Numerical Algorithms (2023) 93:67–83
https://doi.org/10.1007/s11075-022-01403-x

ORIGINAL PAPER

Local and parallel stabilized finite element methods based on two-grid discretizations for the Stokes equations

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Received: 26 July 2022 / Accepted: 19 August 2022 / Published online: 14 September 2022

Abstract
Based on two-grid discretizations, some local and parallel stabilized finite element methods are proposed and investigated for the Stokes problem in this paper. For the finite element discretization, the lowest equal-order finite element pairs are chosen to circumvent the discrete inf-sup condition. In these algorithms, we derive the low-frequency components of the solution for the Stokes problem on a coarse grid and catch the high-frequency components on a fine grid using some local and parallel procedures. Optimal error bounds are demonstrated and some numerical experiments are carried out to support theoretical results.

Keywords Stokes equations · Stabilized finite element method · Two-grid discretizations · Parallel algorithms · Partition of unity

1 Introduction

It is significant but challenging to simulate the motion of the incompressible flow, such as the Stokes problem. It is well known that the inf-sup condition should be satisfied to guarantee the compatibility of the component approximations of the velocity and pressure. Generally speaking, to ensure the inf-sup condition, different spaces for the velocity and pressure are in general considered and investigated in the last decades [20, 33, 36, 40]. Due to the fact that the lowest equal-order finite element pairs are high efficient and convenient for computing, especially in a multigrid context and parallel processing, it is essential to develop some efficient and stable schemes based on the lowest equal-order finite element pairs to study the incompressible flow. On the other hand, the lowest equal-order finite element pairs do not satisfy the inf-sup condition, which usually

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results in nonphysical pressure oscillations in the numerical simulation. Consequently, there have been lots of works [6–8, 22, 23, 32, 37–39, 46] associated with stabilization of the lowest equal-order finite element pairs. The crucial idea of such stabilized schemes is changing the finite element discrete system in order to avoid the inf-sup condition.

As we know, simulation of the incompressible flow often results in large-scale computation needing large computing resources that may only be supported by high-performance parallel computers. Thereby, it is important to devise some parallel algorithms which could be implemented efficiently. In this paper, we focus on developing some local and parallel stabilized finite element methods based on two-grid discretizations for the Stokes problem. Algorithms are motivated by the observation that for a solution of the Stokes problem, low-frequency components could be derived by a relatively coarse grid and high-frequency components could be caught on a fine grid by some local and parallel procedures. This type parallel strategy was firstly proposed by Xu and Zhou for linear and nonlinear elliptic boundary value problems [43, 44], then it was applied for the Stokes problem [20, 21, 33, 36, 45], the Navier-Stokes problem [13, 19, 30, 31, 34, 42], the mixed Stokes-Darcy model [14, 15], the mixed Navier-Stokes-Darcy problem [11, 12, 41], the MHD problem [9, 10, 49], and others [3, 26]. The main superiority of this type parallel strategy is that once the coarse approximation is derived, it requires no data exchange among processors, which makes this method easy to implement with a low communication cost. This is appealing to avoid too much communication cost in today's distributed memory parallel computers.

In this paper, we will consider the lowest equal-order finite element pairs to approximate the velocity and pressure. To offset the lack of discrete inf-sup condition, a local pressure-projection stabilized method based on two local Gauss integrations technique presented in [46] is chosen. This stabilized method has many attractive features, such as the unconditional stability. Firstly, we present and study a local algorithm, then we straightforward generalize this local algorithm to some parallel algorithms. However, the numerical solution by the local and parallel stabilized algorithm is global discontinuous. To improve this algorithm, we study on two steps, introducing the partition of unity and adding a coarse mesh correction. Together with the partition of unity technique, the global continuous solution is derived. Furthermore, constructing a coarse mesh correction could improve the smoothness of the numerical solution. Finally, for these proposed algorithms, optimal error bounds are given, and some numerical experiments are presented to support the theoretical findings.

We organize this paper as follows. In the following section, the Stokes problem is introduced and some preliminaries are presented. In Section 3, we present and analyze some local and parallel stabilized finite element methods. In Section 4, some numerical examples are reported to demonstrate the feasibility and effectiveness of the proposed algorithms.

2 The Stokes model

In this paper, the standard notations for Sobolev space $W^{m,p}(\Omega)^d$ ($d=2,3$) and corresponding norm $\|\cdot\|_{m,p,\Omega}$ used in [1] will be inherited. For $p=2$, we denote $H^s(\Omega)^d = W^{m,2}(\Omega)^d$, thus the norm can be written as $\|\cdot\|_{s,\Omega} = \|\cdot\|_{H^s(\Omega)}$. For m

$=0$, we denote $L^p(\Omega)^d = W^{0,p}(\Omega)^d$ and $\|\cdot\|_{0,\Omega} = \|\cdot\|_{L^2(\Omega)}$ when $p=2$. The space $H^{-1}(\Omega)^d$ with its dual space $H_0^1(\Omega)^d$, and its corresponding norm $\|\cdot\|_{-1,\Omega}$ will be used.

Let Ω be a bounded domain in $\mathbb{R}^d$ $(d = 2, 3)$. The Stokes model we consider in this paper is

$$\begin{aligned} -\nu\Delta u + \nabla p &= f \qquad \text{in } \Omega,\\ \operatorname{div} u &= 0 \qquad \text{in } \Omega,\\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

where u denotes the fluid velocity, p denotes the fluid pressure, ν is the kinematic viscosity, and f is the source term.

Define the following spaces as

$$X = H_0^1(\Omega)^d, \quad Q = L_0^2(\Omega) = \left\{ q \in L^2(\Omega) : \int_\Omega q dx = 0 \right\}.$$

The weak formulation of (2.1) is to seek a pair of $[u,p] \in X \times Q$ such that

$$\begin{aligned} a(u,v) - b(v,p) &= (f,v) \qquad \forall v \in X,\\ b(u,q) &= 0 \qquad \forall q \in Q, \end{aligned} \tag{2.2}$$

where

$$a(u,v) = \nu(\nabla u, \nabla v), \qquad b(v,p) = (\operatorname{div} v, p),$$

and $(\cdot,\cdot)$ represents the standard inner-product in $L^2(\Omega)$.
For the bilinear forms, the following continuous properties hold

$$\begin{aligned} a(u,v) &\le c\|u\|_{1,\Omega}\|v\|_{1,\Omega} \qquad \forall u, v \in X,\\ b(v,p) &\le c\|v\|_{1,\Omega}\|p\|_{0,\Omega}, \qquad \forall [v,p] \in X \times Q, \end{aligned}$$

where the letter c is a generic positive constant which is independent of the mesh size and may stand for different values at different places. For convenience, hereafter we shall use $x \lesssim y$ to denote $x \le cy$ in the rest of the paper.

