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ORIGINAL PAPER

Optimal error estimate of the penalty method for the 2D/3D time-dependent MHD equations

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Abstract
In this article, we mainly consider a first-order decoupling penalty method for the 2D/3D time-dependent incompressible magnetohydrodynamic (MHD) equations in a convex domain. This method applies a penalty term to the constraint "div$\mathbf{u} = 0$," which allows us to transform the saddle point problem into two small problems to solve. The time discretization is based on the backward Euler scheme. Moreover, we derive the optimal error estimate for the penalty method under semi-discretization with the relationship $\epsilon = O(\Delta t)$. Finally, we give abundant of numerical tests to verify the theoretical result and the spatial discretization is based on Lagrange finite element.

Keywords MHD equations · Penalty method · Error estimate · Backward Euler scheme

1 Introduction

In this article, we consider the following 2D/3D time-dependent incompressible MHD equations:

$$\begin{cases} \mathbf{u}_t - R_e^{-1}\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p - S_c \mathrm{curl}\mathbf{B}\times\mathbf{B} = \mathbf{f}, \quad \text{in } \Omega\times[0,T], \\ \mathrm{div}\mathbf{u} = 0, \quad \text{in } \Omega\times[0,T], \\ \mathbf{B}_t + R_m^{-1}\mathrm{curl}(\mathrm{curl}\mathbf{B}) - \mathrm{curl}(\mathbf{u}\times\mathbf{B}) = \mathbf{g}, \quad \text{in } \Omega\times[0,T], \\ \mathrm{div}\mathbf{B} = 0, \quad \text{in } \Omega\times[0,T], \end{cases} \tag{1.1}$$

with the homogeneous boundary conditions and initial conditions

$$\begin{cases} \mathbf{u} = \mathbf{0}, \quad \mathbf{B}\cdot\mathbf{n} = 0, \quad \mathbf{n}\times\mathrm{curl}\mathbf{B} = \mathbf{0}, \quad \text{on } \partial\Omega\times[0,T], \\ \mathbf{u}(\mathbf{x},0) = \mathbf{u}_0, \quad \mathbf{B}(\mathbf{x},0) = \mathbf{B}_0, \quad \text{in } \Omega, \end{cases} \tag{1.2}$$

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where Ω is a bounded, convex, and open domain in $\mathbb{R}^d(d = 2$ or $3)$ with a sufficiently smooth boundary $\partial\Omega$. Here, $\mathbf{u}$ is the fluid velocity, $\mathbf{B}$ is the magnetic field, p is the hydrodynamic pressure, $\mathbf{f}$ denotes the external force term, $\mathbf{g}$ is the known applied current with $\text{div}\mathbf{g} = 0$, $\mathbf{n}$ denotes the outward normal unit vector on $\partial\Omega$, R_e is the hydrodynamic Reynolds number, R_m is the magnetic Reynolds number, S_c is the coupling coefficient, and T is the final time.

The incompressible MHD equations mainly describe the dynamic behavior of an electrically conducting fluid under the effect of an imposed magnetic field. It consists of the Navier-Stokes equations for hydrodynamics and Maxwell's equations for electromagnetism. Its applications involve many branches of physics, such as fusion reactor blankets, liquid metal magnetic pumps, and aluminum electrolysis (see [11, 19, 32]). The detailed information of physical background of the MHD flow, we refer to [12, 20]. In recent years, there have been many research works using finite element method to simulate the MHD equations such as [7, 23, 29–33].

We note that the pressure p does not appear in the incompressible equation, which makes the equations difficult to solve numerically. A popular way to overcome the above difficulty is to relax the incompressibility constraint in an appropriate way. This leads to a number of methods, such as the penalty method, the artificial compressibility method, the pressure stabilization method, and the projection method (see for instance [3–5, 25, 26]). As far as we know, the stabilization method was proposed in [3]. Next, a pressure stability analysis for the Stokes problem was given in [2]. For the projection method, it can be traced back to [4, 26]. Then, a consistent projection finite element method for the incompressible MHD equations was discussed in [30]. Convergence analysis of an unconditionally energy stable projection scheme for MHD equations was proposed in [29]. For the artificial compressibility method, it originated in [4, 5, 25, 26]. Next, the artificial compressibility approximation for MHD equations in unbounded domain was given in [8]. For the penalty method, we can refer to [6]. Then, optimal error estimate of the penalty finite element method for the time-dependent Navier-Stokes equations was given in [13]. A penalty finite element method based on the Euler implicit/explicit scheme for the time-dependent Navier-Stokes equations was proposed in [15]. In [22], the authors study iterative methods in penalty finite element discretization for the steady MHD equations. In [7], the authors given a decoupling penalty finite element method for the stationary incompressible MHD equations. It is worth mentioning that the penalty method is the simplest and the most basic of these mentioned above methods.

In this article, we mainly consider the penalty method to solve the time-dependent incompressible MHD equations. The penalty method applied to (1.1) and (1.2) is to approximate the solution $(\mathbf{u}, p, \mathbf{B})$ by $(\mathbf{u}_\epsilon, p_\epsilon, \mathbf{B}_\epsilon)$ satisfying the following penalty system:

$$\begin{cases} \mathbf{u}_{\epsilon t} - R_e^{-1}\Delta\mathbf{u}_\epsilon + B(\mathbf{u}_\epsilon, \mathbf{u}_\epsilon) + \nabla p_\epsilon - S_c \text{curl}\mathbf{B}_\epsilon \times \mathbf{B}_\epsilon = \mathbf{f}, \quad \text{in } \Omega \times [0, T], \\ \text{div}\mathbf{u}_\epsilon + R_e \epsilon p_\epsilon = 0, \quad \text{in } \Omega \times [0, T], \\ \mathbf{B}_{\epsilon t} + R_m^{-1}\text{curl}(\text{curl}\mathbf{B}_\epsilon) - \text{curl}(\mathbf{u}_\epsilon \times \mathbf{B}_\epsilon) = \mathbf{g}, \quad \text{in } \Omega \times [0, T], \\ \text{div}\mathbf{B}_\epsilon = 0, \quad \text{in } \Omega \times [0, T], \end{cases} \tag{1.3}$$

with the homogeneous boundary conditions and initial conditions

$$\begin{cases} \mathbf{u}_\epsilon = \mathbf{0}, \quad \mathbf{B}_\epsilon \cdot \mathbf{n} = 0, \quad \mathbf{n} \times \mathrm{curl}\mathbf{B}_\epsilon = \mathbf{0}, \quad \text{on } \partial\Omega \times [0, T], \\ \mathbf{u}_\epsilon(\mathbf{x}, 0) = \mathbf{u}_0, \quad \mathbf{B}_\epsilon(\mathbf{x}, 0) = \mathbf{B}_0, \quad \text{in } \Omega, \end{cases} \tag{1.4}$$

where $B(\mathbf{u}, \mathbf{v}) = (\mathbf{u} \cdot \nabla)\mathbf{v} + \frac{1}{2}(\mathrm{div}\mathbf{u})\mathbf{v}$ is the modified bilinear term, and $0 < \epsilon < 1$ is penalty parameter. The penalty method is a decoupled method, which can easily eliminate the p_ϵ in (1.3) by "$\mathrm{div}\mathbf{u}_\epsilon + R_e \epsilon p_\epsilon = 0$" to obtain a penalty system containing only $(\mathbf{u}_\epsilon, \mathbf{B}_\epsilon)$, and then directly get the numerical solution of original equations. This idea has been widely used in many fields of computational fluid dynamics (please refer to [7, 13, 15, 18, 21, 22]).

From [7], we know that $\lim_{\epsilon \to 0} (\mathbf{u}_\epsilon, p_\epsilon, \mathbf{B}_\epsilon) = (\mathbf{u}, p, \mathbf{B})$, the solution of (1.1)–(1.2). The error analysis of the penalty finite element method for the stationary MHD equations is studied in [22–24]. For example, the optimal error estimate of the penalty system for the stationary MHD equations is as follows:

$$\|\mathbf{u} - \mathbf{u}_\epsilon\|_1 + \|\mathbf{B} - \mathbf{B}_\epsilon\|_1 + \|p - p_\epsilon\| \le C\epsilon,$$

where $C > 0$ is a general positive constant, and $\|\cdot\|_1$ and $\|\cdot\|$ denote the norm in $H^1(\Omega)^d$ and $L^2(\Omega)$, respectively. However, there is no literature on the penalty method for time-dependent MHD equations. The purpose of this article is to derive optimal error estimate of the penalty method for the time-dependent MHD equations and its time-discretization. In Theorem 4.1, we derive that the optimal error estimate of the penalty method for time-dependent MHD equations is as follows:

$$\begin{aligned} \sup_{0 \le t \le T} \Big(\|\mathbf{u}(t) - \mathbf{u}_\epsilon(t)\| &+ \|\mathbf{B}(t) - \mathbf{B}_\epsilon(t)\| + R_e^{-1} \|\mathbf{u}(t) - \mathbf{u}_\epsilon(t)\|_1 \\ &+ R_m^{-1} \|\mathbf{B}(t) - \mathbf{B}_\epsilon(t)\|_1 \Big) + \left(\int_0^T \|p(t) - p_\epsilon(t)\|^2 dt \right)^{\frac{1}{2}} \le C\epsilon. \end{aligned}$$

In Theorem 5.1 and (5.17), we derive that the optimal error estimate of the time-discretization scheme of the penalty method for the time-dependent MHD equations is as follows:

$$\begin{aligned} \sqrt{t_n}(\|\mathbf{u}(t_n) - \mathbf{u}_\epsilon^n\| &+ \|\mathbf{B}(t_n) - \mathbf{B}_\epsilon^n\|) + t_n(\|\mathbf{u}(t_n) - \mathbf{u}_\epsilon^n\|_1 + \|\mathbf{B}(t_n) - \mathbf{B}_\epsilon^n\|_1) \\ &+ \left(\Delta t \sum_{n=1}^N t_n^2 \|p(t_n) - p_\epsilon^n\|^2 \right)^{\frac{1}{2}} \le C(\Delta t + \epsilon), \end{aligned}$$

where Δt is the time step, $t_n = n\Delta t$ $(1 \le n \le N)$, $(\mathbf{u}_\epsilon^n, p_\epsilon^n, \mathbf{B}_\epsilon^n)$ is an approximation of $(\mathbf{u}, p, \mathbf{B})$ at time t_n.

The paper is organized as follows. In Section 2, we introduce some notations and preliminary results for the time-dependent penalty MHD equations (1.3) and (1.4). In Section 3, we analyze the error behavior of the linear form for the penalty MHD equations. In Section 4, we consider the penalty method for the MHD equations. In Section 5, we analyze the time-discretized scheme of the penalty MHD equations. In Section 6, we give some numerical tests to verify the theoretical result of the penalty method. Finally, some conclusions are given in Section 7.

2 Preliminaries

In this section, we give some notations. For $1 \leq r \leq \infty$, $L^r(\Omega)$ denotes the usual Lebesgue space on Ω with the norm $\|\cdot\|_{L^r}$. In particular, we will denote $L^2(\Omega)$ norm and $L^2(\Omega)$ inner product by $\|\cdot\|$ and $(\cdot,\cdot)$, respectively. For all non-negative integers k and r, $W^{k,r}(\Omega)$ stands for the standard Sobolev space equipped with the standard Sobolev norm $\|\cdot\|_{k,r}$. The norm of the space $W^{k,2}(\Omega)$ is represented by $\|\cdot\|_k$. The vector functions and vector spaces will be indicated by boldface type. Now, we introduce the following spaces

$$
\begin{aligned}
&\mathbf{X} := \mathbf{H}_0^1(\Omega) = \{\mathbf{v} \in H^1(\Omega)^d : \mathbf{v}\,|_{\partial\Omega} = \mathbf{0}\},\\
&\mathrm{M} := L_0^2(\Omega) = \{q \in L^2(\Omega) : \textstyle\int_\Omega q d\mathbf{x} = 0\},\\
&\mathbf{W} := \mathbf{H}_n^1(\Omega) = \{\mathbf{w} \in H^1(\Omega)^d : \mathbf{w}\cdot\mathbf{n}\,|_{\partial\Omega} = 0\},\\
&\mathbf{X}_0 := \{\mathbf{v} \in \mathbf{X} : \mathrm{div}\mathbf{v} = 0\},\\
&\mathbf{W}_0 := \{\mathbf{w} \in \mathbf{W} : \mathrm{div}\mathbf{w} = 0\},\\
&\tilde{\mathbf{H}} := \{\mathbf{w} \in \mathbf{L}^2(\Omega) : \mathrm{div}\mathbf{w} = 0, \mathbf{w}\cdot\mathbf{n}\,|_{\partial\Omega} = 0\},
\end{aligned}
$$

and the following trilinear form

$$
\begin{aligned}
b(\mathbf{u},\mathbf{v},\mathbf{w}) &= (B(\mathbf{u},\mathbf{v}),\mathbf{w}) = \left((\mathbf{u}\cdot\nabla)\mathbf{v} + \frac{1}{2}(\mathrm{div}\mathbf{u})\mathbf{v},\mathbf{w}\right)\\
&= \frac{1}{2}b(\mathbf{u},\mathbf{v},\mathbf{w}) - \frac{1}{2}b(\mathbf{u},\mathbf{w},\mathbf{v}), \quad \forall \mathbf{u},\mathbf{v},\mathbf{w} \in \mathbf{X}.
\end{aligned}
$$

Therefore, the trilinear form $b(\cdot,\cdot,\cdot)$ satisfies

$$
b(\mathbf{u},\mathbf{v},\mathbf{v}) = 0, \quad \forall \mathbf{u},\mathbf{v} \in \mathbf{X}. \tag{2.1}
$$

Moreover, we have the following two formulas (cf. [22]):

$$
(\mathbf{a}\times\mathbf{b})\times\mathbf{c}\cdot\mathbf{d} = (\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = -(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{d}\times\mathbf{c}),
$$

and

$$
\int_\Omega \mathrm{curl}\mathbf{B}\cdot\mathbf{C}d\mathbf{x} = -\int_{\partial\Omega}(\mathbf{B}\times\mathbf{n})\cdot\mathbf{C}ds + \int_\Omega \mathbf{B}\cdot\mathrm{curl}\mathbf{C}d\mathbf{x},
$$

which imply that for all $\mathbf{B},\mathbf{C} \in \mathbf{W}$ and $\mathbf{u} \in \mathbf{X}$,

$$
(\mathrm{curl}(\mathbf{u}\times\mathbf{B}),\mathbf{C}) = -<(\mathbf{u}\times\mathbf{B})\times\mathbf{n},\mathbf{C}>|_{\partial\Omega} + (\mathbf{u}\times\mathbf{B},\mathrm{curl}\mathbf{C}) = -(\mathrm{curl}\mathbf{C}\times\mathbf{B},\mathbf{u}). \tag{2.2}
$$

Then, we have the following variational formulation of problem (1.1) and (1.2): find $(\mathbf{u},p,\mathbf{B}) \in L^2(0,T;\mathbf{X}) \times L^2(0,T;\mathrm{M}) \times L^2(0,T;\mathbf{W})$ such that for all $(\mathbf{v},q,\mathbf{C}) \in \mathbf{X}\times\mathrm{M}\times\mathbf{W}$

$$
\begin{cases}
(\mathbf{u}_t,\mathbf{v}) + R_e^{-1}(\nabla\mathbf{u},\nabla\mathbf{v}) + b(\mathbf{u},\mathbf{u},\mathbf{v}) - S_c(\mathrm{curl}\mathbf{B}\times\mathbf{B},\mathbf{v}) - (p,\mathrm{div}\mathbf{v}) + (\mathrm{div}\mathbf{u},q) = (\mathbf{f},\mathbf{v}),\\
(\mathbf{B}_t,\mathbf{C}) + R_m^{-1}(\mathrm{curl}\mathbf{B},\mathrm{curl}\mathbf{C}) - (\mathrm{curl}(\mathbf{u}\times\mathbf{B}),\mathbf{C}) = (\mathbf{g},\mathbf{C}),\\
\mathbf{u}(x,0) = \mathbf{u}_0, \quad \mathbf{B}(x,0) = \mathbf{B}_0,
\end{cases} \tag{2.3}
$$

where $\text{div}\mathbf{u}_0 = \text{div}\mathbf{B}_0 = 0$ and the variational formulation of (1.3) and (1.4) reads: find $(\mathbf{u}_\epsilon, p_\epsilon, \mathbf{B}_\epsilon) \in L^2(0,T;\mathbf{X}) \times L^2(0,T;\mathrm{M}) \times L^2(0,T;\mathbf{W})$ such that for all $(\mathbf{v}, q, \mathbf{C}) \in \mathbf{X} \times \mathrm{M} \times \mathbf{W}$

$$\begin{cases} (\mathbf{u}_{\epsilon t}, \mathbf{v}) + R_e^{-1}(\nabla \mathbf{u}_\epsilon, \nabla \mathbf{v}) + b(\mathbf{u}_\epsilon, \mathbf{u}_\epsilon, \mathbf{v}) - S_c(\text{curl}\mathbf{B}_\epsilon \times \mathbf{B}_\epsilon, \mathbf{v}) - (p_\epsilon, \text{div}\mathbf{v}) + (\text{div}\mathbf{u}_\epsilon, q) \\ +R_e\epsilon(p_\epsilon, q) = (\mathbf{f}, \mathbf{v}), \\ (\mathbf{B}_{\epsilon t}, \mathbf{C}) + R_m^{-1}(\text{curl}\mathbf{B}_\epsilon, \text{curl}\mathbf{C}) - (\text{curl}(\mathbf{u}_\epsilon \times \mathbf{B}_\epsilon), \mathbf{C}) = (\mathbf{g}, \mathbf{C}), \\ \mathbf{u}_\epsilon(x,0) = \mathbf{u}_0, \quad \mathbf{B}_\epsilon(x,0) = \mathbf{B}_0. \end{cases} \tag{2.4}$$

Moreover, from the variational formulation (2.3), we can get $\text{div}\mathbf{B}_t = 0$ and $\text{div}\mathbf{B} = 0$ (cf. [14, 34]).

We will use the letter C as a general positive constant depending on coefficients of the equations and the domain Ω, which may have different values at its different occurrences. The following two inequalities will be used repeatedly (cf. [21]):

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) \le C\|\mathbf{u}\|_1 \|\mathbf{v}\|_1^{\frac{1}{2}} \|\mathbf{v}\|_2^{\frac{1}{2}} \|\mathbf{w}\|, \quad \forall \mathbf{v} \in \mathbf{H}^2(\Omega) \cap \mathbf{X}, \quad \mathbf{u}, \mathbf{w} \in \mathbf{X}, \tag{2.5}$$

and

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) \le \begin{cases} \|\mathbf{u}\|_1 \|\mathbf{v}\|_1 \|\mathbf{w}\|_1, & \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{X}, \\ \|\mathbf{u}\|_2 \|\mathbf{v}\| \; \|\mathbf{w}\|_1, & \forall \mathbf{u} \in \mathbf{H}^2(\Omega) \cap \mathbf{X}, \quad \mathbf{v}, \mathbf{w} \in \mathbf{X}, \\ \|\mathbf{u}\|_2 \|\mathbf{v}\|_1 \|\mathbf{w}\|, & \forall \mathbf{u} \in \mathbf{H}^2(\Omega) \cap \mathbf{X}, \quad \mathbf{v}, \mathbf{w} \in \mathbf{X}, \\ \|\mathbf{u}\|_1 \|\mathbf{v}\|_2 \|\mathbf{w}\|, & \forall \mathbf{v} \in \mathbf{H}^2(\Omega) \cap \mathbf{X}, \quad \mathbf{u}, \mathbf{w} \in \mathbf{X}. \end{cases} \tag{2.6}$$

Furthermore, we have the following estimates (cf. [9, 28]):

$$(\text{curl}\mathbf{C} \times \mathbf{B}, \mathbf{v}) + (\mathbf{v} \times \mathbf{B}, \text{curl}\mathbf{C}) = 0, \quad \forall \mathbf{B}, \mathbf{C} \in \mathbf{W}, \quad \mathbf{v} \in \mathbf{X}, \tag{2.7}$$

$$\|\mathbf{v}\| \le \gamma_0 \|\nabla \mathbf{v}\|, \quad \|\mathbf{v}\|_{\mathbf{L}^3} \le C\|\mathbf{v}\|^{\frac{1}{2}} \|\nabla \mathbf{v}\|^{\frac{1}{2}}, \quad \|\mathbf{v}\|_{\mathbf{L}^6} \le C\|\nabla \mathbf{v}\|, \quad \forall \mathbf{v} \in \mathbf{X}, \tag{2.8}$$

$$\|\mathbf{v}\|_{\mathbf{L}^\infty} + \|\nabla \mathbf{v}\|_{\mathbf{L}^3} \le C\|\mathbf{v}\|_1^{\frac{1}{2}} \|\mathbf{v}\|_2^{\frac{1}{2}}, \quad \forall \mathbf{v} \in \mathbf{H}^2(\Omega), \tag{2.9}$$

$$\|\nabla \mathbf{B}\| \le C_0(\|\text{curl}\mathbf{B}\| + \|\text{div}\mathbf{B}\|), \quad \forall \mathbf{B} \in \mathbf{W}, \tag{2.10}$$

$$\|\mathbf{v}\|_{\mathbf{L}^4} \le C_1 \|\nabla \mathbf{v}\|, \quad \forall \mathbf{v} \in \mathbf{X}, \tag{2.11}$$

$$\|\text{curl}\mathbf{B}\| \le \sqrt{2}\|\nabla \mathbf{B}\|, \quad \|\text{div}\mathbf{B}\| \le \sqrt{d}\|\nabla \mathbf{B}\|, \quad \forall \mathbf{B} \in \mathbf{W}, \tag{2.12}$$

$$\text{curl}(\mathbf{u} \times \mathbf{B}) = \mathbf{u}(\text{div}\mathbf{B}) - \mathbf{B}(\text{div}\mathbf{u}) + (\mathbf{B} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{B}, \quad \forall \mathbf{u} \in \mathbf{X}, \quad \mathbf{B} \in \mathbf{W}, \tag{2.13}$$

where γ_0 (only dependent on Ω) is a positive constant, C_0 (only dependent on Ω) is an embedding constant of $\mathbf{H}_n^1(\Omega) \hookrightarrow \mathbf{H}^1(\Omega)$ ($\hookrightarrow$ denotes the continuous embedding), and C_1 (only dependent on Ω) is an embedding constant of $\mathbf{H}^1(\Omega) \hookrightarrow \mathbf{L}^4(\Omega)$.

We define $A_1\mathbf{u} = -\Delta\mathbf{u}$ and $A_{1\epsilon}\mathbf{u} = -\Delta\mathbf{u} - \frac{1}{\epsilon}\nabla\text{div}\mathbf{u}$, which are the operators associated with Navier-Stokes equations and the penalty Navier-Stokes equations. They are the positive self-adjoint operators from $D(A_1) = \mathbf{H}^2(\Omega) \cap \mathbf{X}$ onto $\mathbf{L}^2(\Omega)$ and the powers A_1^α and $A_{1\epsilon}^\alpha$ ($\alpha \in \mathbb{R}$) are well defined. Similarly, we define the Maxwell's operator $A_2 = P_H(\text{curlcurl} - \nabla\text{div}) : D(A_2) \to \tilde{\mathbf{H}}$ and also define $A_{2\epsilon}\mathbf{B} = A_2\mathbf{B}$, where $D(A_2) = \mathbf{H}^2(\Omega) \cap \mathbf{W}$ and P_H is the L^2-orthogonal projection.

