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ORIGINAL PAPER

Convergence and stability of the Milstein scheme for stochastic differential equations with piecewise continuous arguments

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Abstract
This work develops the Milstein scheme for commutative stochastic differential equations with piecewise continuous arguments (SDEPCAs), which can be viewed as stochastic differential equations with time-dependent and piecewise continuous delay. As far as we know, although there have been several papers investigating the convergence and stability for different numerical methods on SDEPCAs, all of these methods are Euler-type methods and the convergence orders do not exceed 1/2. Accordingly, we first construct the Milstein scheme for SDEPCAs in this work and then show its convergence order can reach 1. Moreover, we prove that the Milstein method can preserve the stability of SDEPCAs. In the last section, we provide several numerical examples to verify the theoretical results.

Keywords Stochastic differential equations with piecewise continuous arguments · Commutative noise · The Milstein method · Convergence order · Exponential stability

Mathematics Subject Classification (2010) 60H35 · 65C30 · 60H15

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1 Introduction

Differential equations with piecewise continuous arguments (EPCAs) are well used in control theory and some biomedical models ([1–4]). A typical EPCA is of the form

$$x'(t) = f(t, x(t), x(h(t))),$$

where the argument $h(t)$ has intervals of constancy. A potential application of EPCAs is the stabilization of hybrid control systems with feedback delay [1]. In recent years, some scholars further developed the theory of stabilization for hybrid stochastic differential equations by feedback control based on discrete-time state observations ([5, 6]), and this theory is actually based on the stability of the hybrid stochastic differential equation with piecewise continuous arguments (SDEPCA)

$$\mathrm{d}x(t) = (f(x(t), r(t), t) + u(x([t/\tau]\tau), r(t), t))\mathrm{d}t + g(x(t), r(t), t)\mathrm{d}\omega(t).$$

Therefore, the properties of SDEPCAs have received more and more consideration.

However, most of SDEPCAs do not have explicit solutions; hence, it is extremely important to solve them by numerical methods. Moreover, in order to achieve the required accuracy in many real-world problems, the development of higher-order numerical methods is necessary. But to our knowledge, the numerical methods currently developed for global Lipschitz continuous or highly nonlinear SDEPCAs are all Euler or Euler-type methods (such as the split-step theta method, the tamed Euler method, the truncated Euler method), and the convergence orders of all of these methods do not exceed one-half (see, e.g., [7–12]). Therefore, the main aim of this work is to construct a higher-order numerical scheme for SDEPCAs.

The Milstein scheme is a well-known numerical scheme for stochastic ordinary differential equations (SODEs) with a strong order of convergence one ([13–19]). Several scholars have further derived and analyzed the Milstein scheme for stochastic delay differential equations (SDDEs) [20–29]. However, most of these papers only consider the stochastic differential equations with constant delay [20–28], while an SDEPCA can be viewed as a stochastic differential equation with time-dependent delay, and the delay function is piecewise continuous and not differentiable. Therefore, it is worthwhile to construct the Milstein scheme for SDEPCAs.

In this work, we construct the Milstein scheme for SDEPCAs following the approach used by Kloeden et al. for SODEs [14] and SDDEs [29] and prove that the Milstein solution also converges strongly with order one to the exact solution of commutative SDEPCAs. It is worth mentioning that the Milstein scheme constructed in this paper contains only the derivatives of the coefficients f and g_j to the first component, which is different from the ones derived in the existing publications.

Moreover, whether the numerical method can preserve the stability of the exact solution is also an important criterion for the goodness of the numerical method [30–33]. Therefore, we also consider the stability of the Milstein method in this paper. The rest of this work is arranged as follows. Some basic lemmas and preliminaries are introduced in the second section. The Milstein scheme is developed, and its uniform boundedness in p-th moment is obtained in Sect. 3. Then, the strong convergence order of the Milstein method is proved in Sect. 4. The mean square exponential stability of the Milstein method is given in Sect. 5. Finally, several illustrative examples are given.

2 Notations and preliminaries

Throughout this paper, unless otherwise specified, we will use the following notations. $|x|$ denotes the Euclidean vector norm, and $\langle x, y\rangle$ denotes the inner product of vectors x, y. If A is a vector or matrix, its transpose is denoted by A^{T}. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{\mathrm{trace}(A^{\mathrm{T}}A)}$. For two real numbers a and b, we will use $a \vee b$ and $a \wedge b$ for the max $\{a, b\}$ and min $\{a, b\}$, respectively. $\mathbb{N} := \{0, 1, 2, \ldots, \}$. $[\cdot]$ denotes the greatest-integer function.

Moreover, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets), and let $\mathbb{E}$ denote the expectation corresponding to $\mathbb{P}$. Denote by $\mathcal{L}^p([0, T]; \mathbb{R}^n)$ the family of all $\mathbb{R}^n$-valued, $\mathcal{F}_t$-adapted processes $\{f(t)\}_{0\leq t\leq T}$ such that $\int_0^T |f(t)|^p dt < \infty$, a.s. Denote by $\mathcal{L}^p([0, \infty); \mathbb{R}^n)$ the family of process $\{f(t)\}_{t\geq 0}$ such that for every $T > 0$, $\{f(t)\}_{0\leq t\leq T} \in \mathcal{L}^p([0, T]; \mathbb{R}^n)$.

Let $B(t) = (B^1(t), \ldots, B^d(t))^{\mathrm{T}}$ is a d-dimensional Brownian motion defined on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$; we consider the following SDEPCA:

$$\mathrm{d}x(t) = f(x(t), x([t]))\mathrm{d}t + \sum_{j=1}^{d} g_j(x(t), x([t]))\mathrm{d}B^j(t) \tag{1}$$

on $t \geq 0$ with initial data $x(0) = x_0 \in \mathbb{R}^n$, where $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^{\mathrm{T}} \in \mathbb{R}^n$, $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $g_j : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $j = 1, 2, \ldots, d$. The definition of the exact solution for (1) is as follows.

Definition 1 [34] An $\mathbb{R}^n$-valued stochastic process $\{x(t), t \geq 0\}$ is called a solution of (1) on $[0, \infty)$, if it has the following properties:

- $\{x(t), t \geq 0\}$ is continuous and $\mathcal{F}_t$-adapted;
- $\{f(x(t), x([t]))\} \in \mathcal{L}^1([0, \infty); \mathbb{R}^n)$ and $\left\{g_j(x(t), x([t]))\right\} \in \mathcal{L}^2([0, \infty); \mathbb{R}^n)$;
- (1) is satisfied on each interval $[n, n+1) \subset [0, \infty)$ with integral end points almost surely.

A solution $\{x(t), t \geq 0\}$ is said to be *unique* if any other solution $\{\bar{x}(t), t \geq 0\}$ is indistinguishable from $\{x(t), t \geq 0\}$, that is,

$$\mathbb{P}\{x(t) = \bar{x}(t) \text{ for all } t \geq 0\} = 1.$$

We assume that the coefficients of (1) satisfy the following conditions.

Assumption 2.1 *Suppose $f(x, y)$ and $g_j(x, y)$ are continuously twice differentiable in $x \in \mathbb{R}^n$ with derivatives bounded as follows: for constant $M > 0$*

$$\left|\frac{\partial f(x, y)}{\partial x_k}\right| \vee \left|\frac{\partial g_j(x, y)}{\partial x_k}\right| \vee \left|\frac{\partial^2 f(x, y)}{\partial x_k \partial x_i}\right| \vee \left|\frac{\partial^2 g_j(x, y)}{\partial x_k \partial x_i}\right| \leq M$$

holds for all $x, y \in \mathbb{R}^n$, $k, i = 1, 2, \ldots n$, *and* $j = 1, 2, \ldots, d$, *where*

$$\begin{aligned}
\frac{\partial f(x,y)}{\partial x_k} &= \left(\frac{\partial f_1(x,y)}{\partial x_k}, \frac{\partial f_2(x,y)}{\partial x_k}, \cdots, \frac{\partial f_n(x,y)}{\partial x_k}\right)^T, \\
\frac{\partial g_j(x,y)}{\partial x_k} &= \left(\frac{\partial g_{1j}(x,y)}{\partial x_k}, \frac{\partial g_{2j}(x,y)}{\partial x_k}, \cdots, \frac{\partial g_{nj}(x,y)}{\partial x_k}\right)^T, \\
\frac{\partial^2 f(x,y)}{\partial x_k \partial x_i} &= \frac{\partial}{\partial x_i}\left(\frac{\partial f(x,y)}{\partial x_k}\right) = \left(\frac{\partial^2 f_1(x,y)}{\partial x_k \partial x_i}, \cdots, \frac{\partial^2 f_n(x,y)}{\partial x_k \partial x_i}\right)^T, \\
\frac{\partial^2 g_j(x,y)}{\partial x_k \partial x_i} &= \frac{\partial}{\partial x_i}\left(\frac{\partial g_j(x,y)}{\partial x_k}\right) = \left(\frac{\partial^2 g_{1j}(x,y)}{\partial x_k \partial x_i}, \cdots, \frac{\partial^2 g_{nj}(x,y)}{\partial x_k \partial x_i}\right)^T.
\end{aligned}$$

Remark 1 Under Assumption 2.1, for all $x, y, \bar{x} \in \mathbb{R}^n$,

$$|f(x,y) - f(\bar{x},y)| \vee |g_j(x,y) - g_j(\bar{x},y)| \leq \bar{M}|x - \bar{x}|, \tag{2}$$

where $\bar{M} = \sqrt{n}M$.

Proof For any $x, y, \bar{x} \in \mathbb{R}^n$, according to the mean value theorem of vector-valued function (see [35]), we have

$$\begin{aligned}
|f(x,y) - f(\bar{x},y)| &= \left|\frac{\partial f(\bar{x} + \theta(x - \bar{x}), y)}{\partial x}\right| |x - \bar{x}| \\
&= \sqrt{\sum_{k=1}^{n} \left|\frac{\partial f(\bar{x} + \theta(x - \bar{x}), y)}{\partial x_k}\right|^2} |x - \bar{x}| \\
&\leq \sqrt{n}M|x - \bar{x}|,
\end{aligned}$$

where $\theta \in (0, 1)$, $\frac{\partial f(x,y)}{\partial x} := \left(\frac{\partial f_l(x,y)}{\partial x_k}\right)_{l,k}$, $l, k = 1, 2, \ldots, n$. In the same way, we can also get

$$|g_j(x,y) - g_j(\bar{x},y)| \leq \sqrt{n}M|x - \bar{x}|.$$

The proof is completed. □

Assumption 2.2 *There exists a positive constant* L *such that*

$$|f(x,y) - f(x,\bar{y})| \vee |g_j(x,y) - g_j(x,\bar{y})| \leq L|y - \bar{y}| \tag{3}$$

for all $x, y, \bar{y} \in \mathbb{R}^n$.

Remark 2 Under Assumptions 2.1 and 2.2, there exist a constant $\bar{L} > 0$ such that f and g_j, $j = 1, \ldots, d$ satisfy the following linear growth condition:

$$|f(x,y)| \vee |g_j(x,y)| \leq \bar{L}(1 + |x| + |y|) \tag{4}$$

for all $x, y \in \mathbb{R}^n$.

Proof By (2) and (3), using the fundamental inequality $|a+b| \leq |a|+|b|$, one can obtain

$$
\begin{aligned}
|f(x,y)| \leq & |f(x,y)-f(0,y)|+|f(0,y)-f(0,0)|+|f(0,0)| \\
\leq & \bar{M}|x-0|+L|y-0|+|f(0,0)| \\
\leq & (\bar{M}+L+|f(0,0)|)(1+|x|+|y|).
\end{aligned}
$$

Similarly, it can also be proved that

$$
|g_j(x,y)| \leq (\bar{M}+L+|g_j(0,0)|)(1+|x|+|y|).
$$

Let $\bar{L}=\bar{M}+L+|f(0,0)|+\sum_{j=1}^{d}|g_j(0,0)|$; the proof is completed. □

Based on Theorem 1 in [36], one can obtain the existence and uniqueness of the exact solution for (1) on the interval $[n, n+1)$, $\forall n \in \mathbb{N}$, then the following existence and uniqueness of the solution holds on the whole time interval $[0,\infty)$ according to the continuity. For more details, one can also see Theorem 3.1 in [34]. Moreover, the proof of the following boundedness can be found in [37].

Lemma 2.3 *Under Assumptions 2.1 and 2.2, there is a unique global solution $x(t)$ to (1) on $t \geq 0$ with initial data $x(0)=x_0$. Moreover, for any $p \geq 2$, there is a positive constant C such that*

$$
\mathbb{E}\sup_{t\in[0,T]}|x(t)|^p < C,\ \forall T>0.
$$

Lemma 2.4 *[15, 38] Let $Z_1,\ldots,Z_N : \Omega \to \mathbb{R}$, $N \in \mathbb{N}$ be $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable mapping with $\mathbb{E}|Z_n|^p \leq \infty$ for all $n=1,2,\ldots,N$ and with $\mathbb{E}(Z_{n+1}|Z_1,\ldots,Z_n)=0$ for all $n=1,2,\ldots,N-1$. Then,*

$$
\|Z_1+\cdots+Z_n\|_{L^p} \leq C_p(\|Z_1\|^2_{L^p}+\cdots+\|Z_n\|^2_{L^p})^{\frac{1}{2}},
$$

for every $p\in[2,\infty)$, where $\|\cdot\|_{L^p} := (\mathbb{E}|\cdot|^p)^{1/p}$, C_p is a constant depend on p but independent of n.