Assume the domain Ω be regular enough such that the unique solution $[w,r] \in X \times Q$ satisfies the following steady Stokes system

$$\begin{aligned} -\nu\Delta w + \nabla r &= g \qquad \text{in } \Omega,\\ \operatorname{div} w &= 0 \qquad \text{in } \Omega,\\ w &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{2.3}$$

where $g \in L^2(\Omega)^d$, then there exists

$$\|w\|_{1,\Omega} + \|r\|_{0,\Omega} \lesssim \|g\|_{0,\Omega}. \tag{2.4}$$

Assume that $\tau_h = \{K\}$ is the triangulation of the domain Ω. Let $h = \max_{K\in\tau_h} \operatorname{diam}(K) =$ the longest edge of $\overline{K}$ denote the size of mesh. Define the following finite spaces as

$$X_h(\Omega) = \left\{v \in C^0(\Omega)^d : v|_K \in P_1(K)^d, \forall K \in \tau_h\right\},$$
$$Q_h(\Omega) = \left\{q \in C^0(\Omega) : q|_K \in P_1(K), \forall K \in \tau_h\right\},$$

where $P_1(K)$ denotes the space of piecewise linear polynomial on the element K. Furthermore, we set

$$X_h^0(\Omega) = X_h(\Omega) \cap X, \qquad Q_h^0(\Omega) = Q_h(\Omega) \cap Q.$$

Since we choose the lowest equal-order element pairs to approximate the fluid velocity and the pressure, respectively, the well-known inf-sup condition is not established any more. To solve this problem, a stabilized term is introduced as that in [46]. The L^2-projection is defined as $\Pi : X \to R_0$ with R_0 is the space of piecewise constant on the element K, which has the following properties

$$(p, q) = (\Pi p, q) \qquad \forall p \in L^2(\Omega), q \in R_0, \tag{2.5}$$

$$\|\Pi p\|_{0,\Omega} \le \|p\|_{0,\Omega} \qquad \forall p \in L^2(\Omega), \tag{2.6}$$

$$\|(I - \Pi)p\|_{0,\Omega} \lesssim h\|p\|_{1,\Omega} \qquad \forall p \in H^1(\Omega). \tag{2.7}$$

Define the stabilized term as

$$\begin{aligned} G(p_h, q_h) &= \alpha((I - \Pi)p_h, (I - \Pi)q_h), \\ &= \alpha \sum_{K \in \tau_h(\Omega)} \left(\int_{K,2} p_h q_h dx - \int_{K,1} p_h q_h dx \right), \qquad \forall p_h, q_h \in Q_h^0(\Omega), \end{aligned}$$

where $\int_{K,i} \cdot \, dx$ represents the local Gauss integral in the element K with the polynomials degree i, $i = 1,2$, α is the stabilization parameter satisfies $0 < \alpha < 1$.

With this representation, the stabilized finite element approximation for the Stokes problem is described as follows: Find $[u_h, p_h] \in X_h^0(\Omega) \times Q_h^0(\Omega)$ such that

$$\begin{aligned} a(u_h, v_h) - b(v_h, p_h) &= (f, v_h) \qquad \forall v_h \in X_h^0(\Omega), \\ b(u_h, q_h) + G(p_h, q_h) &= 0 \qquad \forall q_h \in Q_h^0(\Omega). \end{aligned} \tag{2.8}$$

For the sake of simplicity, we set

$$\mathscr{B}(u, p; v, q) = a(u, v) - b(v, p) + b(u, q) + G(p, q) \qquad \forall [v, q] \in X \times Q,$$

it leads to

$$\mathscr{B}(u_h, p_h; v_h, q_h) = (f, v_h) \qquad \forall [v_h, q_h] \in X_h^0(\Omega) \times Q_h^0(\Omega). \tag{2.9}$$

For the problem (2.8), its well-posedness could be found in [22], and there exist the prior error estimates for the stabilized approximate solutions [46]

$$\begin{aligned} \|u - u_h\|_{1,\Omega} + \|p - p_h\|_{0,\Omega} &\lesssim h(\|u\|_{2,\Omega} + \|p\|_{1,\Omega}), \\ \|u - u_h\|_{0,\Omega} + \|p - p_h\|_{-1,\Omega} &\lesssim h^2(\|u\|_{2,\Omega} + \|p\|_{1,\Omega}). \end{aligned} \tag{2.10}$$

Lemma 2.1 For the bilinear form $\mathscr{B}$, there hold

$$|\mathscr{B}(u,p;v,q)| \lesssim |||[u,p]|||_{\Omega}|||[v,q]|||_{\Omega} \qquad \forall [u,p],\ [v,q] \in X \times Q,$$
$$|||[u_h,p_h]|||_{\Omega} \lesssim \sup_{[v_h,q_h]\in X_h^0(\Omega)\times Q_h^0(\Omega)} \frac{|\mathscr{B}(u_h,p_h;v_h,q_h)|}{|||[v_h,q_h]|||_{\Omega}} \qquad \forall [u_h,p_h] \in X_h^0(\Omega)\times Q_h^0(\Omega),$$

where $|||[u,p]|||_{\Omega} = \|u\|_{1,\Omega} + \|p\|_{0,\Omega}$.

3 Local and parallel stabilized finite element methods

In this section, we shall propose some local and parallel stabilized finite element methods based on two-grid discretizations for the Stokes system. Firstly, we study on a local algorithm, then we generalize it to some parallel algorithms. The corresponding error estimates are obtained subsequently.

