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ORIGINAL PAPER

H^1-norm error analysis of a robust ADI method on graded mesh for three-dimensional subdiffusion problems

Ziyi Zhou[1] · Haixiang Zhang[1] · Xuehua Yang[1]

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Abstract
This work proposes a robust ADI scheme on graded mesh for solving three-dimensional subdiffusion problems. The Caputo fractional derivative is discretized by L1 scheme, where the graded mesh is used to eliminate the weak singular behavior of the exact solution at the initial time $t = 0$. The spatial derivatives are approximated by the finite difference method. Based on the improved discrete fractional $Gr\ddot{o}nwall$ inequality, we prove the stability and α-robust H^1-norm convergence, in which the error bound does not blow up when the order of fractional derivative $\alpha \to 1^-$. The 3D numerical examples are proposed to verify the efficiency and accuracy of the ADI method. The CPU time is also provided, which shows the proposed method is very efficient for 3D subdiffusion problems.

Keywords Three-dimensional subdiffusion equation · ADI scheme · L1 scheme · α-Robust · Stability and convergence

1 Introduction

In this article, we consider the following three-dimensional subdiffusion problems

$$D_t^{\alpha} u(x, y, z, t) - \Delta u(x, y, z, t) = f(x, y, z, t), \quad (x, y, z) \in \Upsilon, \quad t \in (0, T], \tag{1}$$
$$u(x, y, z, t) = \psi(x, y, z, t), \quad (x, y, z) \in \partial\Upsilon, \quad t \in (0, T], \tag{2}$$
$$u(x, y, z, 0) = \phi(x, y, z), \quad (x, y, z) \in \Upsilon, \tag{3}$$

✉ Haixiang Zhang
hassenzhang@163.com

Ziyi Zhou
zzy_0802@163.com

Xuehua Yang
hunanshidayang@163.com

[1] College of Science, Hunan University of Technology, Zhuzhou 412008, China

in which $\Upsilon = (0, \mathcal{L}_x) \times (0, \mathcal{L}_y) \times (0, \mathcal{L}_z)$, $\partial\Upsilon$ is the boundary, and f, ψ, and ϕ are smooth functions. $D_t^\alpha u$ represents the Caputo fractional derivative, which can be written as

$$D_t^\alpha u(\cdot, t) := \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\xi)^{-\alpha} \frac{\partial u(\cdot, \xi)}{\partial \xi} d\xi, \quad \alpha \in (0, 1).$$

In recent years, the high-dimensional subdiffusion problems have been used in a wide variety of application fields due to the nonlocal natures, such as physics, finance, and chemistry [1–4], and many numerical schemes for solving subdiffusion equations have been proposed [5–12, 26]. Alikhanov [13] constructed a L2-1_σ formula for time-fractional diffusion equation with variable coefficients. Kopteva et al. [14] considered an initial-boundary value problem of subdiffusion type. Gao et al. [15] constructed finite difference methods for solving a class of time-fractional subdiffusion equations. Zhai et al. [16] constructed the high-precision unconditionally stable finite difference methods for solving time-space fractional diffusion equation. Wu et al. [17] proposed a new high-order finite difference scheme for solving the 2D time-fractional convection-dominated diffusion equation. Du et al. [19] proposed two difference schemes for the multi-dimensional variable-order time-fractional subdiffusion equations. Zeng et al. [20, 21] firstly proposed two fully discrete schemes for the time-fractional subdiffusion equation with space discretized by the finite element method and time discretized by the fractional linear multistep methods. Wang et al. [23] proposed an effective ADI scheme for solving the two-dimensional time-fractional diffusion equation. Balasim et al. [22] developed a new detailed group iterative scheme for 2D time-fractional advection-diffusion equation. Yang et al. [24] constructed a space-time spectral order sinc-collocation method for the fourth-order nonlocal heat model arising in viscoelasticity. Zeng et al. [25] proposed a new alternating direction implicit Galerkin-Legendre spectral method for the two-dimensional Riesz space fractional nonlinear reaction-diffusion equation. Zeng et al. [26] constructed the second-order accurate schemes for the time-fractional subdiffusion equation with unconditional stability. Zhou et al. [27] constructed a fast efficient ADI scheme for 3D nonlocal evolution equation with different weakly singular kernels. Tian et al. [28] proposed an implicit robust difference method for the modified Burgers model with nonlocal dynamic properties. Zhai et al. [29, 30] presented the high-order compact finite difference method for the three-dimensional time-fractional convection-diffusion equation. Roul et al. [31] proposed compact ADI scheme for the two-dimensional time-fractional convection-diffusion equation.

In this paper, we propose a robust ADI scheme for solving three-dimensional subdiffusion problems. We use L1 scheme on graded mesh to approximate the Caputo fractional derivative, where the graded mesh is accustomed to eliminate the weakly singular behavior of the true solution at $t = 0$. By using the standard second-order finite difference for the spatial discretization, we obtain a fully discrete scheme. The stability and α-robust H^1-norm convergence analysis are strictly proved. We obtain a robust ADI scheme based on the ADI algorithm and give two numerical examples. The CPU time is also provided. The main contributions of this paper are shown as follows:

- In this paper, an α-robust H^1-norm convergence analysis for the proposed method based on non-uniform meshes is developed for the three-dimensional subdiffusion equation. Theoretical results prove that the error bound does not blow up when the order of fractional derivative $\alpha \to 1^-$.
- We construct an ADI algorithm to solve the three-dimensional subdiffusion equation, which greatly reduces the computational cost. We compare the difference scheme proposed in this paper with the scheme in [23] in numerical examples. The numerical results verify that our ADI scheme is effective for both 2D and 3D time-fractional subdiffusion problems.

The remainder of this article is organized as follows: In Sect. 2, we construct the ADI scheme for three-dimensional subdiffusion problems. The stability and α-robust H^1-norm convergence analysis are presented in Sect. 3. In Sect. 4, the 3D numerical example is provided. At last, a brief conclusion is provided in Sect. 5.

2 Establishment of the fully discrete ADI scheme

Set $h_x = \frac{\mathcal{L}_x}{\chi_1}$, $h_y = \frac{\mathcal{L}_y}{\chi_2}$, and $h_z = \frac{\mathcal{L}_z}{\chi_3}$ for the positive integers χ_1, χ_2, and χ_3, and $t_n = T(n/N)^\gamma$ for a positive integer N, $\gamma \geq 1$ is a grading parameter. Set $\tau_n = t_n - t_{n-1}$ and $\tau = \max \tau_n$ for $1 \leq n \leq N$.

Let $\overline{\Upsilon}_h = \{(x_r, y_j, z_k) | 0 \leq r \leq \chi_1, 0 \leq j \leq \chi_2, 0 \leq k \leq \chi_3\}$, and $\Upsilon_h = \overline{\Upsilon}_h \cap \Upsilon$, $\partial\Upsilon_h = \Upsilon_h \cap \partial\Upsilon$. Denote u_{rjk}^n be the approximation solution of (2)–(3) at the mesh point (x_r, y_j, z_k, t_n).

