



# Generalized discrete Lotka-Volterra equation, orthogonal polynomials and generalized epsilon algorithm

Xiao-Min Chen<sup>1</sup> · Xiang-Ke Chang<sup>2,3</sup> · Yi He<sup>4,5</sup> · Xing-Biao Hu<sup>2,3</sup>

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## Abstract

In this paper, we propose a generalized discrete Lotka-Volterra equation and explore its connections with symmetric orthogonal polynomials, Hankel determinants and convergence acceleration algorithms. Firstly, we extend the fully discrete Lotka-Volterra equation to a generalized one with a sequence of given constants  $\{u_0^{(n)}\}$  and derive its solution in terms of Hankel determinants. Then, it is shown that the discrete equation of motion is transformed into a discrete Riccati system for a discrete Stieltjes function, hence leading to a complete linearization. Besides, we obtain its Lax pair in terms of symmetric orthogonal polynomials by generalizing the Christoffel transformation for the symmetric orthogonal polynomials. Moreover, a generalization of the famous Wynn's  $\epsilon$ -algorithm is also derived via a Miura transformation to the generalized discrete Lotka-Volterra equation. Finally, the numerical effects of this generalized  $\epsilon$ -algorithm are discussed by applying to some linearly, logarithmically convergent sequences and some divergent series.

**Keywords** Fully discrete Lotka-Volterra lattice · Orthogonal polynomials · Convergence acceleration algorithm · Hankel determinant

**Mathematics Subject Classification (2010)** 37K10 · 33C45 · 65B05

## 1 Introduction

As a class of intriguing systems in the nonlinear world, discrete integrable equations have always been highly concerned by scientists due to their rich mathematical structures and many applications in natural sciences. It is known that discrete integrable systems have some fundamental connections with orthogonal polynomials (OPs) via

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Dedicated to Professor Claude Brezinski on the occasion of his 80th birthday

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✉ Xiang-Ke Chang  
changxk@lsec.cc.ac.cn

Extended author information available on the last page of the article

the discrete Lax pair formalism [16, 18, 41, 44, 45]. (Here, we remark that a *Lax Pair* consists of a pair of linear systems, the compatibility condition of which can lead to a given (partial) differential (or difference) equation (or system) [27]. If a Lax pair exists, it implies that a corresponding equation (or system) is integrable.) Since the theory of formal OPs plays a central role in modern numerical analysis [8–11], especially in the theory of eigenvalue problems, convergence acceleration algorithms, Padé approximations, continued fractions etc., discrete integrable systems are closely related to numerical algorithms [12, 13, 17, 26, 33, 34, 37, 39, 46, 48]. For example, the discrete Toda equation (or called the quotient-difference (QD) algorithm) is nothing but the compatibility condition of spectral transformations of the ordinary OPs [41, 44]. The compatibility of spectral transformations for the symmetric OPs yields the discrete-time Lotka-Volterra (LV) chain, which can be used to compute singular values for bidiagonal matrices [33, 45]. The fully discrete potential KdV equation can be regarded as the Wynn's  $\varepsilon$ -algorithm—one of the most important convergence acceleration algorithms [40].

Furthermore, it is worth pointing out that discrete integrable systems may lead to many unexpected applications in numerical analysis. In particular, some new algorithms for computing eigenvalues and new convergence acceleration algorithms have been proposed via discrete integrable systems. For example, in [25], a new algorithm has been designed based on the discrete hungry Lotka-Volterra system for computing complex eigenvalues of a certain band matrix. Besides, Wynn's  $\varepsilon$ -algorithm has been generalized with the help of the hungry LV equation [13, 46]. Sometimes, an integrable system may bring advantages in some numerical calculations because it possesses a sufficient number of conserved quantities, especially in the case that the problem has something that must be conserved during the calculation, such as energy or the eigenvalues of a matrix [3, 22, 23, 47].

This paper is devoted to exploring the so-called generalized discrete Lotka-Volterra (dLV) equation (see (2.1) for more details)

$$(1 + hu_0^{(n)})u_k^{(n+1)}(1 + hu_{k-1}^{(n+1)}) = (1 + hu_0^{(n+1)})u_k^{(n)}(1 + hu_{k+1}^{(n)}),$$

which is an extension of the ordinary dLV equation

$$u_k^{(n+1)} = u_k^{(n)} + h(u_k^{(n)}u_{k+1}^{(n)} - u_k^{(n+1)}u_{k-1}^{(n+1)}) \quad (1.1)$$

proposed in [30, 31]. Here,  $h$  is a step-size parameter. In fact, the dLV equation is a time discretization of the semi-discrete Lotka-Volterra (LV) equation [4, 20, 35, 38, 42]

$$\dot{u}_k(t) = u_k(t)(u_{k+1}(t) - u_{k-1}(t)),$$

in Euler-type scheme, and  $u_k^{(n)}$  represents the approximation solution of  $u_k(t)$  at  $t = hn$  in the LV equation. Here, we remark that the semi-discrete LV lattice is one of the most famous integrable lattice systems, also known under several different names such as discrete KdV equation, Langmuir lattice, Kac-van Moerbeke lattice, and so on. It has rich applications in biological systems [32], the propagation of

electron density waves [52] and electric network [28, 29], etc. The semi-discrete LV equation together with the boundary condition

$$u_0(t) = 0$$

is referred as the restricted LV equation, while the boundary condition

$$u_0(t) \neq 0$$

is called the unrestricted case, as is used in [42, 49]. Note that the unrestricted case can be considered as a bilaterally infinite lattice [42, 49], but the semi-infinite part is sufficient for us. Compared with the restricted case, the unrestricted case is more general and of course more complicated to deal with.

As for the dLV equation, the boundary condition

$$u_0^{(n)} = 0$$

is usually considered in the literature. It has been shown to be intimately connected with convergence acceleration algorithms etc. And in the finite truncation case

$$u_0^{(n)} = 0, \quad u_{2m}^{(n)} = 0,$$

it has important applications to compute singular values for some types of matrices [33, 48]. From the view of algebraic structure, the underlying reason is that the integrability of the restricted dLV equation is intimately related to some orthogonality. In fact, the restricted dLV equation can be regarded as the compatibility condition of some recurrence relations or spectral transformations pertaining to symmetric OPs, which naturally leads to solutions in terms of Hankel determinants with the moments satisfying linear relations as the entries. The solution indeed means that the restricted dLV can be linearized [45].