3.1 Local algorithm

To describe the following algorithm, we shall introduce some useful notations. Let H denote the size of coarse mesh, and h denotes the size of fine mesh such that $h \ll H$. For a subdomain D, we enlarge it to Ω_0 such that $D \subset\subset \Omega_0 \subset\subset \Omega$ (for $D \subset G \subset \Omega$, $D \subset\subset G$ is used to mean that $\mathrm{dist}(\partial D \setminus \partial\Omega, \partial G \setminus \partial\Omega) > 0$). We assume that Ω_0 aligns with $\tau_H(\Omega)$, $\tau_h(\Omega_0)$ is a regular triangulation on Ω_0. For a global regular triangulation $\tau_h(\Omega)$ that aligns with $\tau_H(\Omega)$, we set $\tau_h(\Omega_0) = \tau_h(\Omega)|_{\Omega_0}$. Similarly, we could define $X_h(\Omega_0)$, $Q_h(\Omega_0)$, $X_h^0(\Omega_0)$, $Q_h^0(\Omega_0)$ on $\tau_h(\Omega_0)$.

Algorithm 0

Step 1. Seek a global coarse solution $[u_H, p_H] \in X_H^0(\Omega) \times Q_H^0(\Omega)$ such that

$$\mathscr{B}(u_H,p_H;v_H,q_H) = (f, v_H) \qquad \forall [v_H,q_H] \in X_H^0(\Omega) \times Q_H^0(\Omega).$$

Step 2. Correct a local fine grid residual $[e_h, \epsilon_h] \in X_h^0(\Omega_0) \times Q_h^0(\Omega_0)$ such that

$$\mathscr{B}(e_h,\epsilon_h;v_h,q_h) = (f, v_h) - \mathscr{B}(u_H,p_H;v_h,q_h) \qquad \forall [v_h,q_h] \in X_h^0(\Omega_0) \times Q_h^0(\Omega_0).$$

Step 3. Set $[u^h, p^h] = [u_H + e_h, p_H + \epsilon_h]$ in D.

For the error estimates of approximate solution, we now describe local a prior estimate for problem (2.9) directly. The reader is referred to [46] for the proof.

Lemma 3.1 Assume $D \subset\subset \Omega_0 \subset \Omega$, $\alpha = \mathcal{O}(h)$, for the solution $[w_h, r_h] \in X_h(\Omega) \times Q_h(\Omega)$, $g \in H^{-1}(\Omega)^d$ satisfying

$$a(w_h,v_h) - b(v_h,r_h) = (g, v_h) \quad \forall v_h \in X_h^0(\Omega_0),$$
$$b(w_h,q_h) + G(r_h,q_h) = 0 \qquad \forall q_h \in Q_h^0(\Omega_0),$$

then there holds

$$\|w_h\|_{1,D} + \|r_h\|_{0,D} \lesssim \|w_h\|_{0,\Omega_0} + \|r_h\|_{-1,\Omega_0} + \|g\|_{-1,\Omega_0}.$$

Theorem 3.1 Assume that $[u^h, p^h]$ is the solution obtained by *Algorithm* 0, then there holds

$$\|u - u^h\|_{1,D} + \|p - p^h\|_{0,D} \lesssim h + H^2. \tag{3.1}$$

Proof. With a simple calculation, we have the following equation of the local algorithm

$$\mathscr{B}(u^h, p^h; v_h, q_h) = (f, v_h) \qquad \forall [v_h, q_h] \in X_h^0(\Omega_0) \times Q_h^0(\Omega_0). \tag{3.2}$$

Subtracting (3.2) from (2.9) results in

$$\mathscr{B}(u_h - u^h, p_h - p^h; v_h, q_h) = 0 \qquad \forall [v_h, q_h] \in X_h^0(\Omega_0) \times Q_h^0(\Omega_0). \tag{3.3}$$

Thanks to Lemma 3.1, we get

$$\begin{aligned} &\|u_h - u^h\|_{1,D} + \|p_h - p^h\|_{0,D} \\ \lesssim\ &\|u_h - u^h\|_{0,\Omega_0} + \|p_h - p^h\|_{-1,\Omega_0} \\ \leq\ &\|u_h - u_H\|_{0,\Omega_0} + \|p_h - p_H\|_{-1,\Omega_0} + \|e_h\|_{0,\Omega_0} + \|\epsilon_h\|_{-1,\Omega_0}. \end{aligned} \tag{3.4}$$

To estimate $\|e_h\|_{0,\Omega_0}$ and $\|\epsilon_h\|_{-1,\Omega_0}$, we introduce a dual Stokes problem as follows. Find $[w, r] \in (H_0^1(\Omega_0)^d \cap H^2(\Omega_0)^d) \times (L_0^2(\Omega_0) \cap H^1(\Omega_0))$, for $[g, \rho] \in L^2(\Omega_0)^d \times (L_0^2(\Omega_0) \cap H^1(\Omega_0))$ such that

$$a(v, w) - b(w, q) + b(v, r) + G(q, r) = (g, v) + (\rho, q) \qquad \forall [v, q] \in H_0^1(\Omega_0)^d \times L_0^2(\Omega_0). \tag{3.5}$$

Based on the regularity of triangulation, we obtain

$$\|w\|_{2,\Omega_0} + \|r\|_{1,\Omega_0} \lesssim \|g\|_{0,\Omega_0} + \|\rho\|_{1,\Omega_0}. \tag{3.6}$$

The standard finite element method for the dual Stokes (3.5) is to find $[w_\mu, r_\mu] \in X_\mu^0(\Omega_0) \times Q_\mu^0(\Omega_0)$ $(\mu = h, H)$ such that

$$a(v, w_\mu) - b(w_\mu, q) + b(v, r_\mu) + G(q, r_\mu) = (g, v) + (\rho, q) \qquad \forall [v, q] \in X_\mu^0(\Omega_0) \times Q_\mu^0(\Omega_0). \tag{3.7}$$

Combining (3.5) with (3.7) yields

$$a(v, w - w_\mu) - b(w - w_\mu, q) + b(v, r - r_\mu) + G(q, r - r_\mu) = 0 \qquad \forall [v, q] \in X_\mu^0(\Omega_0) \times Q_\mu^0(\Omega_0). \tag{3.8}$$

Based on (2.10) and (3.6), we have

$$\begin{aligned} \|w - w_\mu\|_{1,\Omega_0} + \|r - r_\mu\|_{0,\Omega_0} &\lesssim \mu(\|w\|_{2,\Omega_0} + \|r\|_{1,\Omega_0}) \\ &\lesssim \mu(\|g\|_{0,\Omega_0} + \|\rho\|_{1,\Omega_0}). \end{aligned} \tag{3.9}$$

