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ORIGINAL PAPER

Strong and linear convergence of projection-type method with an inertial term for finding minimum-norm solutions of pseudomonotone variational inequalities in Hilbert spaces

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Abstract

The purpose of this paper is to investigate pseudomonotone variational inequalities in real Hilbert spaces. For solving this problem, we introduce a new method. The proposed algorithm combines the advantages of the subgradient extragradient method and the projection and contraction method. We establish the strong convergence of the proposed algorithm under conditions pseudomonotonicity and Lipschitz continuity assumptions. Moreover, under additional strong pseudomonotonicity and Lipschitz continuity assumptions, the linear convergence of the sequence generated by the proposed algorithm is obtained. Numerical examples provide to illustrate the potential of our algorithms as well as compare their performances to several related results.

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1 Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonempty, closed and convex subset of H. Let $F : H \to H$ be a single-valued continuous mapping. We consider classical variational inequality (VI) in the sense of Fichera [14] and Stampacchia [30] (see also Kinderlehrer and Stampacchia [21]) which is formulated as follows: find a point $x^* \in C$ such that

$$\langle Fx^*, x - x^* \rangle \geq 0 \ \ \forall x \in C. \tag{1}$$

We denote by $Sol(C, F)$ the solution set of the VI (1), which is assumed to be nonempty.

In this work, we assume that the following conditions hold:

Condition 1 The solution set $\mathrm{Sol}(C, F)$ is nonempty.

Condition 2 The mapping $F : H \to H$ is pseudomonotone on H, that is,

$$\langle Fx, y - x \rangle \geq 0 \Longrightarrow \langle Fy, y - x \rangle \geq 0 \quad \forall x, y \in H.$$

Condition 3 The mapping $F : H \to H$ is Lipschitz continuous with constant $L > 0$, that is, there exists a number $L > 0$ such that

$$\|Fx - Fy\| \leq L\|x - y\| \quad \forall x, y \in H.$$

Variational inequality (VI) is a very general mathematical model with numerous applications in economics, engineering mechanics, transportation, and many more, see, for example, [2, 13, 21, 23]. During the last decades, many algorithms for solving VIs have been proposed in the literature, see, e.g., [3, 10, 13, 15, 16, 21, 34, 37].

The most well-known one is *extragradient method* proposed by Korpelevich [22] (also by Antipin [1] independently). However, the extragradient method requires the evaluation of two orthogonal projections onto C per iteration. The first method which overcomes this obstacle is the *projection and contraction method* (PC) of He [18] and Sun [33]. Their algorithm is of the form:

$$y_n = P_C(x_n - \tau_n F x_n),$$

and then the next iterate x_{n+1} is generated via the following

$$x_{n+1} = x_n - \gamma \eta_n d(x_n, y_n),$$

where $\gamma \in (0, 2)$,

$$\eta_n := \frac{\langle x_n - y_n, d(x_n, y_n)\rangle}{\|d(x_n, y_n)\|^2},$$

and

$$d(x_n, y_n) := x_n - y_n - \tau_n(Fx_n - Fy_n),$$

where $F : C \to \mathbb{H}$ be monotone and L-Lipschitz continuous operator and $\tau_n \in (0, 1/L)$ or τ_n is updated by some adaptive rule as follows:

$$\tau_n\|Fx_n - Fy_n\| \leq \mu\|x_n - y_n\|, \quad \mu \in (0, 1). \tag{2}$$

Recently, projection and contraction type methods for solving VI have received great attention by many authors, see, e.g., [4, 11, 12, 23, 27, 36].

The second extension of the extragradient method is known as the *subgradient extragradient method* proposed by Censor et al. [6–8]. In this algorithm, the second projection onto the feasible set C is replaced by a projection onto an easy and constructible set that contains C. For each $n \in \mathbb{N}$ generate the following sequences,

$$\begin{cases} y_n = P_C(x_n - \tau Fx_n), \\ T_n = \{x \in H \mid \langle x_n - \tau Fx_n - y_n, x - y_n\rangle \leq 0\}, \\ x_{n+1} = P_{T_n}(x_n - \tau Fy_n), \end{cases}$$

where $\tau \in (0, 1/L)$.

Since the projection and contraction and the subgradient extragradient methods require calculating only one projection onto C per iteration, their computational efforts and performance have an advantage over other existing results in the literature. Recently, [12] introduced a modification of the subgradient extragradient method by using the direction of the projection and contraction method and stepsize rule τ_n satisfying (2).

This paper is motivated and inspired by the work of Censor et al. [6], He [18] and Sun [33], first, we investigate the strong convergence for solving the problem (VI) by our new algorithm which is a combination of the subgradient extragradient method and the projection and contraction method in Hilbert spaces. In the proposed method, we show that an advantage of the proposed algorithm is the computation of only two values of the variational inequality mapping and one projection onto the feasible set per one iteration, which distinguishes our method from most other projection-type methods for variational inequality problems with pseudomonotone mappings. Second, the convergence rate of the algorithm is presented under strong pseudomonotonicity and Lipschitz continuity of the cost operator. Specifically, the proposed algorithm improves the results in the literature in the following ways:

– at each iteration step, a single projection is required to perform;
– an inertial term for speeding up convergence;
– step-sizes are not decreasing;
– without knowledge of the Lipschitz constant of the underline operator;
– without the assumption on the sequential weak continuity of the underline operator;
– strong convergence and moreover, a convergence estimate is established.

This paper is organized as follows: Section 2 consists of the notations and basic definitions which are useful throughout the paper. In Section 3, we propose our algorithm and prove the strong convergence of the iterative sequence to a solution of the variational inequality (1). The convergence rate of the proposed algorithm is presented in Section 4. In Section 5, some numerical results in optimal control problems are reported to demonstrate the performance of the proposed method. Final conclusions are given in Section 6.

2 Preliminaries

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. The weak convergence of $\{x_n\}$ to x is denoted by $x_n \rightharpoonup x$ as $n\to\infty$, while the strong convergence of $\{x_n\}$ to x is written as $x_n \to x$ as $n\to\infty$. For all $x, y\in H$ we have

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle.$$

Definition 2.1 Let $T : H \to H$ be an operator. Then

1. T is called L-Lipschitz continuous with constant $L>0$ if

$$\|Tx-Ty\| \le L\|x-y\| \quad \forall x,y\in H,$$

 if $L=1$ then the operator T is called nonexpansive and if $L\in(0,1)$, T is called a contraction.
2. T is called monotone if

$$\langle Tx-Ty, x-y\rangle \ge 0 \quad \forall x,y\in H;$$

3. T is called pseudomonotone in the sense of Karamardian [19] if

$$\langle Tx, y-x\rangle \ge 0 \Longrightarrow \langle Ty, y-x\rangle \ge 0 \quad \forall x,y\in H; \tag{3}$$

4. T is called α-strongly monotone if there exists a constant $\alpha>0$ such that

$$\langle Tx-Ty, x-y\rangle \ge \alpha\|x-y\|^2 \quad \forall x,y\in H;$$

5. T is called α-strongly pseudomonotone if there exists a constant $\alpha > 0$ such that

$$\langle Tx, y - x\rangle \geq 0 \Longrightarrow \langle Ty, y - x\rangle \geq \alpha\|x - y\|^2 \quad \forall x, y \in H;$$

6. The operator T is called sequentially weakly continuous if for each sequence $\{x_n\}$ we have: x_n converges weakly to x implies Tx_n converges weakly to Tx.

We note that (3) is only one of the definitions of pseudomonotonicity which can be found in the literature. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$ such that $\|x - P_C x\| \leq \|x - y\|\ \forall y \in C$. P_C is called the *metric projection* of H onto C. It is known that P_C is nonexpansive. For properties of the metric projection, the interested reader could be referred to Section 3 in [16].

Lemma 2.1 ([16]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Given* $x \in H$ *and* $z \in C$. *Then* $z = P_C x \Longleftrightarrow \langle x - z, z - y\rangle \geq 0\ \ \forall y \in C$. *Moreover,*

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle\ \forall x, y \in C.$$

Lemma 2.2 *Let H be a real Hilbert space. Then the following results hold*:

i) $\|x + y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2\ \forall x, y \in H$;
ii) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle\ \forall x, y \in H$.

Lemma 2.3 ([9]) *Consider the problem Sol(C, F) with C being a nonempty, closed, convex subset of a real Hilbert space H and* $F : C \to H$ *being pseudomonotone and continuous. Then,* x^* *is a solution of Sol(C, F) if and only if*

$$\langle Fx, x - x^*\rangle \geq 0\ \ \forall x \in C.$$

Lemma 2.4 ([28]) *Let* $\{a_n\}$ *be sequence of nonnegative real numbers,* $\{\alpha_n\}$ *be a sequence of real numbers in* $(0, 1)$ *with* $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\{b_n\}$ *be a sequence of real numbers. Assume that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n b_n\ \ \forall n \geq 1.$$

If $\limsup_{k\to\infty} b_{n_k} \leq 0$ *for every subsequence* $\{a_{n_k}\}$ *of* $\{a_n\}$ *satisfying* $\liminf_{k\to\infty}(a_{n_k+1} - a_{n_k}) \geq 0$ *then* $\lim_{n\to\infty} a_n = 0$.

Definition 2.2 ([26]) Let $\{x_n\}$ be a sequence in H.

i) $\{x_n\}$ is said to converge R-linearly to x^* with rate $\rho \in [0, 1)$ if there is a constant $c > 0$ such that

$$\|x_n - x^*\| \leq c\rho^n\ \ \forall n \in \mathbb{N}.$$

ii) $\{x_n\}$ is said to converge Q-linearly to x^* with rate $\rho \in [0,1)$ if

$$\|x_{n+1} - x^*\| \le \rho \|x_n - x^*\| \quad \forall n \in \mathbb{N}.$$

Initialization: *Given* $\tau_1 > 0, \mu \in (0,1), \gamma \in (0,2), \theta \in [0,1)$. *Let* $u_0, u_1 \in H$ *be arbitrary and* $\{\alpha_n\}$ *be a nonnegative real numbers sequence such that* $\sum_{n=1}^{\infty} \alpha_n < +\infty$. *We assume that* $\{\gamma_n\}$ *and* $\{\varepsilon_n\}$ *are two positive sequences such that* $\varepsilon_n = \circ(\gamma_n)$, *means* $\lim_{n\to\infty} \dfrac{\varepsilon_n}{\gamma_n} = 0$, *where* $\{\gamma_n\} \subset (0,1)$ *satisfies the following conditions:*

$$\lim_{n\to\infty} \gamma_n = 0, \quad \sum_{n=1}^{\infty} \gamma_n = \infty.$$

Iterative Steps: *Given the current iterate* u_n, *calculate* u_{n+1} *as follows:*

Step 1. *Compute* $w_n = (1-\gamma_n)(u_n + \theta_n(u_n - u_{n-1}))$ *and*

$$v_n = P_C(w_n - \tau_n F w_n),$$

where

$$\theta_n = \begin{cases} \min\left\{\theta, \frac{\varepsilon_n}{\|u_n - u_{n-1}\|}\right\} & \text{if } u_n \neq u_{n-1}, \\ \theta & \text{otherwise.} \end{cases} \tag{4}$$

If $w_n = v_n$ *or* $F v_n = 0$ *then stop and* v_n *is a solution of* $Sol(C,F)$. *Otherwise*

Step 2. *Compute*

$$u_{n+1} = P_{T_n}(w_n - \gamma \tau_n \eta_n F v_n),$$

where

$$T_n = \{x \in H \mid \langle w_n - \tau_n F w_n - v_n, x - v_n \rangle \le 0\},$$

$$\eta_n := \begin{cases} \dfrac{\langle w_n - v_n, d_n \rangle}{\|d_n\|^2} & \text{if } d_n \neq 0, \\ 0 & \text{if } d_n = 0, \end{cases}$$

and

$$d_n := w_n - v_n - \tau_n(F w_n - F v_n).$$

Step 3. *Update*

$$\tau_{n+1} := \begin{cases} \min\left\{\mu \dfrac{\|w_n - v_n\|}{\|F w_n - F v_n\|}, \tau_n + \alpha_n\right\} & \text{if } F w_n \neq F v_n, \\ \tau_n + \alpha_n & \text{otherwise.} \end{cases} \tag{5}$$

Set $n := n+1$ *and go to* ***Step 1***.