3 The Milstein scheme

Let us now define the Milstein scheme for (1). Set $\Delta = 1/m$ be a given step size with integer $m \geq 1$, and let the grid points t_k be defined by $t_k = k\Delta (k=0,1,\ldots)$. For $x, y \in \mathbb{R}^n$, $j, r = 1,2,\ldots,d$, define

$$
\begin{aligned}
L^j g_r(x,y) &= \sum_{i=1}^{n} g_{ij}(x,y)\frac{\partial g_r(x,y)}{\partial x_i}, \\
I_{rj}(k) &= \int_{t_k}^{t_{k+1}}\int_{t_k}^{u} \mathrm{d}B^r(v)\mathrm{d}B^j(u).
\end{aligned}
$$

In this work, we only consider the SDEPCAs with diffusion coefficients g_j satisfies the so-called commutativity condition $L^j g_r(x, y) = L^r g_j(x, y)$, $j \neq r$.

Since for arbitrary $k \in \mathbb{N}$, there always exist $s \in \mathbb{N}$ and $l = 0, 1, 2, \ldots, m-1$ such that $k = sm + l$, the discrete Milstein solution $X_{sm+l} \approx x(t_{sm+l})$ is defined by

$$X_{sm+l+1} = X_{sm+l} + f(X_{sm+l}, X_{sm})\Delta + \sum_{j=1}^{d} g_j(X_{sm+l}, X_{sm})\Delta B^j_{sm+l} + \sum_{j,r=1}^{d} L^j g_r(X_{sm+l}, X_{sm}) I_{rj}(sm+l), \tag{5}$$

where $X_0 = x(0) = x_0$, $\Delta B^j_{sm+l} = B^j(t_{sm+l+1}) - B^j(t_{sm+l})$. Due to $I_{rj}(k) + I_{jr}(k) = \Delta B^j_k \Delta B^r_k$ for $r \neq j$, (5) can also be written as

$$X_{sm+l+1} = X_{sm+l} + f(X_{sm+l}, X_{sm})\Delta + \sum_{j=1}^{d} g_j(X_{sm+l}, X_{sm})\Delta B^j_{sm+l} + \frac{1}{2}\sum_{j,r=1}^{d} L^j g_r(X_{sm+l}, X_{sm})\Delta B^j_{sm+l}\Delta B^r_{sm+l} - \frac{1}{2}\sum_{j=1}^{d} L^j g_j(X_{sm+l}, X_{sm})\Delta. \tag{6}$$

Let

$$\bar{X}(t) = \sum_{sm+l=0}^{\infty} X_{sm+l} I_{[t_{sm+l}, t_{sm+l+1})}(t), \ t \geq 0, \tag{7}$$

The continuous version of scheme (5) is given by

$$X(t) = X_0 + \int_0^t f(\bar{X}(u), \bar{X}([u]))\mathrm{d}u + \sum_{j=1}^{d}\int_0^t g_j(\bar{X}(u), \bar{X}([u]))\mathrm{d}B^j(u) + \sum_{j,r=1}^{d}\int_0^t L^j g_r(\bar{X}(u), \bar{X}([u]))\Delta B^r(u)\mathrm{d}B^j(u), \tag{8}$$

where $\Delta B^r(u) = B^r(u) - B^r([u/\Delta]\Delta)$. It can be verified that $X(t_{sm+l}) = \bar{X}(t_{sm+l}) = X_{sm+l}$.

Throughout this paper, let C be a generic constant that varies from one place to another and depends on p, but independent of Δ.

Theorem 3.1 *Let Assumptions 2.1 and 2.2 hold. Then, for any $\Delta \in (0, 1]$ and $p \geq 2$, the Milstein scheme (5) has the following property:*

$$\sup_{0 \leq t_{sm+l} \leq T} \mathbb{E}|X_{sm+l}|^p \leq C, \ \forall T > 0.$$

Proof For any $T > 0$, $t_{sm+l+1} \in [0, T]$, $s \in \mathbb{N}$, $l = 0, 1, \cdots, m-1$, according to (8), one has

$$\begin{aligned}X_{sm+l+1} =& X(t_{sm+l+1})\\ =& X(t_{sm}) + \int_{t_{sm}}^{t_{sm+l+1}} f(\bar{X}(u), \bar{X}([u]))\mathrm{d}u + \sum_{j=1}^{d}\int_{t_{sm}}^{t_{sm+l+1}} g_j(\bar{X}(u), \bar{X}([u]))\mathrm{d}B^j(u)\\ &+ \sum_{j,r=1}^{d}\int_{t_{sm}}^{t_{sm+l+1}} L^j g_r(\bar{X}(u), \bar{X}([u]))\Delta B^r(u)\mathrm{d}B^j(u).\end{aligned}$$

By the inequality $(\sum_{i=1}^{n}|a_i|)^p \le n^{p-1}|a_i|^p$, $p \ge 1$, we have

$$\begin{aligned}\mathbb{E}|X_{sm+l+1}|^p \le\ & 4^{p-1}\mathbb{E}|X_{sm}|^p + 4^{p-1}\mathbb{E}\left|\int_{t_{sm}}^{t_{sm+l+1}} f(\bar{X}(u), \bar{X}([u]))\mathrm{d}u\right|^p\\ &+(4d)^{p-1}\sum_{j=1}^{d}\mathbb{E}\left|\int_{t_{sm}}^{t_{sm+l+1}} g_j(\bar{X}(u), \bar{X}([u]))\mathrm{d}B^j(u)\right|^p\\ &+(4dr)^{p-1}\sum_{j,r=1}^{d}\mathbb{E}\left|\int_{t_{sm}}^{t_{sm+l+1}} L^j g_r(\bar{X}(u), \bar{X}([u]))\Delta B^r(u)\mathrm{d}B^j(u)\right|^p. \quad (9)\end{aligned}$$

According to Hölder's inequality and the Burkholder-Davis-Gundy (B-D-G) inequality, we can deduce that

$$\begin{aligned}\mathbb{E}|X_{sm+l+1}|^p \le\ & C\mathbb{E}|X_{sm}|^p + C((l+1)\Delta)^{p-1}\mathbb{E}\int_{t_{sm}}^{t_{sm+l+1}} \left|f(\bar{X}(u), \bar{X}([u]))\right|^p \mathrm{d}u\\ &+C((l+1)\Delta)^{\frac{p-2}{2}}\sum_{j=1}^{d}\mathbb{E}\int_{t_{sm}}^{t_{sm+l+1}} \left|g_j(\bar{X}(u), \bar{X}([u]))\right|^p \mathrm{d}u\\ &+C((l+1)\Delta)^{\frac{p-2}{2}}\sum_{j,r=1}^{d}\mathbb{E}\int_{t_{sm}}^{t_{sm+l+1}} \left|L^j g_r(\bar{X}(u), \bar{X}([u]))\Delta B^r(u)\right|^p \mathrm{d}u\\ \le\ & C\mathbb{E}|X_{sm}|^p + C\sum_{i=0}^{l}\mathbb{E}\int_{t_{sm+i}}^{t_{sm+i+1}} |f(X_{sm+i}, X_{sm})|^p \mathrm{d}u\\ &+C\sum_{j=1}^{d}\sum_{i=0}^{l}\mathbb{E}\int_{t_{sm+i}}^{t_{sm+i+1}} \left|g_j(X_{sm+i}, X_{sm})\right|^p \mathrm{d}u\\ &+C\sum_{j,r=1}^{d}\sum_{i=0}^{l}\mathbb{E}\int_{t_{sm+i}}^{t_{sm+i+1}} \left|L^j g_r(X_{sm+i}, X_{sm})\Delta B^r(u)\right|^p \mathrm{d}u\\ \le\ & C\mathbb{E}|X_{sm}|^p + C\Delta\sum_{i=0}^{l}\mathbb{E}\,|f(X_{sm+i}, X_{sm})|^p + C\Delta\sum_{j=1}^{d}\sum_{i=0}^{l}\mathbb{E}\left|g_j(X_{sm+i}, X_{sm})\right|^p\\ &+C\sum_{j,r=1}^{d}\sum_{i=0}^{l}\mathbb{E}|L^j g_r(X_{sm+i}, X_{sm})|^p\int_{t_{sm+i}}^{t_{sm+i+1}}\mathbb{E}\left|\int_{t_{sm+i}}^{u}\mathrm{d}B^r(v)\right|^p \mathrm{d}u, \quad (10)\end{aligned}$$

in the last inequality we use the fact that $L^j g_r(X_{sm+i}, X_{sm})$ is $\mathcal{F}_{t_{sm+i}}$-measurable, while $\Delta B^r(u) = B^r(u) - B^r(t_{sm+i})$ is $\mathcal{F}_{t_{sm+i}}$-independent. Applying the B-D-G inequality again, together with (4), we can arrive at

$$\begin{aligned}\mathbb{E}|X_{sm+l+1}|^p \le\ & C\mathbb{E}|X_{sm}|^p + C\Delta\left(l+1+\sum_{i=0}^{l}\mathbb{E}|X_{sm+i}|^p + (l+1)\mathbb{E}|X_{sm}|^p\right)\\ &+C\sum_{j,r=1}^{d}\sum_{i=0}^{l}\mathbb{E}|L^j g_r(X_{sm+i}, X_{sm})|^p\int_{t_{sm+i}}^{t_{sm+i+1}}\Delta^{\frac{p}{2}}\mathrm{d}u\\ \le\ & C\mathbb{E}|X_{sm}|^p + C + C\Delta\sum_{i=0}^{l}\mathbb{E}|X_{sm+i}|^p + C\mathbb{E}|X_{sm}|^p\\ &+C\Delta^{\frac{p}{2}+1}\sum_{j,r=1}^{d}\sum_{i=0}^{l}\mathbb{E}\left|\sum_{k=1}^{n} g_{kj}(X_{sm+i}, X_{sm})\frac{\partial g_r(X_{sm+i}, X_{sm})}{\partial x_k}\right|^p. \qquad (11)\end{aligned}$$

According to Assumption 2.1 and (4), one can obtain

$$\begin{aligned}\mathbb{E}|X_{sm+l+1}|^p \le\ & C\mathbb{E}|X_{sm}|^p + C + C\Delta\sum_{i=0}^{l}\mathbb{E}|X_{sm+i}|^p\\ &+CdM^p n^{p-1}\Delta^{\frac{p}{2}+1}\sum_{j=1}^{d}\sum_{i=0}^{l}\sum_{k=1}^{n}\mathbb{E}\left|g_{kj}(X_{sm+i}, X_{sm})\right|^p\\ \le\ & C\mathbb{E}|X_{sm}|^p + C + C\Delta\sum_{i=0}^{l}\mathbb{E}|X_{sm+i}|^p\\ &+Cd^2M^p n^p L\Delta^{\frac{p}{2}+1}\left(l+1+\sum_{i=0}^{l}\mathbb{E}|X_{sm+i}|^p + (l+1)\mathbb{E}|X_{sm}|^p\right)\\ \le\ & C\mathbb{E}|X_{sm}|^p + C + C\Delta\sum_{i=0}^{l}\mathbb{E}|X_{sm+i}|^p. \qquad (12)\end{aligned}$$

By the discrete Gronwall inequality, we have

$$\mathbb{E}|X_{sm+l+1}|^p \le C(1+\mathbb{E}|X_{sm}|^p)e^{C(l+1)\Delta},$$

hence

$$1+\mathbb{E}|X_{sm+l+1}|^p \le C(1+\mathbb{E}|X_{sm}|^p).$$

In particular, take $l = m-1$, it is easy to see that

$$1+\mathbb{E}|X_{(s+1)m}|^p \le C(1+\mathbb{E}|X_{sm}|^p).$$

Then,

$$\mathbb{E}|X_{sm+l+1}|^p \leq C(1+\mathbb{E}|X_{sm}|^p) \leq C^2(1+\mathbb{E}|X_{(s-1)m}|^p) \leq \cdots \leq C^{s+1}(1+|X_0|^p).$$

Consequently, for any $T > 0$, $t_{sm+l} \in [0, T]$, one can deduce that

$$\mathbb{E}|X_{sm+l}|^p \leq C^{[T]+1}(1+|X_0|^p) \leq C.$$

The proof is completed. □

Lemma 3.2 *Let Assumptions 2.1 and 2.2 hold. Then, for any $T > 0$, $\Delta \in (0, 1]$ and $p \geq 2$,*

$$\sup_{0\leq t\leq T} \mathbb{E}|X(t) - \bar{X}(t)|^p \leq C\Delta^{p/2}, \quad \sup_{0\leq t\leq T} \mathbb{E}|X(t)|^p \leq C.$$