From (2.9), for $[u_\mu, p_\mu] \in X_\mu^0(\Omega_0) \times Q_\mu^0(\Omega_0)$, $\forall [w_H, r_H] \in X_H^0(\Omega_0) \times Q_H^0(\Omega_0)$, it is easy to verify that

$$a(u_h - u_H, w_H) - b(w_H, p_h - p_H) + b(u_h - u_H, r_H) + G(p_h - p_H, r_H) = 0. \tag{3.10}$$

Set $[v,q] = [e_h,\epsilon_h]$ in (3.5), together with (3.3), (3.8), and (3.10), we get

$$\begin{aligned}
&(g, e_h) + (\rho, \epsilon_h) \\
&= a(e_h, w) - b(w, \epsilon_h) + b(e_h, r) + G(\epsilon_h, r) \\
&= a(e_h, w_h) - b(w_h, \epsilon_h) + b(e_h, r_h) + G(\epsilon_h, r_h) \\
&= a(u_h - u_H, w_h) - b(w_h, p_h - p_H) + b(u_h - u_H, r_h) + G(p_h - p_H, r_h) \\
&= a(u_h - u_H, w_h - w) - b(w_h - w, p_h - p_H) + b(u_h - u_H, r_h - r) + G(p_h - p_H, r_h - r) \\
&\quad + a(u_h - u_H, w - w_H) - b(w - w_H, p_h - p_H) + b(u_h - u_H, r - r_H) + G(p_h - p_H, r - r_H).
\end{aligned}$$

From (2.10) and (3.9), there holds

$$\begin{aligned}
&\left|(g, e_h) + (\rho, \epsilon_h)\right| \\
&\lesssim (\|u_h - u_H\|_{1,\Omega_0} + \|p_h - p_H\|_{0,\Omega_0})(\|w - w_h\|_{1,\Omega_0} + \|r - r_h\|_{0,\Omega_0} + \|w - w_H\|_{1,\Omega_0} + \|r - r_H\|_{0,\Omega_0}) \\
&\lesssim H(\|u_h - u_H\|_{1,\Omega_0} + \|p_h - p_H\|_{0,\Omega_0})(\|g\|_{0,\Omega_0} + \|\rho\|_{1,\Omega_0}) \\
&\lesssim H^2(\|g\|_{0,\Omega_0} + \|\rho\|_{1,\Omega_0}),
\end{aligned}$$

which leads to

$$\|e_h\|_{0,\Omega_0} + \|\epsilon_h\|_{-1,\Omega_0} \lesssim H^2.$$

Therefore, we get

$$\|u_h - u^h\|_{1,D} + \|p_h - p^h\|_{0,D} \lesssim H^2.$$

Using the triangle inequality and (2.10), (3.1) is obtained.

3.2 Parallel algorithms

To introduce some parallel algorithms, we divide the domain Ω into a series of disjoint subdomains $D_i, i = 1,2,\ldots,m$, then we enlarge every D_i to Ω_i such that $D_i \subset\subset \Omega_i$. For each Ω_i, we assume that it aligns with $\tau_H(\Omega)$.

Algorithm 1

Step 1. Seek a global coarse solution $[u_H, p_H] \in X_H^0(\Omega) \times Q_H^0(\Omega)$ such that

$$\mathscr{B}(u_H, p_H; v_H, q_H) = (f, v_H) \qquad \forall [v_H, q_H] \in X_H^0(\Omega) \times Q_H^0(\Omega).$$

Step 2. Correct local fine grid residuals $[e_h^i, \epsilon_h^i] \in X_h^0(\Omega_i) \times Q_h^0(\Omega_i), i = 1, 2, \ldots, m$, such that

$$\mathscr{B}(e_h^i, \epsilon_h^i; v_h, q_h) = (f, v_h) - \mathscr{B}(u_H, p_H; v_h, q_h) \qquad \forall [v_h, q_h] \in X_h^0(\Omega_i) \times Q_h^0(\Omega_i).$$

Step 3. Set $[u^h, p^h] = [u_H + e_h^i, p_H + \epsilon_h^i]$ in D_i.

Define the following piecewise norms as

$$|||u-u^h|||_{1,\Omega} = \sqrt{\sum_{i=1}^{m} \|u-u^h\|^2_{1,D_i}},$$
$$|||p-p^h|||_{0,\Omega} = \sqrt{\sum_{i=1}^{m} \|p-p^h\|^2_{0,D_i}}.$$

From what have been discussed above, we may easily get the following error estimate.

Theorem 3.2 Assume that $[u^h,p^h]$ is the solution obtained by *Algorithm* 1, then we obtain the following inequality

$$|||u-u^h|||_{1,\Omega} + |||p-p^h|||_{0,\Omega} \lesssim h + H^2. \tag{3.11}$$

Proof. From Theorem 3.1, we have

$$\|u-u^h\|_{1,D_i} + \|p-p^h\|_{0,D_i} \lesssim h + H^2 \qquad i = 1, 2, \cdots, m.$$

Based on the definition of the piecewise norms, (3.11) is derived.

However, it is obvious that the numerical solution by *Algorithm* 1 is global discontinuous since it is piecewise defined. In the following, by introducing the partition of unity method, we derive a global continuous solution. Suppose that $\{\Omega_i\}_{i=1}^m$ is the open cover of Ω, $\{\phi_i\}_{i=1}^m$ is the partition of unity subordinate to $\{\Omega_i\}_{i=1}^m$ which satisfies

$$\begin{aligned} &\operatorname{supp} \phi_i \subset \overline{\Omega_i} && i = 1, \dots, m, \\ &\sum_{i=1}^m \phi_i = 1 && \text{on } \Omega, \\ &\|\phi_i\|_{L^\infty(\mathbb{R}^n)} \le C && i = 1, \dots, m. \end{aligned}$$

To construct a partition of unity, we need to get a regular triangulation τ_{h_p}. For convenience, we fix h_p such that $h < H \le h_p$, and h_p is independent of h, H. The choice for partition of unity functions is arbitrary, and we fix it as a continuous and piecewise linear Lagrange basis function which satisfies $\phi_i(x_j) = \delta_{i,j}$, $\forall x_j \in \tau_{h_p}$ in this paper.