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3 Strong convergence analysis

First, we introduce the proposed algorithm:

Observe that the projection onto half-space T_n in Step 2 is explicit [5, Section 4.1.3, p. 133], therefore, Algorithm 1 requires only one projection in Step 1. Moreover, the stepsize τ_n is updated adaptively in Step 3 without requiring the knowledge of the Lipschitz constant L. We start the convergence analysis by proving the following Lemmas.

Lemma 3.5 ([24]) *Assume that F is L-Lipschitz continuous on H. Let $\{\tau_n\}$ be the sequence generated by* (5). *Then*

$$\lim_{n\to\infty} \tau_n = \tau \text{ with } \tau \in \left[\min\left\{\tau_1, \frac{\mu}{L}\right\}, \tau_1 + \alpha\right],$$

where $\alpha = \sum_{n=1}^{\infty} \alpha_n$. Moreover

$$\|Fw_n - Fv_n\| \le \frac{\mu}{\tau_{n+1}}\|w_n - v_n\|. \tag{6}$$

Lemma 3.6 *Assume that F is Lipschitz continuous on H and pseudomonotone on C. Then for every $x^* \in Sol(C, F)$, there exists $n_0 > 0$ such that*

$$\|u_{n+1} - x^*\|^2 \le \|w_n - x^*\|^2 - \|w_n - u_{n+1} - \gamma\eta_n d_n\|^2 - (2-\gamma)\gamma\eta_n^2\|d_n\|^2 \quad \forall n \ge n_0.$$

Proof Using (6), we have

$$\begin{aligned}
\|d_n\| =& \|w_n - v_n - \tau_n(Fw_n - Fv_n)\| \\
\ge& \|w_n - v_n\| - \tau_n\|Fw_n - Fv_n\| \\
\ge& \|w_n - v_n\| - \frac{\mu\tau_n}{\tau_{n+1}}\|w_n - v_n\| \\
=& \left(1 - \frac{\mu\tau_n}{\tau_{n+1}}\right)\|w_n - v_n\|.
\end{aligned} \tag{7}$$

Since $\lim_{n\to\infty}\left(1 - \frac{\mu\tau_n}{\tau_{n+1}}\right) = 1 - \mu > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$1 - \frac{\mu\tau_n}{\tau_{n+1}} > \frac{1-\mu}{2} \quad \forall n \ge n_0.$$

Therefore, it follows from (7) that for all $n \ge n_0$ we get

$$\|d_n\| \ge \frac{1-\mu}{2}\|w_n - v_n\| > 0. \tag{8}$$

Since $x^* \in Sol(C,F) \subset C \subset T_n$, using Lemma 2.1 we have

$$\begin{aligned}\|u_{n+1}-x^*\|^2 =&\|P_{T_n}(w_n-\gamma\eta_n\tau_nFv_n)-P_{T_n}x^*\|^2\\ \le&\langle u_{n+1}-x^*, w_n-\gamma\eta_n\tau_nFv_n-x^*\rangle\\ =&\frac{1}{2}\|u_{n+1}-x^*\|^2+\frac{1}{2}\|w_n-\gamma\eta_n\tau_nFv_n-x^*\|^2-\frac{1}{2}\|u_{n+1}-w_n+\gamma\eta_n\tau_nFv_n\|^2\\ =&\frac{1}{2}\|u_{n+1}-x^*\|^2+\frac{1}{2}\|w_n-x^*\|^2+\frac{1}{2}\gamma^2\eta_n^2\tau_n^2\|Fv_n\|^2-\langle w_n-x^*,\gamma\eta_n\tau_nFv_n\rangle\\ &-\frac{1}{2}\|u_{n+1}-w_n\|^2-\frac{1}{2}\gamma^2\eta_n^2\tau_n^2\|Fv_n\|^2-\langle u_{n+1}-w_n,\gamma\eta_n\tau_nFv_n\rangle\\ =&\frac{1}{2}\|u_{n+1}-x^*\|^2+\frac{1}{2}\|w_n-x^*\|^2-\frac{1}{2}\|u_{n+1}-w_n\|^2-\langle u_{n+1}-x^*,\gamma\eta_n\tau_nFv_n\rangle.\end{aligned}$$

This implies that

$$\|u_{n+1}-x^*\|^2\le\|w_n-x^*\|^2-\|u_{n+1}-w_n\|^2-2\gamma\eta_n\tau_n\langle u_{n+1}-x^*,Fv_n\rangle. \tag{9}$$

Since $v_n\in C$ and $x^*\in Sol(C,F)$,we get $\langle Fx^*, v_n-x^*\rangle\ge 0$. By the pseudomonotonicity of F, we have $\langle Fv_n, v_n-x^*\rangle\ge 0$, which implies

$$\langle Fv_n,u_{n+1}-x^*\rangle=\langle Fv_n,u_{n+1}-v_n\rangle+\langle Fv_n,v_n-x^*\rangle\ge\langle Fv_n,u_{n+1}-v_n\rangle.$$

Thus, we obtain

$$-2\gamma\eta_n\tau_n\langle Fv_n,u_{n+1}-x^*\rangle\le-2\gamma\eta_n\tau_n\langle Fv_n,u_{n+1}-v_n\rangle. \tag{10}$$

On the other hand, from $u_{n+1}\in T_n$ we have

$$\langle w_n-\tau_nFw_n-v_n,u_{n+1}-v_n\rangle\le 0.$$

This implies that

$$\langle w_n-v_n-\tau_n(Fw_n-Fv_n),u_{n+1}-v_n\rangle\le\tau_n\langle Fv_n,u_{n+1}-v_n\rangle,$$

thus

$$\langle d_n,u_{n+1}-v_n\rangle\le\tau_n\langle Fv_n,u_{n+1}-v_n\rangle.$$

Hence

$$-2\gamma\eta_n\tau_n\langle Fv_n,u_{n+1}-v_n\rangle\le-2\gamma\eta_n\langle d_n,u_{n+1}-v_n\rangle. \tag{11}$$

On the other hand, we have

$$-2\gamma\eta_n\langle d_n,u_{n+1}-v_n\rangle=-2\gamma\eta_n\langle d_n,w_n-v_n\rangle+2\gamma\eta_n\langle d_n,w_n-u_{n+1}\rangle. \tag{12}$$

From (8), we have $d_n\ne 0\ \ \forall n\ge n_0$, thus $\eta_n=\dfrac{\langle w_n-v_n,d_n\rangle}{\|d_n\|^2}$, which means

$$\langle w_n-v_n,d_n\rangle=\eta_n\|d_n\|^2\ \ \forall n\ge n_0. \tag{13}$$

Moreover

$$2\gamma\eta_n\langle d_n, w_n - u_{n+1}\rangle = 2\langle \gamma\eta_n d_n, w_n - u_{n+1}\rangle$$
$$= \|w_n - u_{n+1}\|^2 + \gamma^2\eta_n^2\|d_n\|^2 - \|w_n - u_{n+1} - \gamma\eta_n d_n\|^2. \tag{14}$$

Substituting (13) and (14) into (12) we get for all $n \geq n_0$ that

$$-2\gamma\eta_n\langle d_n, u_{n+1} - v_n\rangle \leq -2\gamma\eta_n^2\|d_n\|^2 + \|w_n - u_{n+1}\|^2 + \gamma^2\eta_n^2\|d_n\|^2 - \|w_n - u_{n+1} - \gamma\eta_n d_n\|^2$$
$$= \|w_n - u_{n+1}\|^2 - \|w_n - u_{n+1} - \gamma\eta_n d_n\|^2 - (2-\gamma)\gamma\eta_n^2\|d_n\|^2. \tag{15}$$

Combining (11) and (15), we obtain

$$-2\gamma\eta_n\tau_n\langle Fv_n, u_{n+1} - v_n\rangle \leq -2\gamma\eta_n^2\|d_n\|^2 + \|w_n - u_{n+1}\|^2 + \gamma^2\eta_n^2\|d_n\|^2 - \|w_n - u_{n+1} - \gamma\eta_n d_n\|^2$$
$$= \|w_n - u_{n+1}\|^2 - \|w_n - u_{n+1} - \gamma\eta_n d_n\|^2 - (2-\gamma)\gamma\eta_n^2\|d_n\|^2. \tag{16}$$

Again, combining (10) and (16), we get

$$-2\gamma\eta_n\tau_n\langle Fv_n, u_{n+1} - x^*\rangle \leq -2\gamma\eta_n^2\|d_n\|^2 + \|w_n - u_{n+1}\|^2 + \gamma^2\eta_n^2\|d_n\|^2 - \|w_n - u_{n+1} - \gamma\eta_n d_n\|^2$$
$$= \|w_n - u_{n+1}\|^2 - \|w_n - u_{n+1} - \gamma\eta_n d_n\|^2 - (2-\gamma)\gamma\eta_n^2\|d_n\|^2. \tag{17}$$

Substituting (17) into (9) we get

$$\|u_{n+1} - x^*\|^2 \leq \|w_n - x^*\|^2 - \|w_n - u_{n+1} - \gamma\eta_n d_n\|^2 - (2-\gamma)\gamma\eta_n^2\|d_n\|^2 \quad n \geq n_0. \tag{18}$$

Theorem 3.1 *Assume that Conditions* 1–3 *hold. In addition, we assume that the mapping* $F : H \to H$ *satisfies the following condition*

$$\text{whenever } \{u_n\} \subset C, u_n \rightharpoonup z, \text{ one has } \|Fz\| \leq \liminf_{n\to\infty} \|Fu_n\|. \tag{19}$$

Then the sequence $\{u_n\}$ *is generated by Algorithm* 1 *converges strongly to an element* $z \in Sol(C, F)$, *where* $z = P_{Sol(C,F)}(0)$.

