 2 (1), we can conclude that

$$\|w_n - P_{\mathscr{C}}(w_n - \tau_n \mathscr{A} w_n)\|^2 \leq \tau_n \langle \mathscr{A} w_n, w_n - P_{\mathscr{C}}(w_n - \tau_n \mathscr{A} w_n)\rangle,$$

or equivalently

$$\tau_n \langle \mathscr{A} w_n, r(w_n)\rangle \geq \|r(w_n)\|^2. \tag{17}$$

Substituting (16) and (17) into (15), we get that

$$h_n(p) \leq -(1-\mu)\|r(w_n)\|^2.$$

Since $\mu \in (0, 1]$, then $h_n(p) \leq 0$. □

Lemma 12 *Assume that conditions (C1-C3) hold. Let $\{w_n\}$ be a sequence generated by Algorithm 3. If there exists a subsequence $\{w_{n_k}\} \subset \{w_n\}$ converges weakly to $q \in \mathscr{C}$ and $\lim_{k\to\infty} \|w_{n_k} - z_{n_k}\| = 0$, then $q \in Sol(\mathscr{C}, \mathscr{A})$.*

Proof Since $z_{n_k} = P_{\mathscr{C}}(w_{n_k} - \tau_{n_k} \mathscr{A} w_{n_k})$, we have

$$\langle w_{n_k} - \tau_{n_k} \mathscr{A} w_{n_k} - z_{n_k}, x - z_{n_k}\rangle \leq 0, \ \forall x \in \mathscr{C},$$

or equivalently

$$\frac{1}{\tau_{n_k}} \langle w_{n_k} - z_{n_k}, x - z_{n_k}\rangle \leq \langle \mathscr{A} w_{n_k}, x - z_{n_k}\rangle, \ \forall x \in \mathscr{C}.$$

Consequently

$$\frac{1}{\tau_{n_k}} \langle w_{n_k} - z_{n_k}, x - z_{n_k}\rangle + \langle \mathscr{A} w_{n_k}, z_{n_k} - w_{n_k}\rangle \leq \langle \mathscr{A} w_{n_k}, x - w_{n_k}\rangle, \ \forall x \in \mathscr{C}. \tag{18}$$

Next, we prove that

$$\liminf_{k\to\infty} \langle \mathscr{A} w_{n_k}, x - w_{n_k}\rangle \geq 0, \ \forall x \in \mathscr{C}. \tag{19}$$

We consider the following two possible cases.

Case 1 Suppose that $\liminf_{k\to\infty} \tau_{n_k} > 0$. Since $w_{n_k} \rightharpoonup q \in \mathscr{C}$, then $\{w_{n_k}\}$ is bounded. As $\mathscr{A}$ is uniformly continuous on bounded subsets of $\mathscr{C}$, by Lemma 4, we get that $\{\mathscr{A} w_{n_k}\}$ is bounded. Taking $k \to \infty$ in (18), since $\|w_{n_k} - z_{n_k}\| \to 0$, we get

$$\liminf_{k\to\infty} \langle \mathscr{A} w_{n_k}, x - w_{n_k} \rangle \geq 0.$$

Case 2 Assume that $\liminf_{k\to\infty} \tau_{n_k} = 0$. Let $s_{n_k} = P_{\mathscr{C}}(w_{n_k} - \tau_{n_k} l^{-1} \mathscr{A} w_{n_k})$, we have $\tau_{n_k} l^{-1} > \tau_{n_k}$. Applying Lemma 3, we obtain

$$\frac{\|w_{n_k} - P_{\mathscr{C}}(w_{n_k} - \tau_{n_k} l^{-1} \mathscr{A} w_{n_k})\|}{\tau_{n_k} l^{-1}} \leq \frac{\|w_{n_k} - P_{\mathscr{C}}(w_{n_k} - \tau_{n_k} \mathscr{A} w_{n_k})\|}{\tau_{n_k}},$$

that is

$$\|w_{n_k} - s_{n_k}\| \leq \frac{1}{l} \|w_{n_k} - z_{n_k}\| \to 0 \ (k \to \infty). \tag{20}$$

Therefore, $s_{n_k} \rightharpoonup q \in \mathscr{C}$, it folllows that $\{s_{n_k}\}$ is bounded. In addition,

$$\|\mathscr{A} w_{n_k} - \mathscr{A}(w_{n_k} - \tau_{n_k} l^{-1} r(w_{n_k}))\| \to 0 \ (k \to \infty). \tag{21}$$

By $\tau_{n_k} l^{-1} > \tau_{n_k}$, we know that $\tau_{n_k} l^{-1}$ does not satisfy (7), owing to (11), then

$$\|\mathscr{A} w_{n_k} - \mathscr{A}(w_{n_k} - \tau_{n_k} l^{-1} r(w_{n_k}))\| > \frac{\mu}{2} \frac{\|w_{n_k} - P_{\mathscr{C}}(w_{n_k} - \tau_{n_k} l^{-1} \mathscr{A} w_{n_k})\|}{\tau_{n_k} l^{-1}},$$

which implies that

$$\lim_{k\to\infty} \frac{\|w_{n_k} - s_{n_k}\|}{\tau_{n_k} l^{-1}} = 0. \tag{22}$$

Furthermore, it follows from the definition of $\{s_{n_k}\}$ and Lemma 1 that

$$\langle w_{n_k} - \tau_{n_k} l^{-1} \mathscr{A} w_{n_k} - s_{n_k}, x - s_{n_k} \rangle \leq 0, \ \forall x \in \mathscr{C}.$$

It follows that

$$\frac{1}{\tau_{n_k} l^{-1}} \langle w_{n_k} - s_{n_k}, x - s_{n_k} \rangle + \langle \mathscr{A} w_{n_k}, s_{n_k} - w_{n_k} \rangle \leq \langle \mathscr{A} w_{n_k}, x - w_{n_k} \rangle, \ \forall x \in \mathscr{C}. \tag{23}$$