Finally, it is easy to show that (cf. [21, 27])

$$(A_1\mathbf{u},\mathbf{v}) = \left(A_1^{\frac{1}{2}}\mathbf{u}, A_1^{\frac{1}{2}}\mathbf{v}\right) = (\nabla\mathbf{u},\nabla\mathbf{v}), \quad \forall\mathbf{u},\mathbf{v}\in\mathbf{X},$$
$$(A_{1\epsilon}\mathbf{u},\mathbf{v}) = \left(A_{1\epsilon}^{\frac{1}{2}}\mathbf{u}, A_{1\epsilon}^{\frac{1}{2}}\mathbf{v}\right) = (\nabla\mathbf{u},\nabla\mathbf{v}) + \frac{1}{\epsilon}(\mathrm{div}\mathbf{u},\mathrm{div}\mathbf{v}), \quad \forall\mathbf{u},\mathbf{v}\in\mathbf{X},$$
$$(A_2\mathbf{B},\mathbf{C}) = \left(A_2^{\frac{1}{2}}\mathbf{B}, A_2^{\frac{1}{2}}\mathbf{C}\right) = (\mathrm{curl}\mathbf{B},\mathrm{curl}\mathbf{C}) + (\mathrm{div}\mathbf{B},\mathrm{div}\mathbf{C}), \quad \forall\mathbf{B},\mathbf{C}\in\mathbf{W}.$$

Furthermore, we have the following lemmas given in [21].

Lemma 2.1 *There exists a constant $C_2 > 0$ depending only on Ω and such that for ϵ sufficiently small, we have*

$$\|\Delta\mathbf{u}\| \le C_2\|A_{1\epsilon}\mathbf{u}\|, \quad \forall\mathbf{u}\in\mathbf{H}^2(\Omega)\cap\mathbf{X},$$
$$\|\nabla\mathbf{u}\| \le C_2\|A_{1\epsilon}^{\frac{1}{2}}\mathbf{u}\|, \quad \forall\mathbf{u}\in\mathbf{X},$$
$$\|A_{1\epsilon}^{-1}\mathbf{u}\| \le C_2\|\mathbf{u}\|_{-2}, \quad \forall\mathbf{u}\in\mathbf{H}^{-2}(\Omega),$$

where $\mathbf{H}^{-2}(\Omega)$ is the dual space of $\mathbf{H}^2(\Omega)\cap\mathbf{X}$, $\|\cdot\|_{-2}$ is the corresponding norm.

Lemma 2.2 *(Gronwall lemma). Let y(t), h(t), g(t), f(t) be nonnegative functions satisfying*

$$y(t) + \int_0^t h(s)ds \le y(0) + \int_0^t \Big(g(s)y(s) + f(s)\Big)ds, \quad \forall 0\le t\le T, \quad with \int_0^T g(t)dt \le M.$$

Then

$$y(t) + \int_0^t h(s)ds \le \exp(M)\Big(y(0) + \int_0^t f(s)ds\Big), \quad \forall 0\le t\le T.$$

Lemma 2.3 (discrete Gronwall lemma) *Let y^n, h^n, g^n, f^n be nonnegative series satisfying*

$$y^m + \Delta t\sum_{n=0}^m h^n \le B + \Delta t\sum_{n=0}^m (g^n y^n + f^n), \quad with\ \Delta t\sum_{n=0}^N g^n \le M, \quad \forall 0\le m\le N = \left[\frac{T}{\Delta t}\right].$$

Assume $\Delta t g^n < 1$ for every n. Define $\sigma = \max_{0\le n\le N}(1-g^n\Delta t)^{-1}$, then

$$y^m + \Delta t\sum_{n=0}^m h^n \le \exp(\sigma M)\left(B + \Delta t\sum_{n=0}^m f^n\right) \quad \forall 0\le m\le N.$$

In this paper, we assume that the problem (1.1) satisfies the following conditions.

Assumption $\mathcal{A}1$: The initial data $\mathbf{u}_0\in\mathbf{X}_0\cap\mathbf{H}^2(\Omega)$ and $\mathbf{B}_0\in\mathbf{W}_0\cap\mathbf{H}^2(\Omega)$, the external force $\mathbf{f}$, and the applied current $\mathbf{g}$ satisfy the bound

$$\sup_{0\le t\le T}\left\{\|\mathbf{f}(t)\| + \|\mathbf{f}_t(t)\| + \|\mathbf{g}(t)\| + \|\mathbf{g}_t(t)\|\right\} + \|\mathbf{u}_0\|_2 + \|\mathbf{B}_0\|_2 \le C.$$

Suppose that $\mathcal{A}1$ ensures the existence of a unique strong solution to the problem (2.3) on small time interval $[0, T]$ such that (cf. [20])

$$\begin{aligned} &\mathbf{u} \in C(0,T;\mathbf{X}) \cap L^2(0,T;\mathbf{H}^2(\Omega)), \quad p \in L^2(0,T;H^1(\Omega)\cap \mathrm{M}), \\ &\mathbf{B} \in C(0,T;\mathbf{W}_0) \cap L^2(0,T;\mathbf{H}^2(\Omega)), \end{aligned} \tag{2.14}$$

by using the smoothing property of the Navier-Stokes equations at $t = 0$, then (see for instance [16])

$$tp_t \in L^2(0,T;H^1(\Omega)). \tag{2.15}$$

Assumption $\mathcal{A}2$: Assume that the boundary of Ω is smooth so that the unique solution $(\mathbf{v}, q) \in \mathbf{X} \times \mathrm{M}$ of the steady Stokes problem (cf. [14, 34])

$$-\Delta \mathbf{v} + \nabla q = \mathbf{f}, \quad \mathrm{div}\mathbf{v} = 0, \quad \text{in } \Omega, \quad \mathbf{v}|_{\partial\Omega} = \mathbf{0},$$

for prescribed $\mathbf{f} \in \mathbf{L}^2(\Omega)$ satisfies

$$\|\mathbf{v}\|_2 + \|q\|_1 \le c\|\mathbf{f}\|;$$

and Maxwell's equations

$$\mathrm{curl}(\mathrm{curl}\mathbf{C}) = \mathbf{g}, \quad \mathrm{div}\mathbf{C} = 0, \quad \text{in } \Omega, \quad \mathbf{C}\cdot\mathbf{n} = 0, \quad \mathbf{n} \times \mathrm{curl}\mathbf{C} = \mathbf{0}, \quad \text{on } \Omega,$$

for the prescribed $\mathbf{g} \in \mathbf{L}^2(\Omega)$ admit a unique solution $\mathbf{C} \in \mathbf{W}_0$ which satisfies

$$\|\mathbf{C}\|_2 \le c\|\mathbf{g}\|,$$

where c is a positive constant depending only on Ω, and may take different values at its different places.

Using the operators $A_{1\epsilon}$, $A_{2\epsilon}$, we can rewrite the penalized system (2.4) as

$$\begin{cases} \mathbf{u}_{\epsilon t} + R_e^{-1} A_{1\epsilon}\mathbf{u}_\epsilon + B(\mathbf{u}_\epsilon, \mathbf{u}_\epsilon) - S_c \mathrm{curl}\mathbf{B}_\epsilon \times \mathbf{B}_\epsilon = \mathbf{f}, \\ \mathbf{B}_{\epsilon t} + R_m^{-1} A_{2\epsilon}\mathbf{B}_\epsilon - \mathrm{curl}(\mathbf{u}_\epsilon \times \mathbf{B}_\epsilon) = \mathbf{g}. \end{cases} \tag{2.16}$$

Then, we have

$$\|\mathbf{u}_\epsilon(t)\|_1^2 + \|\mathbf{B}_\epsilon(t)\|_1^2 + \int_0^t \left(\|\mathbf{u}_\epsilon(s)\|_2^2 + \|\mathbf{B}_\epsilon(s)\|_2^2\right) ds \le C, \quad t \in [0, T]. \tag{2.17}$$

Taking the L^2 inner product of the first equation of (2.16) with $\mathbf{u}_\epsilon$, the second equation with $S_c\mathbf{B}_\epsilon$, and thanks to (2.1), (2.7), we obtain

$$\frac{1}{2}\frac{d}{dt}(\|\mathbf{u}_\epsilon\|^2 + S_c\|\mathbf{B}_\epsilon\|^2) + R_e^{-1}\|A_{1\epsilon}^{\frac{1}{2}}\mathbf{u}_\epsilon\|^2 + S_c R_m^{-1}\|A_{2\epsilon}^{\frac{1}{2}}\mathbf{B}_\epsilon\|^2 = (\mathbf{f}, \mathbf{u}_\epsilon) + (\mathbf{g}, S_c\mathbf{B}_\epsilon).$$

Due to Lemma 2.1, we have

$$(\mathbf{f}, \mathbf{u}_\epsilon) + (\mathbf{g}, S_c\mathbf{B}_\epsilon) \le C\|\mathbf{f}\|^2 + C\|\mathbf{g}\|^2 + \frac{1}{2R_e}\|A_{1\epsilon}^{\frac{1}{2}}\mathbf{u}_\epsilon\|^2 + \frac{S_c}{2R_m}\|A_{2\epsilon}^{\frac{1}{2}}\mathbf{B}_\epsilon\|^2.$$

Combining the above inequality and integrating from 0 to $t \le T$, we get

$$\|\mathbf{u}_\epsilon\|^2 + S_c\|\mathbf{B}_\epsilon\|^2 + \int_0^t \left(R_e^{-1}\|A_{1\epsilon}^{\frac{1}{2}}\mathbf{u}_\epsilon\|^2 + S_c R_m^{-1}\|A_{2\epsilon}^{\frac{1}{2}}\mathbf{B}_\epsilon\|^2\right) ds \le C. \tag{2.18}$$

Taking the L^2 inner product of the first equation of (2.16) with $A_{1\epsilon}\mathbf{u}_\epsilon$, the second equation with $A_{2\epsilon}\mathbf{B}_\epsilon$, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|A_{1\epsilon}^{\frac{1}{2}}\mathbf{u}_\epsilon\|^2+\|A_{2\epsilon}^{\frac{1}{2}}\mathbf{B}_\epsilon\|^2\right)+R_e^{-1}\|A_{1\epsilon}\mathbf{u}_\epsilon\|^2+R_m^{-1}\|A_{2\epsilon}\mathbf{B}_\epsilon\|^2=(\mathbf{f},A_{1\epsilon}\mathbf{u}_\epsilon)+(\mathbf{g},A_{2\epsilon}\mathbf{B}_\epsilon)$$
$$-b(\mathbf{u}_\epsilon,\mathbf{u}_\epsilon,A_{1\epsilon}\mathbf{u}_\epsilon)+S_c(\mathrm{curl}\mathbf{B}_\epsilon\times\mathbf{B}_\epsilon,A_{1\epsilon}\mathbf{u}_\epsilon)+(\mathrm{curl}(\mathbf{u}_\epsilon\times\mathbf{B}_\epsilon),A_{2\epsilon}\mathbf{B}_\epsilon).$$

By using Lemma 2.1, (2.5) and (2.9)–(2.13), we get

$$\begin{aligned}
(\mathbf{f},A_{1\epsilon}\mathbf{u}_\epsilon)+(\mathbf{g},A_{2\epsilon}\mathbf{B}_\epsilon)&\le\frac{1}{4}(R_e^{-1}\|A_{1\epsilon}\mathbf{u}_\epsilon\|^2+R_m^{-1}\|A_{2\epsilon}\mathbf{B}_\epsilon\|^2)+C\|\mathbf{f}\|^2+C\|\mathbf{g}\|^2,\\
-b(\mathbf{u}_\epsilon,\mathbf{u}_\epsilon,A_{1\epsilon}\mathbf{u}_\epsilon)&\le\|\mathbf{u}_\epsilon\|_1\|\mathbf{u}_\epsilon\|_1^{\frac{1}{2}}\|\mathbf{u}_\epsilon\|_2^{\frac{1}{2}}\|A_{1\epsilon}\mathbf{u}_\epsilon\|\le\frac{1}{8R_e}\|A_{1\epsilon}\mathbf{u}_\epsilon\|^2+C\|\mathbf{u}_\epsilon\|_1^4\|A_{1\epsilon}^{\frac{1}{2}}\mathbf{u}_\epsilon\|^2,\\
S_c(\mathrm{curl}\mathbf{B}_\epsilon\times\mathbf{B}_\epsilon,A_{1\epsilon}\mathbf{u}_\epsilon)&\le C\|\mathrm{curl}\mathbf{B}_\epsilon\|\|\mathbf{B}_\epsilon\|_{\mathbf{L}^\infty}\|A_{1\epsilon}\mathbf{u}_\epsilon\|\\
&\le\frac{1}{8R_e}\|A_{1\epsilon}\mathbf{u}_\epsilon\|^2+\frac{1}{8R_m}\|A_{2\epsilon}\mathbf{B}_\epsilon\|^2+C\|\mathbf{B}_\epsilon\|_1^4\|A_{2\epsilon}^{\frac{1}{2}}\mathbf{B}_\epsilon\|^2\\
(\mathrm{curl}(\mathbf{u}_\epsilon\times\mathbf{B}_\epsilon),A_{2\epsilon}\mathbf{B}_\epsilon)&\le C(\|\mathbf{B}_\epsilon\|_{\mathbf{L}^\infty}\|\mathrm{div}\mathbf{u}_\epsilon\|+\|\mathbf{B}_\epsilon\|_{\mathbf{L}^\infty}\|\nabla\mathbf{u}_\epsilon\|+\|\mathbf{u}_\epsilon\|_{\mathbf{L}^\infty}\|\nabla\mathbf{B}_\epsilon\|)\|A_{2\epsilon}\mathbf{B}_\epsilon\|\\
&\le\frac{1}{8R_e}\|A_{1\epsilon}\mathbf{u}_\epsilon\|^2+\frac{1}{8R_m}\|A_{2\epsilon}\mathbf{B}_\epsilon\|^2+C\left(\|\mathbf{B}_\epsilon\|_1^4\|A_{1\epsilon}^{\frac{1}{2}}\mathbf{u}_\epsilon\|^2+\|\mathbf{u}_\epsilon\|_1^4\|A_{2\epsilon}^{\frac{1}{2}}\mathbf{B}_\epsilon\|^2\right).
\end{aligned}$$

Combining the above inequalities, we arrive at

$$\frac{d}{dt}\left(\|A_{1\epsilon}^{\frac{1}{2}}\mathbf{u}_\epsilon\|^2+\|A_{2\epsilon}^{\frac{1}{2}}\mathbf{B}_\epsilon\|^2\right)+R_e^{-1}\|A_{1\epsilon}\mathbf{u}_\epsilon\|^2+R_m^{-1}\|A_{2\epsilon}\mathbf{B}_\epsilon\|^2$$
$$\le C\|\mathbf{f}\|^2+C\|\mathbf{g}\|^2+C(\|\mathbf{u}_\epsilon\|_1^4+\|\mathbf{B}_\epsilon\|_1^4)\|A_{1\epsilon}^{\frac{1}{2}}\mathbf{u}_\epsilon\|^2+C(\|\mathbf{u}_\epsilon\|_1^4+\|\mathbf{B}_\epsilon\|_1^4)\|A_{2\epsilon}^{\frac{1}{2}}\mathbf{B}_\epsilon\|^2.$$

Integrating the above inequality over $[0,t]$, using (2.18), (2.10), Lemma 2.1 and the Gronwall lemma, we complete the proof of (2.17).

3 Linearized problem

It is difficult to deal with the nonlinear terms of the MHD equations. Therefore, we derive the penalty errors of its linear form as an intermediate step in the analysis of the nonlinear MHD equations in the next section. Next, we will consider the linear form of MHD equations:

$$\begin{cases}\mathbf{u}_t-R_e^{-1}\Delta\mathbf{u}+\nabla p=\mathbf{f},\\ \mathrm{div}\mathbf{u}=0,\quad \mathbf{u}(0)=\mathbf{u}_0,\\ \mathbf{B}_t+R_m^{-1}\mathrm{curl}(\mathrm{curl}\mathbf{B})=\mathbf{g},\\ \mathrm{div}\mathbf{B}=0,\quad \mathbf{B}(0)=\mathbf{B}_0.\end{cases}\tag{3.1}$$

The penalty method applied to (3.1) is

$$\begin{cases}\mathbf{u}_{\epsilon t}-R_e^{-1}\Delta\mathbf{u}_\epsilon+\nabla p_\epsilon=\mathbf{f},\\ \mathrm{div}\mathbf{u}_\epsilon+R_e\epsilon p_\epsilon=0,\quad \mathbf{u}_\epsilon(0)=\mathbf{u}_0,\\ \mathbf{B}_{\epsilon t}+R_m^{-1}\mathrm{curl}(\mathrm{curl}\mathbf{B}_\epsilon)=\mathbf{g},\\ \mathrm{div}\mathbf{B}_\epsilon=0,\quad \mathbf{B}_\epsilon(0)=\mathbf{B}_0.\end{cases}\tag{3.2}$$

Letting $\mathbf{e} = \mathbf{u} - \mathbf{u}_\epsilon$, $\boldsymbol{\eta} = \mathbf{B} - \mathbf{B}_\epsilon$, $\xi = p - p_\epsilon$, and subtracting (3.2) form (3.1), we obtain

$$\mathbf{e}_t - R_e^{-1}\Delta\mathbf{e} + \nabla\xi = \mathbf{0}, \tag{3.3}$$
$$\mathrm{div}\mathbf{e} + R_e\epsilon\xi = R_e\epsilon p, \quad \mathbf{e}(0) = \mathbf{0}, \tag{3.4}$$
$$\boldsymbol{\eta}_t + R_m^{-1}\mathrm{curl}(\mathrm{curl}\boldsymbol{\eta}) = \mathbf{0}, \tag{3.5}$$
$$\mathrm{div}\boldsymbol{\eta} = 0, \quad \boldsymbol{\eta}(0) = \mathbf{0}. \tag{3.6}$$

We can derive from (3.5) and (3.6) that

$$\boldsymbol{\eta}_t + R_m^{-1}A_2\boldsymbol{\eta} = \mathbf{g}, \quad \boldsymbol{\eta}(0) = \mathbf{0}. \tag{3.7}$$

Lemma 3.1 *Under Assumptions $\mathcal{A}1 - \mathcal{A}2$, we have*

$$\|\boldsymbol{e}\|^2 + \|\boldsymbol{\eta}\|^2 + \int_0^t (R_e^{-1}\|\nabla\boldsymbol{e}\|^2 + R_m^{-1}\|\nabla\boldsymbol{\eta}\|^2 + R_e\epsilon\|\xi\|^2)ds \le C\epsilon, \tag{3.8}$$
$$\int_0^t (\|\boldsymbol{e}\|^2 + \|\boldsymbol{\eta}\|^2)ds \le C\epsilon^2. \tag{3.9}$$

Proof Taking the L^2 inner product of (3.3) with $\mathbf{e}$, (3.4) with ξ, (3.7) with $\boldsymbol{\eta}$, and summing up the three relations, we obtain

$$\frac{1}{2}\frac{d}{dt}(\|\mathbf{e}\|^2+\|\boldsymbol{\eta}\|^2)+R_e^{-1}\|\nabla\mathbf{e}\|^2+R_m^{-1}\|A_2^{\frac{1}{2}}\boldsymbol{\eta}\|^2+R_e\epsilon\|\xi\|^2=R_e\epsilon(p,\xi)\le\frac{R_e\epsilon}{2}\|\xi\|^2+\frac{R_e\epsilon}{2}\|p\|^2.$$

Integrating the above inequality from 0 to $t \le T$, thanks to $\mathbf{e}(0) = \mathbf{0}$, $\boldsymbol{\eta}(0) = \mathbf{0}$ and (2.14), we have

$$\|\mathbf{e}\|^2 + \|\boldsymbol{\eta}\|^2 + \int_0^t (R_e^{-1}\|\nabla\mathbf{e}\|^2 + R_m^{-1}\|\nabla\boldsymbol{\eta}\|^2 + R_e\epsilon\|\xi\|^2)ds \le C\epsilon.$$

We now use the standard duality argument. Taking $d = 0$ and adding Lagrange multiplier ∇d to (3.1) for the third equation, we can get

$$\begin{cases} \mathbf{u}_t - R_e^{-1}\Delta\mathbf{u} + \nabla p = \mathbf{f}, \\ \mathrm{div}\mathbf{u} = 0, \quad \mathbf{u}(0) = \mathbf{u}_0, \\ \mathbf{B}_t + R_m^{-1}\mathrm{curl}(\mathrm{curl}\mathbf{B}) + \nabla d = \mathbf{g}, \\ \mathrm{div}\mathbf{B} = 0, \quad \mathbf{B}(0) = \mathbf{B}_0. \end{cases} \tag{3.10}$$

Then, we have the following variational formulation of the problem (3.10): find $(\mathbf{u}, p, \mathbf{B}, d) \in L^2(0, T; \mathbf{X}) \times L^2(0, T; M) \times L^2(0, T; \mathbf{W}) \times L^2(0, T; M)$ such that

$$\begin{cases} (\mathbf{u}_t, \mathbf{v}) + R_e^{-1}(\nabla\mathbf{u}, \nabla\mathbf{v}) - (\mathrm{div}\mathbf{v}, p) + (\mathrm{div}\mathbf{u}, q) = (\mathbf{f}, \mathbf{v}), \quad \mathbf{u}(0) = \mathbf{u}_0, \\ (\mathbf{B}_t, \mathbf{C}) + R_m^{-1}(\mathrm{curl}\mathbf{B}, \mathrm{curl}\mathbf{C}) - (\mathrm{div}\mathbf{C}, d) + (\mathrm{div}\mathbf{B}, s) = (\mathbf{g}, \mathbf{C}), \quad \mathbf{B}(0) = \mathbf{B}_0. \end{cases}$$

Exchanging $(\mathbf{u}, p, \mathbf{B}, d) \leftrightarrow (\mathbf{v}, q, \mathbf{C}, s)$, we get

$$\begin{cases} (\mathbf{v}_t, \mathbf{u}) + R_e^{-1}(\nabla\mathbf{v}, \nabla\mathbf{u}) - (\mathrm{div}\mathbf{u}, q) + (\mathrm{div}\mathbf{v}, p) = (\mathbf{f}, \mathbf{v}), \\ (\mathbf{C}_t, \mathbf{B}) + R_m^{-1}(\mathrm{curl}\mathbf{C}, \mathrm{curl}\mathbf{B}) - (\mathrm{div}\mathbf{B}, s) + (\mathrm{div}\mathbf{C}, d) = (\mathbf{g}, \mathbf{C}). \end{cases}$$

Then, we have

$$\begin{cases} (\mathbf{v}_t, \mathbf{u}) + R_e^{-1}(\nabla \mathbf{v}, \nabla \mathbf{u}) + (\mathrm{div}\mathbf{v}, p) = (\mathbf{f}, \mathbf{v}), \\ -(\mathrm{div}\mathbf{u}, q) = 0, \\ (\mathbf{C}_t, \mathbf{B}) + R_m^{-1}(\mathrm{curl}\mathbf{C}, \mathrm{curl}\mathbf{B}) + (\mathrm{div}\mathbf{C}, d) = (\mathbf{g}, \mathbf{C}), \\ -(\mathrm{div}\mathbf{B}, s) = 0. \end{cases}$$