For $n = 1, 2, \cdots, N$, we use the L1 scheme as follows:

$$
\begin{aligned}
D_N^\alpha u_{rjk}^n &:= \frac{1}{\Gamma(1-\alpha)} \sum_{p=0}^{n-1} \frac{u_{rjk}^{p+1} - u_{rjk}^p}{\tau_{p+1}} \int_{t_p}^{t_{p+1}} (t_n - \xi)^{-\alpha} d\xi \\
&= d_{n,1} u_{rjk}^n - d_{n,n} u_{rjk}^0 - \sum_{p=1}^{n-1} (d_{n,p} - d_{n,p+1}) u_{rjk}^{n-p},
\end{aligned} \tag{4}
$$

where

$$
d_{n,1} = \frac{\tau_n^{-\alpha}}{\Gamma(2-\alpha)}, \quad d_{n,p} = \frac{(t_n - t_{n-p})^{1-\alpha} - (t_n - t_{n-p+1})^{1-\alpha}}{\tau_{n-p+1}\Gamma(2-\alpha)}, \quad 1 \leq p \leq n.
$$

Set $\mu_n = d_{n,1}^{-1} = \tau_n^\alpha \Gamma(2-\alpha)$, the problem (2)–(3) can be approximated by

$$
D_N^\alpha u_{rjk}^n - \Delta_h u_{rjk}^n = f_{rjk}^n, \quad (x_r, y_j, z_k) \in \Upsilon_h, 1 \leq n \leq N, \tag{5}
$$

$$
u_{rjk}^n = \psi(x_r, y_j, z_k, t_n), \quad (x_r, y_j, z_k) \in \partial\Upsilon_h, 1 \leq n \leq N, \tag{6}
$$

$$
u_{rjk}^0 = \phi(x_r, y_j, z_k), \quad (x_r, y_j, z_k) \in \overline{\Upsilon}_h, \tag{7}
$$

where

$$
\Delta_h u_{rjk}^n = \delta_x^2 u_{rjk}^n + \delta_y^2 u_{rjk}^n + \delta_z^2 u_{rjk}^n, \quad \delta_x^2 u_{rjk}^n = \frac{u_{r+1,j,k}^n - 2u_{rjk}^n + u_{r-1,j,k}^n}{h_x^2},
$$

$$\delta_y^2 u_{rjk}^n = \frac{u_{r,j+1,k}^n - 2u_{rjk}^n + u_{r,j-1,k}^n}{h_y^2}, \quad \delta_z^2 u_{rjk}^n = \frac{u_{r,j,k+1}^n - 2u_{rjk}^n + u_{r,j,k-1}^n}{h_z^2}.$$

To get ADI scheme, we add a small term $(\mu_n^2\delta_x^2\delta_y^2 + \mu_n^2\delta_x^2\delta_z^2 + \mu_n^2\delta_y^2\delta_z^2 - \mu_n^3\delta_x^2\delta_y^2\delta_z^2)D_N^\alpha u_{rjk}^n$ into (6), which is order $O(N^{-2\alpha})$. Thus, we can obtain the famous ADI scheme:

$$\begin{aligned}(I + \mu_n^2 \ &\delta_x^2\delta_y^2 + \mu_n^2\delta_x^2\delta_z^2 + \mu_n^2\delta_y^2\delta_z^2 - \mu_n^3\delta_x^2\delta_y^2\delta_z^2)D_N^\alpha u_{rjk}^n - \Delta_h u_{rjk}^n = f_{rjk}^n,\\ &(x_r, y_j, z_k) \in \Upsilon_h, \quad 1 \le n \le N, \qquad (8)\\ &u_{rjk}^n = \psi(x_r, y_j, z_k, t_n), \quad (x_r, y_j, z_k) \in \partial\Upsilon_h, \quad 1 \le n \le N, \qquad (9)\\ &u_{rjk}^0 = \phi(x_r, y_j, z_k), \quad (x_r, y_j, z_k) \in \overline{\Upsilon}_h. \qquad (10)\end{aligned}$$

By substituting (4) into (8), we have

$$\begin{aligned}&u_{rjk}^n - \mu_n\delta_x^2 u_{rjk}^n - \mu_n\delta_y^2 u_{rjk}^n - \mu_n\delta_z^2 u_{rjk}^n + \mu_n^2\delta_x^2\delta_y^2 u_{rjk}^n + \mu_n^2\delta_x^2\delta_z^2 u_{rjk}^n + \mu_n^2\delta_y^2\delta_z^2 u_{rjk}^n\\ &-\mu_n^3\delta_x^2\delta_y^2\delta_z^2 u_{rjk}^n = \mu_n(I + \mu_n^2\delta_x^2\delta_y^2 + \mu_n^2\delta_x^2\delta_z^2 + \mu_n^2\delta_y^2\delta_z^2 - \mu_n^3\delta_x^2\delta_y^2\delta_z^2)(d_{n,n}u_{rjk}^0 +\\ &\sum_{p=1}^{n-1}(d_{n,p} - d_{n,p+1})u_{rjk}^{n-p}) + \mu_n f_{rjk}^n.\end{aligned}$$

Define

$$\begin{aligned}F_{rjk}^n =&\mu_n(I + \mu_n^2\delta_x^2\delta_y^2 + \mu_n^2\delta_x^2\delta_z^2 + \mu_n^2\delta_y^2\delta_z^2 - \mu_n^3\delta_x^2\delta_y^2\delta_z^2)(d_{n,n}u_{rjk}^0 +\\ &\sum_{p=1}^{n-1}(d_{n,p} - d_{n,p+1})u_{rjk}^{n-p} + \mu_n f_{rjk}^n,\end{aligned}$$

then (8) can be rewritten as the following ADI form:

$$(I - \mu_n\delta_x^2)(I - \mu_n\delta_y^2)(I - \mu_n\delta_z^2)u_{rjk}^n = F_{rjk}^n, \quad (x_r, y_j, z_k) \in \Upsilon_h, \quad 1 \le n \le N.$$

Next, we introduce two intermediate variables to get the solution $\{u_{rjk}^n\}$:

$$\begin{aligned}u_{rjk}^{n-\frac{1}{3}} &= (I - \mu_n\delta_z^2)u_{rjk}^n, \quad (x_r, y_j, z_k) \in \Upsilon_h, \quad 1 \le n \le N,\\ u_{rjk}^{n-\frac{2}{3}} &= (I - \mu_n\delta_y^2)u_{rjk}^{n-\frac{1}{3}}, \quad (x_r, y_j, z_k) \in \Upsilon_h, \quad 1 \le n \le N,\end{aligned}$$

from which we can solve the problem (8)–(10) by using the following three algorithm steps.

Algorithm 1 The algorithm of solving $\{u_{rjk}^{n-\frac{2}{3}}\}$.

Input: F_{rmn}^{k}
Output: $\{u_{rjk}^{n-\frac{2}{3}}\}$
Fixing $j \in \{1, 2, \cdots, \chi_2 - 1\}$ and $k \in \{1, 2, \cdots, \chi_3 - 1\}$
for each $1 \le r \le \chi_1 - 1$ and $1 \le n \le N$ **do**
  $(I - \mu_n \delta_x^2) u_{rjk}^{n-\frac{2}{3}} = F_{rjk}^{n}$
  $u_{0jk}^{n-\frac{2}{3}} = (I - \mu_n \delta_y^2) u_{0jk}^{n-\frac{1}{3}} = (I - \mu_n \delta_y^2)(I - \mu_n \delta_z^2) u_{0jk}^{n}$
  $u_{\chi_1,j,k}^{n-\frac{2}{3}} = (I - \mu_n \delta_y^2) u_{\chi_1,j,k}^{n-\frac{1}{3}} = (I - \mu_n \delta_y^2)(I - \mu_n \delta_z^2) u_{\chi_1,j,k}^{n}$
end for
  return $\{u_{rjk}^{n-\frac{2}{3}}\}$

Algorithm 2 The algorithm of solving $\{u_{rjk}^{n-\frac{1}{3}}\}$.