Taking $k \to \infty$ in (23), we get

$$\liminf_{k\to\infty} \langle \mathscr{A} w_{n_k}, x - w_{n_k} \rangle \geq 0, \ \forall x \in \mathscr{C}.$$

By Lemma 9, $q \in Sol(\mathscr{C}, \mathscr{A})$ and the proof is completed. □

Theorem 1 *Assume that conditions (C1-C3) hold and the sequence* $\{\alpha_n\}$ *is chosen such that*

$$\lim_{n\to\infty}\frac{\alpha_n}{\beta_n}\|x_1-x_n\|=0.$$

Then the sequence $\{x_n\}$ *generated by Algorithm 3 converges strongly to an element* $p\in Sol(\mathscr{C},\mathscr{A})$*, where* $p=P_{Sol(\mathscr{C},\mathscr{A})}\circ f(p)$*.*

Proof We divide the proof into five claims.

Claim 1. We prove that the $\{x_n\}$ is bounded. Indeed, let $s_n:=P_{C_n}(w_n)$, then

$$\begin{aligned}\|s_n-p\|^2=\|P_{C_n}(w_n)-p\|^2&\le\|w_n-p\|^2-\|P_{C_n}(w_n)-w_n\|^2\\&=\|w_n-p\|^2-dist^2(w_n,C_n)\\&\le\|w_n-p\|^2.\end{aligned}$$

That is

$$\|s_n-p\|\le\|w_n-p\|. \tag{24}$$

Thus

$$\begin{aligned}\|x_{n+1}-p\|&=\|\beta_nf(w_n)+(1-\beta_n)s_n-p\|\\&=\|\beta_n(f(w_n)-p)+(1-\beta_n)(s_n-p)\|\\&\le\beta_n\|f(w_n)-p\|+(1-\beta_n)\|s_n-p\|\\&\le\beta_n\|f(w_n)-f(p)\|+\beta_n\|f(p)-p\|+(1-\beta_n)\|s_n-p\|\\&\le\beta_n\rho\|w_n-p\|+\beta_n\|f(p)-p\|+(1-\beta_n)\|w_n-p\|\\&=(1-\beta_n(1-\rho))\|w_n-p\|+\beta_n\|f(p)-p\|.\end{aligned} \tag{25}$$

It follows from the definition of $\{w_n\}$ that

$$\begin{aligned}\|w_n-p\|&=\|(1-\alpha_n)x_n+\alpha_nx_1-p\|\\&\le\|x_n-p\|+\alpha_n\|x_1-x_n\|\\&\le\|x_n-p\|+\beta_n\cdot\frac{\alpha_n}{\beta_n}\|x_1-x_n\|.\end{aligned} \tag{26}$$

Since

$$\lim_{n\to\infty}\frac{\alpha_n}{\beta_n}\|x_1-x_n\|=0,$$

there exists $N\in\mathbb{N}$ and a constant $M_1>0$ such that

$$\frac{\alpha_n}{\beta_n}\|x_1-x_n\|\le M_1,\ \forall n\ge N. \tag{27}$$

Combining (26) and (27), we have

$$\|w_n-p\|\le\|x_n-p\|+\beta_nM_1. \tag{28}$$

Substituting (28) into (25), we get

$$\begin{aligned}
\|x_{n+1}-p\| &\le (1-\beta_n(1-\rho))\|x_n-p\|+\beta_n M_1+\beta_n\|f(p)-p\| \\
&\le (1-\beta_n(1-\rho))\|x_n-p\|+\beta_n(1-\rho)\frac{\|f(p)-p\|+M_1}{1-\rho} \\
&\le \max\{\|x_n-p\|, \frac{\|f(p)-p\|+M_1}{1-\rho}\} \\
&\le \cdots \\
&\le \max\{\|x_N-p\|, \frac{\|f(p)-p\|+M_1}{1-\rho}\}.
\end{aligned}$$

This implies that the sequence $\{x_n\}$ is bounded. Consequently, the sequences $\{y_n\}$, $\{f(w_n)\}$ and $\{\mathscr{A}y_n\}$ are bounded too.

Claim 2. We prove that

$$\|s_n-w_n\|^2 \le \|x_n-p\|^2-\|x_{n+1}-p\|^2+\beta_n M_2+2\beta_n\langle f(w_n)-p, x_{n+1}-p\rangle,$$

for some $M_2>0$. Ineeed, from (2) and (25), we have

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|\beta_n(f(w_n)-p)+(1-\beta_n)(s_n-p)\|^2 \\
&\le (1-\beta_n)\|s_n-p\|^2+2\beta_n\langle f(w_n)-p, x_{n+1}-p\rangle \\
&\le \|s_n-p\|^2+2\beta_n\langle f(w_n)-p, x_{n+1}-p\rangle \\
&= \|P_{C_n}(w_n)-p\|^2+2\beta_n\langle f(w_n)-p, x_{n+1}-p\rangle \\
&\le \|w_n-p\|^2-\|s_n-w_n\|^2+2\beta_n\langle f(w_n)-p, x_{n+1}-p\rangle. \qquad (29)
\end{aligned}$$

Using (28), we get

$$\begin{aligned}
\|w_n-p\|^2 &\le (\|x_n-p\|+\beta_n M_1)^2 \\
&= \|x_n-p\|^2+\beta_n(2M_1\|x_n-p\|+\beta_n M_1^2), \qquad (30)
\end{aligned}$$

Substituting (30) into (29), then

$$\|x_{n+1}-p\|^2 \le \|x_n-p\|^2-\|s_n-w_n\|^2+\beta_n M_2+2\beta_n\langle f(w_n)-p, x_{n+1}-p\rangle,$$

where $M_2 := \sup_{n\in\mathbb{N}}\{2M_1\|x_n-p\|+\beta_n M_1^2\}$. Thus,

$$\|s_n-w_n\|^2 \le \|x_n-p\|^2-\|x_{n+1}-p\|^2+\beta_n M_2+2\beta_n\langle f(w_n)-p, x_{n+1}-p\rangle.$$