Proof For any $t \in [0, T]$, there are always $s \in \mathbb{N}$ and $l = 0, 1, \ldots, m-1$ such that $t \in [t_{sm+l}, t_{sm+l+1})$, by (7) and (8), one has

$$\begin{aligned}\mathbb{E}|X(t) - \bar{X}(t)|^p \leq & 3^{p-1}\mathbb{E}\left|\int_{t_{sm+l}}^{t} f(X_{sm+l}, X_{sm})\mathrm{d}u\right|^p \\ & + 3^{p-1}\mathbb{E}\left|\sum_{j=1}^{d}\int_{t_{sm+l}}^{t} g_j(X_{sm+l}, X_{sm})\mathrm{d}B^j(u)\right|^p \\ & + 3^{p-1}\mathbb{E}\left|\sum_{j,r=1}^{d}\int_{t_{sm+l}}^{t} L^j g_r(X_{sm+l}, X_{sm})\Delta B^r(u)\mathrm{d}B^j(u)\right|^p.\end{aligned}$$

Similar to the process of (9)-(12), applying Hölder's inequality, the B-D-G inequality, Assumption 2.1, (4), and Theorem 3.1, one can arrive at

$$\begin{aligned}\mathbb{E}|X(t) - \bar{X}(t)|^p \leq & C\Delta^p\mathbb{E}\,|f(X_{sm+l}, X_{sm})|^p + C\Delta^{\frac{p}{2}}\sum_{j=1}^{d}\mathbb{E}\left|g_j(X_{sm+l}, X_{sm})\right|^p \\ & + C\Delta^{\frac{p-2}{2}}\sum_{j,r=1}^{d}\mathbb{E}|L^j g_r(X_{sm+l}, X_{sm})|^p\int_{t_{sm+l}}^{t}\mathbb{E}\left|\int_{t_{sm+l}}^{u}\mathrm{d}B^r(v)\right|^p\mathrm{d}u \\ \leq & C\Delta^p(1+\mathbb{E}|X_{sm+l}|^p+\mathbb{E}|X_{sm}|^p) + C\Delta^{\frac{p}{2}}(1+\mathbb{E}|X_{sm+l}|^p+\mathbb{E}|X_{sm}|^p) \\ & + C\Delta^p\sum_{j,r=1}^{d}\mathbb{E}\left|\sum_{k=1}^{n} g_{kj}(X_{sm+l}, X_{sm})\frac{\partial g_r(X_{sm+l}, X_{sm})}{\partial x_k}\right|^p \\ \leq & C\Delta^{\frac{p}{2}}(1+\mathbb{E}|X_{sm+l}|^p+\mathbb{E}|X_{sm}|^p) + C\Delta^p\sum_{j,r=1}^{d}\sum_{k=1}^{n}\mathbb{E}\left|g_{kj}(X_{sm+l}, X_{sm})\right|^p \\ \leq & C\Delta^{\frac{p}{2}}(1+\mathbb{E}|X_{sm+l}|^p+\mathbb{E}|X_{sm}|^p) \\ \leq & C\Delta^{\frac{p}{2}}.\end{aligned}$$

Moreover, it is easy to see that

$$\mathbb{E}|X(t)|^p \leq 2^{p-1}\mathbb{E}|\bar{X}(t)|^p + 2^{p-1}\mathbb{E}|X(t) - \bar{X}(t)|^p \leq 2^{p-1} \sup_{0\leq t_{sm+l}\leq T} \mathbb{E}|X_{sm+l}|^p + C\Delta^{\frac{p}{2}} \leq C.$$

□

4 Strong convergence rate of the Milstein scheme

In the following, we sometimes use the notation $(\Phi)_i$ to denote the i-th component of $\Phi \in \mathbb{R}^n$. Let $\varphi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be twice differentiable with respect to the first component, then according to the Taylor formula,

$$\varphi(x, y) - \varphi(\bar{x}, y) = \sum_{i=1}^{n} \frac{\partial\varphi(\bar{x}, y)}{\partial x_i}(x - \bar{x})_i + R(\varphi)(x - \bar{x})$$

for $x, y, \bar{x} \in \mathbb{R}^n$, where

$$R(\varphi)(x - \bar{x}) = \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2\varphi(\bar{x} + \theta(x - \bar{x}), y)}{\partial x_i \partial x_j}(x - \bar{x})_i(x - \bar{x})_j,$$

with $\theta \in (0, 1)$. Note that $X([t]) = \bar{X}([t])$ for all $t \geq 0$, hence

$$\varphi(X(t), X([t])) - \varphi(\bar{X}(t), \bar{X}([t])) = \sum_{i=1}^{n} \frac{\partial\varphi(\bar{X}(t), \bar{X}([t]))}{\partial x_i}(X(t) - \bar{X}(t))_i + R(\varphi)(X(t) - \bar{X}(t)) \quad (13)$$

with

$$R(\varphi)(X(t) - \bar{X}(t)) = \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2\varphi(\bar{X}(t) + \theta(X(t) - \bar{X}(t)), \bar{X}([t]))}{\partial x_i \partial x_j}(X(t) - \bar{X}(t))_i(X(t) - \bar{X}(t))_j.$$

Applying (7) and (8), let $\kappa(t) = [t/\Delta]\Delta$, one has

$$(X(t) - \bar{X}(t))_i = \int_{\kappa(t)}^{t} f_i(\bar{X}(u), \bar{X}([u]))\mathrm{d}u + \sum_{k=1}^{d}\int_{\kappa(t)}^{t} g_{ik}(\bar{X}(u), \bar{X}([u]))\mathrm{d}B^k(u)$$
$$+ \sum_{k,r=1}^{d}\int_{\kappa(t)}^{t} \Big(L^k g_r(\bar{X}(u), \bar{X}([u]))\Big)_i \Delta B^r(u)\mathrm{d}B^k(u).$$

Define

$$\bar{R}(\varphi)(X(t) - \bar{X}(t)) := R(\varphi)(X(t) - \bar{X}(t)) + \sum_{i=1}^{n} \frac{\partial\varphi(\bar{X}(t), \bar{X}([t]))}{\partial x_i}\int_{\kappa(t)}^{t} f_i(\bar{X}(u), \bar{X}([u]))\mathrm{d}u$$
$$+ \sum_{i=1}^{n} \frac{\partial\varphi(\bar{X}(t), \bar{X}([t]))}{\partial x_i}\sum_{k,r=1}^{d}\int_{\kappa(t)}^{t} \Big(L^k g_r(\bar{X}(u), \bar{X}([u]))\Big)_i \Delta B^r(u)\mathrm{d}B^k(u), \quad (14)$$

which gives

$$\begin{aligned}&\varphi(X(t), X([t])) - \varphi(\bar{X}(t), \bar{X}([t]))\\&= \sum_{i=1}^{n} \frac{\partial \varphi(\bar{X}(t), \bar{X}([t]))}{\partial x_i} \sum_{k=1}^{d} \int_{\kappa(t)}^{t} g_{ik}(\bar{X}(u), \bar{X}([u]))\mathrm{d}B^k(u)\\&+\bar{R}(\varphi)(X(t) - \bar{X}(t)).\end{aligned} \tag{15}$$

Lemma 4.1 *Let Assumptions 2.1 and 2.2 hold. Then, for any $T > 0$, $\Delta \in (0, 1]$, and $p \geq 2$,*

$$\mathbb{E}|R(\varphi)(X(t) - \bar{X}(t))|^p \vee \mathbb{E}|\bar{R}(\varphi)(X(t) - \bar{X}(t))|^p \leq C\Delta^p,\ \forall t \in [0, T]$$

for $\varphi = f, g_j$, $j = 1, 2, \ldots, d$.

Proof Take $\varphi = f$, for any $t \in [0, T]$, using Hölder's inequality, one has

$$\begin{aligned}&\mathbb{E}|R(f)(X(t) - \bar{X}(t))|^p\\=&\mathbb{E}\left|\frac{1}{2}\sum_{i,r=1}^{n} \frac{\partial^2 f(\bar{X}(t) + \theta(X(t) - \bar{X}(t)), \bar{X}([t]))}{\partial x_i \partial x_r}(X(t) - \bar{X}(t))_i(X(t) - \bar{X}(t))_r\right|^p\\\leq& n^{2(p-1)}\sum_{i,r=1}^{n} \mathbb{E}\left|\frac{\partial^2 f(\bar{X}(t) + \theta(X(t) - \bar{X}(t)), \bar{X}([t]))}{\partial x_i \partial x_r}(X(t) - \bar{X}(t))_i(X(t) - \bar{X}(t))_r\right|^p\\\leq& n^{2(p-1)}\sum_{i,r=1}^{n} \left(\mathbb{E}\left|\frac{\partial^2 f(\bar{X}(t) + \theta(X(t) - \bar{X}(t)), \bar{X}([t]))}{\partial x_i \partial x_r}\right|^{2p}\right)^{1/2}\\&\times \left(\mathbb{E}|(X(t) - \bar{X}(t))_i|^{4p}\right)^{1/4}\left(\mathbb{E}|(X(t) - \bar{X}(t))_r|^{4p}\right)^{1/4}.\end{aligned}$$

By Assumption 2.1 and Lemma 3.2, one can obtain that

$$\mathbb{E}|R(f)(X(t) - \bar{X}(t))|^p \leq n^{2p}M^p\left(\mathbb{E}|X(t) - \bar{X}(t)|^{4p}\right)^{1/2} \leq C\Delta^p. \tag{16}$$

Moreover, recall that for any $t \in [0, T]$, there always exist $s \in \mathbb{N}$ and $l = 0, 1, \ldots, m-1$ such that $t \in [t_{sm+l}, t_{sm+l+1})$, which gives $\kappa(t) = t_{sm+l}$, hence

$$\begin{aligned}&\mathbb{E}|\bar{R}(f)(X(t) - \bar{X}(t))|^p\\\leq&\ 3^{p-1}\mathbb{E}|R(f)(X(t) - \bar{X}(t))|^p + 3^{p-1}\mathbb{E}\left|\sum_{i=1}^{n}\frac{\partial f(\bar{X}(t), \bar{X}([t]))}{\partial x_i}\int_{t_{sm+l}}^{t} f_i(\bar{X}(u), \bar{X}([u]))\mathrm{d}u\right|^p\\&+3^{p-1}\mathbb{E}\left|\sum_{i=1}^{n}\frac{\partial f(\bar{X}(t), \bar{X}([t]))}{\partial x_i}\sum_{k,r=1}^{d}\int_{t_{sm+l}}^{t}\left(L^k g_r(\bar{X}(u), \bar{X}([u]))\right)_i\Delta B^r(u)\mathrm{d}B^k(u)\right|^p\\=&\ 3^{p-1}\mathbb{E}|R(f)(X(t) - \bar{X}(t))|^p + 3^{p-1}\mathbb{E}\left|\sum_{i=1}^{n}\frac{\partial f(X_{sm+l}, X_{sm})}{\partial x_i}\int_{t_{sm+l}}^{t} f_i(X_{sm+l}, X_{sm})\mathrm{d}u\right|^p\\&+3^{p-1}\mathbb{E}\left|\sum_{i=1}^{n}\frac{\partial f(X_{sm+l}, X_{sm})}{\partial x_i}\sum_{k,r=1}^{d}\int_{t_{sm+l}}^{t}\left(L^k g_r(X_{sm+l}, X_{sm})\right)_i\Delta B^r(u)\mathrm{d}B^k(u)\right|^p.\end{aligned} \tag{17}$$

Using Assumption 2.1, Hölder's inequality, (4) and Theorem 3.1, one can deduce that

$$\begin{aligned}
&\mathbb{E}\left|\sum_{i=1}^{n}\frac{\partial f(X_{sm+l},X_{sm})}{\partial x_i}\int_{t_{sm+l}}^{t}f_i(X_{sm+l},X_{sm})\mathrm{d}u\right|^p\\
\leq\ & n^{p-1}\sum_{i=1}^{n}\mathbb{E}\left|\frac{\partial f(X_{sm+l},X_{sm})}{\partial x_i}\int_{t_{sm+l}}^{t}f_i(X_{sm+l},X_{sm})\mathrm{d}u\right|^p\\
\leq\ & n^{p-1}M^p\sum_{i=1}^{n}\mathbb{E}\left|\int_{t_{sm+l}}^{t}f_i(X_{sm+l},X_{sm})\mathrm{d}u\right|^p\\
\leq\ & n^{p-1}M^p\Delta^{p-1}\sum_{i=1}^{n}\mathbb{E}\int_{t_{sm+l}}^{t}|f_i(X_{sm+l},X_{sm})|^p\,\mathrm{d}u\\
\leq\ & C\Delta^{p-1}\int_{t_{sm+l}}^{t}(1+\mathbb{E}|X_{sm+l}|^p+\mathbb{E}|X_{sm}|^p)\mathrm{d}u\\
\leq\ & C\Delta^p. \qquad (18)
\end{aligned}$$