Letting $(\mathbf{u}, p, \mathbf{B}, d) \leftrightarrow (\boldsymbol{\omega}, \gamma, \boldsymbol{\phi}, k)$, we obtain

$$\begin{cases} (\mathbf{v}_t, \boldsymbol{\omega}) + R_e^{-1}(\nabla \mathbf{v}, \nabla \boldsymbol{\omega}) + (\mathrm{div}\mathbf{v}, \gamma) = (\tilde{\mathbf{f}}, \mathbf{v}), \\ -(\mathrm{div}\boldsymbol{\omega}, q) = 0, \\ (\mathbf{C}_t, \boldsymbol{\phi}) + R_m^{-1}(\mathrm{curl}\mathbf{C}, \mathrm{curl}\boldsymbol{\phi}) + (\mathrm{div}\mathbf{C}, k) = (\tilde{\mathbf{g}}, \mathbf{C}), \\ -(\mathrm{div}\boldsymbol{\phi}, s) = 0. \end{cases} \tag{3.11}$$

Integrating the first equation in (3.11) form 0 to T, we have

$$\int_0^T [(\mathbf{v}_t, \boldsymbol{\omega}) + R_e^{-1}(\nabla \mathbf{v}, \nabla \boldsymbol{\omega}) + (\mathrm{div}\mathbf{v}, \gamma)]dt = \int_0^T (\tilde{\mathbf{f}}, \mathbf{v})dt.$$

By integrating by parts, we get

$$(\mathbf{v}(T), \boldsymbol{\omega}(T)) - (\mathbf{v}(0), \boldsymbol{\omega}(0)) - \int_0^T [(\mathbf{v}, \boldsymbol{\omega}_t) - R_e^{-1}(\nabla \mathbf{v}, \nabla \boldsymbol{\omega}) - (\mathrm{div}\mathbf{v}, \gamma)]dt = \int_0^T (\tilde{\mathbf{f}}, \mathbf{v})dt.$$

Taking $\boldsymbol{\omega}(T) = \mathbf{v}(0) = \mathbf{0}$, we obtain

$$(\mathbf{v}, \boldsymbol{\omega}_t) - R_e^{-1}(\nabla \mathbf{v}, \nabla \boldsymbol{\omega}) - (\mathrm{div}\mathbf{v}, \gamma) = -(\tilde{\mathbf{f}}, \mathbf{v}).$$

Similarly, we can get

$$(\mathbf{C}, \boldsymbol{\phi}_t) - R_m^{-1}(\mathrm{curl}\mathbf{C}, \mathrm{curl}\boldsymbol{\phi}) - (\mathrm{div}\mathbf{C}, k) = -(\tilde{\mathbf{g}}, \mathbf{C}).$$

Taking $\tilde{\mathbf{f}} = -\mathbf{e}$, $\tilde{\mathbf{g}} = -\boldsymbol{\eta}$, thanks to $\nabla k = 0$, we get the following dual problem. For any $0 < t \leq T$, we define $(\boldsymbol{\omega}, \gamma, \boldsymbol{\phi})$ by

$$\boldsymbol{\omega}_s + R_e^{-1}\Delta\boldsymbol{\omega} + \nabla\gamma = \mathbf{e}, \quad \forall 0 < s \leq t, \tag{3.12}$$

$$\mathrm{div}\, \boldsymbol{\omega} = 0, \quad \boldsymbol{\omega}(t) = \mathbf{0}, \tag{3.13}$$

$$\boldsymbol{\phi}_s - R_m^{-1}\mathrm{curl}(\mathrm{curl}\boldsymbol{\phi}) = \boldsymbol{\eta}, \quad \forall 0 < s \leq t, \tag{3.14}$$

$$\mathrm{div}\, \boldsymbol{\phi} = 0, \quad \boldsymbol{\phi}(t) = \mathbf{0}. \tag{3.15}$$

We can derive from (3.14) and (3.15) that

$$\boldsymbol{\phi}_s - R_m^{-1}A_2\boldsymbol{\phi} = \boldsymbol{\eta}, \quad \boldsymbol{\eta}(t) = \mathbf{0}, \quad \forall 0 < s \leq t. \tag{3.16}$$

Let us first establish the following inequality:

$$\begin{aligned} R_e^{-1}\|\boldsymbol{\omega}\|_{L^2(0,T;\mathbf{H}^2(\Omega))} &+ R_m^{-1}\|\boldsymbol{\phi}\|_{L^2(0,T;\mathbf{H}^2(\Omega))} + \|\nabla\gamma\|_{L^2(0,T;L^2(\Omega))} \\ &\leq C\|\mathbf{e}\|_{L^2(0,T;\mathbf{L}^2(\Omega))} + C\|\boldsymbol{\eta}\|_{L^2(0,T;\mathbf{L}^2(\Omega))}. \end{aligned} \tag{3.17}$$

Taking the L^2 inner products of (3.12) with $A_1\boldsymbol{\omega}$, (3.16) with $A_2\boldsymbol{\phi}$, and summing up the two relations, we get

$$-\frac{1}{2}\frac{d}{ds}(\|A_1^{\frac{1}{2}}\boldsymbol{\omega}\|^2 + \|A_2^{\frac{1}{2}}\boldsymbol{\phi}\|^2) + R_e^{-1}\|A_1\boldsymbol{\omega}\|^2 + R_m^{-1}\|A_2\boldsymbol{\phi}\|^2 = -(\mathbf{e}, A_1\boldsymbol{\omega}) - (\boldsymbol{\eta}, A_2\boldsymbol{\phi}).$$

Integrating the above relation from 0 to t, we have

$$\int_0^t (R_e^{-1}\|A_1\boldsymbol{\omega}\|^2+R_m^{-1}\|A_2\boldsymbol{\phi}\|^2)ds+\|\nabla\boldsymbol{\omega}(0)\|^2+\|\nabla\boldsymbol{\phi}(0)\|^2\leq C\int_0^t(\|\mathbf{e}\|^2+\|\boldsymbol{\eta}\|^2)ds.$$

Applying the projection operator P_H on (3.12) and (3.16), we derive

$$\|\boldsymbol{\omega}_s\|_{L^2(0,T;\mathbf{L}^2(\Omega))}+\|\boldsymbol{\phi}_s\|_{L^2(0,T;\mathbf{L}^2(\Omega))}\leq C\|\mathbf{e}\|_{L^2(0,T;\mathbf{L}^2(\Omega))}+C\|\boldsymbol{\eta}\|_{L^2(0,T;\mathbf{L}^2(\Omega))}.$$

From (3.12) and (3.16), we obtain

$$\|\nabla\gamma\|_{L^2(0,T;L^2(\Omega))}\leq C\|\mathbf{e}\|_{L^2(0,T;\mathbf{L}^2(\Omega))}+C\|\boldsymbol{\eta}\|_{L^2(0,T;\mathbf{L}^2(\Omega))}.$$

The proof of (3.17) is thus complete.

Next, taking the L^2 inner product of (3.12) with $\mathbf{e}$ and (3.14) with $\boldsymbol{\eta}$, thanks to (3.3)–(3.5) and div$\boldsymbol{\omega}=0$, we have

$$\begin{aligned}\|\mathbf{e}\|^2+\|\boldsymbol{\eta}\|^2 &= (\boldsymbol{\omega}_s,\mathbf{e})+R_e^{-1}(\Delta\boldsymbol{\omega},\mathbf{e})+(\nabla\gamma,\mathbf{e})+(\boldsymbol{\phi}_s,\boldsymbol{\eta})-R_m^{-1}(\mathrm{curl}(\mathrm{curl}\boldsymbol{\phi}),\boldsymbol{\eta})\\
&=\frac{d}{ds}(\boldsymbol{\omega},\mathbf{e})-(\boldsymbol{\omega},\mathbf{e}_s)-R_e^{-1}(\nabla\boldsymbol{\omega},\nabla\mathbf{e})+(\nabla\gamma,\mathbf{e})\\
&\quad+\frac{d}{ds}(\boldsymbol{\phi},\boldsymbol{\eta})-(\boldsymbol{\phi},\boldsymbol{\eta}_s)-R_m^{-1}(\mathrm{curl}\boldsymbol{\phi},\mathrm{curl}\boldsymbol{\eta})\\
&=\frac{d}{ds}(\boldsymbol{\omega},\mathbf{e})+(\nabla\xi,\boldsymbol{\omega})-(\gamma,\mathrm{div}\mathbf{e})+\frac{d}{ds}(\boldsymbol{\phi},\boldsymbol{\eta})\\
&=\frac{d}{ds}(\boldsymbol{\omega},\mathbf{e})+\frac{d}{ds}(\boldsymbol{\phi},\boldsymbol{\eta})-R_e\epsilon(\gamma,p_\epsilon).\end{aligned}$$

Integrating the above relation from 0 to t, thanks to $\boldsymbol{\omega}(t)=\mathbf{e}(0)=\mathbf{0}$ and $\boldsymbol{\phi}(t)=\boldsymbol{\eta}(0)=\mathbf{0}$, we obtain

$$\int_0^t(\|\mathbf{e}\|^2+\|\boldsymbol{\eta}\|^2)ds=-R_e\epsilon\int_0^t(\gamma,p_\epsilon)ds\leq\int_0^t(\delta\|\gamma\|^2+C_\delta R_e^2\epsilon^2\|p_\epsilon\|^2)ds,$$

where C_δ is a constant depending on δ only. Due to (3.17) and (3.8), we can choose δ sufficiently small such that

$$\int_0^t(\|\mathbf{e}\|^2+\|\boldsymbol{\eta}\|^2)ds\leq\int_0^t CR_e\epsilon^2\|p_\epsilon\|^2ds\leq C\epsilon^2,\quad\forall t\in[0,T].$$

The proof is thus complete. □

Lemma 3.2 *Under Assumptions $\mathcal{A}1-\mathcal{A}2$, we have*

$$\int_0^t s^2\|p_{\epsilon t}\|^2ds\leq C,\quad\forall t\in[0,T].$$

We can refer to [21] to prove this result.

Lemma 3.3 *Under Assumptions $\mathcal{A}1-\mathcal{A}2$, we have*

$$t\|\boldsymbol{e}(t)\|^2+t\|\boldsymbol{\eta}(t)\|^2+\int_0^t s(R_e^{-1}\|\nabla\boldsymbol{e}(s)\|^2+R_m^{-1}\|\nabla\boldsymbol{\eta}(s)\|^2+R_e\epsilon\|\xi(s)\|^2)ds\le C\epsilon^2,\forall t\in[0,T],$$
$$R_e^{-1}t^2\|\nabla\boldsymbol{e}(t)\|^2+R_m^{-1}t^2\|\nabla\boldsymbol{\eta}(t)\|^2+\int_0^t s^2\|\xi(s)\|^2ds\le C\epsilon^2,\quad \forall t\in[0,T].$$

Proof Let us consider the decomposition (see, for instance, [10])

$$\mathbf{X}=\mathbf{X}_0\oplus\mathbf{X}_0^{\perp},\quad \text{where } \mathbf{X}_0^{\perp}=\{(-\Delta)^{-1}\nabla q: q\in L^2(\Omega)\},$$

and $\mathbf{v}=(-\Delta)^{-1}\nabla q$ if $-\Delta\mathbf{v}=\nabla q$ and $\mathbf{v}|_{\partial\Omega}=\mathbf{0}$. It is well known that for $p(t)\in \mathrm{M}$, there exists a unique $\boldsymbol{\varphi}(t)\in\mathbf{X}_0^{\perp}$ such that $\mathrm{div}\boldsymbol{\varphi}(t)=p(t)$ with

$$\|\boldsymbol{\varphi}(t)\|_1\le C\|p(t)\|,\quad \forall t\in[0,T]. \tag{3.18}$$

Moreover, if $p_t(t)\in \mathrm{M}$, we have $\mathrm{div}\varphi_t(t)=p_t(t)$ with

$$\|\boldsymbol{\varphi}_t(t)\|_1\le C\|p_t(t)\|,\quad \forall t\in[0,T]. \tag{3.19}$$

Taking the L^2 inner products of (3.3) with $t\mathbf{e}$, (3.4) with $t\xi$, (3.7) with $t\boldsymbol{\eta}$, summing up the three relations, and using (3.3), we derive

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}t\|\mathbf{e}\|^2\ +\ & R_e^{-1}t\|\nabla\mathbf{e}\|^2+R_e\epsilon t\|\xi\|^2+\frac{1}{2}\frac{d}{dt}t\|\boldsymbol{\eta}\|^2+R_m^{-1}t\|\nabla\boldsymbol{\eta}\|^2\\ &=\frac{1}{2}\|\mathbf{e}\|^2+R_e\epsilon t(p,\xi)+\frac{1}{2}\|\boldsymbol{\eta}\|^2=\frac{1}{2}\|\mathbf{e}\|^2+\frac{1}{2}\|\boldsymbol{\eta}\|^2+R_e\epsilon t(\mathrm{div}\boldsymbol{\varphi},\xi)\\ &=\frac{1}{2}\|\mathbf{e}\|^2+\frac{1}{2}\|\boldsymbol{\eta}\|^2+R_e\epsilon t(\mathbf{e}_t,\boldsymbol{\varphi})+\epsilon t(\nabla\mathbf{e},\nabla\boldsymbol{\varphi})\\ &=\frac{1}{2}\|\mathbf{e}\|^2+\frac{1}{2}\|\boldsymbol{\eta}\|^2+R_e\epsilon\frac{d}{dt}t(\mathbf{e},\boldsymbol{\varphi})-R_e\epsilon(\mathbf{e},\boldsymbol{\varphi})-R_e\epsilon t(\mathbf{e},\boldsymbol{\varphi}_t)+\epsilon t(\nabla\mathbf{e},\nabla\boldsymbol{\varphi}).\end{aligned} \tag{3.20}$$

Using (3.18), we get

$$R_e\epsilon t(\mathbf{e}(t),\boldsymbol{\varphi}(t))\le\frac{t}{4}\|\mathbf{e}(t)\|^2+R_e^2\epsilon^2t\|\boldsymbol{\varphi}(t)\|^2\le\frac{t}{4}\|\mathbf{e}(t)\|^2+C\epsilon^2.$$

Hence, integrating (3.20) from 0 to t, using the above relation, the Cauchy-Schwarz inequality, Lemma 3.1, (3.18), and (3.19), we derive

$$\begin{aligned} t\|\mathbf{e}(t)\|^2 \ + \ t\|\boldsymbol{\eta}(t)\|^2 + \int_0^t (R_e^{-1}s\|\nabla\mathbf{e}\|^2 + R_e\epsilon s\|\xi\|^2 + R_m^{-1}s\|\nabla\boldsymbol{\eta}\|^2)ds \\ \le C\epsilon^2 + C\int_0^t (\|\mathbf{e}(s)\|^2 + \|\boldsymbol{\eta}(s)\|^2)ds + C\epsilon^2\int_0^t \|\boldsymbol{\varphi}(s)\|^2 ds + C\epsilon^2\int_0^t s^2\|\boldsymbol{\varphi}_t(s)\|^2 ds \\ \le C\epsilon^2 + C\epsilon^2\int_0^t (\|p\|^2 + s^2\|p_t\|^2)ds \le C\epsilon^2. \end{aligned}$$

Next, we take the partial derivative with respect to t on (3.4), we obtain

$$\mathrm{div}\boldsymbol{e}_t + R_e\epsilon\xi_t = R_e\epsilon p_t. \tag{3.21}$$

Taking the L^2¡¡ inner product of (3.3) with $t^2\mathbf{e}_t$, (3.21) with $t^2\xi$, (3.7) with $t^2\boldsymbol{\eta}_t$, and summing up the three relations, we obtain

$$\begin{aligned} t^2(\|\mathbf{e}_t\|^2 \ + \ \|\boldsymbol{\eta}_t\|^2) + \frac{1}{2}\frac{d}{dt}t^2(R_e^{-1}\|\nabla\mathbf{e}\|^2 + R_m^{-1}\|\nabla\boldsymbol{\eta}\|^2) + \frac{R_e\epsilon}{2}\frac{d}{dt}t^2\|\xi\|^2 \\ = t(\|\nabla\mathbf{e}\|^2 + \|\nabla\boldsymbol{\eta}\|^2) + R_e\epsilon t\|\xi\|^2 + R_e\epsilon t^2(p_t, \xi). \end{aligned} \tag{3.22}$$

Due to (3.3) and (3.18), we get

$$\begin{aligned} R_e\epsilon t^2(p_t, \xi) \ = \ R_e\epsilon t^2(\mathrm{div}\boldsymbol{\varphi}_t, \xi) = -R_e\epsilon t^2(\boldsymbol{\varphi}_t, \nabla\xi) = R_e\epsilon t^2(\mathbf{e}_t, \boldsymbol{\varphi}_t) + R_e\epsilon t^2(\nabla\mathbf{e}_t, \nabla\boldsymbol{\varphi}_t) \\ \le \frac{t^2}{2}\|\mathbf{e}_t\|^2 + CR_e^2\epsilon^2t^2\|\boldsymbol{\varphi}_t\|^2 + t^2\|\nabla\mathbf{e}\|^2 + CR_e^2\epsilon^2t^2\|\nabla\boldsymbol{\varphi}_t\|^2 \\ \le \frac{t^2}{2}\|\mathbf{e}_t\|^2 + t^2\|\nabla\mathbf{e}\|^2 + CR_e^2\epsilon^2t^2\|p_t\|^2. \end{aligned}$$

Integrating (3.22) and using the Gronwall lemma, we obtain

$$R_e^{-1}t^2\|\nabla\mathbf{e}\|^2 + R_m^{-1}t^2\|\nabla\boldsymbol{\eta}\|^2 + R_e\epsilon t^2\|\xi\|^2 + \int_0^t s^2(\|\mathbf{e}_t\|^2 + \|\boldsymbol{\eta}_t\|^2)ds \le C\epsilon^2, \quad \forall t \in [0, T].$$

Finally, we derive from (3.3) that

$$\|\xi\|^2 \le C\|\nabla\xi\|_{-1}^2 \le C(\|\Delta\mathbf{e}\|_{-1}^2 + \|\mathbf{e}_t\|_{-1}^2) \le C(\|\mathbf{e}\|_1^2 + \|\mathbf{e}_t\|^2).$$

Therefore, we have

$$\int_0^t s^2\|\xi\|^2 ds \le C\int_0^t s^2(\|\mathbf{e}\|_1^2 + \|\mathbf{e}_t\|^2)ds \le C\epsilon^2.$$

The proof is thus complete. □

To summarize, we prove the following theorem.

Theorem 3.1 *Under Assumptions $\mathcal{A}1 - \mathcal{A}2$, we have*

$$t\|\boldsymbol{e}(t)\|^2 + t\|\boldsymbol{\eta}(t)\|^2 + t^2\|\boldsymbol{e}(t)\|_1^2 + t^2\|\boldsymbol{\eta}(t)(t)\|_1^2 + \int_0^t s^2\|\xi\|^2 ds \le C\epsilon^2, \quad \forall t \in [0, T].$$

4 Nonlinear MHD equations

In this section, we transform the nonlinear MHD equations into an intermediate linear equations, and then use the previous result to give an error estimate of the penalty system. Next, let us consider the following intermediate linear equations:

$$\begin{cases} \mathbf{v}_t - R_e^{-1}\Delta\mathbf{v} + \nabla q = \mathbf{f} - (\mathbf{u}\cdot\nabla)\mathbf{u} + S_c\mathrm{curl}\mathbf{B}\times\mathbf{B}, \\ \mathrm{div}\mathbf{v} + R_e\epsilon q_\epsilon = 0, \quad \mathbf{v}(0) = \mathbf{u}_0, \\ \boldsymbol{\Psi}_t + R_m^{-1}\mathrm{curlcurl}\boldsymbol{\Psi} = \mathbf{g} + \mathrm{curl}(\mathbf{u}\times\mathbf{B}), \\ \mathrm{div}\boldsymbol{\Psi} = 0, \quad \boldsymbol{\Psi}(0) = \mathbf{B}_0, \end{cases} \tag{4.1}$$

where $\mathbf{u}$ and $\mathbf{B}$ are the solutions of MHD equations (1.1).