Proof **Claim 1.** The sequence $\{u_n\}$ is bounded. Indeed, we have

$$\begin{aligned}\|w_n - z\| &= \|(1-\gamma_n)(u_n + \theta_n(u_n - u_{n-1})) - z\| \\ &= \|(1-\gamma_n)(u_n - z) + (1-\gamma_n)\theta_n(u_n - u_{n-1}) - \gamma_n z\| \\ &\leq (1-\gamma_n)\|u_n - z\| + (1-\gamma_n)\theta_n\|u_n - u_{n-1}\| + \gamma_n\|z\| \\ &= (1-\gamma_n)\|u_n - z\| + \gamma_n[(1-\gamma_n)\frac{\theta_n}{\gamma_n}\|u_n - u_{n-1}\| + \|z\|].\end{aligned} \tag{20}$$

On the other hand, since (4) we have

$$\frac{\theta_n}{\gamma_n}\|u_n - u_{n-1}\| \leq \frac{\epsilon_n}{\gamma_n} \to 0$$

this implies that $\lim_{n\to\infty}\left[(1-\gamma_n)\frac{\theta_n}{\gamma_n}\|u_n-u_{n-1}\|+\|z\|\right]=\|z\|$, thus there exists $M>0$ such that

$$(1-\gamma_n)\frac{\theta_n}{\gamma_n}\|u_n-u_{n-1}\|+\|z\|\le M. \tag{21}$$

Combining (20) and (21) we obtain

$$\|w_n-z\|\le(1-\gamma_n)\|u_n-z\|+\gamma_n M.$$

Moreover, we have $\lim_{n\to\infty}(1-\mu\frac{\tau_n}{\tau_{n+1}})=1-\mu>\frac{1-\mu}{2}$, thus there exists $n_0\in\mathbb{N}$ such that $1-\mu\frac{\tau_n}{\tau_{n+1}}>0\ \ \forall n\ge n_0$, by Claim 1 we obtain

$$\|u_{n+1}-z\|\le\|w_n-z\|\ \ \forall n\ge n_0. \tag{22}$$

Thus

$$\begin{aligned}\|u_{n+1}-z\|\le&(1-\gamma_n)\|u_n-z\|+\gamma_n M\\ =&\max\{\|u_n-z\|,M\}\le\ldots\le\max\{\|u_{n_0}-z\|,M\}.\end{aligned}$$

Therefore, the sequence $\{u_n\}$ is bounded.

Claim 2.

$$\|w_n-u_{n+1}-\gamma\eta_n d_n\|^2+(2-\gamma)\gamma\eta_n^2\|d_n\|^2\le\|u_n-z\|^2-\|u_{n+1}-z\|^2+\gamma_n M_1.$$

Indeed, we have $\|w_n-z\|\le(1-\gamma_n)\|u_n-z\|+\gamma_n M$, this implies that

$$\begin{aligned}\|w_n-z\|^2\le&(1-\gamma_n)^2\|u_n-z\|^2+2\gamma_n(1-\gamma_n)M\|u_n-z\|+\gamma_n^2M^2\\ \le&\|u_n-z\|^2+\gamma_n[2(1-\gamma_n)M\|u_n-z\|+\gamma_n M^2]\\ \le&\|u_n-z\|^2+\gamma_n M_1,\end{aligned} \tag{23}$$

where $M_1:=\max\{2(1-\gamma_n)M\|u_n-z\|+\gamma_n M^2:\ \ n\in\mathbb{N}\}$. Substituting (23) into (18) we get

$$\|u_{n+1}-x^*\|^2\le\|u_n-z\|^2+\gamma_n M_1-\|w_n-u_{n+1}-\gamma\eta_n d_n\|^2-(2-\gamma)\gamma\eta_n^2\|d_n\|^2.$$

Or equivalently

$$\|w_n-u_{n+1}-\gamma\eta_n d_n\|^2+(2-\gamma)\gamma\eta_n^2\|d_n\|^2\le\|u_n-z\|^2-\|u_{n+1}-z\|^2+\gamma_n M_1.$$

Claim 3.

$$\frac{\left(1-\frac{\mu\tau_n}{\tau_{n+1}}\right)^2}{\left(1+\frac{\mu\tau_n}{\tau_{n+1}}\right)^2}\|w_n-v_n\|^2\le\eta_n^2\|d_n\|^2\ \ \forall n\ge n_0.$$

We have

$$\|d_n\| \leq \|w_n - v_n\| + \tau_n\|Fw_n - Fv_n\| \leq \left(1 + \frac{\mu\tau_n}{\tau_{n+1}}\right)\|w_n - v_n\|.$$

Hence

$$\|d_n\|^2 \leq \left(1 + \frac{\mu\tau_n}{\tau_{n+1}}\right)^2\|w_n - v_n\|^2,$$

or equivalently

$$\frac{1}{\|d_n\|^2} \geq \frac{1}{\left(1 + \frac{\mu\tau_n}{\tau_{n+1}}\right)^2\|w_n - v_n\|^2}.$$

Again, we find

$$\begin{aligned}\langle w_n - v_n, d_n\rangle =& \|w_n - v_n\|^2 - \tau_n\langle w_n - v_n, Fw_n - Fv_n\rangle \\ \geq& \|w_n - v_n\|^2 - \tau_n\|w_n - v_n\|\|Fw_n - Fv_n\| \\ \geq& \|w_n - v_n\|^2 - \frac{\mu\tau_n}{\tau_{n+1}}\|w_n - v_n\|^2 \\ =& \left(1 - \frac{\mu\tau_n}{\tau_{n+1}}\right)\|w_n - v_n\|^2.\end{aligned}$$

Hence for all $n \geq n_0$

$$\eta_n\|d_n\|^2 = \langle w_n - v_n, d_n\rangle \geq \left(1 - \frac{\mu\tau_n}{\tau_{n+1}}\right)\|w_n - v_n\|^2 \tag{24}$$

and

$$\eta_n = \frac{\langle w_n - v_n, d_n\rangle}{\|d_n\|^2} \geq \frac{\left(1 - \frac{\mu\tau_n}{\tau_{n+1}}\right)}{\left(1 + \frac{\mu\tau_n}{\tau_{n+1}}\right)^2}. \tag{25}$$

Combining (24) and (25), we get

$$\eta_n^2\|d_n\|^2 \geq \frac{\left(1 - \frac{\mu\tau_n}{\tau_{n+1}}\right)^2}{\left(1 + \frac{\mu\tau_n}{\tau_{n+1}}\right)^2}\|w_n - v_n\|^2 \quad \forall n \geq n_0. \tag{26}$$

Claim 4.

$$\begin{aligned}\|u_{n+1}-z\|^2 \leq &(1-\gamma_n)\|u_n-z\|^2+\gamma_n\Big[2(1-\gamma_n)\|u_n-z\|\frac{\theta_n}{\gamma_n}\|u_n-u_{n-1}\| \\ &+\theta_n\|u_n-u_{n-1}\|\frac{\theta_n}{\gamma_n}\|u_n-u_{n-1}\|+2\|z\|\|u_n-u_{n+1}\|+2\langle -z,u_{n+1}-z\rangle\Big]\end{aligned}$$

$\forall n\geq n_0$. Indeed, using Lemma 2.2 ii) and (22) we get

$$\begin{aligned}\|x_{n+1}-z\|^2 \leq&\|w_n-z\|^2 \ \ \forall n\geq n_0\\ =&\|(1-\gamma_n)(u_n-z)+(1-\gamma_n)\theta_n(u_n-u_{n-1})-\gamma_n z\|^2 \ \ \forall n\geq n_0\\ \leq&\|(1-\gamma_n)(u_n-z)+(1-\gamma_n)\theta_n(u_n-u_{n-1})\|^2+2\gamma_n\langle -z,w_n-z\rangle \ \ \forall n\geq n_0\\ =&(1-\gamma_n)^2\|u_n-z\|^2+2(1-\gamma_n)\theta_n\|u_n-z\|\|u_n-u_{n-1}\|+\theta_n^2\|u_n-u_{n-1}\|^2\\ &+2\langle -z,u_n-u_{n+1}\rangle+2\langle -z,u_{n+1}-z\rangle \ \ \forall n\geq n_0\\ \leq&(1-\gamma_n)\|u_n-z\|^2+\gamma_n\Big[2(1-\gamma_n)\|u_n-z\|\frac{\theta_n}{\gamma_n}\|u_n-u_{n-1}\|\\ &+\theta_n\|u_n-u_{n-1}\|\frac{\theta_n}{\gamma_n}\|u_n-u_{n-1}\|+2\|z\|\|u_n-u_{n+1}\|+2\langle -z,u_{n+1}-z\rangle\Big] \ \ \forall n\geq n_0.\end{aligned}$$

Claim 5. $\{\|u_n-z\|^2\}$ converges to zero. Indeed, by Lemma 2.4 it suffices to show that $\limsup_{k\to\infty}\langle -z,u_{n_k+1}-z\rangle\leq 0$ and $\limsup_{k\to\infty}\|u_{n_k}-u_{n_k+1}\|\leq 0$ for every subsequence $\{\|u_{n_k}-z\|\}$ of $\{\|u_n-z\|\}$ satisfying

$$\liminf_{k\to\infty}(\|u_{n_k+1}-z\|-\|u_{n_k}-z\|)\geq 0.$$

For this, suppose that $\{\|u_{n_k}-z\|\}$ is a subsequence of $\{\|u_n-z\|\}$ such that $\liminf_{k\to\infty}(\|u_{n_k+1}-z\|-\|u_{n_k}-z\|)\geq 0$. Then

$$\liminf_{k\to\infty}(\|u_{n_k+1}-z\|^2-\|u_{n_k}-z\|^2)=\liminf_{k\to\infty}[(\|u_{n_k+1}-z\|-\|u_{n_k}-z\|)(\|u_{n_k+1}-z\|+\|u_{n_k}-z\|)]\geq 0.$$