However, as far as we know, the unrestricted dLV equation (i.e. that under the boundary condition  $u_0^{(n)} \neq 0$ ) has not yet been investigated. This problem is a significant and fundamental problem. On one hand, in contrast to a large number of literatures on restricted discrete integrable systems, there are still few researches on unrestricted discrete integrable systems. To our knowledge, only one such work was done by Kajiwara et al.[36], where Hankel type determinant formulae for the solution of the unrestricted discrete Toda equation are given. On the other hand, it is not clear whether there are any new phenomena when studying the connection with symmetric OPs, convergent acceleration algorithms, eigenvalue problems etc. And, as is known that the unrestricted semi-discrete LV lattice can be linearized to a Riccati equation via a Stieltjes function, it is natural to ask what happens to the unrestricted dLV equation. To this end, we derive the generalized dLV (2.1) with a sequence of given constants  $\{u_0^{(n)}\}$ . Although, it is not referred as the “unrestricted” dLV equation when  $u_0^{(n)} \neq 0$  in this paper since the equations themselves are not identical, the generalized dLV (2.1) is of sufficient interest.

The first aim of the present work is to explore the solution to the generalized dLV equation and to investigate its connection with the Riccati system. The generalized dLV (2.1) together with its Hankel determinant solution is firstly proposed by using

bilinear and determinantal techniques. Besides, it turns out that the generalized dLV equation is equivalent to a discrete Riccati system by introducing a discrete Stieltjes function. Since the latter is known to be linearizable, we thus achieve a linearization of this nonlinear difference equation. These results are presented in Section 2.

The second goal of this paper is to interpret the generalized dLV equation proposed in this paper from the standpoint of OPs, which is placed in Section 3. It is known that the restricted dLV equation has intimate connections with symmetric OPs [45]. In fact, the restricted dLV equation can be derived from the compatibility condition of the three-term recurrence relation and the Christoffel transformation<sup>1</sup> for the symmetric OPs (see Section 3.1 for more details). As for the generalized dLV equation, we propose a generalized Christoffel transformation for symmetric OPs in Section 3.2 and demonstrate that the generalized Christoffel transformation and the three-term recurrence relation for the symmetric OPs form a Lax pair of the generalized dLV equation.

The third purpose of this paper is to investigate an application of the generalized dLV (2.1) to convergence accelerate algorithms. Wynn's  $\varepsilon$ -algorithm is one of the most famous convergence acceleration algorithms and it is connected with the restricted dLV equation through a Miura transformation<sup>2</sup>. More details for these known results are given in Section 4.1; motivated by which, a generalized  $\varepsilon$ -algorithm connected with the generalized dLV equation is proposed in Section 4.2. Applications of this generalized  $\varepsilon$ -algorithm to some linearly, logarithmically convergent sequences as well as a certain divergent series are presented in Section 4.3.

At the end of the introduction, we remark that all the results of the generalized dLV equation obtained in this paper can be reduced to the restricted case and will cover those for the semi-discrete LV equation in the continuous time limit. For convenience of presentation, we only sketch the proofs of the main results in the main body and put the details of the proofs in the Appendix A.

## 2 Generalized dLV equation

In this section, we shall present the main object of this paper, i.e. the generalized dLV equation and discuss some of its properties.

### 2.1 Generalized dLV and its solution

Based on the work in [21], we extend the restricted dLV equation to a generalized case, which reads

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<sup>1</sup> The so-called *Christoffel transformation* is a nice formula that generates a new class of (bi)OPs with a modified measure  $d\hat{\rho}$  from a given class of (bi)OPs together with the measure  $d\rho$ .

<sup>2</sup> Miura transform is originally known as the transformation between the KdV equation and modified KdV equation, which are two famous integrable systems. Nowadays, the term *Miura transform* has been widely used to indicate a transformation that links certain two integrable systems.

$$(1 + hu_0^{(n)})u_k^{(n+1)}(1 + hu_{k-1}^{(n+1)}) = (1 + hu_0^{(n+1)})u_k^{(n)}(1 + hu_{k+1}^{(n)}), \tag{2.1}$$

where integers  $n \geq 0, k \geq 1$  and  $u_0^{(n)}$  are arbitrary constants. Its Hankel type solution is also obtained and presented in the following theorem. Here, we note that, for convenience, the proofs for the theorems are confined in the main body of the paper and many lengthy details for the proofs are put in the Appendix A.

**Theorem 2.1.1** The generalized dLV (2.1) admits a solution in the form of

$$u_{2k-1}^{(n)} = \frac{H_{k,1}^{(n)}H_{k-1,0}^{(n+1)}}{H_{k,0}^{(n)}H_{k-1,1}^{(n+1)}}, \quad u_{2k}^{(n)} = \frac{H_{k+1,0}^{(n)}H_{k-1,1}^{(n+1)}}{H_{k,0}^{(n+1)}H_{k,1}^{(n)}}, \quad k = 1, 2, \dots, n = 0, 1, \dots, \tag{2.2}$$

where the Hankel determinants<sup>3</sup>  $H_{k,j}^{(n)}$  are defined as

$$H_{k,j}^{(n)} = \begin{vmatrix} c_j^{(n)} & c_{j+1}^{(n)} & \dots & c_{j+k-1}^{(n)} \\ c_{j+1}^{(n)} & c_{j+2}^{(n)} & \dots & c_{j+k}^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ c_{j+k-1}^{(n)} & c_{j+k}^{(n)} & \dots & c_{j+2k-2}^{(n)} \end{vmatrix}, \tag{2.3}$$

with the convention

$$H_{0,j}^{(n)} = 1, \tag{2.4}$$

and the elements  $\{c_j^{(n)}\}$  satisfying

$$c_j^{(n)} = \frac{c_{j-1}^{(n+1)} - c_{j-1}^{(n)}}{h} + \frac{u_0^{(n)}}{c_0^{(n)}} \sum_{i=0}^{j-1} c_i^{(n)} c_{j-1-i}^{(n+1)}, \quad j = 1, 2, \dots, n = 0, 1, \dots \tag{2.5}$$

with arbitrary constants  $c_0^{(n)} \neq 0$ .

**Proof** The case for  $k = 1$  in (2.1) is trivial. We proceed the proof for  $k \geq 2$ , which can be achieved based on Corollary A.3.2. If we use the notation  $\tau_k^{(n)}$  appearing in Corollary A.3.2, it is equivalent to prove

$$u_k^{(n)} = \frac{\tau_{k+2}^{(n)} \tau_{k-1}^{(n+1)}}{\tau_{k+1}^{(n)} \tau_k^{(n+1)}} \tag{2.6}$$

solves the generalized dLV (2.1).