Letting $\boldsymbol{\eta}_{\boldsymbol{u}} = \mathbf{v} - \mathbf{u}$, $\boldsymbol{\eta}_{\boldsymbol{B}} = \boldsymbol{\Psi} - \mathbf{B}$, $\eta_p = q - p$, and subtracting (1.1) from (4.1), we obtain

$$\begin{cases} \boldsymbol{\eta}_{\boldsymbol{u}t} - R_e^{-1}\Delta\boldsymbol{\eta}_{\boldsymbol{u}} + \nabla\eta_p = \mathbf{0}, \\ \mathrm{div}\boldsymbol{\eta}_{\boldsymbol{u}} + R_e\epsilon\eta_p = 0, \quad \boldsymbol{\eta}_{\boldsymbol{u}}(0) = \mathbf{0}, \\ \boldsymbol{\eta}_{\boldsymbol{B}t} + R_m^{-1}\mathrm{curlcurl}\boldsymbol{\eta}_{\boldsymbol{B}} = \mathbf{0}, \\ \mathrm{div}\boldsymbol{\eta}_{\boldsymbol{B}} = 0, \quad \boldsymbol{\eta}_{\boldsymbol{B}}(0) = \mathbf{0}. \end{cases} \tag{4.2}$$

Lemma 4.1 *Under Assumptions $\mathcal{A}1 - \mathcal{A}2$, we have*

$$\int_0^t (\|\boldsymbol{\eta}_{\boldsymbol{u}}(s)\|^2 + \|\boldsymbol{\eta}_{\boldsymbol{B}}(s)\|^2)ds + t(\|\boldsymbol{\eta}_{\boldsymbol{u}}(t)\|^2 + \|\boldsymbol{\eta}_{\boldsymbol{B}}(t)\|^2) + t^2(\|\boldsymbol{\eta}_{\boldsymbol{u}}(t)\|_1^2 + \|\boldsymbol{\eta}_{\boldsymbol{B}}(t)\|_1^2)$$
$$+ \int_0^t s^2\|\eta_p(s)\|^2 ds \le C\epsilon^2, \quad \forall t \in [0, T].$$

Proof Thanks to Assumption $\mathcal{A}1$ and (2.14), we have $\mathbf{f} - \boldsymbol{B}(\mathbf{u},\mathbf{u}) + S_c\mathrm{curl}\mathbf{B}\times\mathbf{B}$, $\mathbf{g} + \mathrm{curl}(\mathbf{u}\times\mathbf{B}) \in L^2(0,T;\mathbf{L}^2(\Omega))$. One the other hand, it can be easily shown that $t\mathbf{u}_t \in L^2(0,T;\mathbf{X})$, $t\mathbf{B}_t \in L^2(0,T;\mathbf{W})$ (cf. [20]). And, since

$$t\frac{\partial}{\partial t}(\mathbf{f} - \boldsymbol{B}(\mathbf{u},\mathbf{u}) + S_c\mathrm{curl}\mathbf{B}\times\mathbf{B}) = t(\mathbf{f}_t - \boldsymbol{B}(\mathbf{u}_t,\mathbf{u}) - \boldsymbol{B}(\mathbf{u},\mathbf{u}_t) + S_c(\mathrm{curl}\mathbf{B}_t\times\mathbf{B} + \mathrm{curl}\mathbf{B}\times\mathbf{B}_t)),$$
$$t\frac{\partial}{\partial t}(\mathbf{g} + \mathrm{curl}(\mathbf{u}\times\mathbf{B})) = t(\mathbf{g}_t + \mathrm{curl}(\mathbf{u}_t\times\mathbf{B}) + \mathrm{curl}(\mathbf{u}\times\mathbf{B}_t)),$$

we have

$$t\frac{\partial}{\partial t}(\mathbf{f} - \boldsymbol{B}(\mathbf{u},\mathbf{u}) + S_c\mathrm{curl}\mathbf{B}\times\mathbf{B}), \quad t\frac{\partial}{\partial t}(\mathbf{g} + \mathrm{curl}(\mathbf{u}\times\mathbf{B})) \in L^2(0,T;\mathbf{L}^2(\Omega)).$$

Then, by applying Lemma 3.1 and Theorem 3.1 to (4.2), we can get Lemma 4.1. □

Next, letting $\boldsymbol{\varphi}_{\boldsymbol{u}} = \mathbf{u}_\epsilon - \mathbf{v}$, $\boldsymbol{\varphi}_{\boldsymbol{B}} = \mathbf{B}_\epsilon - \boldsymbol{\Psi}$, $\varphi_p = p_\epsilon - q$, and subtracting (4.1) from (4.2), we get

$$\boldsymbol{\varphi}_{\boldsymbol{u}t} - R_e^{-1}\Delta\boldsymbol{\varphi}_{\boldsymbol{u}} + \nabla\varphi_p + \boldsymbol{B}(\mathbf{u}_\epsilon,\mathbf{u}_\epsilon) - \boldsymbol{B}(\mathbf{u},\mathbf{u}) - S_c(\mathrm{curl}\mathbf{B}_\epsilon\times\mathbf{B}_\epsilon - \mathrm{curl}\mathbf{B}\times\mathbf{B}) = \mathbf{0}, \tag{4.3}$$
$$\mathrm{div}\boldsymbol{\varphi}_{\boldsymbol{u}} + R_e\epsilon\varphi_p = 0, \quad \boldsymbol{\varphi}_{\boldsymbol{u}}(0) = \mathbf{0}, \tag{4.4}$$
$$\boldsymbol{\varphi}_{\boldsymbol{B}t} + R_m^{-1}\mathrm{curlcurl}\boldsymbol{\varphi}_{\boldsymbol{B}} - \mathrm{curl}(\mathbf{u}_\epsilon\times\mathbf{B}_\epsilon) - \mathrm{curl}(\mathbf{u}\times\mathbf{B}) = \mathbf{0}, \tag{4.5}$$
$$\mathrm{div}\boldsymbol{\varphi}_{\boldsymbol{B}} = 0, \quad \boldsymbol{\varphi}_{\boldsymbol{B}}(0) = \mathbf{0}. \tag{4.6}$$

Since

$$\begin{cases} \boldsymbol{B}(\mathbf{u}_\epsilon,\mathbf{u}_\epsilon)-\boldsymbol{B}(\mathbf{u},\mathbf{u})=\boldsymbol{B}(\mathbf{u},\mathbf{u}_\epsilon-\mathbf{u})+\boldsymbol{B}(\mathbf{u}_\epsilon-\mathbf{u},\mathbf{u}_\epsilon)=\boldsymbol{B}(\mathbf{u},\boldsymbol{\eta_u}+\boldsymbol{\varphi_u})+\boldsymbol{B}(\boldsymbol{\eta_u}+\boldsymbol{\varphi_u},\mathbf{u}_\epsilon),\\ \text{curl}\mathbf{B}_\epsilon\times\mathbf{B}_\epsilon-\text{curl}\mathbf{B}\times\mathbf{B}=\text{curl}\mathbf{B}\times(\boldsymbol{\eta_B}+\boldsymbol{\varphi_B})+\text{curl}(\boldsymbol{\eta_B}+\boldsymbol{\varphi_B})\times\mathbf{B}_\epsilon,\\ \text{curl}(\mathbf{u}_\epsilon\times\mathbf{B}_\epsilon)-\text{curl}(\mathbf{u}\times\mathbf{B})=\text{curl}(\mathbf{u}\times(\boldsymbol{\eta_B}+\boldsymbol{\varphi_B}))+\text{curl}((\boldsymbol{\eta_u}+\boldsymbol{\varphi_u})\times\mathbf{B}_\epsilon), \end{cases} \tag{4.7}$$

we can rewrite (4.3)–(4.6) as

$$\boldsymbol{\varphi_{ut}}+R_e^{-1}A_{1\epsilon}\boldsymbol{\varphi_u}+\boldsymbol{B}(\mathbf{u},\boldsymbol{\eta_u}+\boldsymbol{\varphi_u})+\boldsymbol{B}(\boldsymbol{\eta_u}+\boldsymbol{\varphi_u},\mathbf{u}_\epsilon)-S_c(\text{curl}\mathbf{B}\times(\boldsymbol{\eta_B}+\boldsymbol{\varphi_B}) + \text{curl}(\boldsymbol{\eta_B}+\boldsymbol{\varphi_B})\times\mathbf{B}_\epsilon)=\mathbf{0}, \tag{4.8}$$

$$\boldsymbol{\varphi_{Bt}}+R_m^{-1}A_{2\epsilon}\boldsymbol{\varphi_B}-\text{curl}(\mathbf{u}\times(\boldsymbol{\eta_B}+\boldsymbol{\varphi_B}))-\text{curl}((\boldsymbol{\eta_u}+\boldsymbol{\varphi_u})\times\mathbf{B}_\epsilon)=\mathbf{0}. \tag{4.9}$$

Theorem 4.1 *Under Assumptions $\mathcal{A}1-\mathcal{A}2$, we have the following estimate:*

$$t(\|\boldsymbol{u}(t)-\boldsymbol{u}_\epsilon(t)\|^2+\|\boldsymbol{B}(t)-\boldsymbol{B}_\epsilon(t)\|^2)+t^2(R_e^{-1}\|\boldsymbol{u}(t)-\boldsymbol{u}_\epsilon(t)\|_1^2+R_m^{-1}\|\boldsymbol{B}(t)-\boldsymbol{B}_\epsilon(t)\|_1^2) + \int_0^t s^2\|p(s)-p_\epsilon(s)\|^2ds\le C\epsilon^2,\quad \forall 0<t\le T.$$

Proof Taking the L^2 inner product of (4.8) with $A_{1\epsilon}^{-1}\boldsymbol{\varphi_u}$ and of (4.9) with $A_{2\epsilon}^{-1}\boldsymbol{\varphi_B}$, summing up the two relations, we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|A_{1\epsilon}^{-\frac{1}{2}}\boldsymbol{\varphi_u}\|^2+R_e^{-1}\|\boldsymbol{\varphi_u}\|^2+\frac{1}{2}\frac{d}{dt}\|A_{2\epsilon}^{-\frac{1}{2}}\boldsymbol{\varphi_B}\|^2+R_m^{-1}\|\boldsymbol{\varphi_B}\|^2 &= -b(\mathbf{u},\boldsymbol{\eta_u}+\boldsymbol{\varphi_u},A_{1\epsilon}^{-1}\boldsymbol{\varphi_u})-b(\boldsymbol{\eta_u}+\boldsymbol{\varphi_u},\mathbf{u}_\epsilon,A_{1\epsilon}^{-1}\boldsymbol{\varphi_u})\\ &+ S_c(\text{curl}\mathbf{B}\times(\boldsymbol{\eta_B}+\boldsymbol{\varphi_B}),A_{1\epsilon}^{-1}\boldsymbol{\varphi_u})+S_c(\text{curl}(\boldsymbol{\eta_B}+\boldsymbol{\varphi_B})\times\mathbf{B}_\epsilon,A_{1\epsilon}^{-1}\boldsymbol{\varphi_u})\\ &+ (\text{curl}(\mathbf{u}\times(\boldsymbol{\eta_B}+\boldsymbol{\varphi_B})),A_{2\epsilon}^{-1}\boldsymbol{\varphi_B})+(\text{curl}((\boldsymbol{\eta_u}+\boldsymbol{\varphi_u})\times\mathbf{B}_\epsilon),A_{2\epsilon}^{-1}\boldsymbol{\varphi_B}). \end{aligned}$$

By using (2.6)–(2.13) and Lemma 2.1, we derive that

$$\begin{aligned} &-b(\mathbf{u},\boldsymbol{\eta_u}+\boldsymbol{\varphi_u},A_{1\epsilon}^{-1}\boldsymbol{\varphi_u})-b(\boldsymbol{\eta_u}+\boldsymbol{\varphi_u},\mathbf{u}_\epsilon,A_{1\epsilon}^{-1}\boldsymbol{\varphi_u})\\ &\le C\|\mathbf{u}\|_2\|\boldsymbol{\eta_u}+\boldsymbol{\varphi_u}\|\,\|A_{1\epsilon}^{-1}\boldsymbol{\varphi_u}\|_1+C\|\mathbf{u}_\epsilon\|_2\|\boldsymbol{\eta_u}+\boldsymbol{\varphi_u}\|\,\|A_{1\epsilon}\boldsymbol{\varphi_u}\|_1\\ &\le C\|\mathbf{u}\|_2(\|\boldsymbol{\eta_u}\|+\|\boldsymbol{\varphi_u}\|)\|A_{1\epsilon}^{-\frac{1}{2}}\boldsymbol{\varphi_u}\|+C\|\mathbf{u}_\epsilon\|_2(\|\boldsymbol{\eta_u}\|+\|\boldsymbol{\varphi_u}\|)\|A_{1\epsilon}^{-\frac{1}{2}}\boldsymbol{\varphi_u}\|\\ &\le\frac{1}{4R_e}\|\boldsymbol{\varphi_u}\|^2+C\|\boldsymbol{\eta_u}\|^2+C(\|\mathbf{u}\|_2^2+\|\mathbf{u}_\epsilon\|_2^2)\|A_{1\epsilon}^{-\frac{1}{2}}\boldsymbol{\varphi_u}\|^2,\\ &S_c(\text{curl}\mathbf{B}\times(\boldsymbol{\eta_B}+\boldsymbol{\varphi_B}),A_{1\epsilon}^{-1}\boldsymbol{\varphi_u})+S_c(\text{curl}(\boldsymbol{\eta_B}+\boldsymbol{\varphi_B})\times\mathbf{B}_\epsilon,A_{1\epsilon}^{-1}\boldsymbol{\varphi_u})\\ &\le C\|\mathbf{B}\|_2\|\boldsymbol{\eta_B}+\boldsymbol{\varphi_B}\|\,\|A_{1\epsilon}^{-1}\boldsymbol{\varphi_u}\|_1+C\|\mathbf{B}_\epsilon\|_2\|\boldsymbol{\eta_B}+\boldsymbol{\varphi_B}\|\,\|A_{1\epsilon}^{-1}\boldsymbol{\varphi_u}\|_1\\ &\le\frac{1}{4R_m}\|\boldsymbol{\varphi_B}\|^2+C\|\boldsymbol{\eta_B}\|^2+C(\|\mathbf{B}\|_2^2+\|\mathbf{B}_\epsilon\|_2^2)\|A_{1\epsilon}^{-\frac{1}{2}}\boldsymbol{\varphi_u}\|^2,\\ &(\text{curl}(\mathbf{u}\times(\boldsymbol{\eta_B}+\boldsymbol{\varphi_B})),A_{2\epsilon}^{-1}\boldsymbol{\varphi_B})+(\text{curl}((\boldsymbol{\eta_u}+\boldsymbol{\varphi_u})\times\mathbf{B}_\epsilon),A_{2\epsilon}^{-1}\boldsymbol{\varphi_B})\\ &\le C\|\mathbf{u}\|_2\|\boldsymbol{\eta_B}+\boldsymbol{\varphi_B}\|\,\|A_{2\epsilon}^{-\frac{1}{2}}\boldsymbol{\varphi_B}\|+C\|\mathbf{B}_\epsilon\|_2\|\boldsymbol{\eta_u}+\boldsymbol{\varphi_u}\|\,\|A_{2\epsilon}^{-\frac{1}{2}}\boldsymbol{\varphi_B}\|\\ &\le\frac{1}{4R_e}\|\boldsymbol{\varphi_u}\|^2+\frac{1}{4R_m}\|\boldsymbol{\varphi_B}\|^2+C\|\boldsymbol{\eta_u}\|^2+C\|\boldsymbol{\eta_B}\|^2+C\left(\|\mathbf{u}\|_2^2+\|\mathbf{B}_\epsilon\|_2^2\right)\|A_{2\epsilon}^{\frac{1}{2}}\boldsymbol{\varphi_B}\|^2. \end{aligned}$$

Combining the above inequalities, we arrive at

$$\frac{d}{dt}\left(\|A_{1\epsilon}^{-\frac{1}{2}}\boldsymbol{\varphi_u}\|^2+\|A_{2\epsilon}^{-\frac{1}{2}}\boldsymbol{\varphi_B}\|^2\right)+R_e^{-1}\|\boldsymbol{\varphi_u}\|^2+R_m^{-1}\|\boldsymbol{\varphi_B}\|^2 \tag{4.10}$$
$$\leq C(\|\boldsymbol{\eta_u}\|^2+\|\boldsymbol{\eta_B}\|^2)+C(\|\mathbf{u}\|_2^2+\|\mathbf{u}_\epsilon\|_2^2+\|\mathbf{B}\|_2^2+\|\mathbf{B}_\epsilon\|_2^2)\|A_{1\epsilon}^{-\frac{1}{2}}\boldsymbol{\varphi_u}\|^2+C\left(\|\mathbf{u}\|_2^2+\|\mathbf{B}_\epsilon\|_2^2\right)\|A_{2\epsilon}^{\frac{1}{2}}\boldsymbol{\varphi_B}\|^2.$$

Due to $\int_0^T\left(\|\mathbf{u}\|_2^2+\|\mathbf{u}_\epsilon\|_2^2+\|\mathbf{B}\|_2^2+\|\mathbf{B}_\epsilon\|_2^2\right)dt\leq C$, we can apply the Gronwall lemma, Lemma 4.1, and get

$$\|A_{1\epsilon}^{-\frac{1}{2}}\boldsymbol{\varphi_u}\|^2+\|A_{2\epsilon}^{-\frac{1}{2}}\boldsymbol{\varphi_B}\|^2+\int_0^t(R_e^{-1}\|\boldsymbol{\varphi_u}\|^2+R_m^{-1}\|\boldsymbol{\varphi_B}\|^2)ds\leq C\epsilon^2, \forall t\in[0,T]. \tag{4.11}$$

Next, taking L^2 the inner products of (4.3) with $t\boldsymbol{\varphi_u}$, (4.4) with $t\varphi_p$, (4.9) with $t\boldsymbol{\varphi_B}$, we derive

$$\frac{1}{2}\frac{d}{dt}t\|\boldsymbol{\varphi_u}\|^2+R_e^{-1}t\|\nabla\boldsymbol{\varphi_u}\|^2+R_e\epsilon t\|\varphi_p\|^2\|\boldsymbol{\varphi_u}\|^2+\frac{1}{2}\frac{d}{dt}t\|\boldsymbol{\varphi_B}\|^2R_m^{-1}t\|\nabla\boldsymbol{\varphi_B}\|^2$$
$$=\frac{1}{2}\|\boldsymbol{\varphi_u}\|^2+\frac{S_c}{2}\|\boldsymbol{\varphi_B}\|^2-tb(\mathbf{u},\boldsymbol{\eta_u}+\boldsymbol{\varphi_u},\boldsymbol{\varphi_u})-tb(\boldsymbol{\eta_u}+\boldsymbol{\varphi_u},\mathbf{u}_\epsilon,\boldsymbol{\varphi_u})$$
$$+S_ct(\mathrm{curl}\mathbf{B}\times(\boldsymbol{\eta_B}+\boldsymbol{\varphi_B}),\boldsymbol{\varphi_u})+S_ct(\mathrm{curl}(\boldsymbol{\eta_B}+\boldsymbol{\varphi_B})\times\mathbf{B}_\epsilon,\boldsymbol{\varphi_u})$$
$$+t(\mathrm{curl}(\mathbf{u}\times(\boldsymbol{\eta_B}+\boldsymbol{\varphi_B})),\boldsymbol{\varphi_B})+t(\mathrm{curl}((\boldsymbol{\eta_u}+\boldsymbol{\varphi_u})\times\mathbf{B}_\epsilon),\boldsymbol{\varphi_B}).$$

From (2.6)–(2.11), we get

$$-tb(\mathbf{u},\boldsymbol{\eta_u}+\boldsymbol{\varphi_u},\boldsymbol{\varphi_u})-tb(\boldsymbol{\eta_u}+\boldsymbol{\varphi_u},\mathbf{u}_\epsilon,\boldsymbol{\varphi_u})$$
$$\leq Ct\|\mathbf{u}\|_2\|\boldsymbol{\eta_u}+\boldsymbol{\varphi_u}\|_1\|\boldsymbol{\varphi_u}\|+Ct\|\mathbf{u}_\epsilon\|_2\|\boldsymbol{\eta_u}+\boldsymbol{\varphi_u}\|_1\|\boldsymbol{\varphi_u}\|$$
$$\leq\frac{t}{4R_e}\|\nabla\boldsymbol{\varphi_u}\|^2+Ct\|\boldsymbol{\eta_u}\|_1^2+Ct\left(\|\mathbf{u}\|_2^2+\|\mathbf{u}_\epsilon\|_2^2\right)\|\boldsymbol{\varphi_u}\|^2,$$
$$S_ct(\mathrm{curl}\mathbf{B}\times(\boldsymbol{\eta_B}+\boldsymbol{\varphi_B}),\boldsymbol{\varphi_u})+S_ct(\mathrm{curl}(\boldsymbol{\eta_B}+\boldsymbol{\varphi_B})\times\mathbf{B}_\epsilon,\boldsymbol{\varphi_u})$$
$$\leq Ct\|\mathbf{B}\|_2\|\boldsymbol{\eta_B}+\boldsymbol{\varphi_B}\|_1\|\boldsymbol{\varphi_u}\|+Ct\|\mathbf{B}_\epsilon\|_2\|\boldsymbol{\eta_B}+\boldsymbol{\varphi_B}\|_1\|\boldsymbol{\varphi_u}\|$$
$$\leq\frac{t}{4R_m}\|\nabla\boldsymbol{\varphi_B}\|^2+Ct\|\boldsymbol{\eta_B}\|_1^2+Ct\left(\|\mathbf{B}\|_2^2+\|\mathbf{B}_\epsilon\|_2^2\right)\|\boldsymbol{\varphi_u}\|^2,$$
$$t(\mathrm{curl}(\mathbf{u}\times(\boldsymbol{\eta_B}+\boldsymbol{\varphi_B})),\boldsymbol{\varphi_B})+t(\mathrm{curl}((\boldsymbol{\eta_u}+\boldsymbol{\varphi_u})\times\mathbf{B}_\epsilon),\boldsymbol{\varphi_B})$$
$$\leq Ct\|\mathbf{u}\|_2\|\boldsymbol{\eta_B}+\boldsymbol{\varphi_B}\|_1\|\boldsymbol{\varphi_B}\|+Ct\|\mathbf{B}_\epsilon\|_2\|\boldsymbol{\eta_u}+\boldsymbol{\varphi_u}\|_1\|\boldsymbol{\varphi_B}\|$$
$$\leq\frac{t}{4R_e}\|\boldsymbol{\varphi_u}\|^2+\frac{t}{4R_m}\|\boldsymbol{\varphi_B}\|^2+Ct\left(\|\boldsymbol{\eta_u}\|_1^2+\|\boldsymbol{\eta_B}\|_1^2\right)+Ct\left(\|\mathbf{u}\|_2^2+\|\mathbf{B}_\epsilon\|_2^2\right)\|\boldsymbol{\varphi_B}\|^2.$$

Combining the above inequalities, we arrive at

$$\frac{d}{dt}t(\|\boldsymbol{\varphi_u}\|^2+\|\boldsymbol{\varphi_B}\|^2)+R_e^{-1}t\|\nabla\boldsymbol{\varphi_u}\|^2+R_m^{-1}t\|\nabla\boldsymbol{\varphi_B}\|^2+R_e\epsilon t\|\varphi_p\|^2$$
$$\leq\|\boldsymbol{\varphi_u}\|^2+\|\boldsymbol{\varphi_B}\|^2Ct\left(\|\boldsymbol{\eta_u}\|_1^2+\|\boldsymbol{\eta_B}\|_1^2\right)+Ct\left(\|\mathbf{u}\|_2^2+\|\mathbf{u}_\epsilon\|_2^2+\|\mathbf{B}\|_2^2+\|\mathbf{B}_\epsilon\|_2^2\right)\|\boldsymbol{\varphi_u}\|^2$$
$$+Ct\left(\|\mathbf{u}\|_2^2+\|\mathbf{B}_\epsilon\|_2^2\right)\|\boldsymbol{\varphi_B}\|^2.$$

Integrating the above inequality over $[0, t]$, using (4.10), Lemma 4.1, and the Gronwall lemma, we obtain

$$t(\|\boldsymbol{\varphi_u}\|^2 + \|\boldsymbol{\varphi_B}\|^2) + \int_0^t s(R_e^{-1}\|\nabla\boldsymbol{\varphi_u}\|^2 + R_m^{-1}\|\nabla\boldsymbol{\varphi_B}\|^2)ds \le C\epsilon^2. \quad (4.12)$$

Then, we take the partial derivative with respect to t of (4.4) to get

$$\mathrm{div}\boldsymbol{\varphi}_{ut} + R_e\epsilon\varphi_{pt} = 0. \quad (4.13)$$

Taking the L^2 inner products of (4.3) with $t^2\boldsymbol{\varphi}_{ut}$, (4.13) with $t^2\varphi_p$, (4.9) with $t^2\boldsymbol{\varphi}_{Bt}$ and adding them up, we get

$$\begin{aligned} t^2\|\boldsymbol{\varphi}_{ut}\|^2 &+ \frac{1}{2R_e}\frac{d}{dt}t^2\|\nabla\boldsymbol{\varphi_u}\|^2 + \frac{R_e\epsilon}{2}\frac{d}{dt}t^2\|\varphi_p\|^2 + t^2\|\boldsymbol{\varphi}_{Bt}\|^2 + \frac{1}{2R_m}\frac{d}{dt}t^2\|\nabla\boldsymbol{\varphi_B}\|^2 \\ &= R_e^{-1}t\|\nabla\boldsymbol{\varphi_u}\|^2 + R_m^{-1}t\|\nabla\boldsymbol{\varphi_B}\|^2 + R_e\epsilon t\|\varphi_p\|^2 - t^2 b(\mathbf{u}, \boldsymbol{\eta_u} + \boldsymbol{\varphi_u}, \boldsymbol{\varphi}_{ut}) - t^2 b(\boldsymbol{\eta_u} + \boldsymbol{\varphi_u}, \mathbf{u}_\epsilon, \boldsymbol{\varphi}_{ut}) \\ &+ S_c t^2(\mathrm{curl}\mathbf{B} \times (\boldsymbol{\eta_B} + \boldsymbol{\varphi_B}), \boldsymbol{\varphi}_{ut}) + S_c t^2(\mathrm{curl}(\boldsymbol{\eta_B} + \boldsymbol{\varphi_B}) \times \mathbf{B}_\epsilon, \boldsymbol{\varphi}_{ut}) \\ &+ t^2(\mathrm{curl}(\mathbf{u} \times (\boldsymbol{\eta_B} + \boldsymbol{\varphi_B})), \boldsymbol{\varphi}_{Bt}) + t^2(\mathrm{curl}((\boldsymbol{\eta_u} + \boldsymbol{\varphi_u}) \times \mathbf{B}_\epsilon), \boldsymbol{\varphi}_{Bt}). \end{aligned}$$