By Claim 2 we obtain

$$\begin{aligned}&\limsup_{k\to\infty}\Big[\|w_{n_k}-u_{n_k+1}-\gamma\eta_{n_k}d_{n_k}\|^2+(2-\gamma)\gamma\eta_{n_k}^2\|d_{n_k}\|^2\Big]\\ &\leq\limsup_{k\to\infty}\Big[\|u_{n_k}-z\|^2-\|u_{n_k+1}-z\|^2+\gamma_{n_k}M_1\Big]\\ &\leq\limsup_{k\to\infty}\Big[\|u_{n_k}-z\|^2-\|u_{n_k+1}-z\|^2\Big]+\limsup_{k\to\infty}\gamma_{n_k}M_1\\ &=-\liminf_{k\to\infty}\Big[\|u_{n_k+1}-z\|^2-\|u_{n_k}-z\|^2\Big]\\ &\leq 0.\end{aligned}$$

This implies that

$$\lim_{k\to\infty}\|w_{n_k}-u_{n_k+1}-\gamma\eta_{n_k}d_{n_k}\|=0 \text{ and } \lim_{k\to\infty}\gamma\eta_{n_k}^2\|d_{n_k}\|=0.$$

We have

$$\begin{aligned}\|w_{n_k} - u_{n_k+1}\| \leq &\|w_{n_k} - u_{n_k+1} - \gamma\eta_{n_k} d_{n_k}\| + \gamma\|\eta_{n_k} d_{n_k}\| \\ =&\|w_{n_k} - u_{n_k+1} - \gamma\eta_{n_k} d_{n_k}\| + \gamma\frac{1}{\eta_{n_k}}\eta_{n_k}^2\|d_{n_k}\|.\end{aligned} \tag{27}$$

Combining (25) and (28) we get

$$\begin{aligned}\|w_{n_k} - u_{n_k+1}\| \leq &\|w_{n_k} - u_{n_k+1} - \gamma\eta_{n_k} d_{n_k}\| + \gamma\eta_{n_k}\|d_{n_k}\| \\ =&\|w_{n_k} - u_{n_k+1} - \gamma\eta_{n_k} d_{n_k}\| + \gamma\frac{1}{\eta_{n_k}}\eta_{n_k}^2\|d_{n_k}\| \\ =&\|w_{n_k} - u_{n_k+1} - \gamma\eta_{n_k} d_{n_k}\| + \gamma\frac{\left(1 + \frac{\mu\tau_n}{\tau_{n+1}}\right)^2}{\left(1 - \mu\frac{\tau_n}{\tau_{n+1}}\right)}\eta_{n_k}^2\|d_{n_k}\|.\end{aligned} \tag{28}$$

This implies that

$$\lim_{k\to\infty}\|w_{n_k} - u_{n_k+1}\| = 0. \tag{29}$$

From Claim 3 we also have

$$\lim_{k\to\infty}\|v_{n_k} - w_{n_k}\| = 0. \tag{30}$$

Moreover, we have

$$\|u_{n_k} - w_{n_k}\| = \theta_{n_k}\|u_{n_k} - u_{n_k-1}\| = \gamma_{n_k}.\frac{\theta_{n_k}}{\gamma_{n_k}}\|u_{n_k} - u_{n_k-1}\| \to 0. \tag{31}$$

From (29), (30) and (31), we get

$$\|u_{n_k+1} - u_{n_k}\| \leq \|u_{n_k+1} - w_{n_k}\| + \|w_{n_k} - v_{n_k}\| + \|v_{n_k} - u_{n_k}\| \to 0. \tag{32}$$

Since the sequence $\{u_{n_k}\}$ is bounded, it follows that there exists a subsequence $\{u_{n_{k_j}}\}$ of $\{u_{n_k}\}$, which converges weakly to some $z^* \in H$, such that

$$\limsup_{k\to\infty}\langle -z, u_{n_k} - z\rangle = \lim_{j\to\infty}\langle -z, u_{n_{k_j}} - z\rangle = \langle -z, z^* - z\rangle. \tag{33}$$

Using (31), we get

$$w_{n_k} \rightharpoonup z^* \text{ as } k \to \infty,$$

Using (30), we obtain

$$v_{n_k} \rightharpoonup z^* \text{ as } k \to \infty.$$

Now, we show that $z^* \in Sol(C, F)$. Indeed, since $v_{n_k} = P_C(w_{n_k} - \tau_{n_k} F w_{n_k})$, we have

$$\langle w_{n_k} - \tau_{n_k} F w_{n_k} - v_{n_k}, x - v_{n_k} \rangle \leq 0 \ \ \forall x \in C,$$

or equivalently

$$\frac{1}{\tau_{n_k}} \langle w_{n_k} - v_{n_k}, x - v_{n_j} \rangle \leq \langle F w_{n_k}, x - v_{n_k} \rangle \ \ \forall x \in C.$$

Consequently

$$\frac{1}{\tau_{n_k}} \langle w_{n_k} - v_{n_k}, x - v_{n_k} \rangle + \langle F w_{n_k}, v_{n_k} - w_{n_k} \rangle \leq \langle F w_{n_k}, x - w_{n_k} \rangle \ \ \forall x \in C. \tag{34}$$

Being weakly convergent, $\{w_{n_k}\}$ is bounded. Then, by the Lipschitz continuity of F, $\{F w_{n_k}\}$ is bounded. As $\|w_{n_k} - v_{n_k}\| \to 0$, $\{v_{n_k}\}$ is also bounded and $\tau_{n_k} \geq \min\{\tau_1, \frac{\mu}{L}\}$. Passing (34) to limit as $k \to \infty$, we get

$$\liminf_{k \to \infty} \langle F w_{n_k}, x - w_{n_k} \rangle \geq 0 \ \ \forall x \in C. \tag{35}$$

Moreover, we have

$$\langle F v_{n_k}, x - v_{n_k} \rangle = \langle F v_{n_k} - F w_{n_k}, x - w_{n_k} \rangle + \langle F w_{n_k}, x - w_{n_k} \rangle + \langle F v_{n_k}, w_{n_k} - v_{n_k} \rangle. \tag{36}$$

Since $\lim_{k \to \infty} \|w_{n_k} - v_{n_k}\| = 0$ and F is L-Lipschitz continuous on H, we get

$$\lim_{k \to \infty} \|F w_{n_k} - F v_{n_k}\| = 0$$

which, together with (35) and (36) implies that

$$\liminf_{k \to \infty} \langle F v_{n_k}, x - v_{n_k} \rangle \geq 0.$$

Next, we choose a sequence $\{\epsilon_k\}$ of positive numbers decreasing and tending to 0. For each k, we denote by N_k the smallest positive integer such that

$$\langle F v_{n_j}, x - v_{n_j} \rangle + \epsilon_k \geq 0 \ \ \forall j \geq N_k. \tag{37}$$

Since $\{\epsilon_k\}$ is decreasing, it is easy to see that the sequence $\{N_k\}$ is increasing. Furthermore, for each k, since $\{v_{N_k}\} \subset C$ we can suppose $F v_{N_k} \neq 0$ (otherwise, v_{N_k} is a solution) and, setting

$$t_{N_k} = \frac{F v_{N_k}}{\|F v_{N_k}\|^2},$$

we have $\langle F v_{N_k}, t_{N_k} \rangle = 1$ for each k. Now, we can deduce from (37) that for each k

$$\langle F v_{N_k}, x + \epsilon_k t_{N_k} - v_{N_k} \rangle \geq 0.$$

From F is pseudomonotone on H, we get

$$\langle F(x+\epsilon_k t_{N_k}), x+\epsilon_k t_{N_k} - v_{N_k}\rangle \geq 0.$$

This implies that

$$\langle Fx, x - v_{N_k}\rangle \geq \langle Fx - F(x+\epsilon_k t_{N_k}), x+\epsilon_k t_{N_k} - v_{N_k}\rangle - \epsilon_k \langle Fx, t_{N_k}\rangle. \tag{38}$$

Now, we show that $\lim_{k\to\infty} \epsilon_k t_{N_k} = 0$. Indeed, since $w_{n_k} \rightharpoonup z$ and $\lim_{k\to\infty} \|w_{n_k} - v_{n_k}\| = 0$, we obtain $v_{N_k} \rightharpoonup z$ as $k \to \infty$. By $\{v_n\} \subset C$, we obtain $z^* \in C$. Since F satisfies Condition (19), we have

$$0 < \|Fz^*\| \leq \liminf_{k\to\infty} \|Fv_{n_k}\|.$$

Since $\{v_{N_k}\} \subset \{v_{n_k}\}$ and $\epsilon_k \to 0$ as $k \to \infty$, we obtain

$$0 \leq \limsup_{k\to\infty} \|\epsilon_k t_{N_k}\| = \limsup_{k\to\infty} \left(\frac{\epsilon_k}{\|Fv_{n_k}\|} \right) \leq \frac{\limsup_{k\to\infty} \epsilon_k}{\liminf_{k\to\infty} \|Fv_{n_k}\|} = 0,$$

which implies that $\lim_{k\to\infty} \epsilon_k t_{N_k} = 0$.

Now, letting $k \to \infty$, then the right-hand side of (38) tends to zero by F is uniformly continuous, $\{w_{N_k}\}, \{t_{N_k}\}$ are bounded and $\lim_{k\to\infty} \epsilon_k t_{N_k} = 0$. Thus, we get

$$\liminf_{k\to\infty} \langle Fx, x - v_{N_k}\rangle \geq 0.$$

Hence, for all $x \in C$ we have

$$\langle Fx, x - z^*\rangle = \lim_{k\to\infty} \langle Fx, x - v_{N_k}\rangle = \liminf_{k\to\infty} \langle Fx, x - v_{N_k}\rangle \geq 0.$$

By Lemma 2.3, we get

$$z^* \in \text{Sol(C,F)}.$$

Since (33) and the definition of $z = P_{Sol(C,F)}(0)$, we have

$$\limsup_{k\to\infty} \langle -z, u_{n_k} - z\rangle = \langle -z, z^* - z\rangle \leq 0. \tag{39}$$

Combining (32) and (39), we have

$$\begin{aligned}
\limsup_{k\to\infty} \langle -z, u_{n_k+1} - z\rangle &\leq \limsup_{k\to\infty} \langle -z, u_{n_k} - z\rangle \\
&= \langle -z, z^* - z\rangle \\
&\leq 0.
\end{aligned} \tag{40}$$

Hence, by (40), $\lim_{n\to\infty} \dfrac{\theta_n}{\gamma_n} \|u_n - u_{n-1}\| = 0$, $\lim_{k\to\infty} \|u_{n_k+1} - u_{n_k}\| = 0$, Claim 5 and Lemma 2.4, we have $\lim_{n\to\infty} \|u_n - z\| = 0$. That is the desired result.

4 Convergence rate

In this section, we provide a result on the convergence rate of the iterative sequence generated by Algorithm 1 with $\gamma_n = 0$ and the mapping F is L-Lipschitz continuous on H and κ-strongly pseudomonotone on C. The algorithm is of the form:

Initialization: *Given* $\tau_1 > 0, \mu \in (0,1), \gamma \in (0,2), \theta \in [0,1), \beta \in (0,1)$. *Let* $u_0, u_1 \in H$ *be arbitrary and* $\{\alpha_n\}$ *be a nonnegative real numbers sequence such that* $\sum_{n=1}^{\infty} \alpha_n < +\infty$.