Substituting (2.6) into (2.1), we are left to prove

<sup>3</sup> For fixed  $n, H_{k,j}^{(n)}$  are Hankel determinants of a sequence  $\{c_j^{(n)}\}$ .

$$(1 + hu_0^{(n)}) \frac{\tau_{k+2}^{(n+1)} (\tau_k^{(n+1)} \tau_{k-1}^{(n+2)} + h\tau_{k+1}^{(n+1)} \tau_{k-2}^{(n+2)})}{\tau_k^{(n+2)}} = (1 + hu_0^{(n+1)}) \frac{\tau_{k-1}^{(n+1)} (\tau_{k+2}^{(n)} \tau_{k+1}^{(n+1)} + h\tau_{k+3}^{(n)} \tau_k^{(n+1)})}{\tau_{k+1}^{(n)}}.$$

By employing Corollary A.3.2, we easily arrive at

$$(1 + hu_0^{(n)})\beta^{(n+1)} = (1 + hu_0^{(n+1)})\beta^{(n)},$$

which is obviously valid because of the definition of  $\beta^{(n)}$  in (3.12). Thus, the proof is completed.

**Remark 2.1.2** The (2.1) can be rewritten as

$$u_{k+1}^{(n)} = \frac{1}{h} \left( \frac{(1 + hu_0^{(n)})u_k^{(n+1)}(1 + hu_{k-1}^{(n+1)})}{(1 + hu_0^{(n+1)})u_k^{(n)}} - 1 \right), \quad k = 1, 2, \dots, n = 0, 1, \dots$$

It is not hard to see that all  $u_k^{(n)}$  will be determined iteratively via the initial values  $u_0^{(n)}$  and  $u_1^{(n)}$  with integers  $n \geq 0$ . Since

$$u_1^{(n)} = \frac{c_1^{(n)}}{c_0^{(n)}} = \left( \frac{1}{h} + u_0^{(n)} \right) \frac{c_0^{(n+1)}}{c_0^{(n)}} - \frac{1}{h},$$

all  $u_k^{(n)}$  are actually determined by  $u_0^{(n)}$  and  $c_0^{(n)}$ .

**Remark 2.1.3** If  $u_0^{(n)} = 0$ , (2.1) reduces to the restricted dLV (1.1). Correspondingly, its solution (2.2) with (2.5) reduces to the solution of the restricted dLV (1.1).

The generalized dLV equation can be regarded as a discrete analogue of the unrestricted semi-discrete LV equation discussed in [21, 42]. We will show in Section 2.2 that, as the discrete step size  $h \rightarrow 0$ , the generalized dLV equation tends to the unrestricted semi-discrete LV equation. Moreover, in Section 2.3, we linearize the generalized dLV equation to a Riccati system in terms of a Stieltjes function.

### 2.2 Continuous time limit

In this subsection, we deduce the unrestricted semi-discrete LV equation discussed in [21, 42] by investigating the continuum limits of the results in Theorem 2.1.1 as  $h \rightarrow 0$ , so that the generalized dLV (2.1) can be regarded as a discrete analogue of the unrestricted semi-discrete LV equation.

As is known, the solution of the semi-discrete LV equation [4, 20, 35, 38, 42]

$$\dot{u}_k(t) = u_k(t)(u_{k+1}(t) - u_{k-1}(t)), \tag{2.7}$$

satisfies

$$u_{2k-1} = \frac{H_{k-1}^0 H_k^1}{H_k^0 H_{k-1}^1}, \quad u_{2k} = \frac{H_{k+1}^0 H_{k-1}^1}{H_k^0 H_k^1}, \quad k = 1, 2, \dots, \tag{2.8}$$

where

$$H_k^m(t) = \det(c_{i+j+m}(t))_{i,j=0}^{k-1}, \quad m = 0, 1, \quad k \in \mathbb{Z}_+,$$

and  $\{c_j\}$  is a set of functions of  $t$ . Note that  $H_k^m$  are the Hankel determinants of a sequence of  $\{c_n\}$ .

In the restricted case, namely,  $u_0 = 0$ , the Hankel determinant solutions (2.8) are completely determined by an arbitrary differentiable moment function  $c_0$  of  $t$  with

$$\dot{c}_j(t) = c_{j+1}, \quad j = 0, 1, \dots$$

While for the unrestricted case with  $u_0(t) \neq 0$ , in order to make the Hankel determinant expressions (2.8) still be the solution of (2.7), it requires to add an quadratic term in the evolution equation for the moments  $\{c_j\}$  [21, 42]

$$\dot{c}_j(t) = c_{j+1} - \frac{u_0}{c_0} \sum_{i=0}^j c_i c_{j-i}, \tag{2.9}$$

with arbitrary function  $c_0(t) \neq 0$ .

Now, we will point out that, as the step size  $h \rightarrow 0$ , the generalized dLV (2.1) and its solution (2.2) with determinant elements  $\{c_j^{(n)}\}$  satisfying the recursion relation (2.5) converge to the unrestricted semi-infinite (2.7), and the corresponding solution (2.8) with recursion relation (2.9), respectively.

First, let us rewrite the generalized dLV (2.1) as

$$\frac{\frac{u_k^{(n+1)}}{1+hu_0^{(n+1)}} - \frac{u_k^{(n)}}{1+hu_0^{(n)}}}{h} = \frac{u_k^{(n)}}{1+hu_0^{(n)}} u_{k+1}^{(n)} - \frac{u_k^{(n+1)}}{1+hu_0^{(n+1)}} u_{k-1}^{(n+1)},$$

with a sequence of given constants  $\{u_0^{(n)}\}$ . It is easy to see that, as the step size  $h \rightarrow 0$ , it tends to the unrestricted semi-infinite (2.7) with arbitrary function  $u_0(t)$ . At the same time, the recursion relation (2.5) approaches to

$$c_j = \frac{dc_{j-1}}{dt} + \frac{u_0}{c_0} \sum_{i=0}^{j-1} c_i c_{j-1-i}, \quad j = 1, 2, 3, \dots,$$

which is equivalent to (2.9) with arbitrary function  $c_0(t) \neq 0$ . Consequently, the solution (2.2) of the generalized dLV equation naturally tends to (2.8) as  $h \rightarrow 0$ .

In particular, if letting  $u_0^{(n)} = 0$  and  $u_0(t) = 0$ , the above process also gives the connection between the restricted semi-discrete LV equation and the restricted dLV (1.1).

### 2.3 Riccati system in terms of a Stieltjes function

Some nonlinear difference equation can be linearized via a suitable change of variables. An important example is the discrete Riccati equation, which is a nonlinear difference equation, possessing a quadratic nonlinearity. The discrete Riccati equation is a difference analogue of the famous Riccati differential equation which can—as is well known—be linearized [27, Chapter one, pp 16]. It is shown in [21] that, the unrestricted semi-discrete LV equation could be transformed to a Riccati equation in terms of a Stieltjes function. Inspired by this work, we now pose a discrete Riccati equation pertaining to the generalized dLV equation.