Input: $u_{rjk}^{n-\frac{2}{3}}$
Output: $\{u_{rjk}^{n-\frac{1}{3}}\}$
Fixing $r \in \{1, 2, \cdots, \chi_1 - 1\}$ and $k \in \{1, 2, \cdots, \chi_3 - 1\}$
for each $1 \le j \le \chi_2 - 1$ and $1 \le n \le N$ **do**
  $(I - \mu_n \delta_y^2) u_{rjk}^{n-\frac{1}{3}} = u_{rjk}^{n-\frac{2}{3}}$
  $u_{r0k}^{n-\frac{1}{3}} = (I - \mu_n \delta_z^2) u_{r0k}^{n}$
  $u_{r,\chi_2,k}^{n-\frac{1}{3}} = (I - \mu_n \delta_z^2) u_{r,\chi_2,k}^{n}$
end for
  return $\{u_{rjk}^{n-\frac{1}{3}}\}$

Algorithm 3 The algorithm of solving $\{u_{rjk}^{n}\}$.

Input: $u_{rjk}^{n-\frac{2}{3}}, u_{rjk}^{n-\frac{1}{3}}$
Output: $\{u_{rjk}^{n}\}$
Fixing $r \in \{1, 2, \cdots, \chi_1 - 1\}$ and $j \in \{1, 2, \cdots, \chi_2 - 1\}$
for each $1 \le k \le \chi_3 - 1$ and $1 \le n \le N$ **do**
  $(I - \mu_n \delta_z^2) u_{rjk}^{n} = u_{rjk}^{n-\frac{1}{3}}$
  $u_{rj0}^{n} = \psi(x_r, y_j, z_0, t_n)$
  $u_{r,j,\chi_3}^{n} = \psi(x_r, y_j, z_{\chi_3}, t_n)$
end for
  return $\{u_{rjk}^{n}\}$

3 Stability and convergence analysis

In this section, we shall prove the stability and convergence of the ADI difference scheme (8)–(10) in H^1-norm sense. For grid function $u = \{u_{rjk}^{n} | 0 \le r \le \chi_1, 0 \le$

$j \le \chi_2, 0 \le k \le \chi_3, 0 \le n \le N\}$, we denote

$$\|u^n\|^2 = h_x \sum_{r=1}^{\chi_1-1} h_y \sum_{j=1}^{\chi_2-1} h_z \sum_{k=1}^{\chi_3-1} (u^n_{rjk})^2,$$

$$\|\delta_x u^n\|^2 = h_x \sum_{r=1}^{\chi_1} h_y \sum_{j=1}^{\chi_2-1} h_z \sum_{k=1}^{\chi_3-1} (\delta_x u^n_{r-\frac{1}{2},j,k})^2,$$

$$\|\delta_x\delta_y u^n\|^2 = h_x \sum_{r=1}^{\chi_1} h_y \sum_{j=1}^{\chi_2} h_z \sum_{k=1}^{\chi_3-1} (\delta_x\delta_y u^n_{r-\frac{1}{2},j-\frac{1}{2},k})^2,$$

$$\|\delta_x\delta_y\delta_z u^n\|^2 = h_x \sum_{r=1}^{\chi_1} h_y \sum_{j=1}^{\chi_2} h_z \sum_{k=1}^{\chi_3} (\delta_x\delta_y\delta_z u^n_{r-\frac{1}{2},j-\frac{1}{2},k-\frac{1}{2}})^2,$$

$$\|\delta_x\delta_y\delta_z^2 u^n\|^2 = h_x \sum_{r=1}^{\chi_1} h_y \sum_{j=1}^{\chi_2} h_z \sum_{k=1}^{\chi_3-1} (\delta_x\delta_y\delta_z^2 u^n_{r-\frac{1}{2},j-\frac{1}{2},k})^2,$$

$$\|\Delta_h u^n\|^2 = h_x \sum_{r=1}^{\chi_1-1} h_y \sum_{j=1}^{\chi_2-1} h_z \sum_{k=1}^{\chi_3-1} (\Delta_h u^n_{rjk})^2,$$

where $\delta_x u_{r-\frac{1}{2},j,k} = \frac{1}{h_x}(u_{rjk} - u_{r-1,j,k})$. We can analogously define other norms $\|\delta_y u^n\|$, $\|\delta_z u^n\|$, $\|\delta_y\delta_z u^n\|$, $\|\delta_x\delta_z u^n\|$, $\|\delta_x\delta_z\delta_y^2 u^n\|$, and $\|\delta_y\delta_z\delta_x^2 u^n\|$.

Set $U_h = \{u_{rjk} | u_{rjk} = 0 \text{ if } (x_r, y_j, z_k) \in \partial\Upsilon_h \text{ and } (x_r, y_j, z_k) \in \overline{\Upsilon}_h\}$, for $\forall u$, $w \in U_h$, we define

$$\begin{aligned}
\|\nabla_h u^n\|^2 =& \|\delta_x u^n\|^2 + \|\delta_y u^n\|^2 + \|\delta_z u^n\|^2,\\
\|u^n\|^2_{H^1} =& \|u^n\|^2 + \|\nabla_h u^n\|^2,\\
\|u^n\|^2_A =& \|\nabla_h u^n\|^2 + \mu_n^2(\|\delta_x\delta_y^2 u^n\|^2 + \|\delta_y\delta_x^2 u^n\|^2 + \|\delta_x\delta_z^2 u^n\|^2 + \|\delta_z\delta_x^2 u^n\|^2\\
&+ \|\delta_y\delta_z^2 u^n\|^2 + \|\delta_z\delta_y^2 u^n\|^2 + 3\|\delta_x\delta_y\delta_z u^n\|^2) + \mu_n^3(\|\delta_x^2\delta_y\delta_z u^n\|^2\\
&+ \|\delta_x\delta_y^2\delta_z u^n\|^2 + \|\delta_x\delta_y\delta_z^2 u^n\|^2).
\end{aligned}$$

3.1 Stability analysis

In this subsection, we first prove a lemma in order to get the stability analysis.

Lemma 3.1 *For any grid functions u, $w \in U_h$, we have*

$$\begin{aligned}
&-h_x h_y h_z \sum_{r=1}^{\chi_1-1}\sum_{j=1}^{\chi_2-1}\sum_{k=1}^{\chi_3-1}\left(u^n_{rjk} + \mu_n^2(\delta_x^2\delta_y^2 + \delta_x^2\delta_z^2 + \delta_y^2\delta_z^2)u^n_{rjk} - \mu_n^3\delta_x^2\delta_y^2\delta_z^2 u^n_{rjk}\right)\Delta_h w^n_{rjk}\\
&\le \|u^n\|_A\|w^n\|_A,
\end{aligned} \tag{11}$$

where the equality holds when $u = w$.