Claim 3. We prove that

$$(1-\beta_n)\left[\frac{\mu}{2L}\|r(w_n)\|^2\right]^2 \le \|x_n-p\|^2-\|x_{n+1}-p\|^2+\beta_n M_3,$$

for some $M_3 > 0$. Indeed, from $\{\mathscr{A} y_n\}$ is bounded, there exists $L > 0$ such that $\|\mathscr{A} y_n\| \leq L$. By the definition of $h_n(x)$, for all $u, v \in C_n$,

$$\|h_n(u) - h_n(v)\| = \|\langle \mathscr{A} y_n, u - v\rangle\| \leq \|\mathscr{A} y_n\|\|u - v\| \leq L\|u - v\|,$$

which implies that $h_n(\cdot)$ is L-Lipschitz continuous on C_n. From Lemma 6, we have

$$dist(w_n, C_n) \geq \frac{1}{L} h_n(w_n).$$

Applying Lemma 11, we get

$$dist(w_n, C_n) \geq \frac{\mu}{2L}\|r(w_n)\|^2. \tag{31}$$

Thus,

$$\|s_n - p\|^2 \leq \|w_n - p\|^2 - \left[\frac{\mu}{2L}\|r(w_n)\|^2\right]^2. \tag{32}$$

On the other hand,

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\quad = \|\beta_n f(w_n) + (1 - \beta_n)s_n - p\|^2 \\
&\quad = \|\beta_n(f(w_n) - p) + (1 - \beta_n)(s_n - p)\|^2 \\
&\quad = \beta_n\|f(w_n) - p\|^2 + (1 - \beta_n)\|s_n - p\|^2 - \beta_n(1 - \beta_n)\|f(w_n) - s_n\|^2 \\
&\quad \leq \beta_n\|f(w_n) - p\|^2 + (1 - \beta_n)\|s_n - p\|^2 \\
&\quad \leq \beta_n\|f(w_n) - p\|^2 + (1 - \beta_n)\|w_n - p\|^2 - (1 - \beta_n)\left[\frac{\mu}{2L}\|r(w_n)\|^2\right]^2 \\
&\quad \leq \beta_n\|f(w_n) - p\|^2 + \|w_n - p\|^2 - (1 - \beta_n)\left[\frac{\mu}{2L}\|r(w_n)\|^2\right]^2 \\
&\quad \leq \beta_n\|f(w_n) - p\|^2 + \|x_n - p\|^2 + \beta_n(2M_1\|x_n - p\| + \beta_n M_1^2) \\
&\qquad - (1 - \beta_n)\left[\frac{\mu}{2L}\|r(w_n)\|^2\right]^2 \\
&\quad \leq \|x_n - p\|^2 + \beta_n M_3 - (1 - \beta_n)\left[\frac{\mu}{2L}\|r(w_n)\|^2\right]^2,
\end{aligned}$$

where $M_3 := \sup_{n\in\mathbb{N}}\{\|f(w_n) - p\|^2 + 2M_1\|x_n - p\| + \beta_n M_1^2\}$. Therefore,

$$(1 - \beta_n)\left[\frac{\mu}{2L}\|r(w_n)\|^2\right]^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \beta_n M_3.$$

Claim 4. We prove that

$$\begin{aligned}
&\|x_{n+1}-p\|^2 \\
&\quad \le (1-\beta_n(1-\rho))\|x_n-p\|^2+\beta_n(1-\rho)\Big[\frac{2}{1-\rho}\langle f(p)-p, x_{n+1}-p\rangle \\
&\quad\quad +\frac{2M}{1-\rho}\cdot\frac{\alpha_n}{\beta_n}\|x_1-x_n\|\Big],
\end{aligned}$$

for some $M>0$. In fact, using (2), we have

$$\begin{aligned}
\|w_n-p\|^2 &= \|(1-\alpha_n)x_n+\alpha_n x_1-p\|^2 \\
&\le \|x_n-p\|^2+2\alpha_n\langle x_1-x_n, w_n-p\rangle \\
&\le \|x_n-p\|^2+2\alpha_n\|x_1-x_n\|\|w_n-p\|. \qquad (33)
\end{aligned}$$

By (2), (3) and (24), we obtain

$$\begin{aligned}
&\|x_{n+1}-p\|^2 \\
&\quad = \|\beta_n f(w_n)+(1-\beta_n)s_n-p\|^2 \\
&\quad = \|\beta_n(f(w_n)-f(p))+(1-\beta_n)(s_n-p)+\beta_n(f(p)-p)\|^2 \\
&\quad \le \|\beta_n(f(w_n)-f(p))+(1-\beta_n)(s_n-p)\|^2+2\beta_n\langle f(p)-p, x_{n+1}-p\rangle \\
&\quad \le \beta_n\|f(w_n)-f(p)\|^2+(1-\beta_n)\|s_n-p\|^2+2\beta_n\langle f(p)-p, x_{n+1}-p\rangle \\
&\quad \le \beta_n\rho\|w_n-p\|^2+(1-\beta_n)\|w_n-p\|^2+2\beta_n\langle f(p)-p, x_{n+1}-p\rangle \\
&\quad = (1-\beta_n(1-\rho))\|w_n-p\|^2+2\beta_n\langle f(p)-p, x_{n+1}-p\rangle. \qquad (34)
\end{aligned}$$