Define a set of Stieltjes functions  $\{F^{(n)}\}$  as,

$$F^{(n)} = \sum_{j=0}^{\infty} \frac{c_j^{(n)}}{\lambda^{2j+1}}, \quad (2.10)$$

which acts as a generating function of sequence  $\{c_j^{(n)}\}$ . (Here, we remark that such Stieltjes functions arise in the case of zero odd moments, which naturally appear in the context of symmetric OPs.) Then, it is not hard to see that the recursion relation (2.5) of  $\{c_j^{(n)}\}$  is equivalent to the following discrete Riccati equation in terms of  $F^{(n)}$ ,

$$\frac{F^{(n+1)} - F^{(n)}}{h} = -c_0^{(n)}\lambda + \lambda^2 F^{(n)} - \frac{u_0^{(n)}}{c_0^{(n)}} \lambda F^{(n)} F^{(n+1)}.$$

Obviously, as  $h \rightarrow 0$ , this discrete Riccati equation will tend to

$$\frac{dF(t)}{dt} = -c_0\lambda + \lambda^2 F - \frac{u_0}{c_0} \lambda F^2,$$

and  $F^{(n)}$  in (2.10) will approach to

$$F = \sum_{j=0}^{\infty} \frac{c_j}{\lambda^{2j+1}}.$$

This is the Riccati equation given in [21], and this result coincides with the case of the unrestricted semi-discrete LV equation.

### 3 Connection with OPs

In this section, we discuss connections between the generalized dLV equation and symmetric OPs. Let us first outline some useful knowledge on symmetric OPs and their relationship with the restricted dLV equation in Section 3.1.



### 3.1 On symmetric OPs and restricted dLV

It is known that a set of monic symmetric OPs  $\{P_k(x)\}_{k=0}^\infty$  defined on a symmetric subset of  $\mathbb{R}$  satisfy the three-term recursion relation as follows,

$$xP_k(x) = P_{k+1}(x) + v_k P_{k-1}(x), \quad k = 0, 1, 2, \dots,$$

with the initial conditions  $P_{-1}(x) = 0, P_0(x) = 1$ . From this recursion relation, one can easily find that these polynomials possess the parity property  $P_k(-x) = (-1)^k P_k(x)$ .

Let  $P_k^{(0)}(x) := P_k(x)$ , then, for fixed  $n$ , we define a sequence of symmetric OPs  $\{P_k^{(n)}(x)\}_{k=0}^\infty$  in terms of an infinite sequence  $\{c_j^{(n)}\}$  as

$$P_{2k-1}^{(n)}(x) = \frac{x\mathcal{H}_{2k-2,1}^{(n)}(x)}{H_{k-1,1}^{(n)}}, \quad P_{2k}^{(n)}(x) = \frac{\mathcal{H}_{2k,0}^{(n)}(x)}{H_{k,0}^{(n)}}, \quad k = 1, 2, 3, \dots, \quad (3.1)$$

where

$$H_{k,j}^{(n)} = \det \left( c_{i+l+j}^{(n)} \right)_{i,l=0}^{k-1} \quad (3.2)$$

with  $c_j^{(n)}$  restricted by

$$c_j^{(n)} = \frac{c_{j-1}^{(n+1)} - c_{j-1}^{(n)}}{h}, \quad j = 1, 2, 3, \dots, \quad (3.3)$$

and  $H_{0,j}^{(n)} = 1$ . And,  $\mathcal{H}_{2k,j}^{(n)}(x), j = 0, 1$ , is a polynomial of degree  $2k$  in  $x$ , defined as follows

$$\mathcal{H}_{2k,j}^{(n)}(x) = \begin{vmatrix} c_j^{(n)} & c_{j+1}^{(n)} & \dots & c_{j+k}^{(n)} \\ c_{j+1}^{(n)} & c_{j+2}^{(n)} & \dots & c_{j+k+1}^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ c_{j+k-1}^{(n)} & c_{j+k}^{(n)} & \dots & c_{j+2k-1}^{(n)} \\ 1 & x^2 & \ddots & x^{2k} \end{vmatrix}, \quad k = 1, 2, 3, \dots,$$

together with  $\mathcal{H}_{0,j}^{(n)}(x) = 1$ .

For each set of the OPs  $\{P_k^{(n)}(x)\}_{k=0}^\infty$ , they satisfy the following three-term recurrence relation,

$$xP_k^{(n)}(x) = P_{k+1}^{(n)}(x) + v_k^{(n)} P_{k-1}^{(n)}(x), \quad k = 0, 1, 2, \dots \quad (3.4a)$$

$$P_{-1}^{(n)}(x) = 0, \quad P_0^{(n)}(x) = 1, \quad (3.4b)$$

where

$$v_{2k}^{(n)} = \frac{H_{k+1,0}^{(n)} H_{k-1,1}^{(n)}}{H_{k,1}^{(n)} H_{k,0}^{(n)}}, \quad v_{2k-1}^{(n)} = \frac{H_{k,1}^{(n)} H_{k-1,0}^{(n)}}{H_{k,0}^{(n)} H_{k-1,1}^{(n)}}. \tag{3.5}$$

Actually, one can construct  $\{P_k^{(n+1)}(x)\}_{k=0}^\infty$  from  $\{P_k^{(n)}(x)\}_{k=0}^\infty$  via the Christoffel transformation. Sometimes they are called adjacent families with each other. The corresponding Christoffel transformations between the adjacent families of  $\{P_k^{(n)}(x)\}_{k=0}^\infty$  reads,

$$P_k^{(n+1)} = \frac{hP_{k+2}^{(n)} + w_{k+1}^{(n)} P_k^{(n)}}{hx^2 + 1}, \quad k = 0, 1, 2, \dots, \tag{3.6}$$

where

$$w_{2k}^{(n)} = \frac{H_{k,1}^{(n+1)} H_{k-1,1}^{(n)}}{H_{k,1}^{(n)} H_{k-1,1}^{(n+1)}}, \quad w_{2k-1}^{(n)} = \frac{H_{k,0}^{(n+1)} H_{k-1,0}^{(n)}}{H_{k,0}^{(n)} H_{k-1,0}^{(n+1)}}. \tag{3.7}$$

In the sequel, we also need to use the notion of the associated polynomials. According to the work [42] of Peherstorfer et al., the so-called  $j$ -associated polynomials  $Q_k^{(j,n)}(x)$  of the adjacent OPs  $\{P_k^{(n)}(x)\}$ , satisfy the recurrence relation

$$xQ_k^{(j,n)}(x) = Q_{k+1}^{(j,n)}(x) + v_{k+j}^{(n)} Q_{k-1}^{(j,n)}(x), \quad k = 0, 1, 2, \dots, \tag{3.8}$$

with the initial conditions

$$Q_{-1}^{(j,n)}(x) = 0, \quad Q_0^{(j,n)}(x) = c_0^{(n)},$$

where  $v_k^{(n)}$  is defined by (3.5). In addition, there exists the following recurrence relation among the  $j$ -associated polynomials,

$$c_0^{(n)} Q_k^{(j,n)}(x) = xQ_{k-1}^{(j+1,n)}(x) - v_{j+1}^{(n)} Q_{k-2}^{(j+2,n)}(x), \quad j, n = 0, 1, 2, \dots, \tag{3.9}$$

with initial condition  $Q_k^{(0,n)}(x) = P_k^{(n)}(x)$ .