Iterative Steps: *Given the current iterate* u_n, *calculate* u_{n+1} *as follows:*

Step 1. *Compute* $w_n = u_n + \theta(u_n - u_{n-1})$

$$v_n = P_C(w_n - \tau_n F w_n).$$

If $w_n = v_n$ *or* $F v_n = 0$ *then stop and* v_n *is a solution of* $Sol(C,F)$. *Otherwise*

Step 2. *Compute*

$$z_n = P_{T_n}(w_n - \gamma \tau_n \eta_n F v_n),$$

where

$$T_n = \{x \in H \mid \langle w_n - \tau_n F w_n - v_n, x - v_n \rangle \leq 0\},$$

$$\eta_n := \begin{cases} \dfrac{\langle w_n - v_n, d_n \rangle}{\|d_n\|^2} & \textit{if } d_n \neq 0, \\ 0 & \textit{if } d_n = 0, \end{cases}$$

and

$$d_n := w_n - v_n - \tau_n (F w_n - F v_n).$$

Step 3. *Compute*

$$u_{n+1} = (1-\beta) w_n + \beta z_n,$$

update

$$\tau_{n+1} := \begin{cases} \min\left\{ \mu \dfrac{\|w_n - v_n\|}{\|F w_n - F v_n\|}, \tau_n + \alpha_n \right\} & \textit{if } F w_n \neq F v_n, \\ \tau_n + \alpha_n & \textit{otherwise.} \end{cases}$$

Set $n := n+1$ *and go to* ***Step 1***.

Using the technique in [35] we obtain the following result.

Theorem 4.2 *Assume that* F *is* L*-Lipschitz continuous on* H *and* κ*-strongly pseudomonotone on* C. *Let* $\delta \in (0,1)$ *be arbitrary and* θ *be such that*

$$0 \leq \theta \leq \min\left\{ \frac{\xi}{2+\xi}, \frac{\sqrt{(1+\delta\xi)^2 + 4\delta\xi} - (1+\delta\xi)}{2}, (1-\delta)\left(1 - \frac{(1-\gamma)\gamma(1-\mu)^2}{2(1+\mu)^2}\right) \right\}, \tag{41}$$

where $\xi := \dfrac{1-\beta}{\beta}$. *Then the sequence* $\{u_n\}$ *generated by Algorithm* 2 *converges strongly to the unique solution* x^* *of* (1) *with an R-linear rate.*

Proof First, we will shall that, there exists $\rho, \xi \in (0, 1)$ such that

$$\|u_{n+1} - x^*\|^2 \leq \rho\|w_n - x^*\|^2 - \xi\|u_{n+1} - w_n\|^2.$$

Indeed, under assumptions made, it was proved that (1) has a unique solution [20]. From the κ-strong pseudomonotonicity of F, we have $\langle Fv_n, v_n - x^*\rangle \geq \kappa\|v_n - x^*\|^2$. This implies that

$$\begin{aligned}\langle Fv_n, u_{n+1} - x^*\rangle = \langle Fv_n, u_{n+1} - v_n\rangle + \langle Fv_n, v_n - x^*\rangle \geq \langle Fv_n, u_{n+1} - v_n\rangle \\ \geq \kappa\|v_n - x^*\|^2 + \langle Fv_n, u_{n+1} - v_n\rangle.\end{aligned}$$

Therefore

$$-2\gamma\eta_n\tau_n\langle Fv_n, u_{n+1} - x^*\rangle \leq -2\gamma\eta_n\tau_n\kappa\|v_n - x^*\|^2 - 2\gamma\eta_n\tau_n\langle Fv_n, u_{n+1} - v_n\rangle. \tag{42}$$

Substituting (42) into (9), we get

$$\|z_n - x^*\|^2 \leq \|w_n - x^*\|^2 - \|z_n - w_n\|^2 - 2\gamma\eta_n\tau_n\kappa\|v_n - x^*\|^2 - 2\gamma\eta_n\tau_n\langle Fv_n, z_n - v_n\rangle. \tag{43}$$

Again, substituting (16) into (43), we obtain

$$\|z_n - x^*\|^2 \leq \|w_n - x^*\|^2 - \|w_n - z_n - \gamma\eta_n d_n\|^2 - (2-\gamma)\gamma\eta_n^2\|d_n\|^2 - 2\gamma\eta_n\tau_n\kappa\|v_n - x^*\|^2. \tag{44}$$

Combining (26) and (44), we get

$$\begin{aligned}\|z_n - x^*\|^2 \leq \|w_n - x^*\|^2 - \|w_n - z_n - \gamma\eta_n d_n\|^2 - (2-\gamma)\gamma\frac{\left(1 - \dfrac{\mu\tau_n}{\tau_{n+1}}\right)^2}{\left(1 + \dfrac{\mu\tau_n}{\tau_{n+1}}\right)^2}\|w_n - v_n\|^2 \\ - 2\gamma\eta_n\tau_n\kappa\|v_n - x^*\|^2 \quad \forall n \geq n_0.\end{aligned} \tag{45}$$

It follows from (45) that

$$\|z_n - x^*\|^2 \leq \|w_n - x^*\|^2 - (2-\gamma)\gamma\frac{\left(1 - \dfrac{\mu\tau_n}{\tau_{n+1}}\right)^2}{\left(1 + \dfrac{\mu\tau_n}{\tau_{n+1}}\right)^2}\|w_n - v_n\|^2 - 2\gamma\eta_n\tau_n\kappa\|v_n - x^*\|^2 \quad \forall n \geq n_0. \tag{46}$$

On the other hand, we have

$$\begin{aligned}\|u_{n+1}-x^*\|^2 =&\|(1-\beta)w_n-\beta z_n-x^*\|^2\\ =&(1-\beta)\|w_n-x^*\|^2+\beta\|z_n-x^*\|^2-\beta(1-\beta)\|w_n-z_n\|^2.\end{aligned}$$

Since the definition of $\{u_n\}$ we deduce $w_n-z_n=\frac{1}{\beta}(u_{n+1}-w_n)$. Thus,

$$\|u_{n+1}-x^*\|^2=(1-\beta)\|w_n-x^*\|^2+\beta\|z_n-x^*\|^2-\frac{1}{\beta}(1-\beta)\|u_{n+1}-w_n\|^2. \tag{47}$$

Substituting (46) into (47), we get

$$\begin{aligned}\|u_{n+1}-x^*\|^2 =&\|w_n-x^*\|^2-\beta(2-\gamma)\gamma\frac{\left(1-\frac{\mu\tau_n}{\tau_{n+1}}\right)^2}{\left(1+\frac{\mu\tau_n}{\tau_{n+1}}\right)^2}\|w_n-v_n\|^2-2\beta\gamma\eta_n\tau_n\kappa\|v_n-x^*\|^2\\ &-\frac{1}{\beta}(1-\beta)\|u_{n+1}-w_n\|^2 \ \ \forall n\geq n_0.\end{aligned} \tag{48}$$

Substituting (25) into (48), we deduce

$$\begin{aligned}\|u_{n+1}-x^*\|^2 =&\|w_n-x^*\|^2-\beta(2-\gamma)\gamma\frac{\left(1-\frac{\mu\tau_n}{\tau_{n+1}}\right)^2}{\left(1+\frac{\mu\tau_n}{\tau_{n+1}}\right)^2}\|w_n-v_n\|^2-2\beta\gamma\frac{1-\mu\frac{\tau_n}{\tau_{n+1}}}{(1+\mu\frac{\tau_n}{\tau_{n+1}})^2}\tau_n\kappa\|v_n-x^*\|^2\\ &-\frac{1}{\beta}(1-\beta)\|u_{n+1}-w_n\|^2 \ \ \forall n\geq n_0.\end{aligned} \tag{49}$$

Setting

$$\beta^* := \min\left\{\frac{(2-\gamma)\gamma(1-\mu)^2}{2(1+\mu)^2}, \beta\gamma\frac{1-\mu}{(1+\mu)^2}\kappa\tau\right\} \text{ where } \tau=\lim_{n\to\infty}\tau_n,$$

we have

$$1>\lim_{n\to\infty}\frac{(2-\gamma)\gamma\left(1-\mu\frac{\lambda_n}{\lambda_{n+1}}\right)^2}{\left(1+\mu\frac{\lambda_n}{\lambda_{n+1}}\right)^2}=\frac{(2-\gamma)\gamma(1-\mu)^2}{(1+\mu)^2}\geq 2\beta^*.$$

and

$$\lim_{n\to\infty}\beta\gamma\frac{1-\mu\frac{\tau_n}{\tau_{n+1}}}{(1+\mu\frac{\tau_n}{\tau_{n+1}})^2}\tau_n\kappa=\beta\gamma\frac{1-\mu}{(1+\mu)^2}\kappa\tau\geq\beta^*.$$

Using (49), hence there exists $n_1 > n_0$ such that for all $n \geq n_1$

$$\begin{aligned}\|u_{n+1} - x^*\|^2 \leq &\|w_n - x^*\|^2 - \xi\|u_{n+1} - w_n\|^2 - 2\beta^*\|w_n - v_n\|^2 - 2\beta^*\|v_n - x^*\|^2\\ \leq &(1-\beta^*)\|w_n - x^*\|^2 - \xi\|u_{n+1} - w_n\|^2\\ = &\rho\|w_n - x^*\|^2 - \xi\|u_{n+1} - w_n\|^2,\end{aligned} \tag{50}$$

where $\rho := 1 - \beta^* \in (0,1)$ and $\xi := \dfrac{1-\beta}{\beta}$.