Proof By utilizing the Cauchy-Schwartz inequality, we get

$$
\begin{aligned}
&-h_x h_y h_z \sum_{r=1}^{\chi_1-1}\sum_{j=1}^{\chi_2-1}\sum_{k=1}^{\chi_3-1}\left(u^n_{rjk} + \mu_n^2(\delta_x^2\delta_y^2 + \delta_x^2\delta_z^2 + \delta_y^2\delta_z^2)u^n_{rjk} - \mu_n^3\delta_x^2\delta_y^2\delta_z^2 u^n_{rjk}\right)\delta_x^2 w^n_{rjk}\\
&= h_x h_y h_z \sum_{r=1}^{\chi_1-1}\sum_{j=1}^{\chi_2-1}\sum_{k=1}^{\chi_3-1}(\delta_x u^n_{r-\frac{1}{2},j,k})(\delta_x w^n_{r-\frac{1}{2},j,k}) + \mu_n^2 h_x h_y h_z \sum_{r=1}^{\chi_1-1}\sum_{j=1}^{\chi_2-1}\sum_{k=1}^{\chi_3-1}\\
&\quad(\delta_x\delta_y\delta_z u^n_{r-\frac{1}{2},j-\frac{1}{2},k-\frac{1}{2}})(\delta_x\delta_y\delta_z w^n_{r-\frac{1}{2},j-\frac{1}{2},k-\frac{1}{2}}) + \mu_n^2 h_x h_y h_z \sum_{r=1}^{\chi_1-1}\sum_{j=1}^{\chi_2-1}\sum_{k=1}^{\chi_3-1}(\delta_z\delta_x^2 u^n_{r,j,k-\frac{1}{2}})\\
&\quad(\delta_z\delta_x^2 w^n_{r,j,k-\frac{1}{2}}) + \mu_n^2 h_x h_y h_z \sum_{r=1}^{\chi_1-1}\sum_{j=1}^{\chi_2-1}\sum_{k=1}^{\chi_3-1}(\delta_y\delta_x^2 u^n_{r,j-\frac{1}{2},k})(\delta_y\delta_x^2 w^n_{r,j-\frac{1}{2},k})\\
&\quad+\mu_n^3 h_x h_y h_z \sum_{r=1}^{\chi_1-1}\sum_{j=1}^{\chi_2-1}\sum_{k=1}^{\chi_3-1}(\delta_x^2\delta_y\delta_z u^n_{r,j-\frac{1}{2},k-\frac{1}{2}})(\delta_x^2\delta_y\delta_z w^n_{r,j-\frac{1}{2},k-\frac{1}{2}})\\
&\le \|\delta_x u^n\|\|\delta_x w^n\| + \mu_n^2(\|\delta_y\delta_x^2 u^n\|\|\delta_y\delta_x^2 w^n\| + \|\delta_z\delta_x^2 u^n\|\|\delta_z\delta_x^2 w^n\|\\
&\quad+\|\delta_x\delta_y\delta_z u^n\|\|\delta_x\delta_y\delta_z w^n\|) + \mu_n^3\|\delta_x^2\delta_y\delta_z u^n\|\|\delta_x^2\delta_y\delta_z w^n\|.
\end{aligned}
$$

In addition, taking the inner product with $\delta_y^2 w^n_{rjk}$ and $\delta_z^2 w^n_{rjk}$, we will give similar results as follows:

$$
\begin{aligned}
&-h_x h_y h_z \sum_{r=1}^{\chi_1-1}\sum_{j=1}^{\chi_2-1}\sum_{k=1}^{\chi_3-1}\left(u^n_{rjk} + \mu_n^2(\delta_x^2\delta_y^2 + \delta_x^2\delta_z^2 + \delta_y^2\delta_z^2)u^n_{rjk} - \mu_n^3\delta_x^2\delta_y^2\delta_z^2 u^n_{rjk}\right)\delta_y^2 w^n_{rjk}\\
&\le\|\delta_y u^n\|\|\delta_y w^n\| + \mu_n^2(\|\delta_x\delta_y^2 u^n\|\|\delta_x\delta_y^2 w^n\| + \|\delta_z\delta_y^2 u^n\|\|\delta_z\delta_y^2 w^n\| + \|\delta_x\delta_y\delta_z u^n\|\|\delta_x\delta_y\delta_z w^n\|)\\
&\quad+\mu_n^3\|\delta_y^2\delta_x\delta_z u^n\|\|\delta_y^2\delta_x\delta_z w^n\|,\\
&-h_x h_y h_z \sum_{r=1}^{\chi_1-1}\sum_{j=1}^{\chi_2-1}\sum_{k=1}^{\chi_3-1}\left(u^n_{rjk} + \mu_n^2(\delta_x^2\delta_y^2 + \delta_x^2\delta_z^2 + \delta_y^2\delta_z^2)u^n_{rjk} - \mu_n^3\delta_x^2\delta_y^2\delta_z^2 u^n_{rjk}\right)\delta_z^2 w^n_{rjk}\\
&\le\|\delta_z u^n\|\|\delta_z w^n\| + \mu_n^2(\|\delta_x\delta_z^2 u^n\|\|\delta_x\delta_z^2 w^n\| + \|\delta_y\delta_z^2 u^n\|\|\delta_y\delta_z^2 w^n\| + \|\delta_x\delta_y\delta_z u^n\|\|\delta_x\delta_y\delta_z w^n\|)\\
&\quad+\mu_n^3\|\delta_z^2\delta_x\delta_y u^n\|\|\delta_z^2\delta_x\delta_y w^n\|.
\end{aligned}
$$

By adding the above three inequalities together, and using the Cauchy-Schwartz inequality once again, we get

$$
\begin{aligned}
&-h_x h_y h_z \sum_{r=1}^{\chi_1-1}\sum_{j=1}^{\chi_2-1}\sum_{k=1}^{\chi_3-1}\left(u^n_{rjk} + \mu_n^2(\delta_x^2\delta_y^2+\delta_x^2\delta_z^2+\delta_y^2\delta_z^2)u^n_{rjk} - \mu_n^3\delta_x^2\delta_y^2\delta_z^2 u^n_{rjk}\right)\Delta_h w^n_{rjk}\\
&\le\|\delta_x u^n\|\|\delta_x w^n\| + \mu_n^2(\|\delta_y\delta_x^2 u^n\|\|\delta_y\delta_x^2 w^n\| + \|\delta_z\delta_x^2 u^n\|\|\delta_z\delta_x^2 w^n\| + \|\delta_x\delta_y\delta_z u^n\|\|\delta_x\delta_y\delta_z w^n\|)\\
&\quad+\mu_n^3\|\delta_x^2\delta_y\delta_z u^n\|\|\delta_x^2\delta_y\delta_z w^n\| + \|\delta_y u^n\|\|\delta_y w^n\| + \mu_n^2(\|\delta_x\delta_y^2 u^n\|\|\delta_x\delta_y^2 w^n\| + \|\delta_z\delta_y^2 u^n\|\|\delta_z\delta_y^2 w^n\|
\end{aligned}
$$

$$
\begin{aligned}
&+\|\delta_x\delta_y\delta_z u^n\|\|\delta_x\delta_y\delta_z w^n\|)+\mu_n^3\|\delta_y^2\delta_x\delta_z u^n\|\|\delta_y^2\delta_x\delta_z w^n\|+\|\delta_z u^n\|\|\delta_z w^n\|+\mu_n^2(\|\delta_x\delta_z^2 u^n\|\cdot\\
&\|\delta_x\delta_z^2 w^n\|+\|\delta_y\delta_z^2 u^n\|\|\delta_y\delta_z^2 w^n\|+\|\delta_x\delta_y\delta_z u^n\|\|\delta_x\delta_y\delta_z w^n\|)+\mu_n^3\|\delta_z^2\delta_x\delta_y u^n\|\|\delta_z^2\delta_x\delta_y w^n\|\\
\le&\Big[\|\nabla_h u^n\|^2+\mu_n^2(\|\delta_y\delta_x^2 u^n\|^2+\|\delta_x\delta_y^2 u^n\|^2+\|\delta_x\delta_z^2 u^n\|^2+\|\delta_z\delta_x^2 u^n\|^2+\|\delta_z\delta_y^2 u^n\|^2\\
&+\|\delta_y\delta_z^2 u^n\|^2+3\|\delta_x\delta_y\delta_z u^n\|^2)+\mu_n^3(\|\delta_x^2\delta_y\delta_z u^n\|^2+\|\delta_y^2\delta_x\delta_z u^n\|^2+\|\delta_z^2\delta_x\delta_y u^n\|^2)\Big]^{\frac{1}{2}}\\
&\Big[\|\nabla_h w^n\|^2+\mu_n^2(\|\delta_y\delta_x^2 w^n\|^2+\|\delta_x\delta_y^2 w^n\|^2+\|\delta_x\delta_z^2 w^n\|^2+\|\delta_z\delta_x^2 w^n\|^2+\|\delta_z\delta_y^2 w^n\|^2\\
&+\|\delta_y\delta_z^2 w^n\|^2+3\|\delta_x\delta_y\delta_z w^n\|^2)+\mu_n^3(\|\delta_x^2\delta_y\delta_z w^n\|^2+\|\delta_y^2\delta_x\delta_z w^n\|^2+\|\delta_z^2\delta_x\delta_y w^n\|^2)\Big]^{\frac{1}{2}}\\
\le&\|u^n\|_A\|w^n\|_A.
\end{aligned}
$$

The proof is finished. □

Next, by an improved discrete fractional Grö̈nwall inequality, we yield the following lemmas.