Similarly, applying Assumption 2.1 and the B-D-G inequality, it yields

$$\begin{aligned}
&\mathbb{E}\left|\sum_{i=1}^{n}\frac{\partial f(X_{sm+l},X_{sm})}{\partial x_i}\sum_{k,r=1}^{d}\int_{t_{sm+l}}^{t}\left(L^k g_r(X_{sm+l},X_{sm})\right)_i\Delta B^r(u)\mathrm{d}B^k(u)\right|^p\\
\leq\ & n^{p-1}d^{2(p-1)}M^p\sum_{i=1}^{n}\sum_{k,r=1}^{d}\mathbb{E}\left|\int_{t_{sm+l}}^{t}\left(L^k g_r(X_{sm+l},X_{sm})\right)_i\Delta B^r(u)\mathrm{d}B^k(u)\right|^p\\
\leq\ & n^{p-1}d^{2(p-1)}M^p\sum_{i=1}^{n}\sum_{k,r=1}^{d}C\Delta^{\frac{p-2}{2}}\mathbb{E}\int_{t_{sm+l}}^{t}\left|\left(L^k g_r(X_{sm+l},X_{sm})\right)_i\Delta B^r(u)\right|^p\mathrm{d}u\\
\leq\ & C\Delta^{\frac{p-2}{2}}\sum_{i=1}^{n}\sum_{k,r=1}^{d}\int_{t_{sm+l}}^{t}\mathbb{E}\left|\left(L^k g_r(X_{sm+l},X_{sm})\right)_i\right|^p\mathbb{E}\left|\int_{t_{sm+l}}^{u}\mathrm{d}B^r(v)\right|^p\mathrm{d}u. \qquad (19)
\end{aligned}$$

According to the definition of $L^k g_r(X_{sm+l},X_{sm})$, using Assumptions 2.1, 2.2, and Theorem 3.1, we can know that

$$\begin{aligned}
\mathbb{E}\left|\left(L^k g_r(X_{sm+l},X_{sm})\right)_i\right|^p \leq\ & \left(\mathbb{E}\left|L^k g_r(X_{sm+l},X_{sm})\right|^{2p}\right)^{1/2}\\
\leq\ & \left(\mathbb{E}\left|\sum_{i=1}^{n}g_{ik}(X_{sm+l},X_{sm})\frac{\partial g_r(X_{sm+l},X_{sm})}{\partial x_i}\right|^{2p}\right)^{1/2}
\end{aligned}$$

$$\leq n^{\frac{2p-1}{2}} M^p \left(\sum_{i=1}^{n} \mathbb{E} |g_{ik}(X_{sm+l}, X_{sm})|^{2p} \right)^{1/2}$$
$$\leq C \left(1 + \mathbb{E}|X_{sm+l}|^{2p} + \mathbb{E}|X_{sm}|^{2p} \right)^{1/2}$$
$$\leq C. \tag{20}$$

Substituting (20) into (19), with the help of B-D-G inequality again, we can obtain that

$$\mathbb{E} \left| \sum_{i=1}^{n} \frac{\partial f(X_{sm+l}, X_{sm})}{\partial x_i} \sum_{k,r=1}^{d} \int_{t_{sm+l}}^{t} \left(L^k g_r(X_{sm+l}, X_{sm}) \right)_i \Delta B^r(u) \mathrm{d}B^k(u) \right|^p$$
$$\leq C\Delta^{\frac{p-2}{2}} \sum_{r=1}^{d} \int_{t_{sm+l}}^{t} \mathbb{E} \left| \int_{t_{sm+l}}^{u} \mathrm{d}B^r(v) \right|^p \mathrm{d}u$$
$$\leq C\Delta^p. \tag{21}$$

Combining (16), (17), (18), and (21) yields

$$\mathbb{E}|\bar{R}(f)(X(t) - \bar{X}(t))|^p \leq C\Delta^p, \quad \forall t \in [0, T].$$

Repeating the process above, we can also prove

$$\mathbb{E}|R(g_j)(X(t) - \bar{X}(t))|^p \vee \mathbb{E}|\bar{R}(g_j)(X(t) - \bar{X}(t))|^p \leq C\Delta^p, \quad \forall t \in [0, T]$$

for all $j = 1, 2, \ldots, d$. □

Theorem 4.2 *Let Assumptions 2.1 and 2.2 hold. Then, for any* $\Delta \in (0, 1]$ *and* $p > 0$,

$$\mathbb{E} \sup_{0 \leq t \leq T} |x(t) - X(t)|^p \leq C\Delta^p, \ \forall T > 0.$$

Proof For any $t \in [0, T]$ and $p \geq 2$, according to (1) and (8), using Itô's formula, we can arrive at

$$|x(t) - X(t)|^p \leq \int_0^t p|x(u) - X(u)|^{p-2} \left((x(u) - X(u))^T F(u) + \frac{p-1}{2} \sum_{j=1}^{d} |G_j(u)|^2 \right) \mathrm{d}u$$
$$+ \sum_{j=1}^{d} \int_0^t p|x(u) - X(u)|^{p-2} (x(u) - X(u))^T G_j(u) \mathrm{d}B^j(u),$$

where

$$F(u) = f(x(u), x([u])) - f(\bar{X}(u), \bar{X}([u])),$$

$$G_j(u) = g_j(x(u), x([u])) - g_j(\bar{X}(u), \bar{X}([u])) - \sum_{r=1}^{d} L^j g_r(\bar{X}(u), \bar{X}([u])) \Delta B^r(u).$$

Then, for any $T_1 \in [0, T]$,

$$\mathbb{E} \sup_{0 \le t \le T_1} |x(t) - X(t)|^p \le \sum_{i=1}^{6} A_i, \tag{22}$$

where

$$\begin{aligned}
A_1 =& p\mathbb{E} \int_0^{T_1} |x(t) - X(t)|^{p-2} (x(t) - X(t))^{\mathrm{T}} (f(x(t), x([t])) - f(X(t), X([t]))) \,\mathrm{d}t, \\
A_2 =& p\mathbb{E} \int_0^{T_1} |x(t) - X(t)|^{p-2} (x(t) - X(t))^{\mathrm{T}} \left(f(X(t), X([t])) - f(\bar{X}(t), \bar{X}([t])) \right) \mathrm{d}t, \\
A_3 =& p(p-1)\mathbb{E} \int_0^{T_1} |x(t) - X(t)|^{p-2} \sum_{j=1}^{d} |g_j(x(t), x([t])) - g_j(X(t), X([t]))|^2 \mathrm{d}t, \\
A_4 =& p(p-1)\mathbb{E} \int_0^{T_1} |x(t) - X(t)|^{p-2} \sum_{j=1}^{d} \Bigg| g_j(X(t), X([t])) - g_j(\bar{X}(t), \bar{X}([t])) \\
& - \sum_{r=1}^{d} L^j g_r(\bar{X}(t), \bar{X}([t])) \Delta B^r(t) \Bigg|^2 \mathrm{d}t, \\
A_5 =& p \sum_{j=1}^{d} \mathbb{E} \sup_{0 \le t \le T_1} \int_0^t |x(u) - X(u)|^{p-2} (x(u) - X(u))^{\mathrm{T}} \\
& \times \left(g_j(x(u), x([u])) - g_j(X(u), X([u])) \right) \mathrm{d}B^j(u), \\
A_6 =& p \sum_{j=1}^{d} \mathbb{E} \sup_{0 \le t \le T_1} \int_0^t |x(u) - X(u)|^{p-2} (x(u) - X(u))^{\mathrm{T}} \\
& \times \left(g_j(X(u), X([u])) - g_j(\bar{X}(u), \bar{X}([u])) - \sum_{r=1}^{d} L^j g_r(\bar{X}(u), \bar{X}([u])) \Delta B^r(u) \right) \mathrm{d}B^j(u).
\end{aligned}$$

Applying Young's inequality, (2) and (3), it is easy to get that

$$\begin{aligned}
A_1 \le \;& p\mathbb{E} \int_0^{T_1} |x(t) - X(t)|^{p-1} |f(x(t), x([t])) - f(X(t), X([t]))| \,\mathrm{d}t \\
\le \;& (p-1)\mathbb{E} \int_0^{T_1} |x(t) - X(t)|^p \mathrm{d}t + \mathbb{E} \int_0^{T_1} |f(x(t), x([t])) - f(X(t), X([t]))|^p \,\mathrm{d}t \\
\le \;& C\mathbb{E} \int_0^{T_1} |x(t) - X(t)|^p \mathrm{d}t + C\mathbb{E} \int_0^{T_1} |x([t]) - X([t])|^p \mathrm{d}t \\
\le \;& C \int_0^{T_1} \left(\mathbb{E} \sup_{0 \le u \le t} |x(u) - X(u)|^p \right) \mathrm{d}t.
\end{aligned} \tag{23}$$

Similarly, we can also get

$$
\begin{aligned}
A_3 &= p(p-1)\sum_{j=1}^{d}\mathbb{E}\int_0^{T_1} |x(t)-X(t)|^{p-2}|g_j(x(t),x([t]))-g_j(X(t),X([t]))|^2\mathrm{d}t \\
&\le (p-1)(p-2)\sum_{j=1}^{d}\mathbb{E}\int_0^{T_1} |x(t)-X(t)|^p\mathrm{d}t \\
&\quad +2(p-1)\sum_{j=1}^{d}\mathbb{E}\int_0^{T_1} \left|g_j(x(t),x([t]))-g_j(X(t),X([t]))\right|^p \mathrm{d}t \\
&\le C\int_0^{T_1}\left(\mathbb{E}\sup_{0\le u\le t}|x(u)-X(u)|^p\right)\mathrm{d}t. \qquad (24)
\end{aligned}
$$

Next, we give an estimation for A_4. According to (15), we have

$$
\begin{aligned}
&g_j(X(t),X([t]))-g_j(\bar{X}(t),\bar{X}([t])) \\
&=\sum_{i=1}^{n}\frac{\partial g_j(\bar{X}(t),\bar{X}([t]))}{\partial x_i}\sum_{k=1}^{d}\int_{\kappa(t)}^{t} g_{ik}(\bar{X}(u),\bar{X}([u]))\mathrm{d}B^k(u)+\bar{R}(g_j)(X(t)-\bar{X}(t)).
\end{aligned}
$$

Recall that $L^j g_k(x,y)=L^k g_j(x,y)$, we have

$$
\begin{aligned}
&\sum_{i=1}^{n}\frac{\partial g_j(\bar{X}(t),\bar{X}([t]))}{\partial x_i}\sum_{k=1}^{d}\int_{\kappa(t)}^{t} g_{ik}(\bar{X}(u),\bar{X}([u]))\mathrm{d}B^k(u) \\
&=\sum_{k=1}^{d}\sum_{i=1}^{n} g_{ik}(\bar{X}(t),\bar{X}([t]))\frac{\partial g_j(\bar{X}(t),\bar{X}([t]))}{\partial x_i}\int_{\kappa(t)}^{t}\mathrm{d}B^k(u) \\
&=\sum_{k=1}^{d} L^k g_j(\bar{X}(t),\bar{X}([t]))\int_{\kappa(t)}^{t}\mathrm{d}B^k(u) \\
&=\sum_{k=1}^{d} L^j g_k(\bar{X}(t),\bar{X}([t]))\Delta B^k(t). \qquad (25)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
g_j(X(t),X([t]))-g_j(\bar{X}(t),\bar{X}([t])) &= \sum_{k=1}^{d} L^j g_k(\bar{X}(t),\bar{X}([t]))\Delta B^k(t) \\
&\quad+\bar{R}(g_j)(X(t)-\bar{X}(t)), \qquad (26)
\end{aligned}
$$

and then

$$
\begin{aligned}
A_4 &= p(p-1)\mathbb{E}\int_0^{T_1} |x(t)-X(t)|^{p-2}\sum_{j=1}^{d}\left|\bar{R}(g_j)(X(t)-\bar{X}(t))\right|^2 \mathrm{d}t \\
&\le (p-1)(p-2)\sum_{j=1}^{d}\mathbb{E}\int_0^{T_1}|x(t)-X(t)|^p\mathrm{d}t + 2(p-1)\sum_{j=1}^{d}\mathbb{E}\int_0^{T_1}\left|\bar{R}(g_j)(X(t)-\bar{X}(t))\right|^p \mathrm{d}t \\
&\le C\mathbb{E}\int_0^{T_1}|x(t)-X(t)|^p\mathrm{d}t + C\int_0^{T_1}\mathbb{E}\left|\bar{R}(g_j)(X(t)-\bar{X}(t))\right|^p \mathrm{d}t \\
&\le C\int_0^{T_1}\left(\mathbb{E}\sup_{0\le u\le t}|x(u)-X(u)|^p\right)\mathrm{d}t + C\Delta^p.
\end{aligned} \tag{27}
$$