Since we will particularly concern the 1-associated polynomials in Section 3.2, we give more information on it. We abbreviate  $Q_k^{(1,n)}(x)$  as  $Q_k^{(n)}(x)$  for convenience, which also applies in the sequel. In the literature, the 1-associated polynomials  $\{Q_k^{(n)}\}$  are usually defined as

$$Q_{k-1}^{(n)}(x) = \mathcal{L}^{(n)}\left(\frac{P_k^{(n)}(\xi) - P_k^{(n)}(x)}{\xi - x}\right),$$

where  $\mathcal{L}$  is a linear function of  $\xi$ , satisfying

$$\mathcal{L}^{(n)}(\xi^{2k+1}) = 0, \quad \mathcal{L}^{(n)}(\xi^{2k}) = c_k^{(n)}, \quad k = 0, 1, 2, \dots$$

Similar to the adjacent OPs  $\{P_k^{(n)}(x)\}$ , the 1-associated polynomials  $\{Q_k^{(n)}\}$  enjoy determinant expressions

$$Q_{2k-1}^{(n)}(x) = \frac{1}{H_{k,0}^{(n)}} \begin{vmatrix} c_0^{(n)} & c_1^{(n)} & \dots & c_k^{(n)} \\ c_1^{(n)} & c_2^{(n)} & \dots & c_{k+1}^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ c_{k-1}^{(n)} & c_k^{(n)} & \dots & c_{2k-1}^{(n)} \\ 0 & c_0^{(n)}x & \dots & \sum_{i=0}^{k-1} c_i^{(n)}x^{2k-1-2i} \end{vmatrix}, \tag{3.10a}$$

$$Q_{2k}^{(n)}(x) = \frac{1}{H_{k,1}^{(n)}} \begin{vmatrix} c_1^{(n)} & c_2^{(n)} & \dots & c_{k+1}^{(n)} \\ c_2^{(n)} & c_3^{(n)} & \dots & c_{k+2}^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ c_k^{(n)} & c_{k+1}^{(n)} & \dots & c_{2k}^{(n)} \\ c_0^{(n)} & c_1^{(n)} + c_0^{(n)}x^2 & \dots & \sum_{i=0}^k c_i^{(n)}x^{2k-2i} \end{vmatrix}, \tag{3.10b}$$

where  $H_{k,j}^{(n)}$  is defined by (3.2).

### 3.2 Link between the generalized dLV and symmetric OPs

In this subsection, we will discuss the connection between the generalized dLV equation and symmetric OPs. As mentioned in introduction, it is already known that the restricted dLV equation ( $u_0^{(n)} = 0$ ) can be derived as a compatibility condition of Lax pair in terms of symmetric OPs. For the generalized dLV (2.1) proposed in this paper, correspondingly, we can also construct a Lax pair from the perspective of symmetric OPs.

To this end, we need to use the 1-associated polynomials of  $\{Q_k^{(n)}(x)\}_{k=0}^\infty$  in (3.10a) for the symmetric OPs  $\{P_k^{(n)}(x)\}_{k=0}^\infty$ . In fact, in the case of the generalized recursion relation (2.5) satisfied by  $\{c_j^{(n)}\}$ , we can obtain the generalized Christoffel transformation for  $\{P_k^{(n)}(x)\}_{k=0}^\infty$ , which will reduce to the Christoffel transformation (3.6) when  $u_0^{(n)} = 0$ .

**Theorem 3.2.1** Assume  $\{c_j^{(n)}\}$  satisfy the recursion relation (2.5), then  $\{P_k^{(n)}(x)\}_{k=0}^\infty$  satisfy the following formulae for  $k = 0, 1, 2, \dots$ ,

$$P_k^{(n+1)}(x) = \frac{hP_{k+2}^{(n)}(x) + \beta^{(n)}w_{k+1}^{(n)}P_k^{(n)}(x) + hx\alpha^{(n)}Q_{k-1}^{(n+1)}(x)}{hx^2 + 1}, \tag{3.11}$$

where  $w_k^{(n)}$  is defined by (3.7) and

$$\alpha^{(n)} = \frac{u_0^{(n)}}{c_0^{(n)}}, \quad \beta^{(n)} = 1 + hu_0^{(n)}. \tag{3.12}$$

**Proof** The result follows from the identity (A.9g), and replacing  $E_{k+1,j,j}^{(n+1)}, \tilde{E}_{k,j+1,j}^{(n+1)}, G_{2k,j}^{(n+1)}(x), \tilde{G}_{2k+2,j}^{(n+1)}(x)$ , via (A.6b), (A.6l), (A.6j) and (A.6m).

We end this section by a Lax pair of the generalized dLV (2.1).

**Theorem 3.2.2** The generalized dLV (2.1) may be obtained by the compatibility condition of the three-term recurrence relation (3.4a) and the generalized Christoffel transformation (3.11).

*Proof* First, for  $k = 1, 2, \dots$ , using (3.4a), we obtain that

$$P_{k+1}^{(n+1)}(x) + v_k^{(n+1)}P_{k-1}^{(n+1)}(x) = xP_k^{(n+1)}(x).$$

Then replacing  $P_{k+1}^{(n+1)}$ ,  $P_k^{(n+1)}$ ,  $P_{k-1}^{(n+1)}$ , by using the generalized Christoffel transformation (3.11) into the above relation, we obtain that

$$\begin{aligned} v_k^{(n+1)}w_k^{(n)} &= v_k^{(n)}w_{k+1}^{(n)}, \\ \beta^{(n)}(w_{k+2}^{(n)} - w_{k+1}^{(n)}) + h(v_k^{(n+1)} - v_{k+2}^{(n)}) &= 0, \end{aligned}$$

where the relation (3.8) and the linear independence property of  $\{P_k^{(n)}(x)\}_{k=0}^\infty$  are used.

Under the change of variables

$$\begin{aligned} v_k^{(n)} &= \frac{1}{\beta^{(n)}}u_k^{(n)}(1 + hu_{k-1}^{(n)}), \\ w_k^{(n)} &= \frac{1}{(\beta^{(n)})^2}(1 + hu_k^{(n)})(1 + hu_{k-1}^{(n)}), \end{aligned}$$

the generalized dLV equation (2.1) is immediately deduced. Hence, we complete the proof.