Substituting (33) into (34), we get that

$$\begin{aligned}
&\|x_{n+1}-p\|^2 \\
&\quad \le (1-\beta_n(1-\rho))\|x_n-p\|^2+2\alpha_n\|x_1-x_n\|\|w_n-p\| \\
&\quad\quad +2\beta_n\langle f(p)-p, x_{n+1}-p\rangle \\
&\quad = (1-\beta_n(1-\rho))\|x_n-p\|^2+\beta_n(1-\rho)\cdot\frac{2}{1-\rho}\langle f(p)-p, x_{n+1}-p\rangle \\
&\quad\quad +2\alpha_n\|x_1-x_n\|\|w_n-p\| \\
&\quad \le (1-\beta_n(1-\rho))\|x_n-p\|^2+\beta_n(1-\rho)\cdot\frac{2}{1-\rho}\langle f(p)-p, x_{n+1}-p\rangle \\
&\quad\quad +2M\alpha_n\|x_1-x_n\| \\
&\quad = (1-\beta_n(1-\rho))\|x_n-p\|^2+\beta_n(1-\rho)\Big[\frac{2}{1-\rho}\langle f(p)-p, x_{n+1}-p\rangle \\
&\quad\quad +\frac{2M}{1-\rho}\cdot\frac{\alpha_n}{\beta_n}\|x_1-x_n\|\Big],
\end{aligned}$$

where $M := \sup_{n\in\mathbb{N}}\{\|w_n - p\|\}$.

Claim 5. We prove that $\{\|x_n - p\|\}$ converges to zero by considering two possible cases.

Case 1 There exists $N \in \mathbb{N}$ such that $\|x_{n+1} - p\|^2 \le \|x_n - p\|^2$ for all $n \ge N$, it follows that $\lim_{n\to\infty}\|x_n - p\|^2$ exists. Next we prove that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{35}$$

Indeed, by **Claim 2** we have

$$\lim_{n\to\infty}\|s_n - w_n\| = 0. \tag{36}$$

In addition,

$$\|w_n - x_n\| = \alpha_n\|x_1 - x_n\| = \beta_n \cdot \frac{\alpha_n}{\beta_n}\|x_1 - x_n\| \to 0\ (n \to \infty), \tag{37}$$

and

$$\|x_{n+1} - s_n\| = \beta_n\|f(w_n) - s_n\| \to 0\ (n \to \infty). \tag{38}$$

Combining (36), (37) and (38), we obtain

$$\|x_{n+1} - x_n\| = \|x_{n+1} - s_n\| + \|s_n - w_n\| + \|w_n - x_n\| \to 0\ (n \to \infty).$$

This implies that (35) holds. On the other hand, since the sequence $\{x_n\}$ is bounded, it follows that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$, which converges weakly to $q \in \mathscr{C}$ and

$$\limsup_{n\to\infty}\langle f(p) - p, x_n - p\rangle = \lim_{k\to\infty}\langle f(p) - p, x_{n_k} - p\rangle = \langle f(p) - p, q - p\rangle. \tag{39}$$

According to **Claim 3**, we get

$$\lim_{k\to\infty}\left[\frac{\mu}{2L}\|r(w_{n_k})\|^2\right]^2 = 0.$$

That is

$$\lim_{k\to\infty}\|w_{n_k} - z_{n_k}\| = 0. \tag{40}$$

Using the fact that $x_{n_k} \rightharpoonup q\ (k \to \infty)$, (40) and Lemma 12, we have $q \in Sol(\mathscr{C}, \mathscr{A})$. Since $p = P_{Sol(\mathscr{C},\mathscr{A})} \circ f(p)$, combining (35) and (39), we obtain

$$\begin{aligned}
&\limsup_{n\to\infty}\langle f(p) - p, x_{n+1} - p\rangle \\
&\quad \le \limsup_{n\to\infty}\langle f(p) - p, x_{n+1} - x_n\rangle + \limsup_{n\to\infty}\langle f(p) - p, x_n - p\rangle \\
&\quad \le 0.
\end{aligned} \tag{41}$$

Hence, by **Claim 4**, (41) and Lemma 8, we conclude that $\lim_{n\to\infty}\|x_n-p\|=0$.
Case 2 There exists a subsequence $\{\|x_{n_j}-p\|^2\}\subset\{\|x_n-p\|^2\}$ such that

$$\|x_{n_j}-p\|^2<\|x_{n_j+1}-p\|^2,\ \forall j\in\mathbb{N}.$$

From Lemma 7, there exists a non-decreasing sequence $\{m_k\}$ of $\mathbb{N}$ such that $\lim_{k\to\infty}m_k=\infty$ and

$$\|x_{m_k}-p\|^2\le\|x_{m_k+1}-p\|^2,\ \|x_k-p\|^2\le\|x_{m_k+1}-p\|^2,\ \forall k\in\mathbb{N}. \tag{42}$$

From **Claim 2**,

$$\begin{aligned}&\|s_{m_k}-w_{m_k}\|^2\\&\le\|x_{m_k}-p\|^2-\|x_{m_k+1}-p\|^2+\beta_{m_k}M_2+2\beta_{m_k}\langle f(w_{m_k})-p,x_{m_k+1}-p\rangle\\&\le\|x_{m_k}-p\|^2-\|x_{m_k+1}-p\|^2+\beta_{m_k}M_2+2\beta_{m_k}\|f(w_{m_k})-p\|\|x_{m_k+1}-p\|\\&\to 0\ (k\to\infty).\end{aligned}$$

As proved in **Case 1**, we can conclude that

$$\lim_{k\to\infty}\|x_{m_k+1}-x_{m_k}\|=0$$

and

$$\limsup_{k\to\infty}\langle f(p)-p,x_{m_k+1}-p\rangle\le 0. \tag{43}$$

Applying **Claim 4**, we have

$$\begin{aligned}&\|x_{m_k+1}-p\|^2\\&\le(1-\beta_{m_k}(1-\rho))\|x_{m_k}-p\|^2+\beta_{m_k}(1-\rho)\Big[\frac{2}{1-\rho}\langle f(p)-p,x_{m_k+1}-p\rangle\\&\quad+\frac{2M}{1-\rho}\cdot\frac{\alpha_{m_k}}{\beta_{m_k}}\|x_1-x_{m_k}\|\Big]\\&\le(1-\beta_{m_k}(1-\rho))\|x_{m_k+1}-p\|^2+\beta_{m_k}(1-\rho)\Big[\frac{2}{1-\rho}\langle f(p)-p,x_{m_k+1}-p\rangle\\&\quad+\frac{2M}{1-\rho}\cdot\frac{\alpha_{m_k}}{\beta_{m_k}}\|x_1-x_{m_k}\|\Big].\end{aligned}$$