Using the B-D-G inequality, fundamental inequality $2ab \le a^2+b^2$, (2), and (3), one sees that

$$
\begin{aligned}
A_5 \le\ & C\sum_{j=1}^{d}\mathbb{E}\left(\int_0^{T_1}|x(t)-X(t)|^{2p-2}|g_j(x(t),x([t]))-g_j(X(t),X([t]))|^2\mathrm{d}t\right)^{\frac{1}{2}} \\
\le\ & C\sum_{j=1}^{d}\mathbb{E}\left(\sup_{0\le t\le T_1}|x(t)-X(t)|^p\int_0^{T_1}|x(t)-X(t)|^{p-2}|g_j(x(t),x([t]))-g_j(X(t),X([t]))|^2\mathrm{d}t\right)^{\frac{1}{2}} \\
\le\ & \frac{1}{8}\mathbb{E}\sup_{0\le t\le T_1}|x(t)-X(t)|^p \\
&+ C\sum_{j=1}^{d}\mathbb{E}\int_0^{T_1}|x(t)-X(t)|^{p-2}|g_j(x(t),x([t]))-g_j(X(t),X([t]))|^2\mathrm{d}t \\
\le\ & \frac{1}{8}\mathbb{E}\sup_{0\le t\le T_1}|x(t)-X(t)|^p + C\mathbb{E}\int_0^{T_1}|x(t)-X(t)|^p\mathrm{d}t \\
&+ C\sum_{j=1}^{d}\mathbb{E}\int_0^{T_1}|g_j(x(t),x([t]))-g_j(X(t),X([t]))|^p\mathrm{d}t \\
\le\ & \frac{1}{8}\mathbb{E}\sup_{0\le t\le T_1}|x(t)-X(t)|^p + C\int_0^{T_1}\left(\sup_{0\le u\le t}\mathbb{E}|x(u)-X(u)|^p\right)\mathrm{d}t.
\end{aligned} \tag{28}
$$

Applying the B-D-G inequality again, with the help of (26) and Lemma 4.1, it can be derived that

$$
\begin{aligned}
A_6 \le\ & p\sum_{j=1}^{d}\mathbb{E}\sup_{0\le t\le T_1}\int_0^{t}|x(u)-X(u)|^{p-1}|\bar{R}(g_j)(X(u)-\bar{X}(u))|\mathrm{d}B^j(u) \\
\le\ & C\sum_{j=1}^{d}\mathbb{E}\left(\int_0^{T_1}|x(t)-X(t)|^{2p-2}|\bar{R}(g_j)(X(t)-\bar{X}(t))|^2\mathrm{d}t\right)^{\frac{1}{2}} \\
\le\ & C\sum_{j=1}^{d}\mathbb{E}\left(\sup_{0\le t\le T_1}|x(t)-X(t)|^p\int_0^{T_1}|x(t)-X(t)|^{p-2}|\bar{R}(g_j)(X(t)-\bar{X}(t))|^2\mathrm{d}t\right)^{\frac{1}{2}}
\end{aligned}
$$

$$\leq \frac{1}{8}\mathbb{E}\sup_{0\leq t\leq T_1}|x(t)-X(t)|^p+C\sum_{j=1}^{d}\mathbb{E}\int_0^{T_1}|x(t)-X(t)|^{p-2}|\bar{R}(g_j)(X(t)-\bar{X}(t))|^2\mathrm{d}t$$

$$\leq \frac{1}{8}\mathbb{E}\sup_{0\leq t\leq T_1}|x(t)-X(t)|^p+C\mathbb{E}\int_0^{T_1}|x(t)-X(t)|^p\mathrm{d}t$$

$$+C\sum_{j=1}^{d}\int_0^{T_1}\mathbb{E}|\bar{R}(g_j)(X(t)-\bar{X}(t))|^p\mathrm{d}t$$

$$\leq \frac{1}{8}\mathbb{E}\sup_{0\leq t\leq T_1}|x(t)-X(t)|^p+C\int_0^{T_1}\left(\mathbb{E}\sup_{0\leq u\leq t}|x(u)-X(u)|^p\right)\mathrm{d}t+C\Delta^p. \quad (29)$$

In the following, we give an estimation for A_2. According to (15),

$$f(X(t),X([t]))-f(\bar{X}(t),\bar{X}([t]))=\phi(\bar{X}(t),\bar{X}([t]))+\bar{R}(f)(X(t)-\bar{X}(t)),$$

where

$$\phi(\bar{X}(t),\bar{X}([t])):=\sum_{i=1}^{n}\frac{\partial f(\bar{X}(t),\bar{X}([t]))}{\partial x_i}\sum_{k=1}^{d}\int_{\kappa(t)}^{t}g_{ik}(\bar{X}(u),\bar{X}([u]))\mathrm{d}B^k(u).$$

Using Hölder's inequality and Lemma 4.1, one has

$$A_2=p\mathbb{E}\int_0^{T_1}|x(t)-X(t)|^{p-2}(x(t)-X(t))^{\mathrm{T}}\left(f(X(t),X([t]))-f(\bar{X}(t),\bar{X}([t]))\right)\mathrm{d}t$$

$$\leq B+p\mathbb{E}\int_0^{T_1}|x(t)-X(t)|^{p-1}|\bar{R}(f)(X(t)-\bar{X}(t))|\mathrm{d}t$$

$$\leq B+(p-1)\mathbb{E}\int_0^{T_1}|x(t)-X(t)|^p\mathrm{d}t+\int_0^{T_1}\mathbb{E}|\bar{R}(f)(X(t)-\bar{X}(t))|^p\mathrm{d}t$$

$$\leq B+(p-1)\int_0^{T_1}\left(\mathbb{E}\sup_{0\leq u\leq t}|x(u)-X(u)|^p\right)\mathrm{d}t+C\Delta^p, \quad (30)$$

where

$$B=p\mathbb{E}\int_0^{T_1}|x(t)-X(t)|^{p-2}(x(t)-X(t))^{\mathrm{T}}\phi(\bar{X}(t),\bar{X}([t]))\mathrm{d}t.$$

According to the Young inequality, it is easy to arrive at

$$B\leq p\mathbb{E}\left(\sup_{0\leq t\leq T_1}|x(t)-X(t)|^{p-2}\int_0^{T_1}(x(t)-X(t))^{\mathrm{T}}\phi(\bar{X}(t),\bar{X}([t]))\mathrm{d}t\right)$$

$$\leq\frac{1}{8}\mathbb{E}\sup_{0\leq t\leq T_1}|x(t)-X(t)|^p+C\mathbb{E}\left(\int_0^{T_1}(x(t)-X(t))^{\mathrm{T}}\phi(\bar{X}(t),\bar{X}([t]))\mathrm{d}t\right)^{\frac{p}{2}}.$$

Taking the difference between (1) and (8), one has

$$\begin{aligned} &x(t)-X(t)\\ =&x(\kappa(t))-X(\kappa(t))+\int_{\kappa(t)}^{t}(f(x(u),x([u]))-f(\bar{X}(u),\bar{X}([u])))\mathrm{d}u\\ &+\sum_{j=1}^{d}\int_{\kappa(t)}^{t}\left(g_j(x(u),x([u]))-g_j(\bar{X}(u),\bar{X}([u]))-\sum_{r=1}^{d}L^j g_r(\bar{X}(u),\bar{X}([u]))\Delta B^r(u)\right)\mathrm{d}B_j(u), \end{aligned}$$

where $\kappa(t)=[t/\Delta]\Delta$, then

$$B\leq\frac{1}{8}\mathbb{E}\sup_{0\leq t\leq T_1}|x(t)-X(t)|^p+\sum_{i=1}^{5}B_i \tag{31}$$

with

$$\begin{aligned} B_1=&C\mathbb{E}\left(\int_0^{T_1}(x(\kappa(t))-X(\kappa(t)))^{\mathrm{T}}\phi(\bar{X}(t),\bar{X}([t]))\mathrm{d}t\right)^{\frac{p}{2}},\\ B_2=&C\mathbb{E}\left\{\int_0^{T_1}\left(\int_{\kappa(t)}^{t}(f(x(u),x([u]))-f(X(u),X([u])))\mathrm{d}u\right)^{\mathrm{T}}\phi(\bar{X}(t),\bar{X}([t]))\mathrm{d}t\right\}^{\frac{p}{2}},\\ B_3=&C\mathbb{E}\left\{\int_0^{T_1}\left(\int_{\kappa(t)}^{t}(f(X(u),X([u]))-f(\bar{X}(u),\bar{X}([u])))\mathrm{d}u\right)^{\mathrm{T}}\phi(\bar{X}(t),\bar{X}([t]))\mathrm{d}t\right\}^{\frac{p}{2}},\\ B_4=&C\mathbb{E}\left\{\int_0^{T_1}\left(\sum_{j=1}^{d}\int_{\kappa(t)}^{t}\big(g_j(x(u),x([u]))-g_j(X(u),X([u]))\big)\mathrm{d}B_j(u)\right)^{\mathrm{T}}\phi(\bar{X}(t),\bar{X}([t]))\mathrm{d}t\right\}^{\frac{p}{2}},\\ B_5=&C\mathbb{E}\Bigg\{\int_0^{T_1}\Bigg(\sum_{j=1}^{d}\int_{\kappa(t)}^{t}\Big(g_j(X(u),X([u]))-g_j(\bar{X}(u),\bar{X}([u]))\\ &-\sum_{r=1}^{d}L^j g_r(\bar{X}(u),\bar{X}([u]))\Delta B^r(u)\Big)\mathrm{d}B_j(u)\Bigg)^{\mathrm{T}}\phi(\bar{X}(t),\bar{X}([t]))\mathrm{d}t\Bigg\}^{\frac{p}{2}}. \end{aligned}$$

Let $N=[T_1/\Delta]$,

$$\begin{aligned} B_1=&\underbrace{C\mathbb{E}\left(\sum_{sm+l=0}^{N-1}\int_{t_{sm+l}}^{t_{sm+l+1}}(x(\kappa(t))-X(\kappa(t)))^{\mathrm{T}}\phi(\bar{X}(t),\bar{X}([t]))\mathrm{d}t\right)^{\frac{p}{2}}}_{B_{11}}\\ &+\underbrace{C\mathbb{E}\left(\int_{\kappa(T_1)}^{T_1}(x(\kappa(t))-X(\kappa(t)))^{\mathrm{T}}\phi(\bar{X}(t),\bar{X}([t]))\mathrm{d}t\right)^{\frac{p}{2}}}_{B_{12}}. \end{aligned}$$

Set

$$Z_{sm+l+1} = \int_{t_{sm+l}}^{t_{sm+l+1}} (x(\kappa(t)) - X(\kappa(t)))^{\mathrm{T}} \phi(\bar{X}(t), \bar{X}([t]))\mathrm{d}t, \; sm+l+1 = 1, \ldots, N,$$

it is easy to know that $\mathbb{E}(Z_{sm+l+2}|Z_1, Z_2, \ldots, Z_{sm+l+1}) = 0$ for all $sm + l + 1 = 1, \ldots, N - 1$, then for $p \geq 4$, by Lemma 2.4, we have

$$\begin{aligned}
B_{11} \leq & C \left| \sum_{sm+l=0}^{N-1} Z_{sm+l+1} \right|_{L^{p/2}}^{\frac{p}{2}} \leq C \left(C_p \left(\sum_{sm+l=0}^{N-1} |Z_{sm+l+1}|_{L^{p/2}}^2 \right)^{\frac{1}{2}} \right)^{\frac{p}{2}} \\
\leq & C N^{\frac{p}{4}-1} \sum_{sm+l=0}^{N-1} \mathbb{E}|Z_{sm+l+1}|^{\frac{p}{2}} \\
= & C N^{\frac{p}{4}-1} \sum_{sm+l=0}^{N-1} \mathbb{E} \left| \int_{t_{sm+l}}^{t_{sm+l+1}} (x(\kappa(t)) - X(\kappa(t)))^{\mathrm{T}} \phi(\bar{X}(t), \bar{X}([t]))\mathrm{d}t \right|^{\frac{p}{2}} \\
\leq & C T_1^{\frac{p}{4}-1} \Delta^{\frac{p}{4}} \sum_{sm+l=0}^{N-1} \mathbb{E} \int_{t_{sm+l}}^{t_{sm+l+1}} |x(\kappa(t)) - X(\kappa(t))|^{\frac{p}{2}} |\phi(\bar{X}(t), \bar{X}([t]))|^{\frac{p}{2}} \mathrm{d}t \\
\leq & C \mathbb{E} \int_0^{T_1} |x(\kappa(t)) - X(\kappa(t))|^{\frac{p}{2}} \left(\Delta^{\frac{p}{4}} |\phi(\bar{X}(t), \bar{X}([t]))|^{\frac{p}{2}} \right) \mathrm{d}t \\
\leq & C \mathbb{E} \int_0^{T_1} |x(\kappa(t)) - X(\kappa(t))|^p \mathrm{d}t + C \Delta^{\frac{p}{2}} \int_0^{T_1} \mathbb{E}|\phi(\bar{X}(t), \bar{X}([t]))|^p \mathrm{d}t.
\end{aligned}$$