Using (2.6)–(2.12) and Lemma 4.1, we derive

$$\begin{aligned} &-t^2 b(\mathbf{u}, \boldsymbol{\eta_u} + \boldsymbol{\varphi_u}, \boldsymbol{\varphi}_{ut}) - t^2 b(\boldsymbol{\eta_u} + \boldsymbol{\varphi_u}, \mathbf{u}_\epsilon, \boldsymbol{\varphi}_{ut}) \\ &\quad\le Ct^2\|\mathbf{u}\|_2\|\boldsymbol{\eta_u} + \boldsymbol{\varphi_u}\|_1\|\boldsymbol{\varphi}_{ut}\| + Ct^2\|\mathbf{u}_\epsilon\|_2\|\boldsymbol{\eta_u} + \boldsymbol{\varphi_u}\|_1\|\boldsymbol{\varphi}_{ut}\| \\ &\quad\le \frac{t^2}{4}\|\boldsymbol{\varphi}_{ut}\|^2 + C\epsilon^2\left(\|\mathbf{u}\|_2^2 + \|\mathbf{u}_\epsilon\|_2^2\right) + Ct^2\left(\|\mathbf{u}\|_2^2 + \|\mathbf{u}_\epsilon\|_2^2\right)\|\nabla\boldsymbol{\varphi_u}\|^2, \\ &S_c t^2(\mathrm{curl}\mathbf{B} \times (\boldsymbol{\eta_B} + \boldsymbol{\varphi_B}), \boldsymbol{\varphi}_{ut}) + S_c t^2(\mathrm{curl}(\boldsymbol{\eta_B} + \boldsymbol{\varphi_B}) \times \mathbf{B}_\epsilon, \boldsymbol{\varphi}_{ut}) \\ &\quad\le Ct^2\|\mathbf{B}\|_2\|\boldsymbol{\eta_B} + \boldsymbol{\varphi_B}\|_1\|\boldsymbol{\varphi}_{ut}\| + Ct^2\|\mathbf{B}_\epsilon\|_2\|\boldsymbol{\eta_B} + \boldsymbol{\varphi_B}\|_1\|\boldsymbol{\varphi}_{ut}\| \\ &\quad\le \frac{t^2}{4}\|\boldsymbol{\varphi}_{ut}\|^2 + C\epsilon^2\left(\|\mathbf{B}\|_2^2 + \|\mathbf{B}_\epsilon\|_2^2\right) + Ct^2\left(\|\mathbf{B}\|_2^2 + \|\mathbf{B}_\epsilon\|_2^2\right)\|\nabla\boldsymbol{\varphi_B}\|^2, \\ &t^2(\mathrm{curl}(\mathbf{u} \times (\boldsymbol{\eta_B} + \boldsymbol{\varphi_B})), \boldsymbol{\varphi}_{Bt}) + t^2(\mathrm{curl}((\boldsymbol{\eta_u} + \boldsymbol{\varphi_u}) \times \mathbf{B}_\epsilon), \boldsymbol{\varphi}_{Bt}) \\ &\quad\le Ct^2\|\mathbf{B}\|_2\|\boldsymbol{\eta_B} + \boldsymbol{\varphi_B}\|_1\|\boldsymbol{\varphi}_{Bt}\| + Ct^2\|\mathbf{B}_\epsilon\|_2\|\boldsymbol{\eta_B} + \boldsymbol{\varphi_B}\|_1\|\boldsymbol{\varphi}_{Bt}\| \\ &\quad\le \frac{t^2}{4}\|\boldsymbol{\varphi}_{Bt}\|^2 + C\epsilon^2\left(\|\mathbf{u}\|_2^2 + \|\mathbf{B}_\epsilon\|_2^2\right) + Ct^2\|\mathbf{u}\|_2^2\|\nabla\boldsymbol{\varphi_B}\|^2 + Ct^2\|\mathbf{B}_\epsilon\|_2^2\|\nabla\boldsymbol{\varphi_u}\|^2. \end{aligned}$$

Combining the above inequalities, we obtain

$$\begin{aligned} t^2(\|\boldsymbol{\varphi}_{ut}\|^2 &+ \|\boldsymbol{\varphi}_{Bt}\|^2) + \frac{d}{dt}t^2(R_e^{-1}\|\nabla\boldsymbol{\varphi_u}\|^2 + R_m^{-1}\|\nabla\boldsymbol{\varphi_B}\|^2 + R_e\epsilon\|\varphi_p\|^2) \\ &\le R_e^{-1}t\|\nabla\boldsymbol{\varphi_u}\|^2 + R_m^{-1}\|\nabla\boldsymbol{\varphi_B}\|^2 + R_e\epsilon t\|\varphi_p\|^2 + C\epsilon^2\left(\|\mathbf{u}\|_2^2 + \|\mathbf{u}_\epsilon\|_2^2 + \|\mathbf{B}\|_2^2 + \|\mathbf{B}_\epsilon\|_2^2\right) \\ &+ Ct^2\left(\|\mathbf{u}\|_2^2 + \|\mathbf{u}_\epsilon\|_2^2 + \|\mathbf{B}_\epsilon\|_2^2\right)\|\nabla\boldsymbol{\varphi_u}\|^2 + Ct^2\left(\|\mathbf{u}\|_2^2 + \|\mathbf{B}\|_2^2 + \|\mathbf{B}_\epsilon\|_2^2\right)\|\nabla\boldsymbol{\varphi_B}\|^2. \end{aligned}$$

Integrating over $[0, t]$, using (4.12) and the Gronwall lemma, we get

$$\int_0^t s^2(\|\boldsymbol{\varphi}_{ut}\|^2 + \|\boldsymbol{\varphi}_{Bt}\|^2)ds + t^2(R_e^{-1}\|\nabla\boldsymbol{\varphi_u}\|^2 + R_m^{-1}\|\nabla\boldsymbol{\varphi_B}\|^2 + R_e\epsilon\|\varphi_p\|^2) \le C\epsilon^2. \quad (4.14)$$

We have

$$\|B(\mathbf{u}, \boldsymbol{\eta}_{\boldsymbol{u}} + \boldsymbol{\varphi}_{\boldsymbol{u}})\|_{-1} \le C\|\mathbf{u}\|_1\|\boldsymbol{\eta}_{\boldsymbol{u}} + \boldsymbol{\varphi}_{\boldsymbol{u}}\|_1 \le C\|\mathbf{u}\|_1(\|\boldsymbol{\eta}_{\boldsymbol{u}}\|_1 + \|\boldsymbol{\varphi}_{\boldsymbol{u}}\|_1),$$
$$\|B(\boldsymbol{\eta}_{\boldsymbol{u}} + \boldsymbol{\varphi}_{\boldsymbol{u}}, \mathbf{u}_\epsilon)\|_{-1} \le C\|\mathbf{u}_\epsilon\|_1(\|\boldsymbol{\eta}_{\boldsymbol{u}}\|_1 + \|\boldsymbol{\varphi}_{\boldsymbol{u}}\|_1),$$
$$\|\text{curl}\mathbf{B} \times (\boldsymbol{\eta}_{\boldsymbol{B}} + \boldsymbol{\varphi}_{\boldsymbol{B}})\|_{-1} \le C\|\mathbf{B}\|_1\|\boldsymbol{\eta}_{\boldsymbol{B}} + \boldsymbol{\varphi}_{\boldsymbol{B}}\|_1,$$
$$\|\text{curl}(\boldsymbol{\eta}_{\boldsymbol{B}} + \boldsymbol{\varphi}_{\boldsymbol{B}}) \times \mathbf{B}_\epsilon)\|_{-1} \le C\|\mathbf{B}_\epsilon\|_1(\|\boldsymbol{\eta}_{\boldsymbol{B}}\|_1 + \|\boldsymbol{\varphi}_{\boldsymbol{B}}\|_1).$$

Due to (4.3) and (4.7), we deduce

$$\nabla\varphi_p = -\boldsymbol{\varphi}_{ut} + R_e^{-1}\Delta\boldsymbol{\varphi}_{\boldsymbol{u}} - B(\mathbf{u}_\epsilon, \mathbf{u}_\epsilon) + \boldsymbol{B}(\mathbf{u}, \mathbf{u}) + S_c(\text{curl}\mathbf{B}_\epsilon \times \mathbf{B}_\epsilon - \text{curl}\mathbf{B} \times \mathbf{B}).$$

By using previous estimates on the above equation, we have

$$\int_0^T s^2\|\varphi_p\|^2 ds \le \int_0^T s^2\|\nabla\varphi_p\|_{-1}^2 ds \le C\epsilon^2,$$

which completes the proof of Theorem 4.1. □

5 Time discretizations of the penalized system

In this part, we give the time-discretization of the penalty system and derive its error estimate. Next, we will give some rules for the solutions of the penalty system.

Lemma 5.1 *Suppose* $\mathbf{u}_0, \mathbf{B}_0 \in \mathbf{H}^2(\Omega)$ *and* $\mathcal{A}1 - \mathcal{A}2$ *are valid. Then the solutions* $\mathbf{u}_\epsilon, \mathbf{B}_\epsilon$ *of* (2.18) *satisfy*

$$\mathbf{u}_\epsilon, \mathbf{B}_\epsilon \in C(0, T; \mathbf{H}^2(\Omega)), \tag{5.1}$$
$$\mathbf{u}_{\epsilon t}, \in L^2(0, T; \boldsymbol{X}), \quad \mathbf{B}_{\epsilon t} \in L^2(0, T; \mathbf{W}),$$
$$A_\epsilon^{-\frac{1}{2}}\mathbf{u}_{\epsilon tt}, A^{-\frac{1}{2}}\mathbf{B}_{\epsilon tt} \in L^2(0, T; \mathbf{L}^2(\Omega)), \tag{5.2}$$
$$\sqrt{t}\boldsymbol{u}_{\epsilon tt}, \sqrt{t}\mathbf{B}_{\epsilon tt} \in L^2(0, T; \mathbf{L}^2(\Omega)).$$

Since the proof of (5.1) is standard (cf. [20]), we just need to prove (5.2). Taking the partial derivative with respect to t on (1.3), we have

$$\mathbf{u}_{\epsilon tt} - R_e^{-1}A_{1\epsilon}\mathbf{u}_{\epsilon t} + B(\mathbf{u}_{\epsilon t}, \mathbf{u}_\epsilon) + B(\mathbf{u}_\epsilon, \mathbf{u}_{\epsilon t}) - S_c\text{curl}\mathbf{B}_{\epsilon t} \times \mathbf{B}_\epsilon - S_c\text{curl}\mathbf{B}_\epsilon \times \mathbf{B}_{\epsilon t} = \mathbf{f}_t, \tag{5.3}$$
$$\mathbf{B}_{\epsilon tt} + R_m^{-1}A_{2\epsilon}\mathbf{B}_{\epsilon t} - \text{curl}(\mathbf{u}_{\epsilon t} \times \mathbf{B}_\epsilon) - \text{curl}(\mathbf{u}_\epsilon \times \mathbf{B}_{\epsilon t}) = \mathbf{g}_t. \tag{5.4}$$

Taking the L^2 inner product of (5.3) with $\mathbf{u}_{\epsilon t}$, (5.4) with $S_c\mathbf{B}_{\epsilon t}$, thanks to (2.1) and (2.7), we obtain

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}_{\epsilon t}\|^2 + R_e^{-1}\|A_{1\epsilon}^{\frac{1}{2}}\mathbf{u}_{\epsilon t}\|^2 + \frac{S_c}{2}\frac{d}{dt}\|\mathbf{B}_{\epsilon t}\|^2 + S_c R_m^{-1}\|A_{2\epsilon}^{\frac{1}{2}}\mathbf{B}_{\epsilon t}\|^2 = (\mathbf{f}_t, \mathbf{u}_{\epsilon t}) + (\mathbf{g}_t, \mathbf{B}_{\epsilon t})$$
$$-b(\mathbf{u}_{\epsilon t}, \mathbf{u}_\epsilon, \mathbf{u}_{\epsilon t}) + S_c(\text{curl}\mathbf{B}_\epsilon \times \mathbf{B}_{\epsilon t}, \mathbf{u}_{\epsilon t}) + S_c(\text{curl}(\mathbf{u}_\epsilon \times \mathbf{B}_{\epsilon t}), \mathbf{B}_{\epsilon t}).$$

Using Lemma 2.1 and Young's inequality, we get

$$\begin{aligned}(\mathbf{f}_t,\mathbf{u}_{\epsilon t})+(\mathbf{g}_t,\mathbf{B}_{\epsilon t}) &\le C\|\mathbf{f}_t\|\,\|\mathbf{u}_{\epsilon t}\|_1+C\|\mathbf{g}_t\|\,\|\mathbf{B}_{\epsilon t}\|_1\\ &\le C\|\mathbf{f}_t\|^2+C\|\mathbf{g}_t\|^2+\frac{1}{4R_e}\|A_{1\epsilon}^{\frac12}\mathbf{u}_{\epsilon t}\|^2+\frac{S_c}{4R_m}\|A_{2\epsilon}^{\frac12}\mathbf{B}_{\epsilon t}\|^2.\end{aligned}$$

Due to (2.6) and Young's inequality, we have

$$-b(\mathbf{u}_{\epsilon t},\mathbf{u}_\epsilon,\mathbf{u}_{\epsilon t})\le C\|\mathbf{u}_{\epsilon t}\|\,\|\mathbf{u}_\epsilon\|_2\|\mathbf{u}_{\epsilon t}\|_1\le\frac{1}{8R_e}\|A_{1\epsilon}^{\frac12}\mathbf{u}_{\epsilon t}\|^2+C\|\mathbf{u}_\epsilon\|_2^2\|\mathbf{u}_{\epsilon t}\|^2.$$

By using (2.8), (2.12), and Young's inequality, we have

$$\begin{aligned}&S_c(\mathrm{curl}\mathbf{B}_\epsilon\times\mathbf{B}_{\epsilon t},\mathbf{u}_{\epsilon t})+S_c(\mathrm{curl}(\mathbf{u}_\epsilon\times\mathbf{B}_{\epsilon t}),\mathbf{B}_{\epsilon t})\\ &\le C\|\mathbf{B}_\epsilon\|_2\|\mathbf{B}_{\epsilon t}\|\,\|\mathbf{u}_{\epsilon t}\|_1+C\|\mathbf{u}_\epsilon\|_2\|\mathbf{B}_{\epsilon t}\|\,\|\mathbf{B}_{\epsilon t}\|_1\\ &\le\frac{1}{8R_e}\|A_{1\epsilon}^{\frac12}\mathbf{u}_{\epsilon t}\|^2+\frac{S_c}{4R_m}\|A_{2\epsilon}^{\frac12}\mathbf{B}_{\epsilon t}\|^2+C\left(\|\mathbf{B}_\epsilon\|_2^2+\|\mathbf{u}_\epsilon\|_2^2\right)\|\mathbf{B}_{\epsilon t}\|^2.\end{aligned}$$

Combining the above inequalities, we obtain

$$\begin{aligned}\frac{d}{dt}(\|\mathbf{u}_{\epsilon t}\|^2 &+S_c\|\mathbf{B}_{\epsilon t}\|^2)+R_e^{-1}\|A_\epsilon^{\frac12}\mathbf{u}_{\epsilon t}\|^2+S_cR_m^{-1}\|A^{\frac12}\mathbf{B}_{\epsilon t}\|^2\\ &\le C(\|\mathbf{f}_t\|^2+\|\mathbf{g}_t\|^2)+C\|\mathbf{u}_\epsilon\|_2^2\|\mathbf{u}_{\epsilon t}\|^2+C\left(\|\mathbf{B}_\epsilon\|_2^2+\|\mathbf{u}_\epsilon\|_2^2\right)\|\mathbf{B}_{\epsilon t}\|^2.\end{aligned}$$

Since $\mathbf{u}_\epsilon,\mathbf{B}_\epsilon\in C(0,T;\mathbf{H}^2(\Omega))$, we can show that $\mathbf{u}_{\epsilon t}(0),\mathbf{B}_{\epsilon t}(0)$ is well defined (cf. [20]). Hence, integrating over $[0,t]$, using the Gronwall lemma and Lemma 2.1, we derive

$$\|\mathbf{u}_{\epsilon t}\|^2+S_c\|\mathbf{B}_{\epsilon t}\|^2+\int_0^t\left(R_e^{-1}\|A_{1\epsilon}^{\frac12}\mathbf{u}_{\epsilon t}\|^2+S_cR_m^{-1}\|A_{2\epsilon}^{\frac12}\mathbf{B}_{\epsilon t}\|^2\right)ds\le C.$$

By Lemma 2.1, we get $\|\mathbf{u}_{\epsilon t}\|_{L^2(0,T;\mathbf{H}^1(\Omega))}+\|\mathbf{B}_{\epsilon t}\|_{L^2(0,T;\mathbf{H}^1(\Omega))}\le C$. Then by using (2.6), we get

$$\|A_\epsilon^{-\frac12}B(\mathbf{u}_{\epsilon t},\mathbf{u}_\epsilon)\|\le\|B(\mathbf{u}_{\epsilon t},\mathbf{u}_\epsilon)\|_{-1}\le\sup_{\mathbf{v}\in\mathbf{H}_0^1(\Omega)}\frac{b(\mathbf{u}_{\epsilon t},\mathbf{u}_\epsilon,\mathbf{v})}{\|\mathbf{v}\|_1}\le C\|\mathbf{u}_{\epsilon t}\|\,\|\mathbf{u}_\epsilon\|_2.$$

By the same proof, we have

$$\begin{aligned}&\|A_{1\epsilon}^{-\frac12}B(\mathbf{u}_\epsilon,\mathbf{u}_{\epsilon t})\|\le C\|\mathbf{u}_{\epsilon t}\|\,\|\mathbf{u}_\epsilon\|_2,\\ &\|A_{1\epsilon}^{-\frac12}(\mathrm{curl}\mathbf{B}_{\epsilon t}\times\mathbf{B}_\epsilon)\|+\|A_\epsilon^{-\frac12}(\mathrm{curl}\mathbf{B}_\epsilon\times\mathbf{B}_{\epsilon t})\|\le C\|\mathbf{B}_{\epsilon t}\|\,\|\mathbf{B}_\epsilon\|_2,\\ &\|A_{2\epsilon}^{-\frac12}(\mathrm{curl}\mathbf{u}_{\epsilon t}\times\mathbf{B}_\epsilon)\|\le C\|\mathbf{u}_{\epsilon t}\|\,\|\mathbf{B}_\epsilon\|_2,\quad\|A_{2\epsilon}^{-\frac12}(\mathrm{curl}\mathbf{u}_\epsilon\times\mathbf{B}_{\epsilon t})\|\le C\|\mathbf{B}_{\epsilon t}\|\,\|\mathbf{u}_\epsilon\|_2.\end{aligned}$$

Thus, we get

$$\begin{aligned}A_{1\epsilon}^{-\frac12}\mathbf{u}_{\epsilon tt}&=A_{1\epsilon}^{-\frac12}\{\mathbf{f}_t-R_e^{-1}A_{1\epsilon}\mathbf{u}_{\epsilon t}-B(\mathbf{u}_{\epsilon t},\mathbf{u}_\epsilon)-B(\mathbf{u}_\epsilon,\mathbf{u}_{\epsilon t})+S_c(\mathrm{curl}\mathbf{B}_{\epsilon t}\times\mathbf{B}_\epsilon+\mathrm{curl}\mathbf{B}_\epsilon\times\mathbf{B}_{\epsilon t})\},\\ A_{2\epsilon}^{-\frac12}\mathbf{B}_{\epsilon tt}&=A_{2\epsilon}^{-\frac12}\{\mathbf{g}_t-R_m^{-1}A_{2\epsilon}\mathbf{B}_{\epsilon t}+(\mathrm{curl}(\mathbf{u}_{\epsilon t}\times\mathbf{B}_\epsilon)+\mathrm{curl}(\mathbf{u}_\epsilon\times\mathbf{B}_{\epsilon t}))\}.\end{aligned}$$

Then

$$A_{1\epsilon}^{-\frac{1}{2}}\mathbf{u}_{\epsilon tt},\ A_{2\epsilon}^{-\frac{1}{2}}\mathbf{B}_{\epsilon tt} \in L^2(0,T;\mathbf{L}^2(\Omega)). \tag{5.5}$$

Taking the L^2 inner product of (5.3) with $t\mathbf{u}_{\epsilon tt}$, (5.4) with $t\mathbf{B}_{\epsilon tt}$, we obtain

$$\begin{aligned} t\|\mathbf{u}_{\epsilon tt}\|^2 &+\frac{1}{2R_e}\frac{d}{dt}t\|A_{1\epsilon}^{\frac{1}{2}}\mathbf{u}_{\epsilon t}\|^2+t\|\mathbf{B}_{\epsilon tt}\|^2+\frac{1}{2R_m}\frac{d}{dt}t\|A_{2\epsilon}^{\frac{1}{2}}\mathbf{B}_{\epsilon t}\|^2=\frac{1}{2R_e}\frac{d}{dt}\|A_{1\epsilon}^{\frac{1}{2}}\mathbf{u}_{\epsilon t}\|^2+\frac{1}{2R_m}\frac{d}{dt}\|A_{2\epsilon}^{\frac{1}{2}}\mathbf{B}_{\epsilon t}\|^2\\ &+t(\mathbf{f}_t,\mathbf{u}_{\epsilon tt})+t(\mathbf{g}_t,\mathbf{B}_{\epsilon tt})-tb(\mathbf{u}_{\epsilon t},\mathbf{u}_\epsilon,\mathbf{u}_{\epsilon tt})-tb(\mathbf{u}_\epsilon,\mathbf{u}_{\epsilon t},\mathbf{u}_{\epsilon tt})+S_c(\mathrm{curl}\mathbf{B}_{\epsilon t}\times\mathbf{B}_\epsilon,\mathbf{u}_{\epsilon tt})\\ &+S_c(\mathrm{curl}\mathbf{B}_\epsilon\times\mathbf{B}_{\epsilon t},\mathbf{u}_{\epsilon tt})+(\mathrm{curl}(\mathbf{u}_{\epsilon t}\times\mathbf{B}_\epsilon),\mathbf{B}_{\epsilon tt})+(\mathrm{curl}(\mathbf{u}_\epsilon\times\mathbf{B}_{\epsilon t}),\mathbf{B}_{\epsilon tt}). \end{aligned}$$

Using Schwarz's inequality, we get

$$t(\mathbf{f}_t,\mathbf{u}_{\epsilon tt})+t(\mathbf{g}_t,\mathbf{B}_{\epsilon tt})\le t\|\mathbf{f}_t\|\,\|\mathbf{u}_{\epsilon tt}\|+t\|\mathbf{g}_t\|\,\|\mathbf{B}_{\epsilon tt}\|\le\frac{t}{4}(\|\mathbf{u}_{\epsilon tt}\|^2+\|\mathbf{B}_{\epsilon tt}\|^2)+Ct(\|\mathbf{f}_t\|^2+\|\mathbf{g}_t\|^2).$$