**Remark 3.2.3** We remark that the variable transformations

$$v_k^{(n)} = \frac{1}{\beta^{(n)}}u_k^{(n)}(1 + hu_{k-1}^{(n)}), \tag{3.13a}$$

$$w_k^{(n)} = \frac{1}{(\beta^{(n)})^2}(1 + hu_k^{(n)})(1 + hu_{k-1}^{(n)}), \tag{3.13b}$$

are motivated by those for the restricted dLV equation. Actually, this can be constructed via their connections with the Hankel determinants  $H_{k,j}^{(n)}$ . Substituting the expressions, we have

$$\begin{aligned} \frac{1}{\beta^{(n)}}u_{2j}^{(n)}(1 + hu_{2j-1}^{(n)}) &= \frac{H_{j+1,0}^{(n)}(H_{j,0}^{(n)}H_{j-1,1}^{(n+1)} + hH_{j,1}^{(n)}H_{j-1,0}^{(n+1)})}{\beta^{(n)}H_{j,0}^{(n+1)}H_{j,1}^{(n)}H_{j,0}^{(n)}}, \\ \frac{1}{\beta^{(n)}}u_{2j+1}^{(n)}(1 + hu_{2j}^{(n)}) &= \frac{H_{j+1,1}^{(n)}(H_{j,0}^{(n+1)}H_{j,1}^{(n)} + hH_{j+1,0}^{(n)}H_{j-1,1}^{(n+1)})}{\beta^{(n)}H_{j+1,0}^{(n)}H_{j,1}^{(n+1)}H_{j,1}^{(n)}}. \end{aligned}$$

Then, (3.13a) immediately follows from the bilinear relation (A.10) with respect to  $k = 2j - 2$  and  $k = 2j - 1$ , respectively. Similarly, the validity of the transformation (3.13b) can be verified.

### 3.3 A view of the limit $h \rightarrow 0$

In this subsection, we interpret the Lax pair of the generalized dLV equation given in Theorem 3.2.2, from the point of view of approximation as the discrete step size  $h \rightarrow 0$ .

As is known, after introducing a certain one-parametric deformation of measure that introduces the additional variable  $t$ , the symmetric OPs  $P_k$  consequently become a function of two variables  $x$  and  $t$  [42]. Such symmetric OPs  $P_k(x, t)$  still satisfy the three-term recurrence relation

$$\begin{aligned} xP_k(x, t) &= P_{k+1}(x, t) + u_k(t)P_{k-1}(x, t), & k = 0, 1, 2, \dots, \\ P_{-1}(x, t) &= 0, \quad P_0(x, t) = 1, \end{aligned}$$

where the coefficient  $u_k(t)$  solves the semi-discrete LV (2.7). It was pointed out by Peherstorfer et al. [42] that such a three-term recurrence relation of  $P_k(x, t)$ , together with the following continuous time evolution differential equation,

$$\dot{P}_k(x, t) = -u_k u_{k-1} P_{k-2}(x, t) + \frac{u_0 u_1}{c_0} Q_{k-2}^{(2)}(x, t), \tag{3.14}$$

give a Lax pair of the unrestricted semi-discrete LV (2.7), where the dot represents the differential with respect to  $t$ , and  $Q_{k-2}^{(2)}(x, t)$  is the so-called 2-associated polynomials of  $P_k(x, t)$  satisfying

$$xQ_k^{(2)}(x, t) = Q_{k+1}^{(2)}(x, t) + u_{k+2}(t)Q_{k-1}^{(2)}(x, t).$$

It is noted that there are no simply explicit formulae for  $Q_k^{(2)}(x, t)$ , which are different from the 1-associated polynomials  $Q_k^{(2)}(x)$  in (3.10a) that are shorten for  $Q_k^{(1,n)}(x)$  at  $n = 2$ .

Now, we will show that, the Lax pair given in Theorem 3.2.2 can be regarded as a discrete analogue of the above semi-discrete case. To this end, we first rewrite the generalized Christoffel transformation (3.11) on  $\{P_k^{(n)}(x)\}$  as follows

$$\begin{aligned} \frac{P_k^{(n+1)} - P_k^{(n)}}{h} &= P_{k+2}^{(n)} + \frac{(u_k^{(n)} + u_{k+1}^{(n)}) + hu_k^{(n)} u_{k+1}^{(n)} - u_0^{(n)}}{\beta^{(n)}} P_k^{(n)} \\ &\quad - x^2 P_k^{(n+1)} + \frac{xu_0^{(n)} Q_{k-1}^{(n+1)}}{c_0^{(n)}}. \end{aligned} \tag{3.15}$$

where (3.12) and (3.13b) are used.

With the help of the recurrence relation (3.4a) satisfied by the adjacent OPs  $\{P_k^{(n)}(x)\}$ , we easily see

$$x^2 P_k^{(n+1)} = P_{k+2}^{(n+1)} + (v_k^{(n+1)} + v_{k+1}^{(n+1)}) P_k^{(n+1)} + v_{k-1}^{(n+1)} v_k^{(n+1)} P_{k-2}^{(n+1)}. \tag{3.16}$$

Besides, from the recurrence relation (3.9), we obviously have

$$xQ_{k-1}^{(n+1)}(x) = c_0^{(n+1)} P_k^{(n+1)}(x) + v_1^{(n+1)} Q_{k-2}^{(2,n+1)}(x), \tag{3.17}$$

by using the conventions  $Q_k^{(0,n+1)}(x) = P_k^{(n+1)}(x)$  and  $Q_k^{(1,n+1)}(x) = Q_k^{(n+1)}(x)$ .

Then, the relation (3.15) will become

$$\begin{aligned} \frac{P_k^{(n+1)} - P_k^{(n)}}{h} &= (P_{k+2}^{(n)} - P_{k+2}^{(n+1)}) + \left( \frac{u_k^{(n)} + u_{k+1}^{(n)}}{\beta^{(n)}} P_k^{(n)} - (v_k^{(n+1)} + v_{k+1}^{(n+1)}) P_k^{(n+1)} \right) \\ &\quad + \frac{hu_k^{(n)} u_{k+1}^{(n)}}{\beta^{(n)}} P_k^{(n)} + u_0^{(n)} \left( \frac{c_0^{(n+1)}}{c_0^{(n)}} P_k^{(n+1)} - \frac{1}{\beta^{(n)}} P_k^{(n)} \right) \\ &\quad - v_{k-1}^{(n+1)} v_k^{(n+1)} P_{k-2}^{(n+1)} + \frac{u_0^{(n)} v_1^{(n+1)}}{c_0^{(n)}} Q_{k-2}^{(2,n+1)}. \end{aligned} \tag{3.18}$$

by use of (3.16) and (3.17).