Algorithm 2

Step 1. Seek a global coarse solution $[u_H, p_H] \in X_H^0(\Omega) \times Q_H^0(\Omega)$ such that

$$\mathscr{B}(u_H, p_H; v_H, q_H) = (f, v_H) \qquad \forall [v_H, q_H] \in X_H^0(\Omega) \times Q_H^0(\Omega).$$

Step 2. Correct local fine grid residuals $[e_h^i, \epsilon_h^i] \in X_h^0(\Omega_i) \times Q_h^0(\Omega_i), i = 1, 2, ..., m$, such that

$$\mathscr{B}(e_h^i, \epsilon_h^i; v_h, q_h) = (f, v_h) - \mathscr{B}(u_H, p_H; v_h, q_h) \qquad \forall [v_h, q_h] \in X_h^0(\Omega_i) \times Q_h^0(\Omega_i).$$

Step 3. Set $[u_i^h, p_i^h] = [u_H + e_h^i, p_H + \epsilon_h^i]$ in Ω_i.

Step 4. Derive the global continuous solution as

$$[\tilde{u}^h, \tilde{p}^h] = \left[\sum_{i=1}^{m} \phi_i u_i^h, \sum_{i=1}^{m} \phi_i p_i^h \right].$$

Step 5. Seek a coarse grid correction $[e_H, \epsilon_H] \in X_H^0(\Omega) \times Q_H^0(\Omega)$ satisfying

$$\mathscr{B}(e_H, \epsilon_H; v_H, q_H) = (f, v_H) - \mathscr{B}(\tilde{u}^h, \tilde{p}^h; v_H, q_H) \qquad \forall [v_H, q_H] \in X_H^0(\Omega) \times Q_H^0(\Omega).$$

Step 6. Obtain the final approximate solution

$$[u_H^h, p_H^h] = [\tilde{u}^h + e_H, \tilde{p}^h + \epsilon_H].$$

Theorem 3.3 Assume that $[\tilde{u}^h, \tilde{p}^h] \in X_h^0(\Omega) \times Q_h^0(\Omega)$ is the solution obtained by *Step* 1 to *Step* 4 of *Algorithm* 2, then

$$\|u - \tilde{u}^h\|_{1,\Omega} + \|p - \tilde{p}^h\|_{0,\Omega} \lesssim h + H^2. \tag{3.12}$$

Proof. It is obvious that

$$u_h = (\sum_{i=1}^{m} \phi_i) u_h = \sum_{i=1}^{m} \phi_i u_h, \qquad p_h = (\sum_{i=1}^{m} \phi_i) p_h = \sum_{i=1}^{m} \phi_i p_h.$$

Therefore, we have

$$\begin{aligned}
&\|u_h - \tilde{u}^h\|_{1,\Omega} + \|p_h - \tilde{p}^h\|_{0,\Omega} \\
&= \| \sum_{i=1}^{m} \phi_i u_h - \sum_{i=1}^{m} \phi_i u_i^h \|_{1,\Omega} + \| \sum_{i=1}^{m} \phi_i p_h - \sum_{i=1}^{m} \phi_i p_i^h \|_{0,\Omega} \\
&= \| \sum_{i=1}^{m} \phi_i (u_h - u_i^h) \|_{1,\Omega} + \| \textstyle\sum_{i=1}^{m} \phi_i (p_h - p_i^h) \|_{0,\Omega} \\
&\lesssim \sum_{i=1}^{m} \|\phi_i (u_h - u_i^h)\|_{1,\Omega_i} + \textstyle\sum_{i=1}^{m} \|\phi_i (p_h - p_i^h)\|_{0,\Omega_i} \\
&\lesssim \sum_{i=1}^{m} \|\phi_i\|_{L^\infty(\Omega)} (\|u_h - u_i^h\|_{1,\Omega_i} + \|p_h - p_i^h\|_{0,\Omega_i}) \\
&\lesssim \|u_h - u_i^h\|_{1,\Omega_i} + \|p_h - p_i^h\|_{0,\Omega_i} \\
&\lesssim H^2.
\end{aligned}$$

Combining the triangle inequality with (2.10) yields

$$\|u - \tilde{u}^h\|_{1,\Omega} + \|p - \tilde{p}^h\|_{0,\Omega} \lesssim h + H^2.$$

We finish the proof.

Theorem 3.4 Assume that $[u_H^h, p_H^h] \in X_h^0(\Omega) \times Q_h^0(\Omega)$ is the solution derived from *Algorithm* 2, then the following estimates hold

$$\|u - u_H^h\|_{1,\Omega} + \|p - p_H^h\|_{0,\Omega} \lesssim h + H^2, \tag{3.13}$$

$$\|u-u_H^h\|_{0,\Omega} \lesssim h^2 + H^3. \tag{3.14}$$

Proof. We introduce the following projection operator $(R_H, L_H) : (X, Q) \to (X_H^0, Q_H^0)$ such that

$$\begin{array}{l} a(v_H, u - R_H u) - b(u - R_H u, q_H) + b(v_H, p - L_H p) \\ +G(q_H, p - L_H p) = 0 \qquad \forall [v_H, q_H] \in X_H^0(\Omega) \times Q_H^0(\Omega). \end{array} \tag{3.15}$$

Similar to Corollary 3.5 in [43], the property of projection operator is described as follows

$$\|u - R_H u\|_{1,\Omega} + \|p - L_H p\|_{0,\Omega} \lesssim H(\|u\|_{2,\Omega} + \|p\|_{1,\Omega}). \tag{3.16}$$

From *Algorithm* 2, we know

$$\begin{array}{l} a(e_H, R_H v) - b(R_H v, \epsilon_H) + b(e_H, L_H q) + G(\epsilon_H, L_H q) \\ = (f, R_H v) - [a(\tilde{u}^h, R_H v) - b(R_H v, \tilde{p}^h) + b(\tilde{u}^h, L_H q) + G(\tilde{p}^h, L_H q)] \qquad \forall [v, q] \in X_h^0(\Omega) \times Q_h^0(\Omega). \end{array} \tag{3.17}$$

Combining (3.15) with (3.17), we obtain

$$\begin{array}{l} a(e_H, v) - b(v, \epsilon_H) + b(e_H, q) + G(\epsilon_H, q) \\ = (f, R_H v) - [a(\tilde{u}^h, R_H v) - b(R_H v, \tilde{p}^h) + b(\tilde{u}^h, L_H q) + G(\tilde{p}^h, L_H q)] \qquad \forall [v, q] \in X_h^0(\Omega) \times Q_h^0(\Omega). \end{array} \tag{3.18}$$