Applying Assumption 2.1, the fundamental inequality $(\sum_{i=1}^n a_i)^p \leq n^{p-1} \sum_{i=1}^n a_i^p$, (4) and Lemma 3.2, for any $t \in [0, T_1]$,

$$\begin{aligned}
\mathbb{E}|\phi(\bar{X}(t), \bar{X}([t]))|^p = & \mathbb{E} \left| \sum_{i=1}^n \frac{\partial f(\bar{X}(t), \bar{X}([t]))}{\partial x_i} \sum_{k=1}^d \int_{\kappa(t)}^t g_{ik}(\bar{X}(u), \bar{X}([u]))\mathrm{d}B^k(u) \right|^p \\
\leq & M^p (nd)^{p-1} \sum_{i=1}^n \sum_{k=1}^d \mathbb{E} \left(|g_{ik}(\bar{X}(t), \bar{X}([t]))|^p \left| \int_{\kappa(t)}^t \mathrm{d}B^k(u) \right|^p \right) \\
\leq & M^p n^p d^{p-1} \sum_{k=1}^d \left(\mathbb{E}|g_k(\bar{X}(t), \bar{X}([t]))|^{2p} \right)^{\frac{1}{2}} \left(\mathbb{E} \left| \int_{\kappa(t)}^t \mathrm{d}B^k(u) \right|^{2p} \right)^{\frac{1}{2}} \\
\leq & C \left(1 + \mathbb{E}|\bar{X}(t)|^{2p} + \mathbb{E}|\bar{X}([t]))|^{2p} \right)^{\frac{1}{2}} \left(\mathbb{E} \left| \int_{\kappa(t)}^t \mathrm{d}B^k(u) \right|^{2p} \right)^{\frac{1}{2}} \\
\leq & C \Delta^{\frac{p}{2}} \left(1 + \sup_{0 \leq u \leq t} \mathbb{E}|X(u)|^{2p} \right)^{\frac{1}{2}} \\
\leq & C \Delta^{\frac{p}{2}}.
\end{aligned} \tag{32}$$

Hence,

$$B_{11} \leq C \int_0^{T_1} \left(\mathbb{E} \sup_{0\leq u\leq t} |x(u) - X(u)|^p \right) \mathrm{d}t + C\Delta^p.$$

According to Hölder's inequality, we can get that

$$\begin{aligned} B_{12} \leq & C\Delta^{\frac{p}{2}-1}\mathbb{E} \int_{\kappa(T_1)}^{T_1} |x(\kappa(T_1)) - X(\kappa(T_1))|^{\frac{p}{2}} |\phi(\bar{X}(t), \bar{X}([t]))|^{\frac{p}{2}} \mathrm{d}t \\ \leq & \mathbb{E} \left(|x(\kappa(T_1)) - X(\kappa(T_1))|^{\frac{p}{2}} \cdot C\Delta^{\frac{p}{2}-1} \int_{\kappa(T_1)}^{T_1} |\phi(\bar{X}(t), \bar{X}([t]))|^{\frac{p}{2}} \mathrm{d}t \right) \\ \leq & \frac{1}{8}\mathbb{E}|x(\kappa(T_1)) - X(\kappa(T_1))|^p + C\Delta^{p-2}\mathbb{E}\left(\int_{\kappa(T_1)}^{T_1} |\phi(\bar{X}(t), \bar{X}([t]))|^{\frac{p}{2}} \mathrm{d}t \right)^2 \\ \leq & \frac{1}{8}\mathbb{E} \sup_{0\leq t\leq T_1} |x(t) - X(t)|^p + C\Delta^{p-1} \int_{\kappa(T_1)}^{T_1} \mathbb{E}|\phi(\bar{X}(t), \bar{X}([t]))|^p \mathrm{d}t \\ \leq & \frac{1}{8}\mathbb{E} \sup_{0\leq t\leq T_1} |x(t) - X(t)|^p + C\Delta^p. \end{aligned}$$

Therefore, one can obtain

$$B_1 \leq B_{11} + B_{12} \leq \frac{1}{8}\mathbb{E} \sup_{0\leq t\leq T_1} |x(t) - X(t)|^p + C\Delta^p + C\int_0^{T_1} \left(\mathbb{E} \sup_{0\leq u\leq t} |x(u) - X(u)|^p \right) \mathrm{d}t. \quad (33)$$

Using Hölder's inequality and (32), together with (2)-(3), one can arrive at

$$\begin{aligned} B_2 \leq\ & C\mathbb{E} \int_0^{T_1} \left| \int_{\kappa(t)}^t (f(x(u), x([u])) - f(X(u), X([u])))\mathrm{d}u \right|^{\frac{p}{2}} \left|\phi(\bar{X}(t), \bar{X}([t]))\right|^{\frac{p}{2}} \mathrm{d}t \\ \leq\ & C\int_0^{T_1} \left(\mathbb{E} \left| \int_{\kappa(t)}^t (f(x(u), x([u])) - f(X(u), X([u])))\mathrm{d}u \right|^p \right)^{\frac{1}{2}} \left(\mathbb{E}\left|\phi(\bar{X}(t), \bar{X}([t]))\right|^p \right)^{\frac{1}{2}} \mathrm{d}t \\ \leq\ & C\Delta^{\frac{p}{4}} \int_0^{T_1} \left(\mathbb{E} \left| \int_{\kappa(t)}^t (f(x(u), x([u])) - f(X(u), X([u])))\mathrm{d}u \right|^p \right)^{\frac{1}{2}} \mathrm{d}t \\ \leq\ & C\Delta^{\frac{p}{4}} \int_0^{T_1} \left(\Delta^{p-1}\mathbb{E} \int_{\kappa(t)}^t (|x(u) - X(u)|^p + |x([u]) - X([u])|^p)\mathrm{d}u \right)^{\frac{1}{2}} \mathrm{d}t \\ \leq\ & C\Delta^{\frac{p}{4}} \int_0^{T_1} \left(\Delta^p\mathbb{E} \sup_{0\leq u\leq t} |x(u) - X(u)|^p \right)^{\frac{1}{2}} \mathrm{d}t \\ \leq\ & C\Delta^p + C\int_0^{T_1} \left(\mathbb{E} \sup_{0\leq u\leq t} |x(u) - X(u)|^p \right) \mathrm{d}t. \end{aligned} \quad (34)$$

Similarly,

$$
\begin{aligned}
B_3 \leq & C\mathbb{E}\int_0^{T_1}\left|\int_{\kappa(t)}^t (f(X(u),X([u]))-f(\bar{X}(u),\bar{X}([u])))\mathrm{d}u\right|^{\frac{p}{2}} |\phi(\bar{X}(t),\bar{X}([t]))|^{\frac{p}{2}}\mathrm{d}t\\
\leq & C\Delta^{\frac{p}{2}-1}\mathbb{E}\int_0^{T_1}\left(\int_{\kappa(t)}^t \left|f(X(u),X([u]))-f(\bar{X}(u),\bar{X}([u]))\right|^{\frac{p}{2}}\mathrm{d}u\right)|\phi(\bar{X}(t),\bar{X}([t]))|^{\frac{p}{2}}\mathrm{d}t\\
\leq & C\Delta^{\frac{p}{2}-1}\mathbb{E}\int_0^{T_1}\left(\int_{\kappa(t)}^t \left|X(u)-\bar{X}(u)\right|^{\frac{p}{2}}\mathrm{d}u\right)|\phi(\bar{X}(t),\bar{X}([t]))|^{\frac{p}{2}}\mathrm{d}t\\
\leq & C\Delta^{\frac{p}{2}-1}\int_0^{T_1}\left\{\mathbb{E}\left(\int_{\kappa(t)}^t \left|X(u)-\bar{X}(u)\right|^{\frac{p}{2}}\mathrm{d}u\right)^2\right\}^{\frac{1}{2}}\left\{\mathbb{E}|\phi(\bar{X}(t),\bar{X}([t]))|^{p}\right\}^{\frac{1}{2}}\mathrm{d}t\\
\leq & C\Delta^{\frac{3p-2}{4}}\int_0^{T_1}\left\{\int_{\kappa(t)}^t \mathbb{E}\left|X(u)-\bar{X}(u)\right|^{p}\mathrm{d}u\right\}^{\frac{1}{2}}\mathrm{d}t.
\end{aligned}
$$

It follows from Lemma 3.2 that

$$
B_3 \leq C\Delta^{\frac{3p-2}{4}}\int_0^{T_1}\Delta^{\frac{p+2}{4}}\mathrm{d}t \leq C\Delta^p. \tag{35}
$$

Using Hölder's inequality and the B-D-G inequality again, with the help of (32), yields

$$
\begin{aligned}
B_4 \leq\ & C\mathbb{E}\int_0^{T_1}\left|\sum_{j=1}^d\int_{\kappa(t)}^t \big(g_j(x(u),x([u]))-g_j(X(u),X([u]))\big)\mathrm{d}B(u)\right|^{\frac{p}{2}}|\phi(\bar{X}(t),\bar{X}([t]))|^{\frac{p}{2}}\mathrm{d}t\\
\leq\ & C\sum_{j=1}^d\int_0^{T_1}\left(\mathbb{E}\left|\int_{\kappa(t)}^t \big(g_j(x(u),x([u]))-g_j(X(u),X([u]))\big)\mathrm{d}B(u)\right|^p\right)^{\frac{1}{2}}\\
& \times\big(\mathbb{E}|\phi(\bar{X}(t),\bar{X}([t]))|^p\big)^{\frac{1}{2}}\mathrm{d}t\\
\leq\ & C\Delta^{\frac{p}{4}}\sum_{j=1}^d\int_0^{T_1}\left(\Delta^{\frac{p-2}{2}}\mathbb{E}\int_{\kappa(t)}^t\left|g_j(x(u),x([u]))-g_j(X(u),X([u]))\right|^p\mathrm{d}u\right)^{\frac{1}{2}}\mathrm{d}t\\
\leq\ & C\Delta^{\frac{p-1}{2}}\int_0^{T_1}\left(\mathbb{E}\int_{\kappa(t)}^t(|x(u)-X(u)|^p+|x([u])-X([u]))|^p)\mathrm{d}u\right)^{\frac{1}{2}}\mathrm{d}t\\
\leq\ & C\Delta^{\frac{p}{2}}\int_0^{T_1}\left(\mathbb{E}\sup_{0\leq u\leq t}|x(u)-X(u)|^p\right)^{\frac{1}{2}}\mathrm{d}t\\
\leq\ & C\Delta^p + C\int_0^{T_1}\left(\mathbb{E}\sup_{0\leq u\leq t}|x(u)-X(u)|^p\right)\mathrm{d}t.
\end{aligned} \tag{36}
$$

By (26), (32) and Lemma 4.1, we have

$$
\begin{aligned}
B_5 \leq\ & C\mathbb{E}\int_0^{T_1}\left|\sum_{j=1}^{d}\int_{\kappa(t)}^{t}\bar{R}(g_j)(X(t)-\bar{X}(t))\mathrm{d}B_j(u)\right|^{\frac{p}{2}}|\phi(\bar{X}(t),\bar{X}([t]))|^{\frac{p}{2}}\mathrm{d}t\\
\leq\ & C\sum_{j=1}^{d}\int_0^{T_1}\left(\mathbb{E}\left|\int_{\kappa(t)}^{t}\bar{R}(g_j)(X(t)-\bar{X}(t))\mathrm{d}B_j(u)\right|^{p}\right)^{\frac{1}{2}}\left(\mathbb{E}|\phi(\bar{X}(t),\bar{X}([t]))|^{p}\right)^{\frac{1}{2}}\mathrm{d}t\\
\leq\ & C\Delta^{\frac{p}{4}}\sum_{j=1}^{d}\int_0^{T_1}\left(\Delta^{\frac{p-2}{2}}\int_{\kappa(t)}^{t}\mathbb{E}\left|\bar{R}(g_j)(X(t)-\bar{X}(t))\right|^{p}\mathrm{d}u\right)^{\frac{1}{2}}\mathrm{d}t\\
\leq\ & C\Delta^{p}.
\end{aligned}
\tag{37}
$$

Combining (31) and (33)–(37), it yields

$$
B\leq\frac{1}{4}\mathbb{E}\sup_{0\leq t\leq T_1}|x(t)-X(t)|^{p}+C\Delta^{p}+C\int_0^{T_1}\left(\mathbb{E}\sup_{0\leq u\leq t}|x(u)-X(u)|^{p}\right)\mathrm{d}t.
$$

Substituting this into (30), one has

$$
A_2\leq\frac{1}{4}\mathbb{E}\sup_{0\leq t\leq T_1}|x(t)-X(t)|^{p}+C\Delta^{p}+C\int_0^{T_1}\left(\mathbb{E}\sup_{0\leq u\leq t}|x(u)-X(u)|^{p}\right)\mathrm{d}t. \tag{38}
$$

Combining (22)–(24),(27)–(29), and (38), we have

$$
\mathbb{E}\sup_{0\leq t\leq T_1}|x(t)-X(t)|^{p}\leq C\Delta^{p}+C\int_0^{T_1}\left(\mathbb{E}\sup_{0\leq u\leq t}|x(u)-X(u)|^{p}\right)\mathrm{d}t,\ \forall T_1\in[0,T].
$$

Consequently, it can be deduced from the Gronwall inequality that

$$
\mathbb{E}\sup_{0\leq t\leq T}|x(t)-X(t)|^{p}\leq C\Delta^{p}e^{CT}\leq C\Delta^{p},\ p\geq 4.
$$

Furthermore, for any $q\in(0,4)$, by Hölder's inequality,

$$
\begin{aligned}
\mathbb{E}\sup_{0\leq t\leq T}|x(t)-X(t)|^{q} =&\mathbb{E}\left(\sup_{0\leq t\leq T}|x(t)-X(t)|\right)^{q}\\
\leq&\left(\mathbb{E}\sup_{0\leq t\leq T}|x(t)-X(t)|^{p}\right)^{\frac{q}{p}}\\
\leq& C\Delta^{q}.
\end{aligned}
$$

The proof is completed. □

5 Stability analysis of the Milstein method

In this section, we investigate the exponential stability of the Milstein method for (1). Throughout this section, we shall assume that (1) has a unique global solution for any given initial data x_0. Firstly, we suppose that $f(0,0) = 0$ and $g_j(0,0) = 0$, $j = 1, \ldots, d$ and give the following two definitions of stability.