It follows that

$$\|x_{m_k+1}-p\|^2\le\frac{2}{1-\rho}\langle f(p)-p,x_{m_k+1}-p\rangle+\frac{2M}{1-\rho}\cdot\frac{\alpha_{m_k}}{\beta_{m_k}}\|x_1-x_{m_k}\|.$$

From (42) and (43), we obtain

$$\|x_k-p\|^2\le\|x_{m_k+1}-p\|^2\to 0\ (k\to\infty).$$

Hence, $x_k \to p$ as $k \to \infty$. This completes the proof. □

3.2 Convergence rate

Next, we assume that the operator $\mathscr{A}$ in Algorithm 3 is η-strongly pseudo-monotone. In this case, by modifying Algorithm 3, we obtain that the sequence $\{x_n\}$ strongly converges with a linear rate. The modified algorithm is as follows:

Algorithm 4

Initialization: Let $\mu \in (0, \min\{\frac{1}{\sqrt{2}}, \frac{2l\eta}{L}\}),\ l \in (0,1),\ \gamma > 0$ and $x_1 \in \mathscr{C}$ be arbitrary.
Iterative steps: Given the current iterate x_n $(n \geq 1)$.
Step 1. Compute

$$x_{n+1} = P_{\mathscr{C}}(x_n - \tau_n \mathscr{A} x_n),$$

where $\tau_n := \gamma l^{m_n}$ and m_n is the smallest non-negative integer m satisfying

$$\gamma l^m \|\mathscr{A} x_n - \mathscr{A} x_{n+1}\| \leq \mu \|x_n - x_{n+1}\|. \tag{44}$$

Set $n := n+1$ and return to **Step 1**.

Remark 2 Similar to the proof of Lemma 10, it is easy to get that (44) is well defined.

Theorem 2 *Assume that $\mathscr{A} : \mathscr{C} \to \mathscr{C}$ is η-strongly pseudo-monotone and L-Lipschitz continuous. Then the sequence $\{x_n\}$ generated by Algorithm 4 converges strongly with a Q-linear rate to the unique element p in $Sol(\mathscr{C}, \mathscr{A})$.*

Proof According to the conditions of Theorem 2, $\langle \mathscr{A} p, x_{n+1} - p\rangle \geq 0$ and thus

$$\langle \mathscr{A} x_{n+1}, x_{n+1} - p\rangle \geq \eta \|x_{n+1} - p\|^2.$$

From the definition of $\{x_{n+1}\}$, we have

$$\langle x_n - \tau_n \mathscr{A} x_n - x_{n+1}, p - x_{n+1}\rangle \leq 0.$$

It follows from (44) that

$$\begin{aligned}
2\langle x_n - x_{n+1}, p - x_{n+1}\rangle &\leq 2\tau_n \langle \mathscr{A} x_n, p - x_{n+1}\rangle \\
&= -2\tau_n \langle \mathscr{A} x_{n+1}, x_{n+1} - p\rangle + 2\tau_n \langle \mathscr{A} x_n - \mathscr{A} x_{n+1}, p - x_{n+1}\rangle \\
&\leq -2\tau_n \eta \|x_{n+1} - p\|^2 + 2\tau_n \|\mathscr{A} x_n - \mathscr{A} x_{n+1}\| \|p - x_{n+1}\| \\
&\leq -2\tau_n \eta \|x_{n+1} - p\|^2 + \|x_{n+1} - x_n\|^2 + \mu^2 \|p - x_{n+1}\|^2.
\end{aligned}$$

Moreover,

$$2\langle x_n - x_{n+1}, p - x_{n+1}\rangle = -\|x_n - p\|^2 + \|x_n - x_{n+1}\|^2 + \|x_{n+1} - p\|^2,$$

which implies that

$$(1 + 2\tau_n\eta - \mu^2)\|x_{n+1} - p\|^2 \leq \|x_n - p\|^2.$$

Since $\mu \in (0, \min\{\frac{1}{\sqrt{2}}, \frac{2l\eta}{L}\})$, we have $1+\mu^2 < 2-\mu^2$. Next we show that $\tau_n > \frac{\mu l}{L}$. Indeed, we know that $\frac{\tau_n}{l}$ must violate (44), thus,

$$\frac{\mu l}{\tau_n}\|x_n - x_{n+1}\| < \|\mathscr{A}x_n - \mathscr{A}x_{n+1}\| \leq L\|x_n - x_{n+1}\|,$$

that is $\tau_n > \frac{\mu l}{L}$. It follows that $2\tau_n\eta > 2\frac{\mu l\eta}{L} > \mu^2$ and $\frac{\mu^2}{\min\{1-\mu^2, 2\tau_n\eta\}} < 1$. Fix $\delta \in (\frac{1+\mu^2}{2\mu}, \frac{2-\mu^2}{2\mu})$ and $\varepsilon \in (\frac{\mu^2}{\min\{1-\mu^2,2\tau_n\eta\}}, \frac{2(1-\delta\mu)}{\min\{1-\mu^2,2\tau_n\eta\}})$. Since $\frac{\mu^2}{\min\{1-\mu^2,2\tau_n\eta\}} < 1$, we can choose $\varepsilon \in (0, 1)$ such that

$$\mu^2 < \varepsilon \min\{1 - \mu^2, 2\tau_n\eta\}.$$

Let $\nu := \frac{1}{2}\varepsilon \min\{1 - \mu^2, 2\tau_n\eta\}$, we see that

$$2\varepsilon\tau_n\eta \geq 2\nu > \mu^2.$$

Therefore,

$$1 + 2\tau_n\eta - \mu^2 > 1 + 1 + 2\varepsilon\tau_n\eta - \mu^2 = 1 + \gamma,$$

where $\gamma := 2\varepsilon\tau_n\eta - \mu^2 > 0$. It follows that

$$\|x_{n+1} - p\|^2 \leq \frac{1}{1+\gamma}\|x_n - p\|^2,$$

that is,

$$\|x_{n+1} - p\| \leq \sqrt{\frac{1}{1+\gamma}}\|x_n - p\|.$$

This implies that the sequence $\{x_n\}$ generated by Algorithm 4 converges strongly to p with a Q-linear rate. □