Although the upper index  $n$  of  $P_k^{(n)}(x)$ , as mentioned in the beginning of Section 3.1, stands for  $n$  repeated applications of the Christoffel transformation to the initial symmetric OPs  $P_k(x)$ , quite interestingly, one can still interpret it in terms of the discretization index of time  $t$ .

Now, consider  $P_k^{(n)}(x)$  as the approximation of  $P_k(x, t)$  at  $t = hn$ . As the discrete step size  $h \rightarrow 0$ , we have

$$P_k^{(n)}(x) \rightarrow P_k(x, t), \quad Q_{k-2}^{(2,n)}(x) \rightarrow Q_{k-2}^{(2)}(x, t), \quad u_k^{(n)} \rightarrow u_k(t),$$

and also

$$\beta^{(n)} \rightarrow 1, \quad v_k^{(n)} \rightarrow u_k(t)$$

obtained from (3.12) and (3.13a). Thus, as  $h \rightarrow 0$ , the (3.18) approaches to the continuous time evolution (3.14) of the symmetric OPs  $P_k(x, t)$ .

We summarize the above interpretation into the following theorem.

**Theorem 3.3.1** As the discrete step size  $h \rightarrow 0$ , the Lax pair of the generalized dLV equation given in Theorem 3.2.2,

$$\begin{aligned} xP_k^{(n)}(x) &= P_{k+1}^{(n)}(x) + v_k^{(n)} P_{k-1}^{(n)}(x), \quad P_{-1}^{(n)}(x) \equiv 0, \quad P_0^{(n)}(x) \equiv 1, \quad k = 0, 1, 2, \dots \\ P_k^{(n+1)}(x) &= \frac{hP_{k+2}^{(n)}(x) + \beta^{(n)} w_{k+1}^{(n)} P_k^{(n)}(x) + hx\alpha^{(n)} Q_{k-1}^{(n+1)}(x)}{hx^2 + 1}, \end{aligned}$$

tends to the Lax pair of the unrestricted semi-discrete LV (2.7), appeared in [42],

$$\begin{aligned} xP_k(x, t) &= P_{k+1}(x, t) + u_k(t) P_{k-1}(x, t), \quad P_{-1}(x, t) \equiv 0, \quad P_0(x, t) \equiv 1, \quad k = 0, 1, 2, \dots \\ \dot{P}_k(x, t) &= -u_k u_{k-1} P_{k-2}(x, t) + \frac{u_0 u_1}{c_0} Q_{k-2}^{(2)}(x, t), \end{aligned}$$

where  $\{Q_{k-2}^{(2)}(x, t)\}$  are the 2-associated polynomials of the symmetric OPs  $\{P_k(x, t)\}$ .

**Remark 3.3.2** Take the boundary condition  $u_0^{(n)} = 0$ , then Theorem 3.3.1 gives a correspondence between the Lax pairs of the dLV equation and the semi-discrete LV equation in the restricted case.

## 4 Connection with $\varepsilon$ -algorithm

In this section, we will investigate the application of the generalized dLV (2.1) to the field of numerical computation. Based on the generalized recursion relation (2.5) of  $\{c_j^{(n)}\}$ , we derive a generalized  $\varepsilon$ -algorithm. It turns out that this generalized  $\varepsilon$ -algorithm is connected with the generalized dLV (2.1) via a Miura transformation. We then analyse the numerical effect of this generalized  $\varepsilon$ -algorithm through several examples.

To begin with, let us first sketch some relevant knowledge about Wynn’s  $\varepsilon$ -algorithm in Section 4.1.

### 4.1 Review of Shanks transformation and Wynn’s $\varepsilon$ -algorithm

Sequences and series are two quite common notions which appear frequently in mathematics and engineering. Unfortunately, many sequences or series converge so slowly that they almost have no effective use in practice. Moreover, certain divergent sequences and series can also be summed by applying a suitable summation approach. Sequence transformation, in such situation, is an effective way to accelerate the rate of convergence or to achieve a summation of certain divergent sequences.

Formally, a sequence transformation  $\mathcal{T}$  is simply a map

$$\mathcal{T} : \{S_n\} \mapsto \{T_n\},$$

which transforms the original unsatisfactory sequence  $\{S_n\}$  to a new sequence  $\{T_n\}$  with hopefully better numerical properties. It is normally required that the new sequence  $\{T_n\}$  first must be convergent, then it is supposed to have the same (generalized) limit  $S$  as the  $\{S_n\}$ . A sequence transformation  $\mathcal{T}$  is called to accelerate the convergence of the sequence  $\{S_n\}$  or that the sequence  $\{T_n\}$  converges faster than  $\{S_n\}$ , if

$$\lim_{n \rightarrow \infty} \frac{T_n - S}{S_n - S} = 0.$$

So far, there exist many sequence transformations in the literature and many of them can be expressed as ratios of determinants. The most well-known sequence transformation is the Aitken’s  $\Delta^2$ -transformation [2] and its high order extension, i.e. the Shanks transformation [14, 43],

$$e_k(S_n) = \frac{\begin{vmatrix} S_n & S_{n+1} & \cdots & S_{n+k} \\ \Delta S_n & \Delta S_{n+1} & \cdots & \Delta S_{n+k} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta^k S_n & \Delta^k S_{n+1} & \cdots & \Delta^k S_{n+k} \end{vmatrix}}{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \Delta S_n & \Delta S_{n+1} & \cdots & \Delta S_{n+k} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta^k S_n & \Delta^k S_{n+1} & \cdots & \Delta^k S_{n+k} \end{vmatrix}},$$

where  $\Delta$  is the usual forward difference operator

$$\Delta f(n) = f(n+1) - f(n), \quad (4.1a)$$

and its higher powers are defined by

$$\Delta^i f(n) = \Delta(\Delta^{i-1} f(n)), \quad i \in \mathbb{N}, \quad (4.1b)$$

$$\Delta^0 f(n) = f(n). \quad (4.1c)$$

Here, we remark that Aitken's  $\Delta^2$ -transformation [2] corresponds to the special case  $k=2$  of the Shanks transformation.