Lemma 3.2 [34, 35] *Suppose that the sequences* $\{s_n\}_{n=1}^{\infty}$, $\{l_n\}_{n=1}^{\infty}$ *are nonnegative, if the grid function* $\{u^n : n=0,1,\cdots,N\}$ *satisfies* $u_0 \ge 0$ *and for* $n=1,2,\cdots,N$ *satisfying* $(D_N^{\alpha}u^n)u^n \le s^n u^n + (l^n)^2$, *then it holds that*

$$
u^n \le u^0 + \mu_n \sum_{i=1}^{n} \vartheta_{n,i}(s_i + l_i) + \max_{1\le i\le n}\left\{l^i\right\},
$$

where $\vartheta_{n,n}=1$, $\vartheta_{n,i}=\sum_{p=1}^{n-i}\frac{1}{d_{n,p}-d_{n,p+1}}\vartheta_{n-p,i}>0$, $n=1,2,\cdots,N$, $i=1,2,\cdots,n-1$.

Lemma 3.3 [36] *For* $n=1,\cdots,N$, *we obtain*

$$
\mu_n \sum_{i=1}^{n} \vartheta_{n,i} \le C t_n^{\alpha}.
$$

Theorem 3.4 *Suppose* $\{u_{rjk}^n | 0\le r\le \chi_1, 0\le j\le \chi_2, 0\le k\le \chi_3, 1\le n\le N\}$ *is the solution of discrete problem* (8)–(10), *where* $u_{rjk}^n=0$ *if* $(x_r,y_j,z_k)\in\partial\Psi_h$, *then we have*

$$
\|u^n\|_A \le \|u^0\|_A + C(t_n^{\alpha}+1)\max_{1\le i\le n}\{\|f^i\|\}. \tag{12}
$$

Proof At first, multiplying both sides of (8) by $-h_xh_yh_z\Delta_h u_{rjk}^n$, and summing over r, j, k for $(x_r,y_j,z_k)\in\Upsilon_h$, we get

$$
\begin{aligned}
&-d_{n,1}h_xh_yh_z\sum_{r=1}^{\chi_1-1}\sum_{j=1}^{\chi_2-1}\sum_{k=1}^{\chi_3-1}\left(u_{rjk}^n+\mu_n^2(\delta_x^2\delta_y^2+\delta_x^2\delta_z^2+\delta_y^2\delta_z^2)u_{rjk}^n-\mu_n^3\delta_x^2\delta_y^2\delta_z^2u_{rjk}^n\right)\Delta_h u_{rjk}^n\\
&+\|\Delta_h u^n\|^2=d_{n,n}[-h_xh_yh_z\sum_{r=1}^{\chi_1-1}\sum_{j=1}^{\chi_2-1}\sum_{k=1}^{\chi_3-1}(u_{rjk}^0+\mu_n^2(\delta_x^2\delta_y^2+\delta_x^2\delta_z^2+\delta_y^2\delta_z^2)u_{rjk}^0-
\end{aligned}
$$

$$\mu_n^3\delta_x^2\delta_y^2\delta_z^2u_{rjk}^0)\Delta_h u_{rjk}^n] + \sum_{p=1}^{n-1}(d_{n,p}-d_{n,p+1})[-h_xh_yh_z\sum_{r=1}^{\chi_1-1}\sum_{j=1}^{\chi_2-1}\sum_{k=1}^{\chi_3-1}(u_{rjk}^{n-p}+$$

$$\mu_n^2(\delta_x^2\delta_y^2+\delta_x^2\delta_z^2+\delta_y^2\delta_z^2)u_{rjk}^{n-p}-\mu_n^3\delta_x^2\delta_y^2\delta_z^2u_{rjk}^{n-p})\Delta_h u_{rjk}^n]-h_xh_yh_z\sum_{r=1}^{\chi_1-1}\sum_{j=1}^{\chi_2-1}\sum_{k=1}^{\chi_3-1}f_{rjk}^n\Delta_h u_{rjk}^n.$$

Then, by Lemma 3.1 and using the Cauchy-Schwartz inequality and Young's inequality, we obtain

$$d_{n,1}\|u^n\|_A^2+\|\Delta_h u^n\|^2 \le d_{n,n}\|u^0\|_A\|u^n\|_A+\sum_{p=1}^{n-1}(d_{n,p}-d_{n,p+1})\|u^{n-p}\|_A\|u^n\|_A$$
$$+\frac{1}{4}\|f^n\|^2+\|\Delta_h u^n\|^2.$$

Thus, we yield

$$d_{n,1}\|u^n\|_A^2 \le d_{n,n}\|u^0\|_A\|u^n\|_A+\sum_{p=1}^{n-1}(d_{n,p}-d_{n,p+1})\|u^{n-p}\|_A\|u^n\|_A+\frac{1}{4}\|f^n\|^2,$$

which obtains

$$(D_N^\alpha\|u^n\|_A)\|u^n\|_A \le \frac{1}{4}\|f^n\|^2. \tag{13}$$

Finally by applying Lemma 3.2 to (13) and Lemma 3.3, we have

$$\begin{aligned}\|u^n\|_A &\le \|u^0\|_A+\mu_n\sum_{i=1}^{n}\vartheta_{n,i}\max_{1\le i\le n}\{\|f^i\|\}+\frac{1}{2}\max_{1\le i\le n}\{\|f^i\|\}\\ &\le \|u^0\|_A+C(t_n^\alpha+1)\max_{1\le i\le n}\{\|f^i\|\}.\end{aligned}$$

The proof is finished. □

3.2 Convergence analysis

In this subsection, the H^1-norm convergence of the ADI scheme (8)-(10) will be considered.

Define the notation

$$e_{rjk}^n := u(x_r,y_j,z_k,t_n)-u_{rjk}^n,\quad (x_r,y_j,z_k)\in\Psi_h,\quad 0\le n\le N.$$

Next, for further analysis, several significant lemmas are presented as follows.