Taking $[v_h, q_h] = [R_H v, L_H q]$ into (2.9), then

$$a(u_h, R_H v) - b(R_H v, p_h) + b(u_h, L_H q) + G(p_h, L_H q) = (f, R_H v) \qquad \forall [v, q] \in X_h^0(\Omega) \times Q_h^0(\Omega). \tag{3.19}$$

Together with (3.18), we obtain

$$\begin{array}{l} a(e_H, v) - b(v, \epsilon_H) + b(e_H, q) + G(\epsilon_H, q) \\ = a(u_h - \tilde{u}^h, R_H v) - b(R_H v, p_h - \tilde{p}^h) + b(u_h - \tilde{u}^h, L_H q) + G(p_h - \tilde{p}^h, L_H q) \quad \forall [v, q] \in X_h^0(\Omega) \times Q_h^0(\Omega). \end{array} \tag{3.20}$$

Based on the triangle inequality and Theorem 3.3, there holds

$$\begin{aligned} \|u_h - u_H^h\|_{1,\Omega} + \|p_h - p_H^h\|_{0,\Omega} &\leq \|u_h - \tilde{u}^h\|_{1,\Omega} + \|p_h - \tilde{p}^h\|_{0,\Omega} + \|e_H\|_{1,\Omega} + \|\epsilon_H\|_{0,\Omega} \\ &\lesssim H^2 + \|e_H\|_{1,\Omega} + \|\epsilon_H\|_{0,\Omega}. \end{aligned}$$

From Lemma 2.1, Theorem 3.3 and (3.20), we have

$$\begin{aligned} \|e_H\|_{1,\Omega} + \|\epsilon_H\|_{0,\Omega} &\lesssim \sup_{[v,q] \in X_h^0 \times Q_h^0} \frac{|\mathscr{B}(e_H, \epsilon_H; v, q)|}{|||[v,q]|||_\Omega} \\ &= \sup_{[v,q] \in X_h^0 \times Q_h^0} \frac{|\mathscr{B}(u_h - \tilde{u}^h, p_h - \tilde{p}^h; R_H v, L_H q)|}{|||[v,q]|||_\Omega} \\ &\lesssim \|u_h - \tilde{u}^h\|_{1,\Omega} + \|p_h - \tilde{p}^h\|_{0,\Omega} \\ &\lesssim H^2. \end{aligned}$$

With the triangle inequality and (2.10), (3.13) is derived.
To estimate $\|u - u_H^h\|_{0,\Omega}$, for $u_h - u_H^h \in L^2(\Omega)$, we introduce a dual problem to find $[\Phi, \Psi] \in X \times Q$ such that

$$a(v,\Phi) - b(\Phi,q) + b(v,\Psi) + G(q,\Psi) = (v, u_h - u_H^h) \qquad \forall [v,q] \in X \times Q. \tag{3.21}$$

Since the triangulation is regular, there holds

$$\|\Phi\|_{2,\Omega} + \|\Psi\|_{1,\Omega} \lesssim \|u_h - u_H^h\|_{0,\Omega}. \tag{3.22}$$

Taking $[v,q] = [u_h - u_H^h, p_h - p_H^h]$ into (3.21) yields

$$\begin{aligned}(u_h - u_H^h, u_h - u_H^h) &= \|u_h - u_H^h\|_{0,\Omega}^2 \\ &= a(u_h - u_H^h, \Phi) - b(\Phi, p_h - p_H^h) + b(u_h - u_H^h, \Psi) + G(p_h - p_H^h, \Psi) \\ &= a(u_h - u_H^h, (I - R_H)\Phi) - b((I - R_H)\Phi, p_h - p_H^h) + b(u_h - u_H^h, (I - L_H)\Psi) \\ &\quad + G(p_h - p_H^h, (I - L_H)\Psi) + a(u_h - u_H^h, R_H\Phi) - b(R_H\Phi, p_h - p_H^h) \\ &\quad + b(u_h - u_H^h, L_H\Psi) + G(p_h - p_H^h, L_H\Psi).\end{aligned}$$

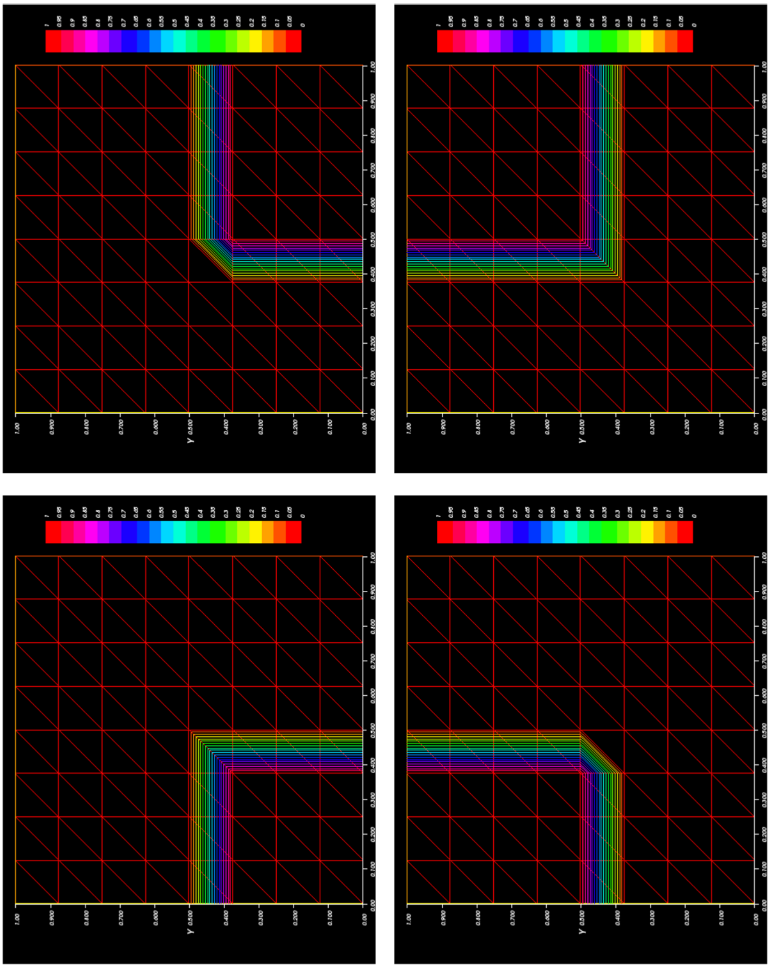


Fig. 1 The partition of unity functions

Table 1 The errors of velocity in H^1 norm by SFEM, *Algorithms* 1, 2

$1/h$	$\|u-u_h\|_{1,\Omega}$	*rate*	$\|\|\|u-u^h\|\|\|_{1,\Omega}$	*rate*	$\|u-u_H^h\|_{1,\Omega}$	*rate*
64	1.25162×10^{-2}	–	1.24454×10^{-2}	–	1.25152×10^{-2}	–
144	5.54700×10^{-3}	1.00350	5.53302×10^{-3}	0.999614	5.54691×10^{-3}	1.00342
256	3.11692×10^{-3}	1.00182	3.11249×10^{-3}	0.999907	3.11690×10^{-3}	1.00180
400	1.99384×10^{-3}	1.00111	1.99203×10^{-3}	0.999959	1.99383×10^{-3}	1.00111