Definition 2 The SDEPCA (1) is said to be exponentially stable in mean square if there exist positive constants λ and H_1 such that for any given initial value $x_0 \in \mathbb{R}^n$,

$$\mathbb{E}|x(t)|^2 \leq H_1|x_0|^2 e^{-\lambda t}, \quad \forall t \geq 0.$$

Definition 3 For a given step-size $\Delta > 0$, the Milstein method is said to be exponentially stable in mean square if there exist positive constants γ and H_2 such that for any given initial value $x_0 \in \mathbb{R}^n$,

$$\mathbb{E}|X_k|^2 \leq H_2|x_0|^2 e^{-\gamma k\Delta}$$

for all $k \in \mathbb{N}$.

Remark 3 Under Assumptions 2.1, 2.2, and $f(0,0) = g_j(0,0) = 0$, similar to the Remark 2, it follows

$$|f(x,y)| \vee |g_j(x,y)| \leq \tilde{L}(|x| + |y|) \tag{39}$$

for all $x, y \in \mathbb{R}^n$, where $\tilde{L} = \bar{M} + L$, $\bar{M}$, and L are defined in Assumptions 2.1 and 2.2.

Let $g = (g_1, g_2, \ldots, g_d)$, we assume the following condition holds to obtain the stability.

Assumption 5.1 *Assume that there are positive constants $\lambda_1 > \lambda_2 > 0$ such that*

$$\langle x, f(x,y)\rangle + \frac{1}{2}|g(x,y)|^2 \leq -\lambda_1|x|^2 + \lambda_2|y|^2, \quad \forall x, y \in \mathbb{R}^n.$$

By Theorem 4.1 in [7], we can obtain the exponentially stability in the mean square of (1).

Theorem 5.2 *Let Assumption 5.1 holds. Then, (1) is exponentially stable in mean square, i.e.,*

$$\mathbb{E}|x(t)|^2 \leq H_1|x_0|^2 e^{-\lambda t}, \quad \forall t \geq 0,$$

where $\lambda = -\log r(1)$ and $H_1 = r(1)^{-1}$ with $r(1) = \frac{\lambda_2}{\lambda_1} + (1 - \frac{\lambda_2}{\lambda_1})e^{-2\lambda_1}$.

To obtain the stability of the Milstein method, we introduce the following lemmas (for details of the proofs, see [7]).

Lemma 5.3 *Let z_{sm+l} be a sequence of numbers, $s, m \in \mathbb{N}, l = 0, 1, \ldots, m-1$. If there are constants $\alpha > \beta > 0$ such that $1 - \alpha\Delta > 0$ and*

$$z_{sm+l+1} \leq (1-\alpha\Delta) z_{sm+l} + \beta\Delta z_{sm},$$

then

$$z_{sm+l+1} \leq \left(\frac{\beta}{\alpha} + \left(1 - \frac{\beta}{\alpha}\right) e^{-\alpha(l+1)\Delta}\right) z_{sm}.$$

Lemma 5.4 *Assume that α, β are two positive constants. If $\alpha > \beta$, then for all $t \geq 0$, we have*

$$0 < \frac{\beta}{\alpha} + \left(1 - \frac{\beta}{\alpha}\right) e^{-\alpha t} < 1.$$

Let $K = nd^2(d^2+2)M^2\tilde{L}^2 + 2\tilde{L}^2$, $\alpha = 2\lambda_1 - K\Delta$, $\beta = 2\lambda_2 + K\Delta$, $\Gamma(m) = \frac{\beta}{\alpha} + \left(1 - \frac{\beta}{\alpha}\right) e^{-\alpha}$, where M and $\tilde{L}$ are defined in Assumptions 2.1 and Remark 3, respectively. Then, we obtain the exponential stability of the Milstein method.

Theorem 5.5 *Let Assumptions 2.1, 2.2, and 5.1 hold. Then, for any step-size $0 < \Delta < \bar{\Delta} \wedge 1$, the Milstein scheme (5) is exponentially stable in mean square, i.e.,*

$$\mathbb{E}|X_k|^2 \leq H_2 |x_0|^2 e^{-\gamma k\Delta}$$

for all $k \in \mathbb{N}$, where $H_2 = \frac{1}{\Gamma(m)}$, $\gamma = -\log\Gamma(m)$,

$$\bar{\Delta} = \begin{cases} \frac{\lambda_1 - \lambda_2}{K}, & \text{if } \lambda_1^2 \leq K, \\ \left(\frac{\lambda_1 - \lambda_2}{K}\right) \wedge \left(\frac{\lambda_1 - \sqrt{\lambda_1^2 - K}}{K}\right), & \text{otherwise}. \end{cases}$$

Moreover, $\lim_{\Delta \to 0} \gamma = \lambda$, where λ is defined in Theorem 5.2.

Proof For any $s \in \mathbb{N}, l = 0, 1, \ldots, m-1$, according to (6), using Assumption 5.1,

$$\begin{aligned} \mathbb{E}|X_{sm+l+1}|^2 = & \mathbb{E}|X_{sm+l}|^2 + \mathbb{E}|f(X_{sm+l}, X_{sm})|^2 \Delta^2 + \mathbb{E}|g(X_{sm+l}, X_{sm}) \Delta B_{sm+l}|^2 \\ & + \mathbb{E}|H_{sm+l}|^2 + 2\mathbb{E}\langle X_{sm+l}, f(X_{sm+l}, X_{sm}) \Delta\rangle \\ & + 2\mathbb{E}\langle X_{sm+l} + f(X_{sm+l}, X_{sm}) \Delta, g(X_{sm+l}, X_{sm}) \Delta B_{sm+l}\rangle \\ & + 2\mathbb{E}\langle X_{sm+l} + f(X_{sm+l}, X_{sm}) \Delta + g(X_{sm+l}, X_{sm}) \Delta B_{sm+l}, H_{sm+l}\rangle, \end{aligned} \tag{40}$$

where

$$
\begin{aligned}
H_{sm+l} =& \frac{1}{2}\sum_{j,r=1}^{d} L^j g_r(X_{sm+l}, X_{sm})\Delta B^j_{sm+l}\Delta B^r_{sm+l} - \frac{1}{2}\sum_{j.=1}^{d} L^j g_j(X_{sm+l}, X_{sm})\Delta \\
=& \frac{1}{2}\sum_{j,r=1, j\neq r}^{d} L^j g_r(X_{sm+l}, X_{sm})\Delta B^j_{sm+l}\Delta B^r_{sm+l} \\
&+ \frac{1}{2}\sum_{j=1}^{d} L^j g_j(X_{sm+l}, X_{sm})\left((\Delta B^j_{sm+l})^2 - \Delta\right).
\end{aligned}
$$

Note that $L^j g_r(X_{sm+l}, X_{sm})$ is $\mathcal{F}_{t_{sm+l}}$-measurable, ΔB^j_{sm+l} and ΔB^r_{sm+l} are $\mathcal{F}_{t_{sm+l}}$-independent; moreover, ΔB^j_{sm+l} and ΔB^r_{sm+l} are independent, and using the fundamental inequality, we can arrive at

$$
\begin{aligned}
\mathbb{E}|H_{sm+l}|^2 \leq& \frac{d^2}{2}\sum_{j,r=1, j\neq r}^{d} \mathbb{E}\left|L^j g_r(X_{sm+l}, X_{sm})\Delta B^j_{sm+l}\Delta B^r_{sm+l}\right|^2 \\
&+ \frac{d}{2}\sum_{j=1}^{d}\mathbb{E}\left|L^j g_j(X_{sm+l}, X_{sm})\left((\Delta B^j_{sm+l})^2 - \Delta\right)\right|^2 \\
\leq& \frac{d^2}{2}\sum_{j,r=1, j\neq r}^{d} \mathbb{E}|L^j g_r(X_{sm+l}, X_{sm})|^2\mathbb{E}|\Delta B^j_{sm+l}|^2\mathbb{E}|\Delta B^r_{sm+l}|^2 \\
&+ \frac{d}{2}\sum_{j=1}^{d}\mathbb{E}|L^j g_j(X_{sm+l}, X_{sm})|^2\mathbb{E}|(\Delta B^j_{sm+l})^2 - \Delta|^2.
\end{aligned}
$$

Recall the definition of $L^j g_r(x, y)$, using Assumption 2.1 and (39), we have

$$
\begin{aligned}
\mathbb{E}|L^j g_r(X_{sm+l}, X_{sm})|^2 =& \mathbb{E}\left|\sum_{i=1}^{n} g_{ij}(X_{sm+l}, X_{sm})\frac{\partial g_r(X_{sm+l}, X_{sm})}{\partial x_i}\right|^2 \\
\leq& nM^2\mathbb{E}\left|g_j(X_{sm+l}, X_{sm})\right|^2 \\
\leq& 2nM^2\tilde{L}^2(\mathbb{E}|X_{sm+l}|^2 + \mathbb{E}|X_{sm}|^2).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\mathbb{E}|H_{sm+l}|^2 \le\ & nd^2M^2\tilde{L}^2 \sum_{j,r=1,j\neq r}^{d} (\mathbb{E}|X_{sm+l}|^2 + \mathbb{E}|X_{sm}|^2)\mathbb{E}|\Delta B^j_{sm+l}|^2\mathbb{E}|\Delta B^r_{sm+l}|^2 \\
& + ndM^2\tilde{L}^2 \sum_{j=1}^{d} (\mathbb{E}|X_{sm+l}|^2 + \mathbb{E}|X_{sm}|^2)\mathbb{E}|(\Delta B^j_{sm+l})^2 - \Delta|^2 \\
\le\ & nd^4M^2\tilde{L}^2(\mathbb{E}|X_{sm+l}|^2 + \mathbb{E}|X_{sm}|^2)\Delta^2 \\
& + ndM^2\tilde{L}^2 \sum_{j=1}^{d} (\mathbb{E}|X_{sm+l}|^2 + \mathbb{E}|X_{sm}|^2)(\mathbb{E}|\Delta B^j_{sm+l}|^4 + \Delta^2 - 2\Delta\mathbb{E}|\Delta B^j_{sm+l}|^2) \\
\le\ & nd^2(d^2+2)M^2\tilde{L}^2(\mathbb{E}|X_{sm+l}|^2 + \mathbb{E}|X_{sm}|^2)\Delta^2.
\end{aligned} \tag{41}
$$

In addition, using the independence again, one has

$$
\begin{aligned}
& \mathbb{E}\langle X_{sm+l} + f(X_{sm+l}, X_{sm})\,\Delta, H_{sm+l}\rangle \\
= & \frac{1}{2}\mathbb{E}\left\langle X_{sm+l} + f(X_{sm+l}, X_{sm})\,\Delta, \sum_{j,r=1,j\neq r}^{d} L^j g_r(X_{sm+l}, X_{sm})\Delta B^j_{sm+l}\Delta B^r_{sm+l}\right\rangle \\
& + \frac{1}{2}\mathbb{E}\left\langle X_{sm+l} + f(X_{sm+l}, X_{sm})\,\Delta, \sum_{j=1}^{d} L^j g_j(X_{sm+l}, X_{sm})\left((\Delta B^j_{sm+l})^2 - \Delta\right)\right\rangle \\
= & \frac{1}{2}\sum_{j,r=1,j\neq r}^{d} \mathbb{E}\left\{(X_{sm+l} + f(X_{sm+l}, X_{sm})\,\Delta)^{\mathrm{T}} L^j g_r(X_{sm+l}, X_{sm})\Delta B^j_{sm+l}\Delta B^r_{sm+l}\right\} \\
& + \frac{1}{2}\sum_{j=1}^{d} \mathbb{E}\left\{(X_{sm+l} + f(X_{sm+l}, X_{sm})\,\Delta)^{\mathrm{T}} L^j g_j(X_{sm+l}, X_{sm})(\Delta B^j_{sm+l})^2\right\} \\
& - \frac{1}{2}\sum_{j=1}^{d} \mathbb{E}\left\{(X_{sm+l} + f(X_{sm+l}, X_{sm})\,\Delta)^{\mathrm{T}} L^j g_j(X_{sm+l}, X_{sm})\Delta\right\} \\
= & 0.
\end{aligned} \tag{42}
$$