Lemma 3.5 *[33] Suppose that $|\partial_t^q(\cdot,t)| \le C(1+t^{\alpha-1})$, where $q=0,1,2$. Then for all $(x_r, y_j, z_k, t_n) \in \Upsilon$, one has*

$$|D_N^\alpha u_{rjk}^n - D_n^\alpha u(x_r, y_j, z_k, t_n)| \le C t_n^{-\alpha} N^{-\min\{\alpha+1,(2-\alpha)/\gamma\}}. \tag{14}$$

Lemma 3.6 *[36] Set $\mathcal{E}^n = N^{-\gamma} t_n^{\alpha-1}$ for $1 < \gamma < 2-\alpha$, $\mathcal{E}^n = N^{\alpha-2} t_n^{\alpha-1}[1+\ln(t_n/t_1)]$ for $\gamma = 2-\alpha$, $\mathcal{E}^n = N^{\alpha-2} t_n^{\alpha-(2-\alpha)/\gamma}$ for $\gamma > 2-\alpha$. Then for $n = 1, 2, \cdots, N$, we have*

$$\mu_n \sum_{i=1}^{n} i^{-\min\{\alpha+1,(2-\alpha)/\gamma\}} \vartheta_{n,i} \le C\mathcal{E}^n, \tag{15}$$

in which the constant C is α-robust.

Denote

$$R_t u_{rjk}^n := (I + \mu_n^2\delta_x^2\delta_y^2 + \mu_n^2\delta_x^2\delta_z^2 + \mu_n^2\delta_y^2\delta_z^2 - \mu_n^3\delta_x^2\delta_y^2\delta_z^2) D_N^\alpha u_{rjk}^n - D_t^\alpha u(x_r, y_j, z_k),$$
$$R_h u_{rjk}^n := \Delta u(x_r, y_j, z_k, t_n) - \Delta_h u_{rjk}^n,$$

where the small term $(\mu_n^2\delta_x^2\delta_y^2 + \mu_n^2\delta_x^2\delta_z^2 + \mu_n^2\delta_y^2\delta_z^2 - \mu_n^3\delta_x^2\delta_y^2\delta_z^2) D_N^\alpha u_{rjk}^n$ have the truncation error $O(N^{-2\alpha})$, $\|R_h u^n\| = O(h_x^2 + h_y^2 + h_z^2)$.

Then subtracting (2)–(3) from (8)–(10), the error equation are obtained as follows:

$$\begin{aligned} & (I + \mu_n^2\delta_x^2\delta_y^2 + \mu_n^2\delta_x^2\delta_z^2 + \mu_n^2\delta_y^2\delta_z^2 - \mu_n^3\delta_x^2\delta_y^2\delta_z^2) D_N^\alpha e_{rjk}^n - \Delta_h e_{rjk}^n \\ = \; & R_t u_{rjk}^n + R_h u_{rjk}^n, \quad (x_r, y_j, z_k) \in \Upsilon_h, \quad 1 \le n \le N. \end{aligned} \tag{16}$$
$$e_{rjk}^n = 0 \quad (x_r, y_j, z_k) \in \partial\Psi_h, \quad 1 \le n \le N, \tag{17}$$
$$e_{rjk}^0 = 0, \quad (x_r, y_j, z_k) \in \overline{\Upsilon}_h. \tag{18}$$

Theorem 3.7 *Suppose that $|\partial_t^q(\cdot,t)| \le C(1+t^{\alpha-1})$ for $q=0,1,2$. Then there exists a α-robust constant C such that*

$$\|u(\cdot,t_n) - u^n\|_{H^1} \le C(h_x^2 + h_y^2 + h_z^2 + N^{-\min\{\gamma\alpha, 2-\alpha, 2\alpha\}}), \tag{19}$$

in which the constant C does not blow up when $\alpha \to 1^-$.

Proof Multiplying $-h_x h_y h_z \Delta_h e_{rjk}^n$ to both sides of (17) and summing over r, j, k for $(x_r, y_j, z_k) \in \Upsilon_h$, we can get

$$d_{n,1}\|e^n\|_A^2 + \|\Delta_h e^n\|^2 \le d_{n,n}\|e^0\|_A \|e^n\|_A + \sum_{p=1}^{n-1}(d_{n,p} - d_{n,p+1})\|e^{n-p}\|_A \|e^n\|_A$$

$$-h_xh_yh_z\sum_{r=1}^{\chi_1-1}\sum_{j=1}^{\chi_2-1}\sum_{k=1}^{\chi_3-1}R_tu_{rjk}^n\Delta_he_{rjk}^n-h_xh_yh_z\sum_{r=1}^{\chi_1-1}\sum_{j=1}^{\chi_2-1}\sum_{k=1}^{\chi_3-1}R_hu_{rjk}^n\Delta_he_{rjk}^n. \tag{20}$$

Since

$$-h_xh_yh_z\sum_{r=1}^{\chi_1-1}\sum_{j=1}^{\chi_2-1}\sum_{k=1}^{\chi_3-1}R_tu_{rjk}^n\Delta_he_{rjk}^n\le\|\nabla_hR_tu^n\|\cdot\|\nabla_he^n\|\le\|\nabla_hR_tu^n\|\cdot\|e^n\|_A, \tag{21}$$

and

$$-h_xh_yh_z\sum_{r=1}^{\chi_1-1}\sum_{j=1}^{\chi_2-1}\sum_{k=1}^{\chi_3-1}R_hu_{rjk}^n\Delta_he_{rjk}^n\le\|\Delta_he^n\|^2+\frac{1}{4}\|R_hu^n\|^2. \tag{22}$$

Then substituting (21) and (22) into (20), we have

$$\begin{aligned}d_{n,1}\|e^n\|_A^2\le d_{n,n}\|e^0\|_A\|e^n\|_A&+\sum_{p=1}^{n-1}(d_{n,p}-d_{n,p+1})\|e^{n-p}\|_A\|e^n\|_A\\&+\|\nabla_hR_tu^n\|\cdot\|e^n\|_A+\frac{1}{4}\|R_hu^n\|^2,\end{aligned}$$

that is,

$$(D_N^\alpha\|e^n\|_A)\|e^n\|_A\le\|\nabla_hR_tu^n\|\cdot\|e^n\|_A+\frac{1}{4}\|R_hu^n\|^2. \tag{23}$$

From (23), using Lemma 3.2 and noting that $\|e^0\|_A=0$, one has

$$\|e^n\|_A\le C\mu_n\sum_{i=1}^{n}\vartheta_{n,i}(\|\nabla_hR_tu^i\|+\frac{1}{4}\|R_hu^i\|)+\max_{1\le i\le n}\left\{\frac{1}{4}\|R_hu^i\|\right\}. \tag{24}$$

Then utilizing Lemmas 3.3, 3.5, and 3.6, one gets

$$\begin{aligned}\|e^n\|_A\le&C\mu_n\sum_{i=1}^{n}\vartheta_{n,i}(h_x^2+h_y^2+h_z^2+t_n^{-\alpha}N^{-\min\{\alpha+1,(2-\alpha)/\gamma\}}+t_n^{-\alpha}N^{-2\alpha})\\&+C(h_x^2+h_y^2+h_z^2)\\\le&C(h_x^2+h_y^2+h_z^2+N^{-\min\{\gamma\alpha,2-\alpha,2\alpha\}}).\end{aligned} \tag{25}$$

At last, utilizing the definition of $\|\cdot\|_A$ and from [32, Lemma 2.2], we know $\|e^n\|_{H^1}\le C\|e^n\|_A$. The proof of the theorem is finished. □

4 Numerical experiment

In this section, we conduct two numerical experiments on the proposed ADI difference scheme, which show that the numerical results are in agreement with the theoretical analysis results. Set parameters $\mathcal{L}_x = \mathcal{L}_y = \mathcal{L}_z = \pi$, $\chi_1 = \chi_2 = \chi_3$, and $T = 1$.