Subtracting (3.17) from (3.19), we get

$$a(u_h-u_H^h,R_H\Phi)-b(R_H\Phi,p_h-p_H^h)+b(u_h-u_H^h,L_H\Psi)+G(p_h-p_H^h,L_H\Psi)=0. \tag{3.23}$$

Thus, together with (3.16), (3.22), and (3.23), there holds

$$\begin{aligned}\|u_h-u_H^h\|_{0,\Omega}^2 &\lesssim (\|u_h-u_H^h\|_{1,\Omega}+\|p_h-p_H^h\|_{0,\Omega})(\|(I-R_H)\Phi\|_{1,\Omega}+\|(I-L_H)\Psi\|_{0,\Omega})\\ &\lesssim H(\|u_h-u_H^h\|_{1,\Omega}+\|p_h-p_H^h\|_{0,\Omega})(\|\Phi\|_{2,\Omega}+\|\Psi\|_{1,\Omega})\\ &\lesssim H(\|u_h-u_H^h\|_{1,\Omega}+\|p_h-p_H^h\|_{0,\Omega})\|u_h-u_H^h\|_{0,\Omega}.\end{aligned}$$

Noting (3.13), then

$$\begin{aligned}\|u_h-u_H^h\|_{0,\Omega} &\lesssim H(\|u_h-u_H^h\|_{1,\Omega}+\|p_h-p_H^h\|_{0,\Omega})\\ &\lesssim H^3.\end{aligned}$$

Therefore, together with the triangle inequality and (2.10), we get

$$\begin{aligned}\|u-u_H^h\|_{0,\Omega} &\lesssim \|u-u_h\|_{0,\Omega}+\|u_h-u_H^h\|_{0,\Omega}\\ &\lesssim h^2+H^3.\end{aligned}$$

4 Numerical tests

This section will show some numerical tests to verify the theoretical results. The computational domain is $\Omega=(0,1)\times(0,1)$, linear polynomial functions are used both for the velocity and pressure fields, and all the numerical results are derived by using the public domain software FreeFem++ [17]. We decompose Ω into four subdomains D_i $(i=1,2,3,4)$ as follows

$$\begin{aligned}D_1 &= (0,0.5)\times(0,0.5),\ D_2=(0.5,1)\times(0,0.5),\\ D_3 &= (0,0.5)\times(0.5,1),\ D_4=(0.5,1)\times(0.5,1),\end{aligned}$$

Table 2 The errors of pressure in L^2 norm by SFEM, *Algorithms* 1, 2

$1/h$	$\|p-p_h\|_{0,\Omega}$	*rate*	$\|\|\|p-p^h\|\|\|_{0,\Omega}$	*rate*	$\|p-p_H^h\|_{0,\Omega}$	*rate*
64	5.66985×10^{-3}	–	2.15853×10^{-3}	–	4.37666×10^{-3}	–
144	1.57005×10^{-3}	1.58343	5.94220×10^{-4}	1.59068	1.19798×10^{-3}	1.59773
256	6.39818×10^{-4}	1.56019	2.41561×10^{-4}	1.56445	4.85534×10^{-4}	1.56969
400	3.20776×10^{-4}	1.54708	1.20936×10^{-4}	1.55026	2.42711×10^{-4}	1.55366

Table 3 The errors of velocity in L^2 norm by SFEM and *Algorithm* 2

$1/h$	$\|u-u_h\|_{0,\Omega}$	*rate*	$\|u-u_H^h\|_{0,\Omega}$	*rate*
8	4.39938×10^{-3}	–	4.30545×10^{-3}	–
64	7.07656×10^{-5}	1.98604	7.05634×10^{-5}	1.97703
216	6.21475×10^{-6}	1.99972	6.21119×10^{-6}	1.99784
512	1.10612×10^{-6}	1.99997	1.10591×10^{-6}	1.99953

then we enlarge them to

$$\begin{aligned}\Omega_1 &= (0,0.75)\times(0,0.75),\ \Omega_2=(0.25,1)\times(0,0.75),\\ \Omega_3 &= (0,0.75)\times(0.25,1),\ \Omega_4=(0.25,1)\times(0.25,1).\end{aligned}$$

We choose the continuous piecewise linear Lagrange basis functions as the partition of unity functions, which are presented in Fig. 1.

4.1 Example 1

In this example, we consider the Stokes problem (2.1) with the following exact solution

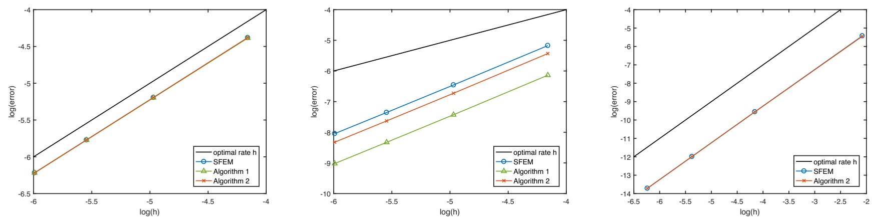
$$\begin{aligned}u &= [u_1,u_2]=[10x^2(x-1)^2y(y-1)(2y-1),-10x(x-1)(2x-1)y^2(y-1)^2],\\ p &= 10(2x-1)(2y-1).\end{aligned}$$

For simplicity, we set $\nu=1$, then the source term f can be derived.