From (2.6), we obtain

$$-tb(\mathbf{u}_{\epsilon t},\mathbf{u}_\epsilon,\mathbf{u}_{\epsilon tt})-tb(\mathbf{u}_\epsilon,\mathbf{u}_{\epsilon t},\mathbf{u}_{\epsilon tt})\le Ct\|\mathbf{u}_\epsilon\|_2\|\mathbf{u}_{\epsilon t}\|_1\|\mathbf{u}_{\epsilon tt}\|\le\frac{t}{8}\|\mathbf{u}_{\epsilon tt}\|^2+Ct\|\mathbf{u}_\epsilon\|_2^2\|\mathbf{u}_{\epsilon t}\|_1^2.$$

By using (2.12) and Young's inequality, we obtain

$$S_c(\mathrm{curl}\mathbf{B}_{\epsilon t}\times\mathbf{B}_\epsilon,\mathbf{u}_{\epsilon tt})+S_c(\mathrm{curl}\mathbf{B}_\epsilon\times\mathbf{B}_{\epsilon t},\mathbf{u}_{\epsilon tt})\le Ct\|\mathbf{B}_\epsilon\|_2\|\mathbf{B}_{\epsilon t}\|_1\|\mathbf{u}_{\epsilon tt}\|\le\frac{t}{8}\|\mathbf{u}_{\epsilon tt}\|^2+Ct\|\mathbf{B}_\epsilon\|_2^2\|\mathbf{B}_{\epsilon t}\|_1^2.$$

Using (2.8)–(2.13), we can derive

$$\begin{aligned} (\mathrm{curl}(\mathbf{u}_{\epsilon t}\times\mathbf{B}_\epsilon),\mathbf{B}_{\epsilon tt})+(\mathrm{curl}(\mathbf{u}_\epsilon\times\mathbf{B}_{\epsilon t}),\mathbf{B}_{\epsilon tt}) &\le Ct\|\mathbf{B}_\epsilon\|_2\|\mathbf{u}_{\epsilon t}\|_1\|\mathbf{B}_{\epsilon tt}\|+Ct\|\mathbf{u}_\epsilon\|_2\|\mathbf{B}_{\epsilon t}\|_1\|\mathbf{B}_{\epsilon tt}\|\\ &\le\frac{t}{4}\|\mathbf{B}_{\epsilon tt}\|^2+Ct\|\mathbf{B}_\epsilon\|_2^2\|\mathbf{u}_{\epsilon t}\|_1^2+Ct\|\mathbf{u}_\epsilon\|_2^2\|\mathbf{B}_{\epsilon t}\|_1^2. \end{aligned}$$

Combining the above inequalities, integrating over $[0,T]$, and using Assumption $\mathcal{A}1$ and (5.6), we derive

$$\int_0^T t(\|\mathbf{u}_{\epsilon tt}\|^2+\|\mathbf{B}_{\epsilon tt}\|^2)dt\le C.$$

Let us consider the time discretization of the penalized system (2.16) by the backward Euler scheme

$$\begin{cases} d_t\mathbf{u}_\epsilon^n-R_e^{-1}A_{1\epsilon}\mathbf{u}_\epsilon^n+B(\mathbf{u}_\epsilon^n,\mathbf{u}_\epsilon^n)-S_c\mathrm{curl}\mathbf{B}_\epsilon^n\times\mathbf{B}_\epsilon^n=\mathbf{f}(t_n),\\ d_t\mathbf{B}_\epsilon^n+\ R_m^{-1}A_{2\epsilon}\mathbf{B}_\epsilon^n-\mathrm{curl}(\mathbf{u}_\epsilon^n\times\mathbf{B}_\epsilon^n)=\mathbf{g}(t_n), \end{cases} \tag{5.6}$$

where $0<\Delta t<1$ is the time-step size, $t_n=n\Delta t$, $t_N=T$, $(\mathbf{u}_\epsilon^0,\mathbf{B}_\epsilon^0)=(\mathbf{u}_0,\mathbf{B}_0)$, and $d_t\mathbf{u}_\epsilon^n=\frac{1}{\Delta t}(\mathbf{u}_\epsilon^n-\mathbf{u}_\epsilon^{n-1})$ for $1\le n\le N$.

Lemma 5.2 *Under the assumption of Lemma 5.1, we have*

$$t_m(\|\boldsymbol{u}_\epsilon(t_n)-\boldsymbol{u}_\epsilon^n\|_1^2+\|\boldsymbol{B}_\epsilon(t_n)-\boldsymbol{B}_\epsilon^n\|_1^2)+\Delta t\sum_{n=1}^{m}t_n\left(\|\boldsymbol{u}_\epsilon(t_n)-\boldsymbol{u}_\epsilon^n\|_2^2+\|\boldsymbol{B}_\epsilon(t_n)-\boldsymbol{B}_\epsilon^n\|_2^2\right)\le C\Delta t^2,\ \forall m\le N.$$

Proof Letting $\boldsymbol{e}_{\boldsymbol{u}}^n=\mathbf{u}_\epsilon(t_n)-\mathbf{u}_\epsilon^n$, $\boldsymbol{e}_{\boldsymbol{B}}^n=\mathbf{B}_\epsilon(t_n)-\mathbf{B}_\epsilon^n$ and subtracting (5.6) from (2.18) at $t=t_n$, we get

$$d_t\boldsymbol{e}_{\boldsymbol{u}}^n+R_e^{-1}A_{1\epsilon}\boldsymbol{e}_{\boldsymbol{u}}^n+B(\mathbf{u}_\epsilon^n,\boldsymbol{e}_{\boldsymbol{u}}^n)+B(\boldsymbol{e}_{\boldsymbol{u}}^n,\mathbf{u}_\epsilon(t_n))-S_c(\mathrm{curl}\mathbf{B}_\epsilon(t_n)\times\boldsymbol{e}_{\boldsymbol{B}}^n+\mathrm{curl}\boldsymbol{e}_{\boldsymbol{B}}^n\times\mathbf{B}_\epsilon^n)=\mathbf{R}_1^n,\quad(5.7)$$
$$d_t\boldsymbol{e}_{\boldsymbol{B}}^n+\ R_m^{-1}A_{2\epsilon}\boldsymbol{e}_{\boldsymbol{B}}^n-(\mathrm{curl}(\mathbf{u}_\epsilon(t_n)\times\boldsymbol{e}_{\boldsymbol{B}}^n)+\mathrm{curl}(\boldsymbol{e}_{\boldsymbol{u}}^n\times\mathbf{B}_\epsilon^n))=\mathbf{R}_2^n,\quad(5.8)$$

where

$$\mathbf{R}_1^n=-(\mathbf{u}_{\epsilon t}(t_n)-\frac{1}{\Delta t}(\mathbf{u}_\epsilon(t_n)-\mathbf{u}_\epsilon(t_{n-1})))=-\frac{1}{\Delta t}\int_{t_{n-1}}^{t_n}(t-t_{n-1})\mathbf{u}_{\epsilon tt}dt,\quad(5.9)$$
$$\mathbf{R}_2^n=-(\mathbf{B}_{\epsilon t}(t_n)-\frac{1}{\Delta t}(\mathbf{B}_\epsilon(t_n)-\mathbf{B}_\epsilon(t_{n-1})))=-\frac{1}{\Delta t}\int_{t_{n-1}}^{t_n}(t-t_{n-1})\mathbf{B}_{\epsilon tt}dt.\ (5.10)$$

Taking the L^2 inner product of (5.7) with $2\Delta t\boldsymbol{e}_{\boldsymbol{u}}^n$, (5.8) with $2S_c\Delta t\boldsymbol{e}_{\boldsymbol{B}}^n$, thanks to (2.1) and (2.7), we obtain

$$\begin{aligned}\|\boldsymbol{e}_{\boldsymbol{u}}^n\|^2\ -\ \|\boldsymbol{e}_{\boldsymbol{u}}^{n-1}\|^2+\|\boldsymbol{e}_{\boldsymbol{u}}^n-\boldsymbol{e}_{\boldsymbol{u}}^{n-1}\|^2+2R_e^{-1}\Delta t\|A_{1\epsilon}^{\frac{1}{2}}\boldsymbol{e}_{\boldsymbol{u}}^n\|^2&\\ +S_c(\|\boldsymbol{e}_{\boldsymbol{B}}^n\|^2-\|\boldsymbol{e}_{\boldsymbol{B}}^{n-1}\|^2+\|\boldsymbol{e}_{\boldsymbol{B}}^n-\boldsymbol{e}_{\boldsymbol{B}}^{n-1}\|^2)+2S_cR_m^{-1}\Delta t\|A_{2\epsilon}^{\frac{1}{2}}\boldsymbol{e}_b^n\|^2&\\ =2\Delta t(\mathbf{R}_1^n,\boldsymbol{e}_{\boldsymbol{u}}^n)+2\Delta t(\mathbf{R}_2^n,\boldsymbol{e}_{\boldsymbol{B}}^n)-2\Delta tb(\boldsymbol{e}_{\boldsymbol{u}}^n,\mathbf{u}_\epsilon(t_n),\boldsymbol{e}_{\boldsymbol{u}}^n)&\\ +2S_c\Delta t(\mathrm{curl}\mathbf{B}_\epsilon(t_n)\times\boldsymbol{e}_{\boldsymbol{B}}^n,\boldsymbol{e}_{\boldsymbol{u}}^n)+2S_c\Delta t(\mathrm{curl}(\mathbf{u}_\epsilon(t_n)\times\boldsymbol{e}_{\boldsymbol{B}}^n),\boldsymbol{e}_{\boldsymbol{B}}^n).&\end{aligned}$$

Using the Schwarz inequality, (5.9), and (5.10), we get

$$\begin{aligned}2\Delta t(\mathbf{R}_1^n,\boldsymbol{e}_{\boldsymbol{u}}^n)\ +\ 2\Delta t(\mathbf{R}_2^n,\boldsymbol{e}_{\boldsymbol{B}}^n)&=2\Delta t\left(A_{1\epsilon}^{-\frac{1}{2}}\mathbf{R}_1^n,A_{1\epsilon}^{\frac{1}{2}}\boldsymbol{e}_{\boldsymbol{u}}^n\right)+2\Delta t\left(A_{2\epsilon}^{-\frac{1}{2}}\mathbf{R}_2^n,A_{2\epsilon}^{\frac{1}{2}}\boldsymbol{e}_{\boldsymbol{B}}^n\right)\\ &\le\frac{C}{\Delta t}\left(\int_{t_{n-1}}^{t_n}(t-t_{n-1})\|A_{1\epsilon}^{-\frac{1}{2}}\mathbf{u}_{\epsilon tt}\|dt\right)^2+\frac{\Delta t}{2R_e}\|A_{1\epsilon}^{\frac{1}{2}}\boldsymbol{e}_{\boldsymbol{u}}^n\|^2\\ &+\frac{C}{\Delta t}\left(\int_{t_{n-1}}^{t_n}(t-t_{n-1})\|A_{2\epsilon}^{-\frac{1}{2}}\mathbf{B}_{\epsilon tt}\|dt\right)^2+\frac{\Delta t}{2R_m}\|A_{2\epsilon}^{\frac{1}{2}}\boldsymbol{e}_{\boldsymbol{B}}^n\|^2\\ &\le\frac{\Delta t}{2}\left(R_e^{-1}\|A_{1\epsilon}^{\frac{1}{2}}\boldsymbol{e}_{\boldsymbol{u}}^n\|^2+S_cR_m^{-1}\|A_{2\epsilon}^{\frac{1}{2}}\boldsymbol{e}_{\boldsymbol{B}}^n\|^2\right)\\ &+C\Delta t^2\int_{t_{n-1}}^{t_n}\left(\|A_{1\epsilon}^{-\frac{1}{2}}\mathbf{u}_{\epsilon tt}\|^2+\|A_{2\epsilon}^{-\frac{1}{2}}\mathbf{B}_{\epsilon tt}\|^2\right)dt.\end{aligned}$$

We can derive from (2.6) and (5.1) that

$$2\Delta tb(\boldsymbol{e}_{\boldsymbol{u}}^n,\mathbf{u}_\epsilon(t_n),\boldsymbol{e}_{\boldsymbol{u}}^n)\le C\Delta t\|\boldsymbol{e}_{\boldsymbol{u}}^n\|\ \|\boldsymbol{e}_{\boldsymbol{u}}^n\|_1\|\mathbf{u}_\epsilon(t_n)\|_2\le\frac{\Delta t}{4R_e}\|A_{1\epsilon}^{\frac{1}{2}}\boldsymbol{e}_{\boldsymbol{u}}\|^2+C\Delta t\|\boldsymbol{e}_{\boldsymbol{u}}^n\|^2.$$

Due to (2.8)–(2.11) and (5.1), we have

$$
\begin{aligned}
&2S_c\Delta t(\mathrm{curl}\mathbf{B}_\epsilon(t_n)\times \boldsymbol{e}_{\boldsymbol{B}}^n, \boldsymbol{e}_{\boldsymbol{u}}^n) + 2S_c\Delta t(\mathrm{curl}(\mathbf{u}_\epsilon(t_n)\times \boldsymbol{e}_{\boldsymbol{B}}^n), \boldsymbol{e}_{\boldsymbol{B}}^n)\\
&\quad \le C\Delta t\|\mathbf{B}_\epsilon(t_n)\|_2\|\boldsymbol{e}_{\boldsymbol{B}}^n\|_1\|\boldsymbol{e}_{\boldsymbol{u}}^n\| + C\Delta t\|\mathbf{u}_\epsilon(t_n)\|_2\|\boldsymbol{e}_{\boldsymbol{B}}^n\|_1\|\boldsymbol{e}_{\boldsymbol{B}}^n\|\\
&\quad \le \frac{\Delta t}{4R_e}\|A_{1\epsilon}^{\frac{1}{2}}\boldsymbol{e}_{\boldsymbol{u}}\|^2 + \frac{S_c\Delta t}{2R_m}\|A_{2\epsilon}^{\frac{1}{2}}\boldsymbol{e}_{\boldsymbol{B}}\|^2 + C\Delta t\|\boldsymbol{e}_{\boldsymbol{u}}\|^2 + C\Delta t\|\boldsymbol{e}_{\boldsymbol{B}}\|^2.
\end{aligned}
$$

Combining the above inequalities, we obtain

$$
\begin{aligned}
&\|\boldsymbol{e}_{\boldsymbol{u}}^n\|^2 - \|\boldsymbol{e}_{\boldsymbol{u}}^{n-1}\|^2 + S_c(\|\boldsymbol{e}_{\boldsymbol{B}}^n\|^2 - \|\boldsymbol{e}_{\boldsymbol{B}}^{n-1}\|^2) + R_e^{-1}\Delta t\|A_{1\epsilon}^{\frac{1}{2}}\boldsymbol{e}_{\boldsymbol{u}}^n\|^2 + S_cR_m^{-1}\Delta t\|A_{2\epsilon}^{\frac{1}{2}}\boldsymbol{e}_b^n\|^2\\
&\quad \le C\Delta t^2\int_{t_{n-1}}^{t_n}\left(\|A_{1\epsilon}^{-\frac{1}{2}}\mathbf{u}_{\epsilon tt}\|^2 + \|A_{2\epsilon}^{-\frac{1}{2}}\mathbf{B}_{\epsilon tt}\|^2\right)dt + C\Delta t\|\boldsymbol{e}_{\boldsymbol{u}}\|^2 + C\Delta t\|\boldsymbol{e}_{\boldsymbol{B}}\|^2. \quad (5.11)
\end{aligned}
$$

Summing (5.11) from 1 to m, and using the discrete Gronwall lemma, Lemma 5.1, we derive

$$
\|\boldsymbol{e}_{\boldsymbol{u}}^n\|^2 + S_c\|\boldsymbol{e}_{\boldsymbol{B}}^n\|^2 + \Delta t\sum_{n=1}^{m}\left(R_e^{-1}\|A_{1\epsilon}^{\frac{1}{2}}\boldsymbol{e}_{\boldsymbol{u}}^n\|^2 + S_cR_m^{-1}\|A_{2\epsilon}^{\frac{1}{2}}\boldsymbol{e}_b^n\|^2\right) \le C\Delta t^2. \quad (5.12)
$$

Thus,

$$
\|\mathbf{u}_\epsilon^n\|_1 + \|\mathbf{B}_\epsilon^n\|_1 \le \|\boldsymbol{e}_{\boldsymbol{u}}^n\|_1 + \|\mathbf{u}_\epsilon(t_n)\|_1 + \|\boldsymbol{e}_{\boldsymbol{B}}^n\|_1 + \|\mathbf{B}_\epsilon(t_n)\|_1 \le C\|A_{1\epsilon}^{\frac{1}{2}}\boldsymbol{e}_{\boldsymbol{u}}^n\| + C\|A_{2\epsilon}^{\frac{1}{2}}\boldsymbol{e}_{\boldsymbol{B}}^n\| + C \le C. \quad (5.13)
$$

Taking the L^2 inner product of (5.7) with $2t_n\Delta tA_{1\epsilon}\boldsymbol{e}_{\boldsymbol{u}}^n$, (5.8) with $2t_n\Delta tA_{2\epsilon}\boldsymbol{e}_{\boldsymbol{B}}^n$, we have

$$
\begin{aligned}
&t_n\left\{\|A_{1\epsilon}^{\frac{1}{2}}\boldsymbol{e}_{\boldsymbol{u}}^n\|^2 - \|A_{1\epsilon}^{\frac{1}{2}}\boldsymbol{e}_{\boldsymbol{u}}^{n-1}\|^2 + \|A_{1\epsilon}^{\frac{1}{2}}\left(\boldsymbol{e}_{\boldsymbol{u}}^n - \boldsymbol{e}_{\boldsymbol{u}}^{n-1}\right)\right\} + 2R_e^{-1}t_n\Delta t\|A_{1\epsilon}\boldsymbol{e}_{\boldsymbol{u}}^n\|^2 + 2R_m^{-1}t_n\Delta t\|A_{2\epsilon}\boldsymbol{e}_{\boldsymbol{B}}^n\|^2\\
&\quad + t_n\left\{\|A_{2\epsilon}^{\frac{1}{2}}\boldsymbol{e}_{\boldsymbol{B}}^n\|^2 - \|A_{2\epsilon}^{\frac{1}{2}}\boldsymbol{e}_{\boldsymbol{B}}^{n-1}\|^2 + \|A_{2\epsilon}^{\frac{1}{2}}(\boldsymbol{e}_{\boldsymbol{B}}^n - \boldsymbol{e}_{\boldsymbol{B}}^{n-1})\right\} = 2\Delta t(t_nR_1^n, A_{1\epsilon}\boldsymbol{e}_{\boldsymbol{u}}^n)\\
&\quad + 2\Delta t(t_nR_2^n, A_{2\epsilon}\boldsymbol{e}_{\boldsymbol{B}}^n) - 2t_n\Delta tb(\mathbf{u}_\epsilon^n, \boldsymbol{e}_{\boldsymbol{u}}^n, A_{1\epsilon}\boldsymbol{e}_{\boldsymbol{u}}^n) - 2t_n\Delta tb(\boldsymbol{e}_{\boldsymbol{u}}^n, \mathbf{u}_\epsilon(t_n), A_{1\epsilon}\boldsymbol{e}_{\boldsymbol{u}}^n)\\
&\quad + 2S_ct_n\Delta t(\mathrm{curl}(\mathbf{B}_\epsilon(t_n)\times \boldsymbol{e}_{\boldsymbol{B}}^n, A_{1\epsilon}\boldsymbol{e}_{\boldsymbol{u}}^n) + 2S_ct_n\Delta t(\mathrm{curl}\boldsymbol{e}_{\boldsymbol{B}}^n\times \mathbf{B}_\epsilon^n, A_{1\epsilon}\boldsymbol{e}_{\boldsymbol{u}}^n)\\
&\quad + 2t_n\Delta t(\mathrm{curl}(\mathbf{u}_\epsilon(t_n)\times \boldsymbol{e}_{\boldsymbol{B}}^n), A_{2\epsilon}\boldsymbol{e}_{\boldsymbol{B}}^n) + 2t_n\Delta t(\mathrm{curl}(\boldsymbol{e}_{\boldsymbol{u}}^n\times \mathbf{B}_\epsilon^n), A_{2\epsilon}\boldsymbol{e}_{\boldsymbol{B}}^n).
\end{aligned}
$$

Using Young's inequality, we can derive

$$
\begin{aligned}
&2\Delta t(t_n R_1^n, A_{1\epsilon}\boldsymbol{e}_{\boldsymbol{u}}^n) + 2\Delta t(t_n R_2^n, A_{2\epsilon}\boldsymbol{e}_{\boldsymbol{B}}^n)\\
&\le \frac{Ct_n}{\Delta t}\left(\int_{t_{n-1}}^{t_n}(t-t_{n-1})\|\mathbf{u}_{\epsilon tt}\|dt\right)^2 + \frac{t_n\Delta t}{2R_e}\|A_{1\epsilon}\boldsymbol{e}_{\boldsymbol{u}}^n\|^2\\
&\quad + \frac{Ct_n}{\Delta t}\left(\int_{t_{n-1}}^{t_n}(t-t_{n-1})\|\mathbf{B}_{\epsilon tt}\|dt\right)^2 + \frac{t_n\Delta t}{2R_m}\|A_{2\epsilon}\boldsymbol{e}_{\boldsymbol{B}}^n\|^2\\
&\le \frac{Ct_n}{\Delta t}\int_{t_{n-1}}^{t_n}\frac{(t-t_{n-1})^2}{t}dt\cdot\int_{t_n}^{t_{n-1}} t\|\mathbf{u}_{\epsilon tt}\|^2dt + \frac{t_n\Delta t}{2R_e}\|A_{1\epsilon}\boldsymbol{e}_{\boldsymbol{u}}^n\|^2\\
&\quad + \frac{Ct_n}{\Delta t}\int_{t_{n-1}}^{t_n}\frac{(t-t_{n-1})^2}{t}dt\cdot\int_{t_n}^{t_{n-1}} t\|\mathbf{B}_{\epsilon tt}\|^2dt + \frac{t_n\Delta t}{2R_m}\|A_{2\epsilon}\boldsymbol{e}_{\boldsymbol{B}}^n\|^2\\
&\le \frac{t_n\Delta t}{2}(R_e^{-1}\|A_{1\epsilon}\boldsymbol{e}_{\boldsymbol{u}}^n\|^2 + R_m^{-1}\|A_{2\epsilon}\boldsymbol{e}_{\boldsymbol{B}}^n\|^2) + C\Delta t^2\int_{t_{n-1}}^{t_n}(t\|\mathbf{u}_{\epsilon tt}\|^2 + t\|\mathbf{B}_{\epsilon tt}\|^2)dt.
\end{aligned}
$$