Similarly,

$$\begin{aligned}
&\mathbb{E}\langle g\left(X_{sm+l}, X_{sm}\right) \Delta B_{sm+l}, H_{sm+l}\rangle \\
&= \frac{1}{2}\mathbb{E}\left\{\left(\sum_{k=1}^{d} g_k\left(X_{sm+l}, X_{sm}\right) \Delta B_{sm+l}^k\right)^{\mathrm{T}} \sum_{j,r=1, j\neq r}^{d} L^j g_r(X_{sm+l}, X_{sm})\Delta B_{sm+l}^j \Delta B_{sm+l}^r\right\} \\
&\quad+\frac{1}{2}\mathbb{E}\left\{\left(\sum_{k=1}^{d} g_k\left(X_{sm+l}, X_{sm}\right) \Delta B_{sm+l}^k\right)^{\mathrm{T}} \sum_{j=1}^{d} L^j g_j(X_{sm+l}, X_{sm}) \left((\Delta B_{sm+l}^j)^2 - \Delta\right)\right\} \\
&= \frac{1}{2}\mathbb{E}\left\{\sum_{k,j,r=1, k\neq j\neq r}^{d} g_k\left(X_{sm+l}, X_{sm}\right)^{\mathrm{T}} L^j g_r(X_{sm+l}, X_{sm})\Delta B_{sm+l}^k \Delta B_{sm+l}^j \Delta B_{sm+l}^r\right\} \\
&\quad+\frac{1}{2}\mathbb{E}\left\{\sum_{j,r=1, j\neq r}^{d} g_j(X_{sm+l}, X_{sm})^{\mathrm{T}} L^j g_r(X_{sm+l}, X_{sm})(\Delta B_{sm+l}^j)^2 \Delta B_{sm+l}^r\right\} \\
&\quad+\frac{1}{2}\mathbb{E}\left\{\sum_{j,r=1, j\neq r}^{d} g_r(X_{sm+l}, X_{sm})^{\mathrm{T}} L^j g_r(X_{sm+l}, X_{sm})\Delta B_{sm+l}^j (\Delta B_{sm+l}^r)^2\right\} \\
&\quad+\frac{1}{2}\mathbb{E}\left\{\sum_{k,j=1}^{d} g_k\left(X_{sm+l}, X_{sm}\right)^{\mathrm{T}} L^j g_j(X_{sm+l}, X_{sm}) \left((\Delta B_{sm+l}^j)^2 - \Delta\right) \Delta B_{sm+l}^k\right\} \\
&= 0.
\end{aligned} \tag{43}$$

Moreover, it is easy to know that

$$\mathbb{E}\langle X_{sm+l} + f\left(X_{sm+l}, X_{sm}\right) \Delta, g\left(X_{sm+l}, X_{sm}\right) \Delta B_{sm+l}\rangle = 0. \tag{44}$$

Substituting (41)–(44) into (40), using (39) and Assumption 5.1, one can obtain that

$$\begin{aligned}
\mathbb{E}|X_{sm+l+1}|^2 =& \mathbb{E}|X_{sm+l}|^2 + \mathbb{E}|f\left(X_{sm+l}, X_{sm}\right)|^2\Delta^2 + \mathbb{E}|g\left(X_{sm+l}, X_{sm}\right) \Delta B_{sm+l}|^2 \\
&+ nd^2(d^2+2)M^2\tilde{L}^2(\mathbb{E}|X_{sm+l}|^2 + \mathbb{E}|X_{sm}|^2)\Delta^2 \\
&+ 2\mathbb{E}\langle X_{sm+l}, f\left(X_{sm+l}, X_{sm}\right)\Delta\rangle \\
=& \mathbb{E}|X_{sm+l}|^2 + \left(nd^2(d^2+2)M^2\tilde{L}^2 + 2\tilde{L}^2\right)(\mathbb{E}|X_{sm+l}|^2 + \mathbb{E}|X_{sm}|^2)\Delta^2 \\
&+ 2\Delta\mathbb{E}\left(\langle X_{sm+l}, f\left(X_{sm+l}, X_{sm}\right)\rangle + \frac{1}{2}|g\left(X_{sm+l}, X_{sm}\right)|^2\right) \\
\leq& \mathbb{E}|X_{sm+l}|^2 + K(\mathbb{E}|X_{sm+l}|^2 + \mathbb{E}|X_{sm}|^2)\Delta^2 - 2\lambda_1\mathbb{E}|X_{sm+l}|^2\Delta + 2\lambda_2\mathbb{E}|X_{sm}|^2\Delta \\
&= (1-\alpha\Delta)\mathbb{E}|X_{sm+l}|^2 + \beta\Delta\mathbb{E}|X_{sm}|^2.
\end{aligned}$$

Since $\Delta < \bar{\Delta}$, we have $\alpha > \beta > 0$ and $1 - \alpha\Delta > 0$, by Lemma 5.3, yields

$$\mathbb{E}|X_{sm+l+1}|^2 \leq \Gamma(l+1)\mathbb{E}|X_{sm}|^2.$$

where $\Gamma(l+1) = \left(\frac{\beta}{\alpha} + \left(1 - \frac{\beta}{\alpha}\right)e^{-\alpha(l+1)\Delta}\right)$. In particular, if $l = m-1$, it follows

$$\mathbb{E}|X_{(s+1)m}|^2 \leq \Gamma(m)\mathbb{E}|X_{sm}|^2.$$

Therefore

$$\begin{aligned}\mathbb{E}|X_{sm+l+1}|^2 \le& \Gamma(l+1)\mathbb{E}|X_{sm}|^2\\ \le& \Gamma(l+1)\Gamma(m)\mathbb{E}|X_{(s-1)m}|^2\\ &\vdots\\ \le& \Gamma(l+1)\Gamma(m)^s|x_0|^2.\end{aligned}$$

According to Lemma 5.4, we know that $\Gamma(l+1) \in (0,1)$ for all $l = 0, 1, \ldots, m-1$. Hence,

$$\begin{aligned}\mathbb{E}|X_{sm+l+1}|^2 \le& \frac{\Gamma(l+1)}{\Gamma(m)^{(l+1)\Delta}} e^{(sm+l+1)\Delta \log \Gamma(m)}|x_0|^2\\ \le& \frac{1}{\Gamma(m)} e^{(sm+l+1)\Delta \log \Gamma(m)}|x_0|^2.\end{aligned}$$

Let $H_2 = \frac{1}{\Gamma(m)} > 1$, $\gamma = -\log\Gamma(m) > 0$, we can get that

$$\mathbb{E}|X_k|^2 \le H_2 e^{-\gamma k\Delta}|x_0|^2, \quad \forall k \in \mathbb{N}.$$

Furthermore,

$$\begin{aligned}\lim_{\Delta\to 0}\gamma =& -\lim_{\Delta\to 0}\log\Gamma(m)\\ =& -\lim_{\Delta\to 0}\log\left(\frac{\beta}{\alpha} + \left(1-\frac{\beta}{\alpha}\right)e^{-\alpha}\right)\\ =& -\log\left(\frac{\lambda_2}{\lambda_1} + \left(1-\frac{\lambda_2}{\lambda_1}\right)e^{-2\lambda_1}\right)\\ =& \lambda.\end{aligned}$$

The proof is completed. □

6 Numerical examples

In this section, two numerical examples are given to show the convergence rate obtained in the previous section.

Example 1 In this example, we consider the scalar SDEPCA

$$\mathrm{d}x(t) = 2x([t])\mathrm{d}t - x(t)\mathrm{d}B(t)$$

on $t \ge 0$ with the initial value $x_0 = 1$, $B(t)$ is a scalar Brownian motion. We generate 3000 different Brownian paths. Let $T = 1$, Fig. 1 depicts p-th moment errors $\mathbb{E}|x(1) -$

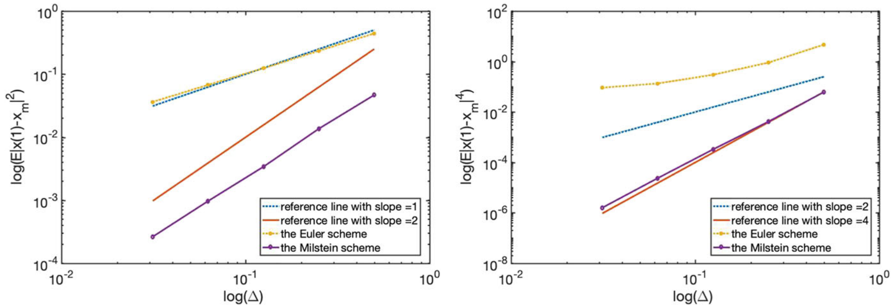

Fig. 1 Log-log plot of errors against step sizes (left: $p = 2$; right: $p = 4$)

$X_m|^p$ as a function of the step size Δ in log-log plot, where we use the numerical solutions produced by Euler and Milstein methods with step sizes 2^{-3}, 2^{-4}, 2^{-5}, 2^{-6}, and 2^{-7}. The simulation using the Euler scheme with step size $\Delta = 2^{-16}$ is regarded as the "true solution." It can be seen from Fig. 1 that the convergence order of the Euler method is around $\frac{1}{2}$, while the convergence order of the Milstein method is close to 1.

Example 2 In the following, we consider the 2-dim SDEPCA

$$\begin{cases} \mathrm{d}x_1(t) = (-x_1(t) + \frac{1}{2}x_2(t) + \sin(x_1([t])))\mathrm{d}t + (x_1(t) + x_2([t]) + \cos(x_2([t])))\mathrm{d}B(t), \\ \mathrm{d}x_2(t) = (\frac{1}{2}x_1(t) - x_2(t) + \cos(x_1([t])))\mathrm{d}t + (\sin(x_1(t)) + x_2([t]))\mathrm{d}B(t) \end{cases}$$

on $t \geq 0$ with the initial value $x_0 = (1, 2)^{\mathrm{T}}$. We use the numerical solution of the Euler method with step-size $\Delta = 2^{-15}$ as the "exact solution," and the step sizes for numerical solutions are taken to be 2^{-4}, 2^{-5}, 2^{-7}, 2^{-8}, and 2^{-9}. The convergence rates for Euler and Milstein methods are shown in Fig. 2.

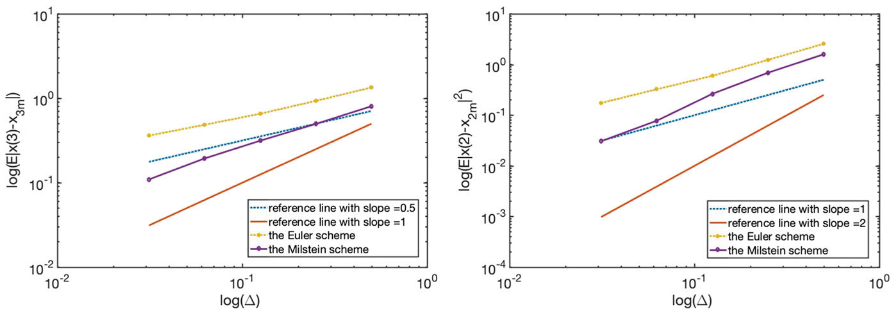

Fig. 2 Log-log plot of errors against step sizes (left: $T = 3$, $p = 1$; right: $T = 2$, $p = 2$)

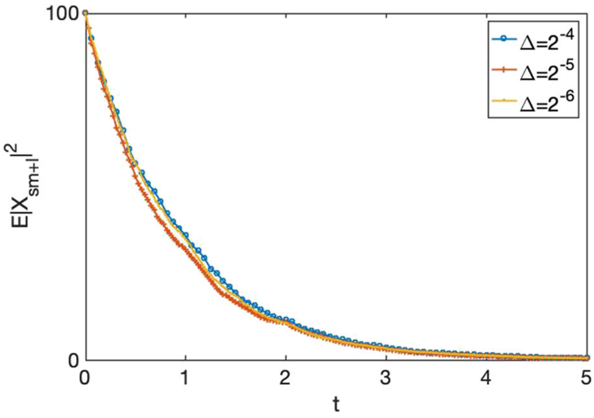

Fig. 3 The mean square stability of the Milstein solutions for (45)

Example 3 In this example, we consider the stability of the Milstein method for the following scalar SDEPCA

$$\mathrm{d}x(t) = (-x(t) + \frac{1}{4}x([t]))\mathrm{d}t + \frac{1}{2}x(t)\mathrm{d}B(t) \tag{45}$$

on $t \geq 0$ with the initial value $x_0 = 10$. It is easy to get that $n = d = 1$, $M = 1$, $L = \frac{1}{4}$, hence $\tilde{L} = M + L = \frac{5}{4}$ and $K = \frac{125}{16}$. On the other hand, we can obtain that $\lambda_1 = \frac{3}{4}$, $\lambda_2 = \frac{1}{8}$ by Assumption 5.1. Since $\lambda_1^2 = \frac{9}{16} < K$, according to Theorem 5.5, $\bar{\Delta} = \frac{\lambda_1 - \lambda_2}{K} = \frac{2}{25}$. Therefore, we choose three step sizes $\Delta = 2^{-4}, 2^{-5}$, and 2^{-6} to show the stability of the Milstein method. The mean square stability of the numerical solutions can be observed from Fig. 3.

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Declarations

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