Next, we prove that the sequence $\{u_n\}$ converges strongly to x^* with an R linear rate. Indeed, we have

$$\begin{aligned}\|w_n - x^*\|^2 = &\|(1+\theta)(u_n - x^*) - \theta(u_{n-1} - x^*)\|^2\\ = &(1+\theta)\|u_n - x^*\|^2 - \theta\|u_{n-1} - x^*\|^2 + \theta(1+\theta)\|u_n - u_{n-1}\|^2\end{aligned}$$

and

$$\begin{aligned}\|u_{n+1} - w_n\|^2 = &\|u_{n+1} - u_n - \theta(u_n - u_{n-1})\|^2\\ = &\|u_{n+1} - u_n\|^2 + \theta^2\|u_n - u_{n-1}\|^2 - 2\theta\left\langle u_{n+1} - u_n, u_n - u_{n-1}\right\rangle\\ \geq &\|u_{n+1} - u_n\|^2 + \theta^2\|u_n - u_{n-1}\|^2 - 2\theta\|u_{n+1} - u_n\|\|u_n - u_{n-1}\|\\ \geq &\|u_{n+1} - u_n\|^2 + \theta^2\|u_n - u_{n-1}\|^2 - \theta\|u_{n+1} - u_n\|^2 - \theta\|u_n - u_{n-1}\|^2\\ \geq &(1-\theta)\|u_{n+1} - u_n\|^2 - \theta(1-\theta)\|u_n - u_{n-1}\|^2.\end{aligned}$$

Combining these inequalities with (50) we obtain for all $n \geq n_1$ that

$$\begin{aligned}\|u_{n+1} - x^*\|^2 \leq &\rho(1+\theta)\|u_n - x^*\|^2 - \rho\theta\|u_{n-1} - x^*\|^2 + \rho\theta(1+\theta)\|u_n - u_{n-1}\|^2\\ &- \xi(1-\theta)\|u_{n+1} - u_n\|^2 + \xi\theta(1-\theta)\|u_n - u_{n-1}\|^2,\end{aligned}$$

or equivalently

$$\begin{aligned}&\|u_{n+1} - x^*\|^2 - \rho\theta\|u_n - x^*\|^2 + \xi(1-\theta)\|u_{n+1} - u_n\|^2\\ \leq &\rho\left[\|u_n - x^*\|^2 - \theta\|u_{n-1} - x^*\|^2 + \xi(1-\theta)\|u_n - u_{n-1}\|^2\right]\\ &- (\rho\xi(1-\theta) - \rho\theta(1+\theta) - \xi\theta(1-\theta))\|u_n - u_{n-1}\|^2.\end{aligned}$$

Setting

$$a_n := \|u_n - x^*\|^2 - \theta\|u_{n-1} - x^*\|^2 + \xi(1-\theta)\|u_n - u_{n-1}\|^2,$$

since $\rho \in (0,1)$ we can write

$$\begin{aligned}a_{n+1} \leq &\|u_{n+1} - x^*\|^2 - \rho\theta\|u_n - x^*\|^2 + \xi(1-\theta)\|u_{n+1} - u_n\|^2\\ \leq &\rho a_n - (\rho\xi(1-\theta) - \rho\theta(1+\theta) - \xi\theta(1-\theta))\|u_n - u_{n-1}\|^2.\end{aligned}$$

Note that from (41) we have

$$\begin{aligned}\theta \leq &(1-\delta)\left(1-\frac{(1-\gamma)\gamma(1-\mu)}{2(1+\mu)}\right)\\ \leq &(1-\delta)\left(1-\min\left\{\frac{(2-\gamma)\gamma(1-\mu)^2}{2(1+\mu)^2}, \beta\gamma\frac{1-\mu}{(1+\mu)^2}\kappa\tau\right\}\right)=(1-\delta)\rho,\end{aligned}$$

which implies

$$\xi\theta(1-\theta) \leq (1-\delta)\rho\xi(1-\theta). \tag{51}$$

Since

$$\theta \leq \frac{\sqrt{(1+\delta\xi)^2+4\delta\xi}-(1+\delta\xi)}{2}$$

it holds

$$\theta^2+(1+\delta\xi)\theta-\delta\xi \leq 0,$$

or equivalently

$$\theta(1+\theta) \leq \delta\xi(1-\theta).$$

Hence

$$\rho\theta(1+\theta) \leq \delta\rho\xi(1-\theta). \tag{52}$$

From (51) and (52) we deduce

$$\rho\xi(1-\theta)-\rho\theta(1+\theta)-\xi\theta(1-\theta) \geq 0.$$

Moreover, since $\theta \leq \frac{\xi}{2+\xi}$, we have $\theta \leq \frac{\xi(1-\theta)}{2}$, which implies

$$\begin{aligned}a_n =&(1-\xi(1-\theta))\|u_n-x^*\|^2+\xi(1-\theta)\left(\|u_n-x^*\|^2+\|u_n-u_{n-1}\|^2\right)-\theta\|u_{n-1}-x^*\|^2\\ \geq&(1-\xi(1-\theta))\|u_n-x^*\|^2+\frac{\xi(1-\theta)}{2}\|u_{n-1}-x^*\|^2-\theta\|u_{n-1}-x^*\|^2\\ \geq&(1-\xi(1-\theta))\|u_n-x^*\|^2 \geq 0.\end{aligned}$$

Hence for all $n \geq n_1$ it holds

$$a_{n+1} \leq \rho a_n \leq \ldots \leq \rho^{n-n_1+1}a_{n_1}.$$

This follows that

$$\|u_n-x^*\|^2 \leq \frac{a_{n_1}}{\rho^{n_1}(1-\xi(1-\theta))}\rho^n,$$

which implies that $\{u_n\}$ converges R-linearly to x^*.

5 Numerical experiments

In this section, we first present computational experiments to illustrate our newly proposed algorithm for solving variational inequality arising in optimal control problem.

Let T be a positive number. Denote by $L_2([0,T],\mathbb{R}^m)$ the Hilbert space of square integrable, measurable vector functions $u:[0,T]\to\mathbb{R}^m$ with inner product $\langle u,v\rangle=\int_0^T\langle u(t),v(t)\rangle dt$, and norm $\|u\|_2=\sqrt{\langle u,u\rangle}$. We consider the following optimal control problem:

$$u^*(t)=\operatorname{argmin}\{z(u):u\in U\},$$

on the interval [0, T] and assume that such a control exists. Here U is the set of admissible controls, which has the form of an m-dimensional box and consists of piecewise continuous function:

$$U=\left\{u(t)\in L_2([0,T],\mathbb{R}^m):u_i(t)\in[u_i^-,u_i^+],i=1,2,\dots,m\right\}.$$

Specially, the control can be bang-bang (piecewise constant function). The terminal objective has the form

$$z(u)=\phi(x(T)),$$

where ϕ is a convex and differentiable function, defined on the attainability set.

Suppose that the trajectory $x(t)\in L_2([0,T])$ satisfies constraints in the form of a system of linear differential equation:

$$\dot{x}(t)=A(t)x(t)+B(t)u(t),\quad x(0)=x_0,\quad t\in[0,T],$$

where $A(t)\in\mathbb{R}^{n\times n}$, $B(t)\in\mathbb{R}^{n\times m}$ are given continuous matrices for every $t\in[0,T]$. By the Pontryagin Maximum Principle there exists a function $p^*\in L_2([0,T])$ such that the triple (x^*,p^*,u^*) solves for a.e. $t\in[0,T]$ the system

$$\begin{cases}\dot{x}^*(t) & =A(t)x^*(t)+B(t)u^*(t)\\ x^*(0) & =x_0,\end{cases}$$

$$\begin{cases}\dot{p}^*(t) & =-A(t)^\top p^*(t)\\ p^*(T) & =\nabla\phi(x(T)),\end{cases}$$

$$0\in B(t)^\top p^*(t)+N_U(u^*(t)),$$

where $N_U(u)$ is the normal cone to U at u. Denoting $Gu(t):=B(t)^\top p(t)$, it is known that Gu is the gradient of the objective cost function z (see [38] and the references contained therein). We can write the inclusion problem (54) as a variational inequality problem: find $u\in U$ such that

$$\langle Gu,v-u\rangle\geq 0,\ \forall v\in U.$$

Discretizing the continuous functions and choosing a natural number N with the *mesh size* $h := T/N$, we identify any discretized control $u^N := (u_0, u_1, \dots, u_{N-1})$ with its piecewise constant extension:

$$u^N(t) = u_i \text{ for } t \in \left[t_i, t_{i+1}\right),\ i = 0, 1, \dots, N.$$

Moreover, we identity any discretized state $x^N := (x_0, x_1, \dots, x_N)$ with its piecewise linear interpolation

$$x^N(t) = x_i + \frac{t - t_i}{h}\left(x_{i+1} - x_i\right),\ \text{for } t \in \left[t_i, t_{i+1}\right),\ i = 0, 1, \dots, N-1.$$

Similarly we get the co-state variable $p^N := (p_0, p_1, \dots, p_N)$ (see [38] for more details).

Next, we consider three examples in which the terminal function is not linear.

In the numerical results listed in the following tables, 'Iter.' denotes the number of iterations and 'CPU in s' denotes the execution time in seconds. Besides, '–' means that the algorithm can't reach the error conditions because the inner loop of the algorithm reaches infinite loop after some steps.

In the following three examples, we take $N = 100$. The initial control $u_0(t)$ is chosen randomly in $[-1, 1]$, and the termination condition is controlled by the relative solution error, defined by (Table 1)

$$\text{RSE} = \frac{\|u_n - u^*\|}{\|u_n\|}$$

at the current u_n.

Example 1 (See [39, Rocket Car])

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}\left(\left(x_1(5)\right)^2 + \left(x_2(5)\right)^2\right) \\ \text{subject to} & \dot{x_1}(t) = x_2(t), \\ & \dot{x_2}(t) = u(t),\ \forall t \in [0, 5], \\ & x_1(0) = 6,\ x_2(0) = 1, \\ & u(t) \in [-1, 1]. \end{array}$$

The corresponding exact optimal control is

$$u^*(t) = \begin{cases} 1 & \text{if } t \in (\tau, 5] \\ -1 & \text{if } t \in (0, \tau], \end{cases}$$

where $\tau = 3.5174292$.

In Fig. 1 and Fig. 2, we take RSE ≤ 0.3. Figure 1 displays the approximate optimal control and the corresponding trajectories. Other parameters are selected as follows:

Algorithm 1: $\tau_1 = 0.5, \mu = 0.7, \gamma = 1.9, \theta = 0.5, \gamma_n = \frac{1}{3000(n+1)}, \epsilon_n = \frac{1}{3000(n+1)^{1.1}}$ and $\alpha_n = \frac{1}{n^2}$;

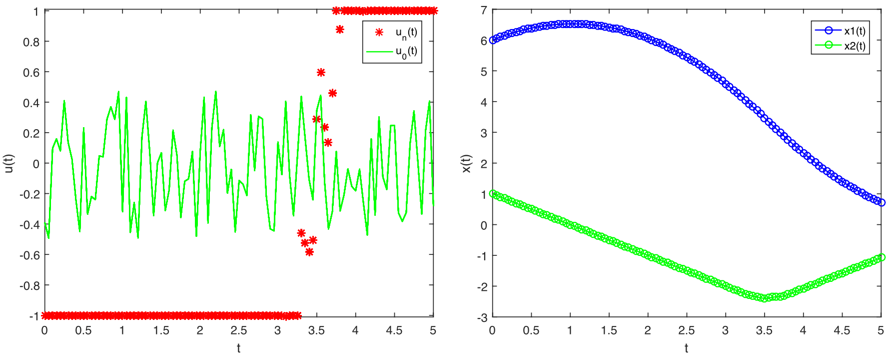

Fig. 1 Random initial control (green) and optimal control (red) on the left and optimal trajectories on the right computed by Alg. 1 for Example 1

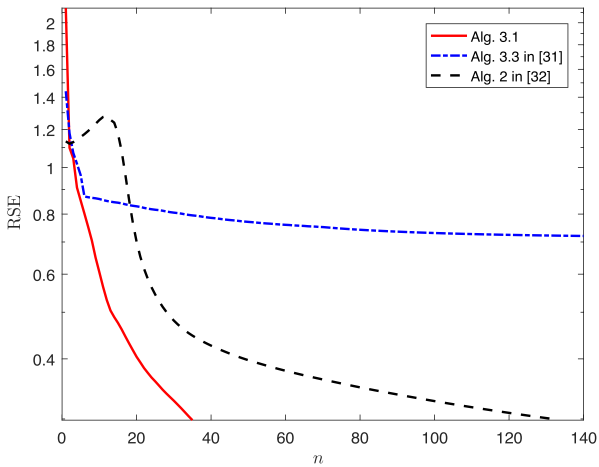

Fig. 2 Comparison of Alg. 1 with Alg. 3.3 in [31] and Alg. 2 in [32] for Example 1

Algorithm 3.3 in [31]: $\sigma = 0.1, \gamma = 0.6$ and $\alpha_n = \frac{1}{3000(n+1)}$;
Algorithm 2 in [32]: $\lambda_0 = 0.6, \mu = 0.9$ and $\alpha_n = \frac{1}{3000(n+1)}$.