4 Numerical experiments

We give some numerical examples to show performances of our proposed Algorithm 3 and compare with [15, Algorithm SD], [14, Algorithm PY], [28, Algorithm DY] and [12, Algorithm DA].

Example 1 Assume that $\mathscr{A} : \mathbb{R}^m \to \mathbb{R}^m$ is defined by $\mathscr{A}(x) := Mx + q$ with $M = NN^T + S + D$, N is an $m \times m$ matrix, S is an $m \times m$ skew-symmetric matrix, D is an $m \times m$ diagonal matrix, whose diagonal entries are positive (so $\mathscr{A}$ is positive definite).

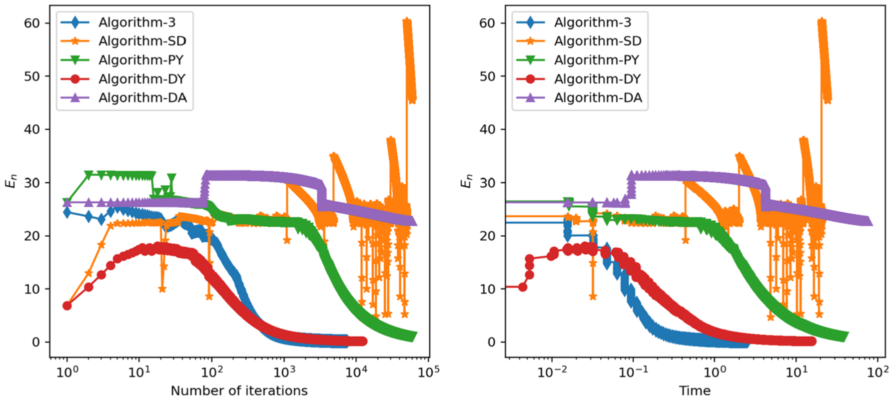

Fig. 1 Example 1 with $m = 5$

All the entries of N, S and D are randomly generated in the interval (0, 20). Consider $Sol(\mathscr{C}, \mathscr{A})$ with the feasible set $\mathscr{C} = \{x \in \mathbb{R}^m : -10 \le x_i \le 10, i = 1, 2, \cdots, m\}$. The parameters are taken as follows.

- Algorithm 3: $\alpha_n = \begin{cases} \frac{1}{(n+1)^2}, & \text{if } \|x_n - x_1\| = 0 \\ \min\left\{\frac{1}{(n+1)^2}, \frac{1}{(n+1)^2 \times \|x_n - x_1\|}\right\}, & \text{otherwise} \end{cases}$, $\mu = 0.9$, $l = 0.9$, $\gamma = 0.05$, $\beta_n = \frac{1}{n+1}$ and $\rho = 0.5$;
- Algorithm SD: $\alpha_n = \frac{1}{n+1}$, $\mu = 0.9$, $l = 0.9$, $\rho = 0.5$ and λ is randomly generated in the interval $(0, 1/\mu)$;
- Algorithm PY: $\alpha_n = \frac{1}{n+1}$, $\mu = 0.9$ and $l = 0.9$;
- Algorithms DY, DA: $\alpha_n = \frac{1}{n+1}$, $\mu = 0.9$, $l = 0.9$, $\gamma = 0.05$.

In our experiment, the starting point x_1 is generated randomly in $(-10, 10)^m$, where $m = 5, 10$. We use the stopping rule $E_n = \|x_n - P_{\mathscr{C}}(I - \mathscr{A})x_n\| \le 10^{-1}$ and we also stop if the number of iterations $N = 60000$ for all algorithms. Figures 1, 2 and

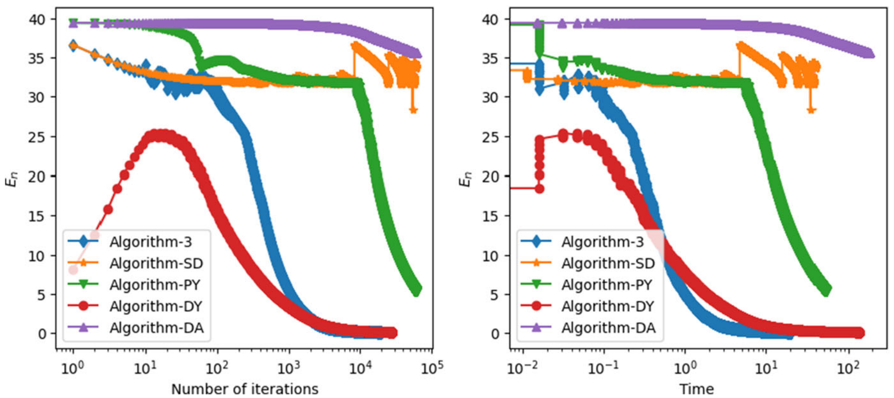

Fig. 2 Example 1 with $m = 10$

Table 1 The number of termination iterations and execution time of all algorithms

Method	Algorithm 3	Algorithm SD	Algorithm PY	Algorithm DY	Algorithm DA
Iter	7504	60000	60000	12301	60000
CPU	2.5121	24.2943	38.5131	15.4500	76.0455

Tables 1, 2 show the computational results for Example 1 by using Algorithms 3, SD, PY, DY and DA.