Define the H^1-norm error $E(\tau, h)$ of the numerical solutions by

$$E_{H^1}(\tau, h) = \sqrt{\|e^n\|^2 + \|\nabla_h e^n\|^2}.$$

The spatial and temporal convergence orders are calculated respectively by

$$Rate^x = \log_2\left[\frac{E_{H^1}(\tau, 2h)}{E_{H^1}(\tau, h)}\right], \quad Rate^t = \log_2\left[\frac{E_{H^1}(2\tau, h)}{E_{H^1}(\tau, h)}\right].$$

Example 1 In the first example, we consider the following problem:

$$\begin{aligned}
&D_t^\alpha u(x, y, z, t) - \Delta u(x, y, z, t) = f(x, y, z, t), \quad (x, y, z) \in \Upsilon, \quad t \in (0, T],\\
&u(x, y, z, t) = 0, \quad (x, y, z) \in \partial\Upsilon, \quad t \in (0, T],\\
&u(x, y, z, 0) = 0, \quad (x, y, z) \in \Upsilon,
\end{aligned}$$

where the source term is

$$f(x, y, z, t) = \Gamma(1+\alpha)\sin x \sin y \sin z + 3t^\alpha \sin x \sin y \sin z.$$

The exact solution is $u(x, y, z, t) = t^\alpha \sin x \sin y \sin z$.

Table 1 lists the numerical results about the H^1-norm errors, the temporal convergence rates, and the CPU time with different α and N when $\gamma = 2 - \alpha$ and $\chi_1 = \chi_2 = \chi_3 = \chi = 128$. We can clearly see that the ADI scheme (8)–(10)

Table 1 H^1-norm errors and the temporal convergence rates for $\chi = 128$ with $\gamma = 2 - \alpha$ for Example 1

	$\alpha = 0.2$			$\alpha = 0.4$		
N	$E_{H^1}(\tau, h)$	$Rate^t$	CPU(s)	$E_{H^1}(\tau, h)$	$Rate^t$	CPU(s)
4	1.3661e-0	–	32.513	9.1094e-1	–	32.497
8	1.1232e-0	0.2824	87.607	5.6976e-1	0.6770	88.205
16	9.0493e-1	0.3117	268.63	3.4179e-1	0.7372	269.64
32	7.1699e-1	0.3359	917.34	1.9994e-1	0.7736	912.70
	$\alpha = 0.6$			$\alpha = 0.8$		
N	$E_{H^1}(\tau, h)$	$Rate^t$	CPU(s)	$E_{H^1}(\tau, h)$	$Rate^t$	CPU(s)
4	5.9914e-1	–	32.060	3.8784e-1	–	31.942
8	2.7754e-1	1.1102	86.150	1.3675e-1	1.5040	87.370
16	1.2272e-1	1.1774	266.48	4.6830e-2	1.5460	266.81
32	5.3271e-2	1.2039	912.08	1.6058e-2	1.5441	904.85

Table 2 H^1-norm errors and the temporal convergence rates with $\gamma = 2(2-\alpha)$ for Example 1 when $\chi = 128$ and $t = T$

χ	$\alpha = 0.15$		$\alpha = 0.25$		$\alpha = 0.30$		$\alpha = 0.45$	
	$E_{H^1}(\tau, h)$	$Rate^t$	$E_{H^1}(\tau, h)$	$Rate^t$	$E_{H^1}(\tau, h)$	$Rate^t$	$E_{H^1}(\tau, h)$	$Rate^t$
4	1.6300e-0	–	1.4504e-0	–	1.3351e-0	–	1.0809e-0	–
8	1.4648e-0	0.1542	1.1921e-0	0.2830	1.0315e-0	0.3721	6.9242e-1	0.6425
16	1.2855e-0	0.1884	9.3610e-1	0.3487	7.5521e-1	0.4498	4.0928e-1	0.7586
32	1.1085e-0	0.2137	7.1016e-1	0.3985	5.3135e-1	0.5072	2.2977e-1	0.8329

Table 3 H^1-norm errors and the spatial convergence rates for $N = 512$ with $\gamma = \frac{2-\alpha}{\alpha}$ for Example 1

χ	$\alpha = 0.6$			$\alpha = 0.8$		
	$E_{H^1}(\tau, h)$	$Rate^x$	CPU(s)	$E_{H^1}(\tau, h)$	$Rate^x$	CPU(s)
4	1.2262e-1	–	37.124	1.2046e-1	–	36.695
8	3.1420e-2	1.9645	200.25	3.2878e-2	1.8733	198.80
16	5.9406e-3	2.4030	963.63	8.4240e-3	1.9645	953.88
32	8.6023e-4	2.7878	4354.0	1.9254e-3	2.1294	4403.7

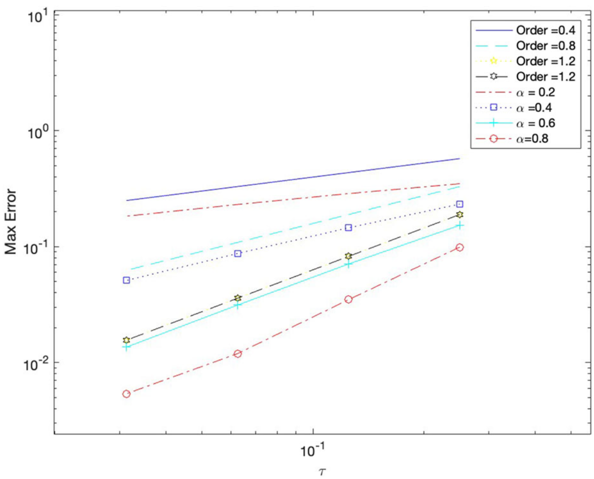

Fig. 1 The temporal convergence orders for Example 1

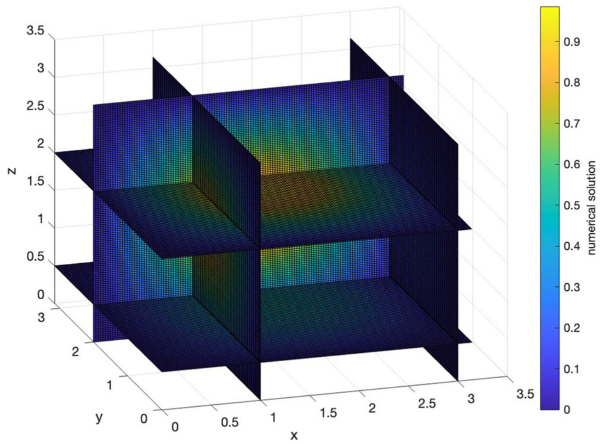

Fig. 2 The numerical solution surface with $\alpha = 0.6$, $\chi = 128$, and $N = 32$ when $\gamma = 2 - \alpha$ for Example 1

obtain $O(N^{-\min\{\gamma\alpha,2-\alpha,2\alpha\}})$ accuracy in time as predicted. In Table 2, we give the H^1-norm errors and the temporal convergence rates of different small fractional order α for $\gamma = 2(2-\alpha)$, $\chi = 128, andt = T$. It can be seen that when α is a small value, the convergence order of 2α is basically satisfied. In Table 3, fixing $N = 128$, $\gamma = (2-\alpha)/\alpha$ and choosing different α, the second-order accuracy can be clearly observed in space. In addition, the CPU times in seconds of Tables 1 and 3 are given. Figure 1 shows the order of temporal convergence. Figures 2 and 3 show the values of the numerical solutions and the exact solutions obtained at different cross sections of the three-dimensional space when $\alpha = 0.6$, $\chi = 128$, $N = 32$, and $\gamma = 2 - \alpha$, respectively, where the color represents the calculated amplitude.

In Tables 4, 5, and 6, we compare the H^1-norm errors and the temporal convergence rates of our 3D ADI scheme (8)–(10) with the 2D ADI scheme in [23] for different γ.