The corresponding results reported in Fig. 2 and Table 1 illustrate that Algorithm 1 behaves better than Algorithm 3.3 in [31] and Algorithm 2 in [32] in term of time and steps.

Table 1 Numerical results for Algorithm 1, Algorithm 3.3 in [31] and Algorithm 2 in [32] in Example 1

RSE	Iter.			CPU in s		
	Alg. 1	Alg. 3.3 in [31]	Alg. 2 in [32]	Alg. 3.1	Alg. 3.3 in [31]	Alg. 2 in [32]
0.8	7	33	19	0.0030	0.0413	0.0069
0.5	14	–	29	0.0055	–	0.0104
0.3	35	–	132	0.0240	–	0.0465

Example 2 (See [40, Example 6.3])

$$\begin{array}{ll} \text{minimize} & -x_1(2) + \left(x_2(2)\right)^2 \\ \text{subject to} & \dot{x_1}(t) = x_2(t), \\ & \dot{x_2}(t) = u(t),\ \forall t \in [0, 2], \\ & x_1(0) = 0,\ x_2(0) = 0, \\ & u(t) \in [-1, 1]. \end{array}$$

The corresponding exact optimal control is

$$u^* = \begin{cases} 1 & \text{if } t \in [0, 6/5) \\ -1 & \text{if } t \in (6/5, 2], \end{cases}$$

In Fig. 3 and Fig. 4, we take RSE $\leq 10^{-5}$. Figure 3 displays the approximate optimal control and the corresponding trajectories. Other parameters are selected as follows:

Algorithm 1: $\tau_1 = 0.2, \mu = 0.8, \gamma = 1.9, \theta = 0.5, \gamma_n = \frac{1}{10000(n+1)}, \epsilon_n = \frac{1}{10000(n+1)^{1.1}}$ and $\alpha_n = \frac{1}{n^2}$;

Algorithm 3.3 in [31]: $\sigma = 0.1, \gamma = 0.3$ and $\alpha_n = \frac{1}{10000(n+1)}$;

Algorithm 2 in [32]: $\lambda_0 = 0.3, \mu = 0.9$ and $\alpha_n = \frac{1}{10000(n+1)}$.

In Fig. 4 and Table 2, we can obtain that the performance of our Algorithm 1 in time and steps is better than Algorithm 2 in [32] and Algorithm 3.3 in [31].

Example 3 (control of a harmonic oscillator)

$$\begin{array}{ll} \text{minimize} & x_2(3\pi) \\ \text{subject to} & \dot{x_1}(t) = x_2(t), \\ & \dot{x_2}(t) = -x_1(t) + u(t),\ \forall t \in [0, 3\pi], \\ & x(0) = 0, \\ & u(t) \in [-1, 1]. \end{array}$$

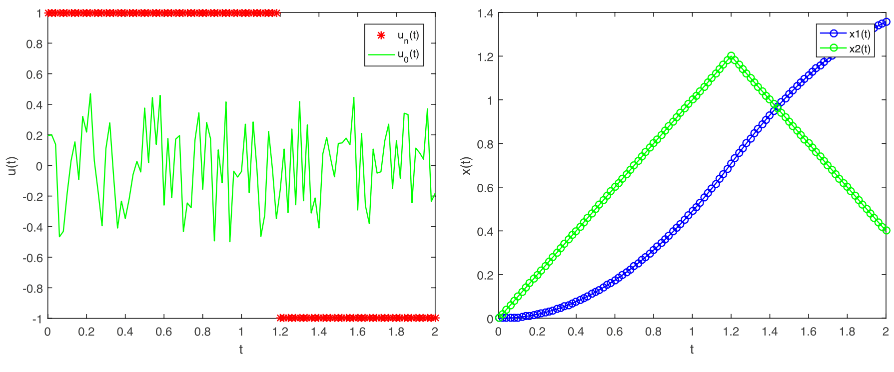

Fig. 3 Random initial control (green) and optimal control (red) on the left and optimal trajectories on the right computed by Algorithm 1 for Example 2

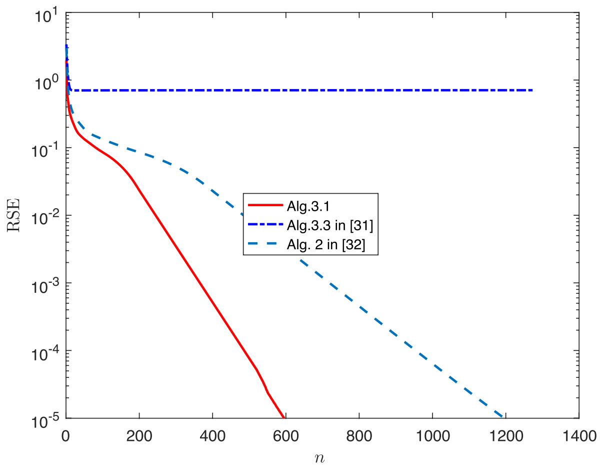

Fig. 4 Comparison of Alg. 1 with Alg. 3.3 in [31] and Alg. 2 in [32] for Example 2

Table 2 Numerical results for Algorithm 1, Algorithm 3.3 in [31] and Algorithm 2 in [32] in Example 2

RSE	Iter.			CPU in s		
	Alg. 1	Alg. 3.3 in [31]	Alg. 2 in [32]	Alg. 3.1	Alg. 3.3 in [31]	Alg. 2 in [32]
10^{-1}	82	–	161	0.0270	–	0.0523
10^{-3}	366	–	719	0.1209	–	0.2348
10^{-5}	595	–	1196	0.2153	–	0.3976

The exact optimal control in this problem is known:

$$u^* = \begin{cases} 1 & \text{if } t \in [0, \pi/2) \cup (3\pi/2, 5\pi/2) \\ -1 & \text{if } t \in (\pi/2, 3\pi/2) \cup (5\pi/2, 3\pi]. \end{cases}$$

In Fig. 5 and Fig. 6, we take RSE ≤ 0.1. Figure 5 displays the approximate optimal control and the corresponding trajectories. Other parameters are selected as follows:

Algorithm 1: $\tau_1 = 1.9, \mu = 0.8, \gamma = 1.2, \theta = 0.5, \gamma_n = \frac{1}{100(n+1)}, \epsilon_n = \frac{1}{100(n+1)^{1.1}}$ and $\alpha_n = \frac{1}{n^2}$;

Algorithm 3.3 in [31]: $\sigma = 0.0001, \gamma = 0.1$ and $\alpha_n = \frac{1}{100(n+1)}$;

Algorithm 2 in [32]: $\lambda_0 = 1.9, \mu = 0.9$ and $\alpha_n = \frac{1}{100(n+1)}$.

As shown in Fig. 6 and Table 3, Algorithm 1 performs better than Algorithm 3.3 in [31] and Algorithm 2 in [32].

Next we provide several numerical examples to demonstrate the efficiency of the proposed algorithm compared to some known algorithms.

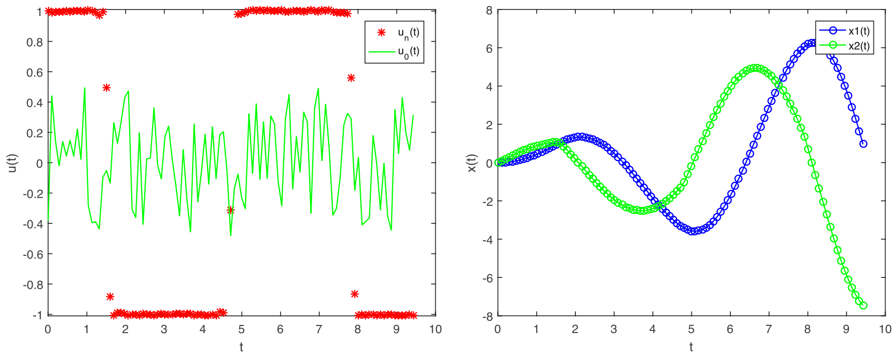

Fig. 5 Random initial control (green) and optimal control (red) on the left and optimal trajectories on the right computed by Algorithm 1 for Example 3

Fig. 6 Comparison of Alg. 1 with Alg. 3.3 in [31] and Alg. 2 in [32] for Example 3

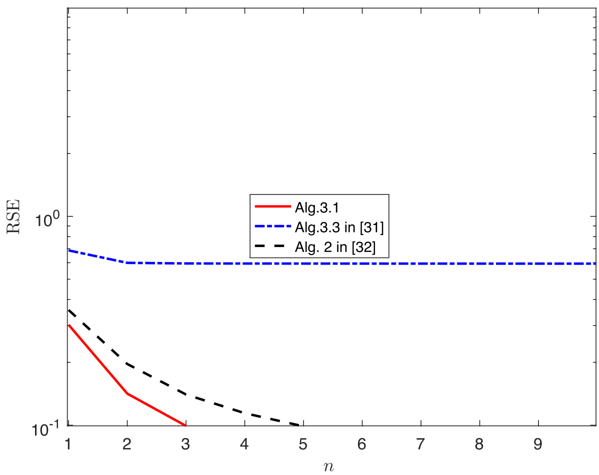

Table 3 Numerical results for Algorithm 1, Algorithm 3.3 in [31] and Algorithm 2 in [32] in Example 3

RSE	Iter.			CPU in s		
	Alg. 1	Alg. 3.3 in [31]	Alg. 2 in [32]	Alg. 3.1	Alg. 3.3 in [31]	Alg. 2 in [32]
0.7	1	3	1	0.0005	0.0009	0.0007
0.2	2	–	3	0.0014	–	0.0016
0.1	4	–	7	0.0077	–	0.0096