Due to (2.5)–(2.11) and (5.13), we obtain

$$
\begin{aligned}
&-2t_n\Delta t b(\mathbf{u}_\epsilon^n, \boldsymbol{e}_{\boldsymbol{u}}^n, A_{1\epsilon}\boldsymbol{e}_{\boldsymbol{u}}^n) - 2t_n\Delta t b(\boldsymbol{e}_{\boldsymbol{u}}^n, \mathbf{u}_\epsilon(t_n), A_{1\epsilon}\boldsymbol{e}_{\boldsymbol{u}}^n)\\
&\quad\le Ct_n\Delta t\|A_{1\epsilon}\boldsymbol{e}_{\boldsymbol{u}}^n\|\ \|\mathbf{u}_\epsilon^n\|_1\|\boldsymbol{e}_{\boldsymbol{u}}^n\|_1^{\frac12}\|\boldsymbol{e}_{\boldsymbol{u}}^n\|_2^{\frac12} + Ct_n\Delta t\|\mathbf{u}_\epsilon(t_n)\|_2\|\boldsymbol{e}_{\boldsymbol{u}}^n\|_1\|A_{1\epsilon}\boldsymbol{e}_{\boldsymbol{u}}^n\|\\
&\quad\le Ct_n\Delta t\|A_{1\epsilon}\boldsymbol{e}_{\boldsymbol{u}}^n\|^{\frac32}\|\boldsymbol{e}_{\boldsymbol{u}}^n\|_1^{\frac12} + Ct_n\Delta t\|\boldsymbol{e}_{\boldsymbol{u}}^n\|_1\|A_{1\epsilon}\boldsymbol{e}_{\boldsymbol{u}}^n\| \le \frac{t_n\Delta t}{8R_e}\|A_{1\epsilon}\boldsymbol{e}_{\boldsymbol{u}}^n\|^2 + Ct_n\Delta t\|\boldsymbol{e}_{\boldsymbol{u}}^n\|_1^2,\\
&2S_ct_n\Delta t(\mathrm{curl}(\mathbf{B}_\epsilon(t_n)\times\boldsymbol{e}_{\boldsymbol{B}}^n), A_{1\epsilon}\boldsymbol{e}_{\boldsymbol{u}}^n) + 2S_ct_n\Delta t(\mathrm{curl}\boldsymbol{e}_{\boldsymbol{B}}^n\times\mathbf{B}_\epsilon^n, A_{1\epsilon}\boldsymbol{e}_{\boldsymbol{u}}^n)\\
&\quad\le Ct_n\Delta t\|\mathbf{B}_\epsilon(t_n)\|_2\|\boldsymbol{e}_{\boldsymbol{B}}^n\|_1\|A_{1\epsilon}\boldsymbol{e}_{\boldsymbol{u}}^n\| + Ct_n\Delta t\|A_{1\epsilon}\boldsymbol{e}_{\boldsymbol{u}}^n\|\ \|\mathbf{B}_\epsilon^n\|_1\|\boldsymbol{e}_{\boldsymbol{B}}^n\|_1^{\frac12}\|\boldsymbol{e}_{\boldsymbol{B}}^n\|_2^{\frac12}\\
&\quad\le Ct_n\Delta t\|\boldsymbol{e}_{\boldsymbol{B}}^n\|_1\|A_{1\epsilon}\boldsymbol{e}_{\boldsymbol{u}}^n\| + Ct_n\Delta t\|A_{1\epsilon}\boldsymbol{e}_{\boldsymbol{u}}^n\|\|A_{2\epsilon}\boldsymbol{e}_{\boldsymbol{B}}^n\|^{\frac12}\|\boldsymbol{e}_{\boldsymbol{B}}^n\|_1^{\frac12}\\
&\quad\le \frac{t_n\Delta t}{4R_e}\|A_{1\epsilon}\boldsymbol{e}_{\boldsymbol{u}}^n\|^2 + \frac{t_n\Delta t}{4R_m}\|A_{2\epsilon}\boldsymbol{e}_{\boldsymbol{B}}^n\|^2 + Ct_n\Delta t\|\boldsymbol{e}_{\boldsymbol{B}}^n\|_1^2,\\
&2t_n\Delta t(\mathrm{curl}(\mathbf{u}_\epsilon(t_n)\times\boldsymbol{e}_{\boldsymbol{B}}^n), A_{2\epsilon}\boldsymbol{e}_{\boldsymbol{B}}^n) + 2t_n\Delta t(\mathrm{curl}(\boldsymbol{e}_{\boldsymbol{u}}^n\times\mathbf{B}_\epsilon^n), A_{2\epsilon}\boldsymbol{e}_{\boldsymbol{B}}^n)\\
&\quad\le Ct_n\Delta t\|\mathbf{u}_\epsilon(t_n)\|_2\|\boldsymbol{e}_{\boldsymbol{B}}^n\|_1\|A_{2\epsilon}\boldsymbol{e}_{\boldsymbol{B}}^n\| + Ct_n\Delta t\|A_{2\epsilon}\boldsymbol{e}_{\boldsymbol{B}}^n\|\ \|\mathbf{B}_\epsilon^n\|_1\|\boldsymbol{e}_{\boldsymbol{u}}^n\|_1^{\frac12}\|\boldsymbol{e}_{\boldsymbol{u}}^n\|_2^{\frac12}\\
&\quad\le \frac{t_n\Delta t}{8R_e}\|A_{1\epsilon}\boldsymbol{e}_{\boldsymbol{u}}^n\|^2 + \frac{t_n\Delta t}{4R_m}\|A_{2\epsilon}\boldsymbol{e}_{\boldsymbol{B}}^n\|^2 + Ct_n\Delta t\|\boldsymbol{e}_{\boldsymbol{B}}^n\|_1^2 + Ct_n\Delta t\|\boldsymbol{e}_{\boldsymbol{u}}^n\|_1^2.
\end{aligned}
$$

Combining the above inequalities, we obtain

$$
\begin{aligned}
&t_n\left\{\|A_{1\epsilon}^{\frac12}\boldsymbol{e}_{\boldsymbol{u}}^n\|^2 - \|A_{1\epsilon}^{\frac12}\boldsymbol{e}_{\boldsymbol{u}}^{n-1}\|^2 + \|A_{2\epsilon}^{\frac12}\boldsymbol{e}_{\boldsymbol{B}}^n\|^2 - \|A_{2\epsilon}^{\frac12}\boldsymbol{e}_{\boldsymbol{B}}^{n-1}\|^2\right\} + t_n\Delta t(R_e^{-1}\|A_{1\epsilon}\boldsymbol{e}_{\boldsymbol{u}}^n\|^2 + R_m^{-1}\|A_{2\epsilon}\boldsymbol{e}_{\boldsymbol{B}}^n\|^2)\\
&\qquad\le t_n\Delta t(\|\boldsymbol{e}_{\boldsymbol{u}}^n\|_1^2 + \|\boldsymbol{e}_{\boldsymbol{B}}^n\|_1^2) + C\Delta t^2\int_{t_{n-1}}^{t_n}(t\|\mathbf{u}_{\epsilon tt}\|^2 + t\|\mathbf{B}_{\epsilon tt}\|^2)dt. \qquad (5.14)
\end{aligned}
$$

Taking the summation of (5.14) for n form 1 to m, using the discrete Gronwall lemma and the relation

$$
t_n\left\{\|A_{1\epsilon}^{\frac12}\boldsymbol{e}_{\boldsymbol{u}}^n\|^2 - \|A_{1\epsilon}^{\frac12}\boldsymbol{e}_{\boldsymbol{u}}^{n-1}\|^2\right\} = t_n\|A_{1\epsilon}^{\frac12}\boldsymbol{e}_{\boldsymbol{u}}^n\|^2 - t_{n-1}\|A_{1\epsilon}^{\frac12}\boldsymbol{e}_{\boldsymbol{u}}^{n-1}\|^2 - \Delta t\|A_{1\epsilon}^{\frac12}\boldsymbol{e}_{\boldsymbol{u}}^{n-1}\|^2,
$$

we get

$$t_m\left(\|A_{1\epsilon}^{\frac{1}{2}}\boldsymbol{e}_{\boldsymbol{u}}^n\|^2+\|A_{2\epsilon}^{\frac{1}{2}}\boldsymbol{e}_{\boldsymbol{B}}^n\|^2\right)+\Delta t\sum_{n=1}^{m}t_n(R_e^{-1}\|A_{1\epsilon}\boldsymbol{e}_{\boldsymbol{u}}^n\|^2+R_m^{-1}\|A_{2\epsilon}\boldsymbol{e}_{\boldsymbol{B}}^n\|^2)\le C\Delta t^2.$$

The proof is thus complete. □

Thus, we have

$$\|\mathbf{u}_\epsilon^n\|_2+\|\mathbf{B}_\epsilon^n\|_2\le\|\boldsymbol{e}_{\boldsymbol{u}}^n\|_2+\|\mathbf{u}_\epsilon(t_n)\|_2+\|\boldsymbol{e}_{\boldsymbol{B}}^n\|_2+\|\mathbf{B}_\epsilon(t_n)\|_2\le C\|A_{1\epsilon}\boldsymbol{e}_{\boldsymbol{u}}^n\|+C\|A_{2\epsilon}\boldsymbol{e}_{\boldsymbol{B}}^n\|+C\le C. \quad (5.15)$$

Finally, combining Theorem 4.1 and Lemma 5.2, we prove the following theorem.

Theorem 5.1 *Under the assumption of Lemma 5.1, we have*

$$t_n(\|\boldsymbol{u}(t_n)-\boldsymbol{u}_\epsilon^n\|^2+\|\boldsymbol{B}(t_n)-\boldsymbol{B}_\epsilon^n\|^2)+t_n^2(\|\boldsymbol{u}(t_n)-\boldsymbol{u}_\epsilon^n\|_1^2+\|\boldsymbol{B}(t_n)-\boldsymbol{B}_\epsilon^n\|_1^2)\le C(\Delta t^2+\epsilon^2),\ \forall n\le N.$$

Taking the L^2 inner product of (5.7) with $2t_n^2 d_t\boldsymbol{e}_{\boldsymbol{u}}^n\Delta t$, (5.8) with $2t_n^2 d_t\boldsymbol{e}_{\boldsymbol{B}}^n\Delta t$, we obtain

$$\begin{aligned}2t_n^2\|d_t\boldsymbol{e}_{\boldsymbol{u}}^n\|\Delta t\ &+\ 2t_n^2R_e^2(\|A_{1\epsilon}^{\frac{1}{2}}\boldsymbol{e}_{\boldsymbol{u}}^n\|^2-\|A_{1\epsilon}^{\frac{1}{2}}\boldsymbol{e}_{\boldsymbol{u}}^{n-1}\|^2+\|A_{1\epsilon}^{\frac{1}{2}}(\boldsymbol{e}_{\boldsymbol{u}}^n-\boldsymbol{e}_{\boldsymbol{u}}^{n-1})\|^2)\\&+\ 2t_n^2\|d_t\boldsymbol{e}_{\boldsymbol{B}}^n\|\Delta t+2t_n^2R_m^2(\|A_{2\epsilon}^{\frac{1}{2}}\boldsymbol{e}_{\boldsymbol{B}}^n\|^2-\|A_{2\epsilon}^{\frac{1}{2}}\boldsymbol{e}_{\boldsymbol{B}}^{n-1}\|^2+\|A_{2\epsilon}^{\frac{1}{2}}(\boldsymbol{e}_{\boldsymbol{B}}^n-\boldsymbol{e}_{\boldsymbol{B}}^{n-1})\|^2)\\&=2t_n^2(\mathbf{R}_1^n,d_t\boldsymbol{e}_{\boldsymbol{u}}^n)\Delta t+2t_n^2(\mathbf{R}_2^n,d_t\boldsymbol{e}_{\boldsymbol{B}}^n)\Delta t-2t_n^2b(\boldsymbol{e}_{\boldsymbol{u}}^n,\mathbf{u}_\epsilon(t_n),d_t\boldsymbol{e}_{\boldsymbol{u}}^n)\Delta t-2t_n^2b(\mathbf{u}_\epsilon^n,\boldsymbol{e}_{\boldsymbol{u}}^n,d_t\boldsymbol{e}_{\boldsymbol{u}}^n)\Delta t\\&+\ 2S_ct_n^2(\mathrm{curl}\mathbf{B}_\epsilon(t_n)\times\boldsymbol{e}_{\boldsymbol{B}}^n,d_t\boldsymbol{e}_{\boldsymbol{u}}^n)\Delta t+2S_ct_n^2(\mathrm{curl}\boldsymbol{e}_{\boldsymbol{B}}^n\times\mathbf{B}_\epsilon^n,d_t\boldsymbol{e}_{\boldsymbol{u}}^n)\Delta t\\&+\ 2t_n^2(\mathrm{curl}(\mathbf{u}_\epsilon(t_n)\times\boldsymbol{e}_{\boldsymbol{B}}^n),d_t\boldsymbol{e}_{\boldsymbol{B}}^n)\Delta t+2t_n^2(\mathrm{curl}(\boldsymbol{e}_{\boldsymbol{u}}^n\times\mathbf{B}_\epsilon^n),d_t\boldsymbol{e}_{\boldsymbol{B}}^n)\Delta t.\end{aligned}\quad (5.16)$$

Using Young's inequality, we have

$$\begin{aligned}&2t_n^2(\mathbf{R}_1^n,d_t\boldsymbol{e}_{\boldsymbol{u}}^n)\Delta t+2t_n^2(\mathbf{R}_2^n,d_t\boldsymbol{e}_{\boldsymbol{B}}^n)\Delta t\\&\le\frac{t_n^2\Delta t}{2}\|d_t\boldsymbol{e}_{\boldsymbol{u}}^n\|^2+\frac{Ct_n}{\Delta t}\left(\int_{t_{n-1}}^{t_n}(t-t_n)\|\mathbf{u}_{\epsilon tt}\|dt\right)^2+\frac{t_n^2\Delta t}{2}\|d_t\boldsymbol{e}_{\boldsymbol{B}}^n\|^2+\frac{Ct_n}{\Delta t}\left(\int_{t_{n-1}}^{t_n}(t-t_n)\|\mathbf{B}_{\epsilon tt}\|dt\right)^2\\&\le\frac{t_n^2\Delta t}{2}(\|d_t\boldsymbol{e}_{\boldsymbol{u}}^n\|^2+\|d_t\boldsymbol{e}_{\boldsymbol{B}}^n\|^2)+\frac{Ct_n}{\Delta t}\int_{t_{n-1}}^{t_n}\frac{(t-t_n)^2}{t^2}dt\int_{t_{n-1}}^{t_n}t^2(\|\mathbf{u}_{\epsilon tt}\|^2+\|\mathbf{B}_{\epsilon tt}\|^2)dt\\&\le\frac{t_n^2\Delta t}{2}(\|d_t\boldsymbol{e}_{\boldsymbol{u}}^n\|^2+\|d_t\boldsymbol{e}_{\boldsymbol{B}}^n\|^2)+C\Delta t^2\int_{t_{n-1}}^{t_n}t^2(\|\mathbf{u}_{\epsilon tt}\|^2+\|\mathbf{B}_{\epsilon tt}\|^2)dt.\end{aligned}$$

Due to (2.6)–(2.12), we get

$$
\begin{aligned}
&-\ 2t_n^2 b(\boldsymbol{e}_{\boldsymbol{u}}^n, \mathbf{u}_\epsilon(t_n), d_t\boldsymbol{e}_{\boldsymbol{u}}^n)\Delta t - 2t_n^2 b(\mathbf{u}_\epsilon^n, \boldsymbol{e}_{\boldsymbol{u}}^n, d_t\boldsymbol{e}_{\boldsymbol{u}}^n)\Delta t\\
&\quad\le Ct_n^2\Delta t\|\boldsymbol{e}_{\boldsymbol{u}}^n\|_1\|\mathbf{u}_\epsilon(t_n)\|_2\|d_t\boldsymbol{e}_{\boldsymbol{u}}^n\| + Ct_n^2\Delta t\|\mathbf{u}_\epsilon^n\|_2\|\boldsymbol{e}_{\boldsymbol{u}}^n\|_1\|d_t\boldsymbol{e}_{\boldsymbol{u}}^n\|\\
&\quad\le \frac{t_n^2\Delta t}{4}\|d_t\boldsymbol{e}_{\boldsymbol{u}}^n\|^2 + Ct_n^2\Delta t(\|\mathbf{u}_\epsilon(t_n)\|_2^2 + \|\mathbf{u}_\epsilon^n\|_2^2)\|A_{1\epsilon}^{\frac{1}{2}}\boldsymbol{e}_{\boldsymbol{u}}^n\|^2,\\
&2S_ct_n^2(\mathrm{curl}\mathbf{B}_\epsilon(t_n)\times\boldsymbol{e}_{\boldsymbol{B}}^n, d_t\boldsymbol{e}_{\boldsymbol{u}}^n)\Delta t + 2S_ct_n^2(\mathrm{curl}\boldsymbol{e}_{\boldsymbol{B}}^n\times\mathbf{B}_\epsilon^n, d_t\boldsymbol{e}_{\boldsymbol{u}}^n)\Delta t\\
&\quad\le Ct_n^2\Delta t\|\mathbf{B}_\epsilon(t_n)\|_2\|\boldsymbol{e}_{\boldsymbol{B}}^n\|_1\|d_t\boldsymbol{e}_{\boldsymbol{u}}^n\| + Ct_n^2\Delta t\|\boldsymbol{e}_{\boldsymbol{B}}^n\|_1\|\mathbf{B}_\epsilon^n\|_2\|d_t\boldsymbol{e}_{\boldsymbol{u}}^n\|\\
&\quad\le \frac{t_n^2\Delta t}{4}\|d_t\boldsymbol{e}_{\boldsymbol{u}}^n\|^2 + Ct_n^2\Delta t(\|\mathbf{B}_\epsilon(t_n)\|_2^2 + \|\mathbf{B}_\epsilon^n\|_2^2)\|A_{2\epsilon}^{\frac{1}{2}}\boldsymbol{e}_{\boldsymbol{B}}^n\|^2,\\
&2t_n^2(\mathrm{curl}(\mathbf{u}_\epsilon(t_n)\times\boldsymbol{e}_{\boldsymbol{B}}^n), d_t\boldsymbol{e}_{\boldsymbol{B}}^n)\Delta t + 2t_n^2(\mathrm{curl}(\boldsymbol{e}_{\boldsymbol{u}}^n\times\mathbf{B}_\epsilon^n), d_t\boldsymbol{e}_{\boldsymbol{B}}^n)\Delta t\\
&\quad\le Ct_n^2\Delta t\|\mathbf{u}_\epsilon(t_n)\|_2\|\boldsymbol{e}_{\boldsymbol{B}}^n)\|_1\|d_t\boldsymbol{e}_{\boldsymbol{B}}^n\| + Ct_n^2\Delta t\|\boldsymbol{e}_{\boldsymbol{u}}^n\|_1\|\mathbf{B}_\epsilon^n\|_2\|d_t\boldsymbol{e}_{\boldsymbol{B}}^n\|\\
&\quad\le \frac{t_n^2\Delta t}{2}\|d_t\boldsymbol{e}_{\boldsymbol{B}}^n\|^2 + Ct_n^2\Delta t\|\mathbf{u}_\epsilon\left(t_n)\|_2^2\|A_{2\epsilon}^{\frac{1}{2}}\boldsymbol{e}_{\boldsymbol{B}}^n\right)\|^2 + Ct_n^2\Delta t\|\mathbf{B}_\epsilon^n\|_2^2\|A_{1\epsilon}^{\frac{1}{2}}\boldsymbol{e}_{\boldsymbol{u}}^n\|^2.
\end{aligned}
$$

Combining the above inequalities, we obtain

$$
\begin{aligned}
&t_n^2(\|d_t\boldsymbol{e}_{\boldsymbol{u}}^n\| + \|d_t\boldsymbol{e}_{\boldsymbol{B}}^n\|)\Delta t + 2t_n^2R_e^2\left(\|A_{1\epsilon}^{\frac{1}{2}}\boldsymbol{e}_{\boldsymbol{u}}^n\|^2 - \|A_{1\epsilon}^{\frac{1}{2}}\boldsymbol{e}_{\boldsymbol{u}}^{n-1}\|^2\right) + 2t_n^2R_m^2\left(\|A_{2\epsilon}^{\frac{1}{2}}\boldsymbol{e}_{\boldsymbol{B}}^n\|^2 - \|A_{2\epsilon}^{\frac{1}{2}}\boldsymbol{e}_{\boldsymbol{B}}^{n-1}\|^2\right)\\
&\le C\Delta t^2\int_{t_{n-1}}^{t_n} t^2(\|\mathbf{u}_{\epsilon tt}\|^2 + \|\mathbf{B}_{\epsilon tt}\|^2)dt + Ct_n^2\Delta t\left(\|\mathbf{u}_\epsilon(t_n)\|_2^2 + \|\mathbf{u}_\epsilon^n\|_2^2 + \|\mathbf{B}_\epsilon^n\|_2^2\right)\|A_{1\epsilon}^{\frac{1}{2}}\boldsymbol{e}_{\boldsymbol{u}}^n\|^2\\
&+Ct_n^2\Delta t(\|\mathbf{B}_\epsilon(t_n)\|_2^2 + \|\mathbf{B}_\epsilon^n\|_2^2 + \|\mathbf{u}_\epsilon(t_n)\|_2^2)\|A_{2\epsilon}^{\frac{1}{2}}\boldsymbol{e}_{\boldsymbol{B}}^n\|^2.
\end{aligned}
$$

Summing (5.16) from 1 to m and using the discrete Gronwall lemma, we have

$$
\Delta t\sum_{n=1}^{m} t_n^2(\|d_t\boldsymbol{e}_{\boldsymbol{u}}^n\|^2 + \|d_t\boldsymbol{e}_{\boldsymbol{B}}^n\|^2) \le C\Delta t^2, \quad \forall m \le N.
$$

From Eq. (5.6) and the available estimates for $\boldsymbol{e}_{\boldsymbol{u}}^n, \boldsymbol{e}_{\boldsymbol{B}}^n$, we can prove

$$
\Delta t\sum_{n=1}^{N} t_n^2\|p(t_n) - p_\epsilon^n\|^2 \le C(\Delta t^2 + \epsilon^2), \tag{5.17}
$$

which is similar to the estimate in the continuous case (see Theorem 4.1).

6 Numerical results

In this section, we present several numerical experiments to illustrate the accuracy and performance of our proposed method. All finite element calculations are carried out by (P_1^b, P_1, P_1^b) finite element pair. The penalty parameter ϵ is selected as $\epsilon = O(h)$.

6.1 Accuracy test

This section will test the convergence rates of our proposed penalty finite element method. We set domain $\Omega = [0, 1]^2$, physical parameters $R_e = R_m = S_c = 1$, and final time $T = 1$. We prescribe the exact solution $(\mathbf{u}, p, \mathbf{B})$ as follows:

$$\begin{cases} u_1 = x^2(x-1)^2y(y-1)(2y-1)\cos(t), \quad u_2 = -y^2(y-1)^2x(x-1)(2x-1)\cos(t), \\ B_1 = \sin(\pi x)\cos(\pi y)\cos(t), \quad B_2 = -\sin(\pi y)\cos(\pi x)\cos(t), \\ p = (2x-1)(2y-1)\cos(t). \end{cases}$$

The time step is chosen by $\Delta t = O(h^2)$. The error of velocity, pressure, and magnetic field are presented in Table 1. We observe that the first-order accuracy for $\|\mathbf{u}-\mathbf{u}_{\epsilon h}^n\|_1$, $\|\mathbf{B}-\mathbf{B}_{\epsilon h}^n\|_1$ and the second-order accuracy asymptotically for $\|\mathbf{u}-\mathbf{u}_{\epsilon h}^n\|$, $\|\mathbf{B}-\mathbf{B}_{\epsilon h}^n\|$, which agree with the theoretical results. Notice that $\|p-p_{\epsilon h}^n\|$ has faster convergence rate than the theoretical result.

6.2 Island coalescence

Let us consider an example of driving magnetic reconnection, the island coalescence problem. Understanding fast magnetic reconnection is one of the important issues in plasma physics. We set two magnetic islands as the initial conditions of the perturbed Harris sheet magnetic field configuration in the island coalescence problem. The combination of the two magnetic islands produces Lorentz forces, pulling the two islands together. The detailed information of physical background of this issue, we can refer to [31].