Let "Iter" denote number of iterations, "CPU" denote the CPU time seconds. Example 1 shows that our proposed Algorithm 3 is fast, efficient and easy to implement. We notice that from Figs. 1, 2 and Tables 1, 2 that our Algorithm 3 outperforms SD, PY, DY and DA, in terms of CPU time and required number of iterations for each case of different dimensions given as follows: $m = 5, 10$.

Example 2 Consider $\mathcal{H} = L^2([0, 1])$ with inner product $\langle x, y\rangle := \int_0^1 x(t)y(t)dt$ and norm $\|x\|_2 := (\int_0^1 |x(t)|^2 dt)^{\frac{1}{2}}$. Suppose $\mathscr{C} := \{x \in \mathcal{H} : \|x\|_2 \leq 2\}$. Let $g : \mathscr{C} \to \mathbb{R}$ be defined by $g(u) := \frac{1}{1+\|u\|_2^2}$. Observe that g is L_g-Lipschitz continuous with $L_g = \frac{16}{25}$ and $\frac{1}{5} \leq g(u) \leq 1, \forall u \in \mathscr{C}$. Define the Volterra integral mapping $F : \mathcal{H} \to \mathcal{H}$ by

$$F(u)(t) := \int_0^t u(s)ds, \ \forall u \in \mathcal{H}, t \in [0, 1].$$

Then F is bounded linear monotone, see [29]. Now define $\mathscr{A} : \mathscr{C} \to \mathcal{H}$ by

$$\mathscr{A}(u)(t) := g(u)F(u)(t), \ \forall u \in \mathscr{C}, t \in [0, 1].$$

As given in [30], the mapping $\mathscr{A}$ is pseudo-monotone but not monotone. Now take

$$\mathscr{C} := \{x \in \mathcal{H} : \langle a, x\rangle \geq b\},$$

where $a \in \mathcal{H}$ and $b \in R$. Then we define the metric projection $P_{\mathscr{C}}$ as

$$P_{\mathscr{C}}(x) = \begin{cases} \frac{b-\langle a,x\rangle}{\|a\|_2^2} a + x, & \text{if } \langle a, x\rangle < b \\ x, & \text{otherwise.} \end{cases}$$

Let $Q = \{x \in \mathcal{H} : \|x\|_2 \leq r\}$ be a closed ball centered at 0 with radius $r = 2$, then Q is a nonempty closed and convex subset of $L^2([0, 1])$. Thus, the projection onto Q

Table 2 The number of termination iterations and execution time of all algorithms

Method	Algorithm 3	Algorithm SD	Algorithm PY	Algorithm DY	Algorithm DA
Iter	19076	60000	60000	27687	60000
CPU	19.6727	39.4768	52.8476	136.1701	183.1247

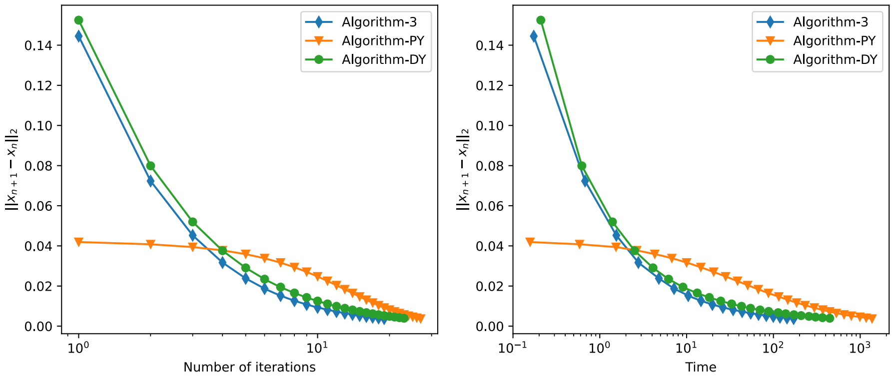

Fig. 3 Example 2 with $x_1 = t$

is easily computed as

$$P_Q(x) = \begin{cases} r\frac{x}{\|x\|^2}, & x \notin Q \\ x, & \text{otherwise.} \end{cases}$$

During this experiment, the parameters are taken as follows.

- Algorithm 3: $\alpha_n = \begin{cases} \frac{1}{(n+1)^2}, & \text{if } \|x_n - x_1\|_2 = 0 \\ \min\left\{\frac{1}{(n+1)^2}, \frac{1}{(n+1)^2 \times \|x_n - x_1\|_2}\right\}, & \text{otherwise} \end{cases}$, $\mu = 0.4$, $l = 0.1$, $\gamma = 0.5$, $\beta_n = \frac{1}{n+1}$ and $\rho = 0.5$;
- Algorithm SD: $\alpha_n = \frac{1}{n+1}$, $\mu = 0.4$, $l = 0.1$, $\rho = 0.5$ and λ is randomly generated in the interval $(0, 1/\mu)$;
- Algorithm PY: $\alpha_n = \frac{1}{n+1}$, $\mu = 0.4$ and $l = 0.1$;
- Algorithms DY, DA: $\alpha_n = \frac{1}{n+1}$, $\mu = 0.4$, $l = 0.1$, $\gamma = 0.5$.

We test Algorithms 3, PY, DY for different cases of the initial point $x_1 \in L^2([0, 1])$. Let the initial point be $x_1 = t$. We take $E_n = \|x_{n+1} - x_n\|_2 \leq 4 \times 10^{-3}$ as a termination criterion. The results of this test are displayed in Fig. 3 and Table 3.

Let the initial point be $x_1 = t^3$. We terminate the iterations if $E_n = \|x_{n+1} - x_n\|_2 \leq 10^{-3}$. The numerical results are presented in Fig. 4 and Table 3.