Example 4 In the example, we study an important Nash-Cournot oligopolistic market equilibrium model, which was proposed originally by Murphy et. al. [25] as a convex optimization problem. Later, Harker reformulated it as a monotone variational inequality in [17]. We provide only a short description of the problem, for

more details we refer to [13, 17, 25]. There are N firms, each of them supplies a homogeneous product in a non-cooperative fashion. Let $q_i \geq 0$ be the ith firm's supply at cost $f_i(q_i)$ and $p(Q)$ be the inverse demand curve, where $Q \geq 0$ is the total supply in the market, i.e., $Q = \sum_{i=1}^{N} q_i$. A Nash equilibrium solution for the market defined above is a set of nonnegative output levels $(q_1^*, q_2^*, ..., q_N^*)$ such that q_i^* is an optimal solution to the following problem for all $i = 1, 2..., N$:

$$\max_{q_i \geq 0} q_i p(q_i + Q_i^*) - f_i(q_i) \tag{53}$$

where

$$Q_i^* = \sum_{j=1, j\neq i}^{N} q_j^*.$$

A variational inequality equivalent to (53) is (see [17])

$$\text{find} \quad (q_1^*, q_2^*, ..., q_N^*) \in \mathbb{R}_+^N \quad \text{such that} \quad \langle F(q^*), q - q^* \rangle \geq 0 \quad \forall q \in \mathbb{R}_+^N, \tag{54}$$

where $F(q^*) = (F_1(q^*), F_2(q^*), ..., F_N(q^*))$ and

$$F_i(q^*) = f_i'(q_i^*) - p\left(\sum_{j=1}^{N} q_j^*\right) - q_i^* p'\left(\sum_{j=1}^{N} q_j^*\right).$$

As in the classical example of the Nash-Cournot equilibrium [17, 25], we consider an oligopoly with $N = 5$ firms, each with the inverse demand function p and the cost function f_i take the form

$$p(Q) = 5000^{1/1.1} Q^{-1/1.1} \quad \text{and} \quad f_i(q_i) = c_i q_i + \frac{\beta_i}{\beta_i + 1} L_i^{\frac{-1}{\beta_i}} q_i^{\frac{\beta_i+1}{\beta_i}}$$

and the parameters c_i, L_i, β_i as in [25], see Table 4.

The process is started with the initial $x_0 = (10, 10, 10, 10)^T$ and $x_1 = 0.9 * x_0$ and stopping conditions is Residual $:= \|w_n - v_n\| \leq 10^{-9}$ or the number of iterations $\geq$ 200 for all algorithms. Other parameters are selected as follows:

Algorithm 1: $\tau_1 = 1.8, \mu = 0.7, \gamma = 1.99, \theta = 0.5, \gamma_n = \frac{1}{10000000*(n+1)}$ and $\alpha_n = \frac{1}{n^2}$

Table 4 Parameters for experiment

Firm i	c_i	L_i	β_i
1	10	5	1.2
2	8	5	1.1
3	6	5	1.0
4	4	5	0.9
5	2	5	0.8

Algorithm 1 in [29]: $\nu_0 = 1, \theta = 0.9$ and $\rho_n = \rho = 0.4$.

Algorithm 2 in [32]: $\lambda_0 = 1, \mu = 0.7$ and $\alpha_n = \frac{1}{10000000*(n+1)}$. The numerical results are described in Fig. 7.

Example 5 Consider the following fractional programming problem:

$$\min f(x) = \frac{x^T Qx + a^T x + a_0}{b^T x + b_0}$$

subject to $x \in X := \{x \in \mathbb{R}^m : b^T x + b_0 > 0\}$,

where Q is an $m \times m$ symmetric matrix, $a, b \in \mathbb{R}^m$, and $a_0, b_0 \in \mathbb{R}$. It is well known that f is pseudo-convex on X when Q is positive-semidefinite. We consider the following cases:

Case 1:

$$Q = \begin{pmatrix} 5 & -1 & 2 & 0 \\ -1 & 5 & -1 & 3 \\ 2 & -1 & 3 & 0 \\ 0 & 3 & 0 & 5 \end{pmatrix}, a = \begin{pmatrix} 1 \\ -2 \\ -2 \\ 1 \end{pmatrix}, b = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}, a_0 = -2, b_0 = 4.$$

We minimize f over $C := \{x \in \mathbb{R}^4 : 1 \le x_i \le 10, i = 1, ..., 4\} \subset X$. It is easy to verify that Q is symmetric and positive definite in $\mathbb{R}^4$ and consequently f is pseudo-convex on X.

The process is started with the initial $x_0 = (20, -20, 20, -20)^T$ and $x_1 = 0.9 * x_0$ and stopping conditions is Residual $:= \|u_n - q_n\| \le 10^{-9}$ or the number of iterations ≥ 200 for all algorithms. Other parameters are selected as follows:

Algorithm 1: $\tau_1 = 0.6, \mu = 0.6, \gamma = 1.5, \theta = 0.5, \alpha_n = \frac{1}{n^2}$ and $\gamma_n = \frac{1}{C*(n+1)}$ with $C = 10^7$.

Algorithm 1 in [29]: $\nu_0 = 1, \theta = 0.9.\mu = 0.6$ and $\rho_n = \rho = 0.4$.

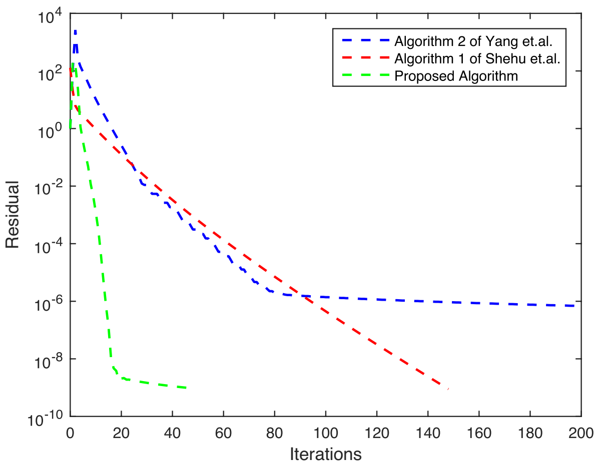

Fig. 7 Comparison of Alg. 1 with Alg. 1 in [29] and Alg. 2 in [32] for Example 4. Execution times of the Algorithms respectively are 0.019, 0.048, 0.07 (seconds)

Algorithm 2 in [32]: $\lambda_0 = 0.6, \mu = 0.6$ and $\alpha_n = \frac{1}{C*(n+1)}$. The numerical results are described in Fig. 8.

Case 2: In the second experiment, to make the problem even more challenging. Let matrix $A : \mathbb{R}^{m\times m} \to \mathbb{R}^{m\times m}$, vectors $c, d, y_0 \in \mathbb{R}^m$ and c_0, d_0 are generated from a normal distribution with mean zero and unit variance. We put $e = (1, 1, ..., 1)^T \in \mathbb{R}^m, Q = A^T A + I, a := e + c, b := e + d, a_0 = 1 + c_0, b_0 = 1 + d_0$ We minimize f over $C := \{x \in \mathbb{R}^m : 1 \le x_i \le 10, i = 1, ..., m\} \subset X$. Because Matrix Q is symmetric and positive definite in $\mathbb{R}^m$ and consequently f is pseudo-convex on X. The process is started with the initial $x_0 := m * y_0$ and $x_1 = 0.9 * x_0$, stopping conditions and parameters as Case 1. The numerical results are described in Fig. 9.

The corresponding results reported in Figs. 7, 8, and 9 show that Alg. 1 behaves better than Algorithm 1 in [29] and Algorithm 2 in [32].

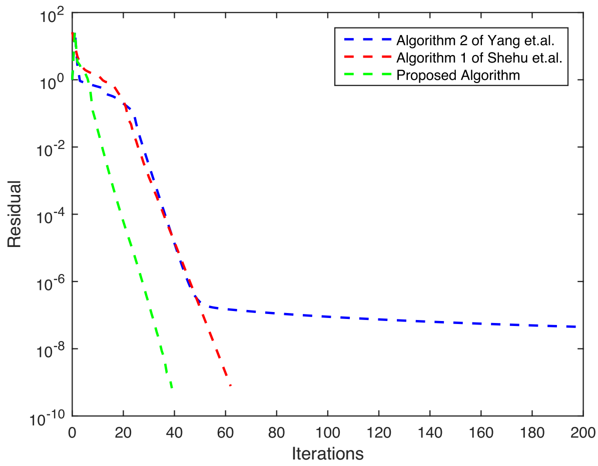

Fig. 8 Comparison of Alg. 1 with Alg. 1 in [29] and Alg. 2 in [32] for Example 5. Execution times of the Algorithms respectively are 0.18, 0.24, 0.8 (seconds)

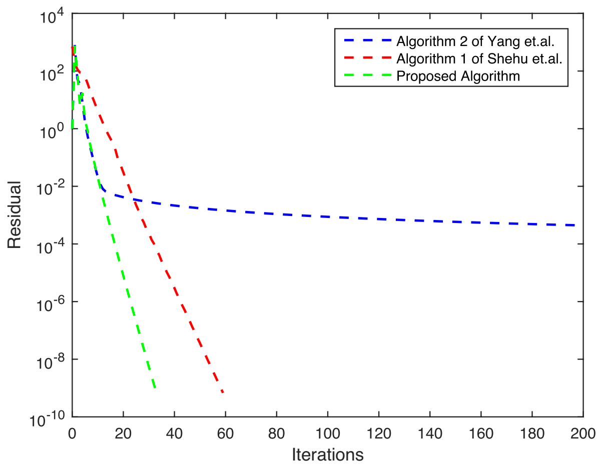

Fig. 9 Comparison of Alg. 1 with Alg. 1 in [29] and Alg. 2 in [32] for Example 5 with $n = 100$. Execution times of the Algorithms respectively are 0.24, 0.37, 1.25 (seconds)

6 Conclusions

In this paper, we introduce some results of the modified subgradient extragradient method for solving pseudomonotone variational inequalities in real Hilbert spaces. The algorithm only needs to calculate one projection onto the feasible set C per iteration and does not require the prior information of the Lipschitz constant of the cost mapping. First, we prove a sufficient condition for a strong convergence of the proposed algorithm under relaxed assumptions. Second, the proposed algorithm is proved to converge strongly with an R-linear convergence rate, under Lipschitz continuity and strong pseudomonotonicity assumptions. Finally, several numerical results are presented to illustrate the efficiency and advantages of the proposed method.

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Data availability The datasets generated during and/or analyzed during the current study are available from the corresponding author on reasonable request.

Declarations

Conflict of interest The authors declare no competing interests.

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