In this example, we set $\Omega = [-1, 1] \times [-0.5, 0.5]$, $R_e = R_m = 1000$, $S_c = 1$. The source terms are taken as

$$\begin{cases} \mathbf{f} = (0, 0), \\ \mathbf{g} = \left(\frac{2\zeta(1-\tau^2)}{\delta^2} \frac{\sinh(\frac{y}{\delta})}{(\cosh(\frac{y}{\delta})+\tau\cos(\frac{x}{\delta}))^3}, \frac{2\tau\zeta(1-\tau^2)}{\delta^2} \frac{\sin(\frac{x}{\delta})}{(\cosh(\frac{y}{\delta})+\tau\cos(\frac{x}{\delta}))^3} \right). \end{cases}$$

The initial conditions are set as

$$\begin{cases} \mathbf{u}_0 = (0, 0), \\ \mathbf{B}_0 = \left(\frac{\sinh(\frac{y}{\delta})}{\cosh(\frac{y}{\delta})+\tau\cos(\frac{x}{\delta})} + \delta_1, \frac{\tau\sin(\frac{x}{\delta})}{\cosh(\frac{y}{\delta})+\tau\cos(\frac{x}{\delta})} + \delta_2 \right). \end{cases}$$

Table 1 The convergence rates of our scheme at $t_n = 1s$ (2D)

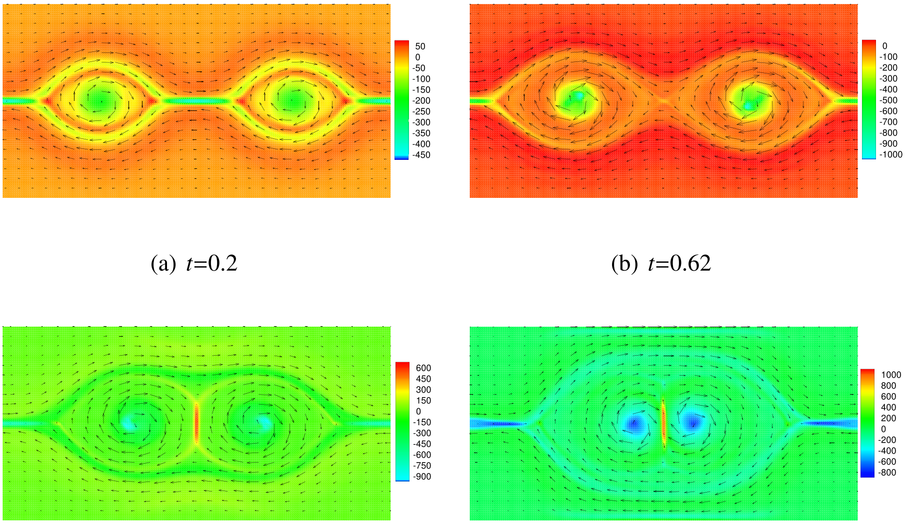
h	$\|\mathbf{u}-\mathbf{u}_{\epsilon h}^n\|$	Ratio	$\|\mathbf{u}-\mathbf{u}_{\epsilon h}^n\|_1$	Ratio	$\|p-p_{\epsilon h}^n\|$	Ratio	$\|\mathbf{B}-\mathbf{B}_{\epsilon h}^n\|$	Ratio	$\|\mathbf{B}-\mathbf{B}_{\epsilon h}^n\|_1$	Ratio
1/8	2.46e−4		5.34e−3		4.68e−3		1.26e−2		3.13e−1	
1/16	6.18e−5	1.99	2.59e−3	1.04	1.36e−3	1.79	3.22e−3	1.97	1.58e−1	0.99
1/32	1.53e−5	2.01	1.28e−3	1.02	4.14e−4	1.71	8.11e−4	1.99	7.90e−2	1.00
1/64	3.80e−6	2.01	6.34e−4	1.01	1.34e−4	1.63	2.03e−4	2.00	3.95e−2	1.00
1/128	9.45e−7	2.01	3.16e−4	1.00	4.51e−5	1.57	5.08e−5	2.00	1.98e−2	1.00

Here $\delta_1 = -\frac{\gamma}{\pi}\cos(\pi x)\sin(\frac{1}{2}\pi y)$, $\delta_2 = \frac{\gamma}{2\pi}\cos(\frac{1}{2}\pi y)\sin(\pi x)$ are perturbations, $\zeta = 1.0$, $\tau = 0.2$, $\delta = \frac{1}{2\pi}$, and $\gamma = -0.01$. The velocity field $\mathbf{u}$ and magnetic field $\mathbf{B}$ are periodic boundary conditions on the left and right boundaries, and zeros tangential stress ($u_2 = 0$) and perfect conducting wall ($B_2 = 0$) on the top and bottom boundaries.

We choose $h = \frac{1}{100}$ and $\Delta t = \frac{1}{5000}$. Figure 1 gives the vector field of the magnetic field $\mathbf{B}$ and the magnitude of the current density J ($J = \text{curl}\mathbf{B}$). We observe the dynamic reconnection behaviors of current density and magnetic island during the coalescence process, and a sharp peak in current density is appeared at the reconnection point. Figure 2 shows the magnitude of pressure at different moments, and we find pressure and magnetic field have the same the coalescence process.

6.3 Hydromagnetic Kelvin-Helmholtz instability

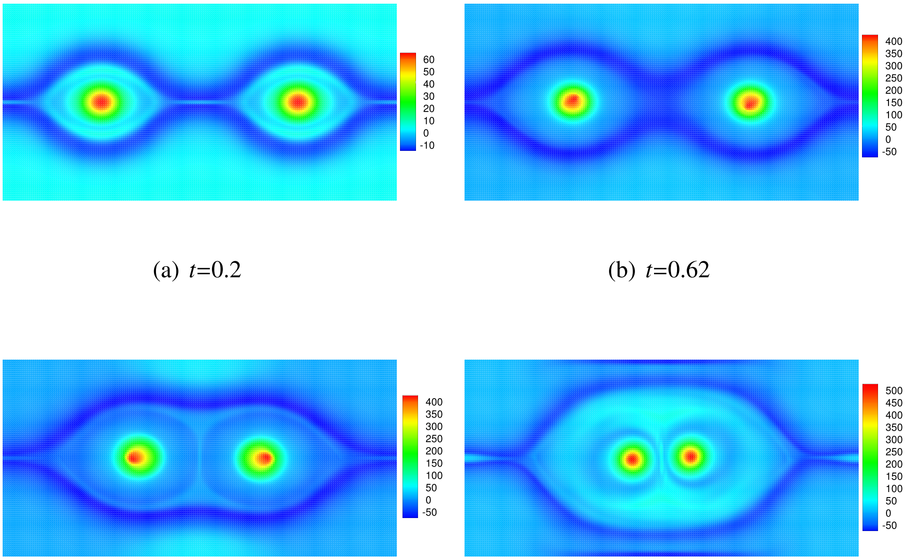
The Kelvin-Helmholtz (K-H) instability in sheared flow configuration is an effective mechanism to initiate fluid mixing, momentum and energy transfer, and turbulence development, we can refer to [1]. This question is of great significance when studying various spaces, astrophysical, and geophysical environments involving shear plasma flows. Including the interface between the solar wind and the magnetosphere, coronal streamers moving through the solar wind and so on. Since most astrophysical environments are conductive, related fluids may be magnetized. Therefore, it is important to understand the role of magnetic field in K-H instability. The detailed information of physical background of this issue, we can refer to [32].



(a) t=0.2 (b) t=0.62

(c) t=0.7 (d) t=0.78

Fig. 1 Snapshots of the magnetic field $\mathbf{B}$ (in arrows) and the magnitude of the current density J (in colormap) at $t = 0.2, 0.62, 0.7, 0.78s$



(a) t=0.2 (b) t=0.62

(c) t=0.7 (d) t=0.78

Fig. 2 Snapshots of the pressure p at $t = 0.2, 0.62, 0.7, 0.78s$

In this example, the domain of our calculation is $\Omega = [0, 2] \times [0, 1]$. The initial velocity is $\mathbf{u}_0 = (1.5, 0)$ in the top half domain and $\mathbf{u}_0 = (-1.5, 0)$ in the bottom half domain. The initial magnetic field is $\mathbf{B}_0 = (\tanh(y/\tau), 0)$, where $\tau = 0.07957747154595$ (see [32]). The boundary conditions of velocity $\mathbf{u}$ are zero tangential stress ($u_2 = 0$) on the top and bottom boundaries, and periodic boundary conditions on the left and right boundaries. The magnetic field $\mathbf{B}$ are $\mathbf{B} \times \mathbf{n} = \mathbf{B}_0 \times \mathbf{n}$ on the top boundary, $\mathbf{B} \times \mathbf{n} = -\mathbf{B}_0 \times \mathbf{n}$ on the bottom boundary, and periodic boundary conditions on the left and right boundaries. We set $R_e = R_m = 1000$, $S_c = 0.095$, $h = \frac{1}{60}$, $\Delta t = \frac{1}{600}$.

Figure 3 shows the contour of the first component B_1 of the magnetic field $\mathbf{B} = (B_1, B_2)$ and the velocity $\mathbf{u}$ at different moments. We observe the profiles of vortexes and the magnetic field show the typical structure of K-H instability, and it deforms and rotates along with the flow. The magnitude of the pressure p at the time corresponding to B_1 is presented in Fig. 4. We find the obtained numerical results are consistent with the experimental results discussed in [33].

6.4 Flow around a cylinder

This example is about the calculation of the flow around a cylinder, which is a well-known problem in [17]. We set the computational domain as $\Omega = [0, 2.2] \times$

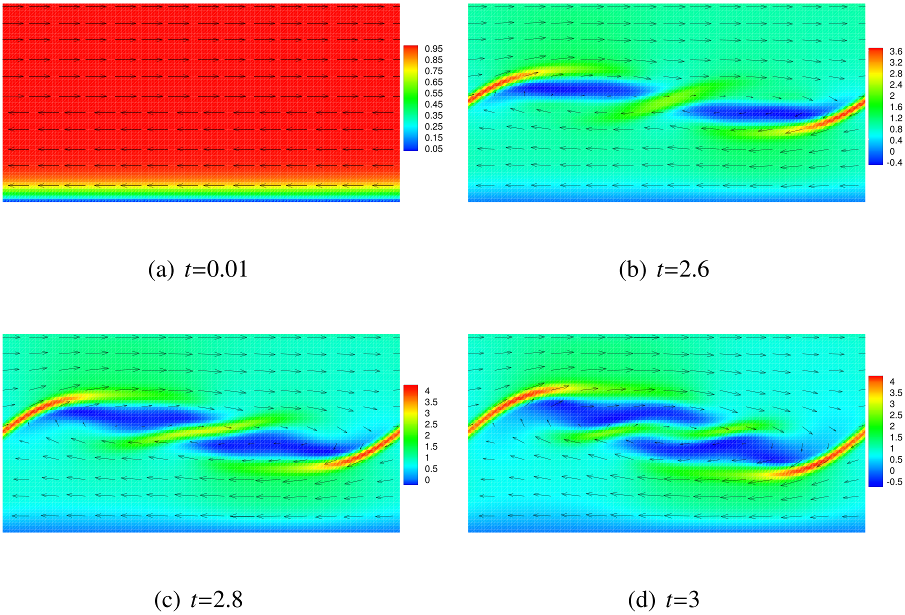


Fig. 3 The velocity $\mathbf{u}$ with the filled contour of B_1 that shows the hydromagnetic K-H instability. Snapshots are taken at $t = 0.01, 2.6, 2.8, 3s$

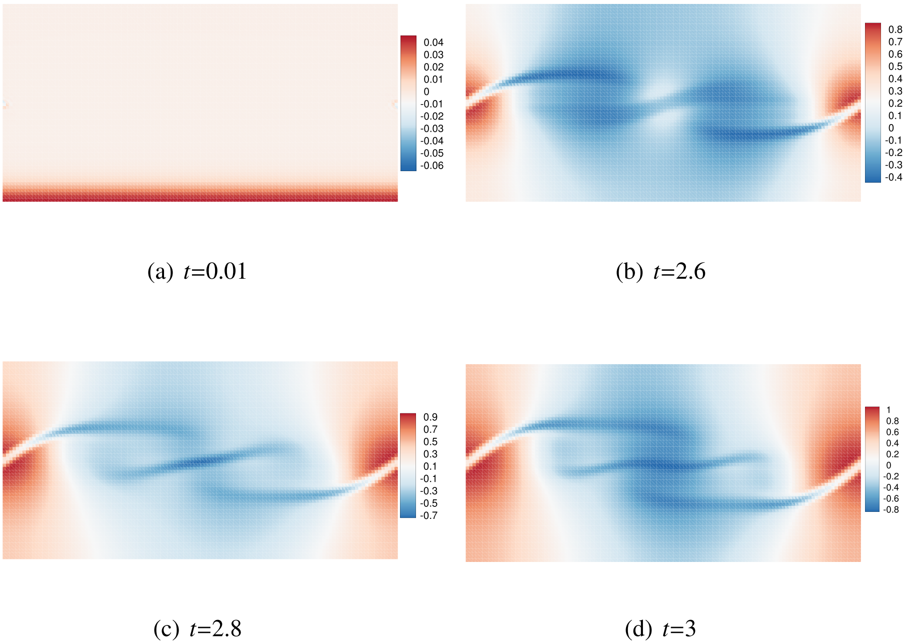


Fig. 4 Snapshots of the pressure p taken at $t = 0.01, 2.6, 2.8, 3s$

$[0, 0.41]$. A disk of radius 0.05 is placed at $(0.2, 0.2)$. We take the initial values $\mathbf{u}_0 = \mathbf{B}_0 = \mathbf{0}$ and the source terms $\mathbf{f} = \mathbf{g} = \mathbf{0}$. The velocity $\mathbf{u}$ are $\mathbf{u} = \left(\frac{6}{0.41^2}\sin(\frac{\pi t}{8})y(0.41 - y), 0\right)$ on the left and right boundaries, and no-slip boundary conditions at the other boundaries.

In Fig. 5, we plot the velocity for the MHD equations without magnetic field with $R_e = 1000$. We find that with the increase of time, the two vortices behind the cylinder gradually separated into vortex streets. The numerical results coincide well with the phenomenon in [17]. In order to show the influence of the magnetic field on the fluid, the boundary condition of magnetic field is $\mathbf{B} \times \mathbf{n} = (1, 0) \times \mathbf{n}$. We set $R_e = 1000$, $R_m = 1$ and change the coupling coefficient S_c to simulate this problem. Figures 6, 7, and 8 describe the plot the velocity at $S_c = 0.5, 1, 100$. We find that with the increase of S_c, the magnetic field inhibits the formation of vortex streets.

6.5 2D-driven cavity flow

In this example, we simulate the 2D-driven cavity flow, we can refer to [31]. Our calculation domain is $\Omega = [0, 1]^2$. We take the initial values $\mathbf{u}_0 = \mathbf{B}_0 = \mathbf{0}$ and the source terms $\mathbf{f} = \mathbf{g} = \mathbf{0}$.

The boundary condition of velocity is $\mathbf{u} = (1, 0)$ on the top side, and no-flow boundary conditions on the bottom, left, and right sides. The boundary condition of magnetic field is $\mathbf{B} \times \mathbf{n} = (-1, 0) \times \mathbf{n}$.

We set $h = \frac{1}{40}$, $\Delta t = \frac{1}{400}$, fix $R_m = S_c = 1$ and change the fluid Reynolds numbers R_e or fix $R_e = R_m = 1$ and change the coupling coefficient S_c to simulate this problem. The velocity field $\mathbf{u}$ under different fluid Reynolds numbers is presented in Fig. 9. The velocity field $\mathbf{u}$ under different coupling coefficient is presented in Fig. 10. We can see the main vortex split into two small vortexes as the fluid Reynolds number increases. The numerical results are similar to the experimental results discussed in [31, 35], which shows that our algorithm is effective.

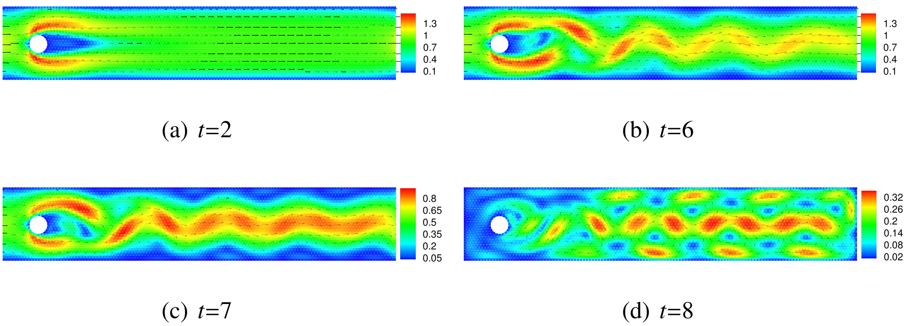
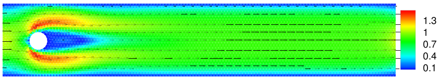
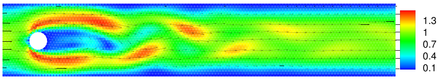


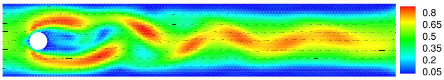
Fig. 5 The velocity $\mathbf{u}$ at $t = 2, 6, 7, 8s$ for the MHD equations without magnetic field with $R_e = 1000$



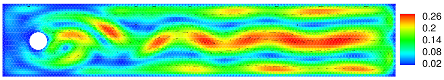
(a) t=2



(b) t=6

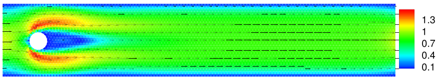


(c) t=7

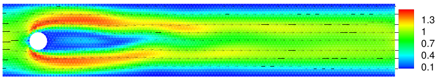


(d) t=8

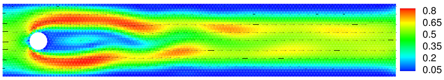
Fig. 6 The velocity **u** at $t = 2, 6, 7, 8s$ with $R_e = 1000$, $R_m = 1$, $S_c = 0.5$



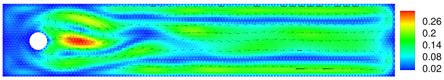
(a) t=2



(b) t=6

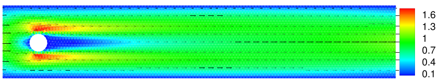


(c) t=7

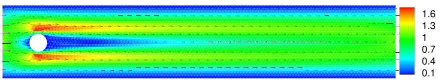


(d) t=8

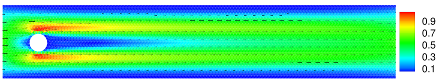
Fig. 7 The velocity **u** at $t = 2, 6, 7, 8s$ with $R_e = 1000$, $R_m = 1$, $S_c = 1$



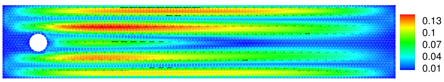
(a) t=2



(b) t=6

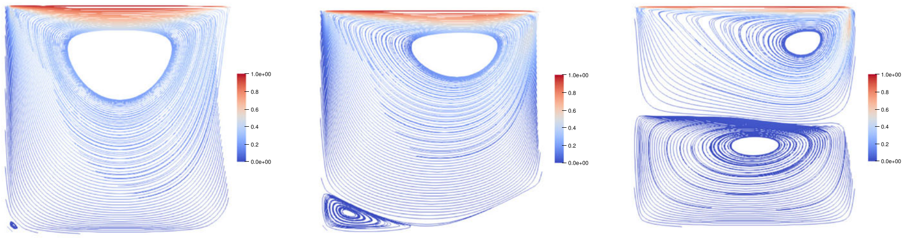


(c) t=7



(d) t=8

Fig. 8 The velocity **u** at $t = 2, 6, 7, 8s$ with $R_e = 1000$, $R_m = 1$, $S_c = 100$



(a) R_e=1 (b) R_e=100 (c) R_e=1000

Fig. 9 The streamlines of velocity at R_e = 1, 100, 1000

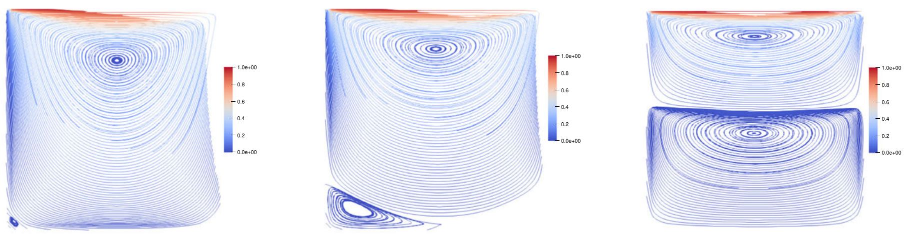
6.6 3D-driven cavity flow

In this example, we test the 3D-driven cavity flow problem, we can refer to [31]. We set a cubic domain $\Omega = [0, 1]^3$. We take the initial values $\mathbf{u}_0 = \mathbf{B}_0 = \mathbf{0}$ and the source terms $\mathbf{f} = \mathbf{g} = \mathbf{0}$. The boundary condition of velocity field is $\mathbf{u} = (1, 0, 0)$ on the top wall, and no-slip boundary conditions on the bottom, front, back, left, and right walls of the domain. The boundary condition of magnetic field is $\mathbf{B} \times \mathbf{n} = (-1, 0, 0) \times \mathbf{n}$ on the walls.

We set $h = \frac{1}{15}$, $\Delta t = \frac{1}{150}$, fixed $R_m = 1$, $S_c = 0.1$, and change the fluid Reynolds numbers R_e to simulate this problem. Figure 11 shows the velocity field $\mathbf{u}$ at plane $y = 0.5$ for fluid Reynolds number $Re = 1, 100, 1000$. We find the bigger R_e, the vortex becomes larger in the cavity.

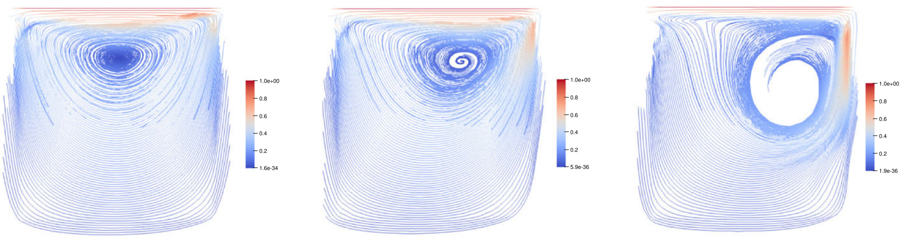
7 Conclusions

In this paper, we present the penalty method for the 2D/3D time-dependent MHD equations. The main idea of this method is to decouple the MHD equations into two equations, one is the equation of velocity and magnetic field $(\mathbf{u}, \mathbf{B})$, and the other is the equation pressure p. What's more, we derive the optimal error estimate of the



(a) S_c=1 (b) S_c=100 (c) S_c=1000

Fig. 10 The streamlines of velocity at S_c = 1, 100, 1000



(a) R_e=1 (b) R_e=100 (c) R_e=1000

Fig. 11 The streamlines of velocity at plane $y = 0.5$ of R_e = 1, 100, 1000

time-discretization of the penalty equations. Several 2D and 3D numerical experiments verify the theoretical result. The fully discrete scheme of the time-dependent MHD equations will be given in our on going work.

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Data availability All data generated or analyzed during this study are included in this published article.

Declarations

Conflict of interest The authors declare no competing interests.

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