Let "Iter" denote number of iterations, "CPU" denote the CPU time seconds. Figures 3, 4 and Table 3 show that the numerical results of Example 2 with initial points $x_0 = t$ and $x_1 = t^3$, respectively. They show that the performance of

Table 3 The number of termination iterations and execution time of all algorithms

x_1	Method	Algorithm-3	Algorithm-PY	Algorithm-DY
t	Iter.	18	22	26
	CPU	171.0379	445.7557	1368.5464
t^3	Iter.	39	41	52
	CPU	581.5528	1132.5577	5692.5919

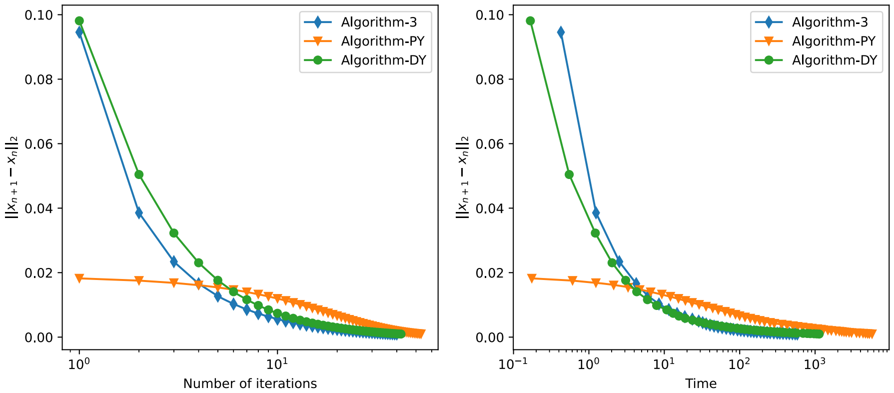

Fig. 4 Example 2 with $x_1 = t^3$

Algorithm 3 is better than that of Algorithms PY, DY, in terms of the number of iterations and CPU time required to reach the stopping criterion.

Example 3 Let us consider the variational inequality problem. Let

$$\mathscr{A}(x) = \begin{pmatrix} (x_1^2 + (x_2 - 1)^2)(1 + x_2) \\ -x_1^3 - x_1(x_2 - 1)^2 \end{pmatrix}$$

and $\mathscr{C} := \{x \in \mathbb{R}^2 : \ -10 \leq x_i \leq 10, \ i = 1, 2\}$. This problem has unique solution $x^* = (0, -1)^T$. It is easy to see that $\mathscr{A}$ is not a monotone map on $\mathscr{C}$. However, using the Monte Carlo approach (see [31]), it can be shown that $\mathscr{A}$ is pseudo-monotone on $\mathscr{C}$. Let $f(x) = 20/x$. Parameters in different algorithms are selected as follows:

Algorithm 3: $\beta_n = \frac{1}{50n+1}$, $\gamma = 1.35$, $l = 0.22$, $\mu = 0.99$ and $\alpha = 0.8$;

Algorithm SD: $\alpha_n = \frac{1}{50n+1}$, $l = 0.22$, $\mu = 0.99$ and $\lambda = 0.9/\mu$;

Table 4 Comparison of Algorithm 3, Algorithm SD and Algorithm PY for Example 3

		Algorithm 3		Algorithm SD		Algorithm PY	
Initial point	ε	Iter.	CPU	Iter.	CPU	Iter.	CPU
	0.01	29	0.0001	33	0.0001	32	0.0001
$[10, 10]^T$	0.001	41	0.0005	36	0.0002	304	0.0006
	0.0001	390	0.0015	394	0.0010	3025	0.0051
	0.01	21	0.0003	16	0.0001	14	0.0001
$[-5, -5]^T$	0.001	41	0.0005	42	0.0002	132	0.0003
	0.0001	390	0.0014	394	0.0009	1304	0.0023
	0.01	35	0.0004	28	0.0002	25	0.0005
$[-10, 5]^T$	0.001	41	0.0004	41	0.0002	239	0.0005
	0.0001	390	0.0014	394	0.0009	2374	0.0041

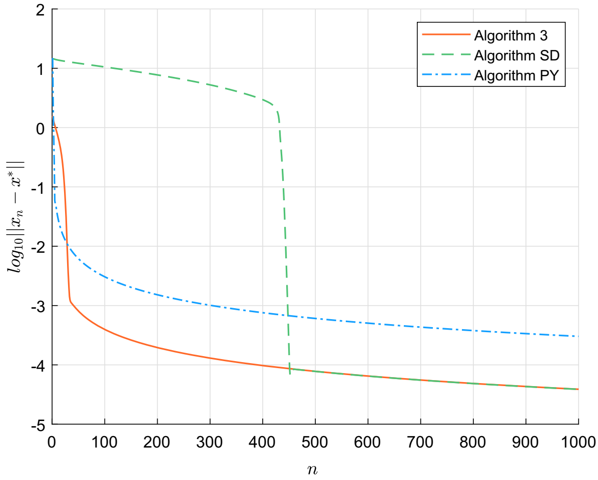

Fig. 5 The value of error versus the iteration numbers for Example 3

Algorithm PY: $\alpha_n = \frac{1}{50n+1}$, $l = 0.22$ and $\mu = 0.99$.

We take some initial points and different stopping criteria conditions in the Table 4. The following statistical data are obtained by averaging the number of iterations and CPU costs from 10 independent trials. In the Fig. 5, the initial point is $[10, 10]^T$ and the number of steps to stop criteria is 1000.

5 Conclusions

In this paper, we have given an improved projection-type method for solving classical variational inequalities in Hilbert spaces. We have proposed a novel line-search rule that removes the reliance on Lipschitz continuity. Furthermore, a strong convergence theorem is obtained through the combination of viscosity iteration and the projection method. In numerical experiments, we have compared our Algorithm 3 with some recent related results, and it can be found from the figures and tables that our new scheme has better convergence performance.

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Availability of supporting data Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Ethical Approval and Consent to participate Not Applicable.

Consent for publication Not Applicable.

Human and Animal Ethics Not Applicable.

Competing interests The authors declare no competing interests.

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