The convergence rates in tables are obtained by

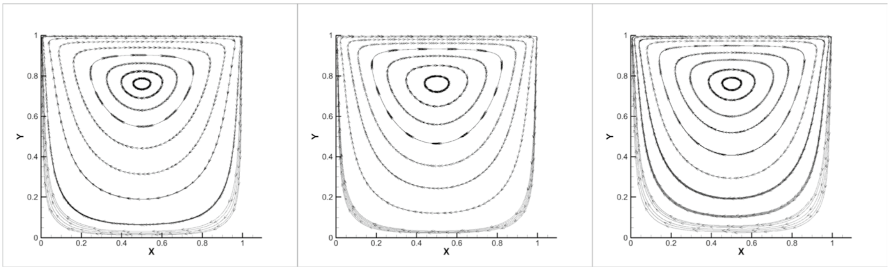
$$rate=\frac{ln(E_i)/ln(E_{i+1})}{ln(h_i)/ln(h_{i+1})},$$

where E_i denotes the error with the mesh size h_i. Choose $H=\frac{1}{8},\frac{1}{12},\frac{1}{16},\frac{1}{20}$, $h=H^2$, $\alpha=h$, using the lowest equal-order finite element pair (P1-P1), with the uniform triangulation. For the errors of velocity in H^1 norm by SFEM (stabilized finite element



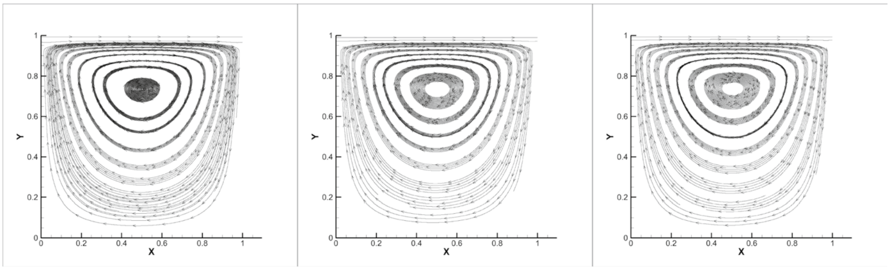
(a) The H^1 error for the velocity by three methods. (b) The L^2 error for the pressure by three methods. (c) The L^2 error for the velocity by SFEM and *Algorithm* 2.

Fig. 2 Rates analysis for velocity and pressure by three methods



(a) ν=1 (b) ν=0.1 (c) ν=0.01

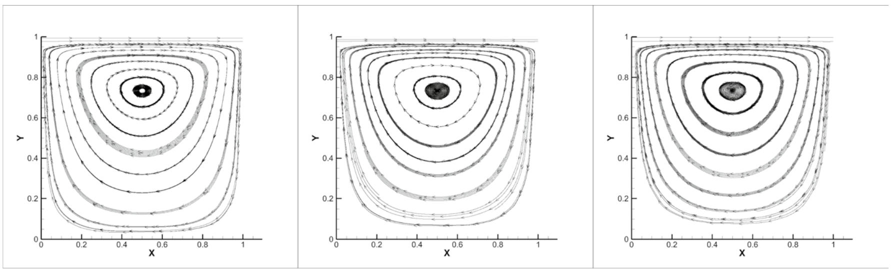
Fig. 3 The streamlines of velocity by SFEM with different ν



(a) ν=1 (b) ν=0.1 (c) ν=0.01

Fig. 4 The streamlines of velocity by *Algorithm* 1 with different ν

method), *Algorithms* 1, 2, we take the same mesh size h to compare these three algorithms. The results are presented in Table 1. Table 1 indicates that the convergence rates of velocity in H^1 norm by SFEM, *Algorithms* 1, 2, all agree with theoretical analysis. Table 2 shows that the errors of pressure in L^2 norm by SFEM, *Algorithms* 1, 2, arrive at the theoretical analysis above. In particular, we see that *Algorithms* 1, 2 perform better than SFEM, namely, the local and parallel algorithm is necessary and the coarse grid correction is important for the Stokes problem.



(a) ν=1 (b) ν=0.1 (c) ν=0.01

Fig. 5 The streamlines of velocity by *Algorithm* 2 with different ν

Then, we set $H = \frac{1}{4}, \frac{1}{16}, \frac{1}{36}, \frac{1}{64}$, $h^2 = H^3$. From Table 3, it is easy to see that SFEM and *Algorithm* 2 all yield optimal convergence rates which are in accordance with theorems, namely, the convergence rate of velocity in L^2 norm could arrive at $\mathcal{O}(h^2)$.

In order to describe visually the convergence rates of velocity and pressure obtained by SFEM, *Algorithms* 1, 2, the following pictures are presented. From Fig. 2(a), we notice that the convergence rates of velocity in H^1 norm by all three algorithms are consistent with theoretical analysis. Figure 2(b) shows that the pressure by *Algorithm* 1 is better than that of SFEM which is in accord with Table 2. For Fig. 2(c), it illustrates that the convergence rate of velocity in L^2 norm by *Algorithm* 2 is similar to that of SFEM.

Example 2. In this example, we consider the cavity flow in [0,1] × [0,1] with different values of ν and $\alpha = h$. In this square region, the exact solution is unknown with the following boundary conditions.

$$\begin{aligned} u_1 &= 1 \quad \text{on } [0,1]\times 1,\\ u_1 &= 0 \quad \text{on } [0,1]\times 0 \cup 0\times[0,1]\cup 1\times[0,1],\\ u_2 &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Set $f = 0$, $h = \frac{1}{144}$, we obtain the following streamlines of velocity by SFEM, *Algorithms* 1, 2. From Figs. 3, 4, and 5, we can see that the streamlines of velocity obtained by three algorithms do not change much as the value of ν is taken, and the streamlines of *Algorithms* 1, 2 are close to those of SFEM.

Acknowledgements The authors would like to thank the reviewers for their constructive comments, which allowed for the improvement of the presentation of the results.

Author contributions Xinhui Wang: formal analysis, visualization, writing, review. Guangzhi Du: conceptualization, methodology, validation, review. All authors reviewed the manuscript.

Funding This work is subsidized by the National Natural Science Foundation of China (No. 12172202), the Natural Science Foundation of Shandong Province (No. ZR2021MA063), the Support Plan for Outstanding Youth Innovation Team in Shandong Higher Education Institutions (No. 2021KJ037), the Natural Science Foundation of Shaanxi Province (No. 2021JQ-426), and the Scientific Research Program of Shaanxi Provincial Education Department (No. 21JK0935).

Data availability Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Competing interests The authors declare no competing interests.

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