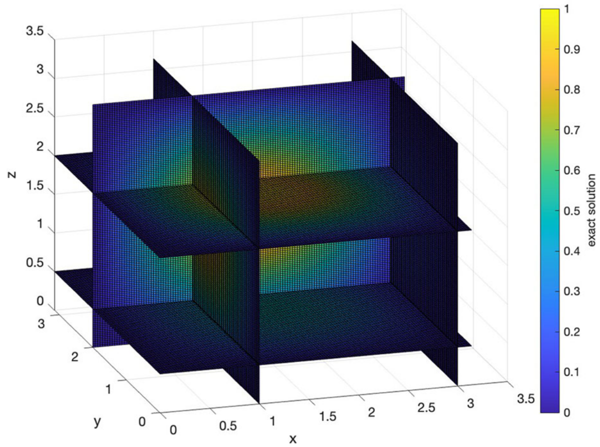

Fig. 3 The exact solution surface with $\alpha = 0.6$, $\chi = 128$, and $N = 32$ when $\gamma = 2 - \alpha$ for Example 1

Table 4 Comparison in time with $\gamma = 2 - \alpha$ and $\chi = 128$ for Example 1

α	N	The present 3D ADI scheme		The 2D ADI scheme in [23]	
		$E_{H^1}(\tau, h)$	$Rate^t$	$E_{H^1}(\tau, h)$	$Rate^t$
	4	1.3661e-0	–	4.3891e-1	–
$\alpha = 0.2$	8	1.1232e-0	0.2824	3.3492e-1	0.3349
	16	9.0493e-1	0.3117	2.7352e-1	0.3474
	32	7.1699e-1	0.3359	2.1317e-1	0.3596

Numerical results show that our ADI scheme is effective for both three-dimensional and two-dimensional, and there is no significant increase in CPU time for 3D.

Example 2 In this example, we consider the following problem:

$$\begin{aligned} & D_t^\alpha u(x, y, z, t) - \Delta u(x, y, z, t) = f(x, y, z, t), \quad (x, y, z) \in \Upsilon, \quad t \in (0, T], \\ & u(x, y, z, t) = 0, \quad (x, y, z) \in \partial\Upsilon, \quad t \in (0, T], \\ & u(x, y, z, 0) = \sin x \sin y \sin z, \quad (x, y, z) \in \Upsilon, , \end{aligned} \tag{26}$$

where the forcing term is

$$f(x, y, z, t) = \left[3(t^\alpha + t^3) + \Gamma(1+\alpha) + \frac{6t^{3-\alpha}}{\Gamma(4-\alpha)}\right] \sin x \sin y \sin z.$$

The exact solution u is unknown; we take different α of the numerical solution to verify the feasibility of the algorithm when spacial node number $\chi = 128$, temporal node number $N = 32$, and the grid mesh $\gamma = 2 - \alpha$. From Figs. 4 and 5, we can see that the method still works properly in this case.

5 Conclusion

This study proposes an efficient ADI scheme for three-dimensional subdiffusion problems on graded mesh. We obtain a fully discrete scheme with space discretized by the standard second-order finite difference and the Caputo fractional derivative discretized

Table 5 Comparison in time with $\gamma = 2$ and $\chi = 128$ for Example 1

α	N	The present 3D ADI scheme		The 2D ADI scheme in [23]	
		$E_{H^1}(\tau, h)$	$Rate^t$	$E_{H^1}(\tau, h)$	$Rate^t$
$\alpha = 0.2$	4	1.3925e-0	–	4.4872e-1	–
	8	1.1527e-0	0.2727	3.5819e-1	0.3251
	16	9.3289e-1	0.3052	2.8267e-1	0.3416
	32	7.4146e-1	0.3313	2.2083e-2	0.3562

Table 6 Comparison in time with $\gamma = \frac{2-\alpha}{\alpha}$ and $\chi = 128$ for Example 1

α	N	The present 3D ADI scheme		The 2D ADI scheme in [23]	
		$E_{H^1}(\tau, h)$	$Rate^t$	$E_{H^1}(\tau, h)$	$Rate^t$
	4	2.1452e-1	–	2.8041e-1	–
$\alpha = 0.6$	8	1.1060e-1	0.9557	1.3696e-2	1.0338
	16	5.1856e-2	1.0928	6.2911e-2	1.1224
	32	2.3154e-2	1.1633	2.7959e-2	1.1700

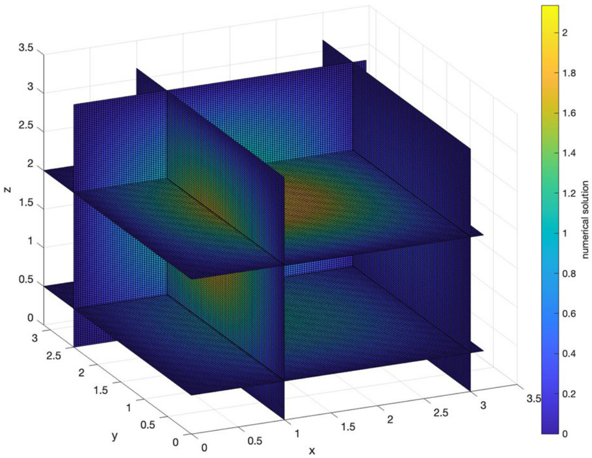

Fig. 4 The numerical solution surface with $\alpha = 0.6$, $\chi = 128$, and $N = 32$ when $\gamma = 2 - \alpha$ for Example 2

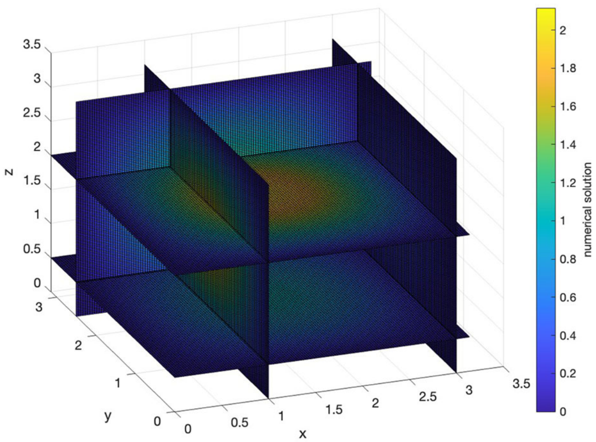

Fig. 5 The numerical solution surface with $\alpha = 0.8$, $\chi = 128$, and $N = 32$ when $\gamma = 2 - \alpha$ for Example 2

by L1 scheme. ADI algorithm is used to transform the three-dimensional problem into three one-dimensional problems, which greatly reduces the computational cost. The theoretical analysis shows that our scheme is unconditionally stable with the spatial convergence order of two and the temporal convergence order of $\min\{\gamma\alpha, 2-\alpha, 2\alpha\}$. We give two numerical examples and also compare our present 3D ADI scheme with the 2D ADI scheme in [23]. Numerical results show that our ADI scheme is very effective in solving the three-dimensional subdiffusion problems. In further work, we will consider applying this method to nonlinear problems, such as phase field equations.

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Author Contributions Haixiang Zhang provided the methodology along with the model problem. Ziyi Zhou implemented the scheme, obtained the error analysis and the numerical experiments and writing the manuscript. Xuehua Yang revised the manuscript for important content.

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Data Availability The datasets generated during and/or analyzed during the current study are available from the corresponding author on reasonable request.

Declarations

Ethical approval The authors consent to participate and consent to publish of this manuscript.

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