To avoid evaluating the determinants, it is very necessary to implement recursive algorithms called convergence acceleration algorithms to compute the sequence transformation. In fact, the Shanks transformation can be computed recursively via the following nonlinear recursion algorithm

$$\varepsilon_{k+1}^{(n)} - \varepsilon_{k-1}^{(n+1)} = \frac{1}{\varepsilon_k^{(n+1)} - \varepsilon_k^{(n)}}, \quad (4.2a)$$

$$\varepsilon_{-1}^{(n)} = 0, \quad \varepsilon_0^{(n)} = S_n, \quad n = 0, 1, 2, \dots, \quad (4.2b)$$

which is nothing but Wynn's  $\varepsilon$ -algorithm [51]. The boundary value problem (4.1) admits the unique solution

$$\varepsilon_{2k}^{(n)} = \frac{H_{k+1}(S_n)}{H_k(\Delta^2 S_n)}, \quad \varepsilon_{2k+1}^{(n)} = \frac{H_k(\Delta^3 S_n)}{H_{k+1}(\Delta S_n)}, \quad (4.3)$$

where  $H_k(g_n)$  is a  $k$ -order Hankel determinant defined as

$$H_k(g_n) = \begin{vmatrix} g_n & g_{n+1} & \cdots & g_{n+k-1} \\ g_{n+1} & g_{n+2} & \cdots & g_{n+k} \\ \vdots & \vdots & \ddots & \vdots \\ g_{n+k-1} & g_{n+k} & \cdots & g_{n+2k-2} \end{vmatrix}.$$

Obviously, the connection between the  $\varepsilon$ -algorithm and Shanks transformation is given by

$$\varepsilon_{2k}^{(n)} = e_k(S_n), \quad \varepsilon_{2k+1}^{(n)} = \frac{1}{e_k(\Delta S_n)}, \quad k, n = 0, 1, \dots \quad (4.4)$$

which can be found in Wynn's paper [51, Theorem on p. 91].

In addition, the E-transformation, Levin's transformation, the  $\rho$ -algorithm,  $\theta$ -algorithm and  $\eta$ -algorithm etc. are also well-known examples in the theory of sequence transformations and convergence acceleration algorithms. For an extensive investigation on sequence transformations and convergence acceleration algorithms, one might consult the book [14] and a recent review [15] and their reference in.

There exist interesting relations between convergence acceleration algorithms and integrable systems. The  $\varepsilon$ -algorithm is nothing but the fully discrete potential KdV equation, which is related to restricted dLV equation, while the  $\rho$ -algorithm



corresponds to the cylindrical KdV equation [40]. Based on these facts, many new convergence acceleration algorithms have been proposed. For instance, a new convergence accelerate algorithm was derived in [26] from the lattice Boussinesq equation. Brezinski et al. in [13] obtained a multi-step extension of the  $\varepsilon$ -algorithm to implement a multi-step extension of the Shanks sequence transformation. For more recent developments, see, e.g. [15, 17, 46].

Now, we present more details on the link between Wynn’s  $\varepsilon$ -algorithm and the restricted dLV equation. To this end, we first claim that the solution (4.3) of Wynn’s  $\varepsilon$ -algorithm is equivalent to the following form

$$\varepsilon_{2k}^{(n)} = \frac{H_{k+1,-1}^{(n)}}{H_{k,1}^{(n)}}, \quad \varepsilon_{2k+1}^{(n)} = \frac{H_{k,2}^{(n)}}{H_{k+1,0}^{(n)}}, \tag{4.5}$$

where  $H_{k,j}^{(n)}$  is a Hankel determinant defined by (3.2) with its elements  $\{c_j^{(n)}\}$  satisfying (3.3). Noting that we introduce additionally  $c_{-1}^{(n)}$  in (4.5), we hence extend the recursion relation (3.3) for  $\{c_j^{(n)}\}$  valid to  $c_0^{(n)}$ , namely,

$$c_0^{(n)} = \frac{c_{-1}^{(n+1)} - c_{-1}^{(n)}}{h}, \quad n = 0, 1, 2, \dots \tag{4.6}$$

In fact, the claim easily follows from the fact

$$H_{k,j}^{(n)} = H_k(\Delta^{j+1}S_n),$$

and

$$c_j^{(n)} = \Delta^{j+1}S_n, \quad j = -1, 0, 1, \dots,$$

if we set  $c_{-1}^{(n)} = S_n$ .

Under the relations (3.3) and (4.6), one can prove

$$H_{k+1,-1}^{(n+1)}H_{k,1}^{(n)} - H_{k+1,-1}^{(n)}H_{k,1}^{(n+1)} - H_{k+1,0}^{(n)}H_{k,0}^{(n+1)} = 0, \tag{4.7a}$$

$$H_{k,2}^{(n)}H_{k,0}^{(n+1)} - H_{k,1}^{(n+1)}H_{k,1}^{(n)} - H_{k+1,0}^{(n)}H_{k-1,2}^{(n+1)} = 0, \tag{4.7b}$$

$$H_{k+1,0}^{(n+1)}H_{k,2}^{(n)} - H_{k+1,0}^{(n)}H_{k,2}^{(n+1)} - H_{k+1,1}^{(n)}H_{k,1}^{(n+1)} = 0, \tag{4.7c}$$

$$H_{k,1}^{(n)}H_{k,-1}^{(n+1)} - H_{k,0}^{(n+1)}H_{k,0}^{(n)} - H_{k+1,-1}^{(n)}H_{k-1,1}^{(n+1)} = 0, \tag{4.7d}$$

by using the Jacobi identity for the determinants (A.8). Actually, this confirms the solution (4.5) of Wynn’s  $\varepsilon$ -algorithm (4.1). With the help of the bilinear relations (4.7a), and using the determinant expressions (2.2) for  $u_j^{(n)}$ , and (4.5) for  $\varepsilon_k^{(n)}$ , it is not hard to see that there exists the following Miura transformation between the restricted dLV equation and Wynn’s  $\varepsilon$ -algorithm [13]

$$\frac{u_{k+1}^{(n)}}{u_k^{(n+1)}} = \frac{\varepsilon_{k+1}^{(n+1)} - \varepsilon_{k+1}^{(n)}}{\varepsilon_{k-1}^{(n+2)} - \varepsilon_{k-1}^{(n+1)}}.$$

### 4.2 Generalized $\varepsilon$ -algorithm

Based on the connection between the restricted dLV equation and Wynn’s  $\varepsilon$ -algorithm introduced in Section 4.1, the purpose of this subsection is to acquire a convergence accelerate algorithm connected with the generalized dLV (2.1).

First, for a given sequence  $\{S_n\}$ , we denote  $c_{-1}^{(n)} = S_n$ . Then, we define a generalized Shanks transformation  $\tilde{\varepsilon}_k : \{S_n\} \mapsto \{\tilde{\varepsilon}_k(S_n)\}$  by,

$$\tilde{\varepsilon}_k(S_n) = \frac{H_{k+1,-1}^{(n)}}{H_{k,1}^{(n)}}, \tag{4.8}$$

where  $H_{k,j}^{(n)}$  is a Hankel determinant defined by (2.3), and its determinant elements  $c_j^{(n)}$  satisfying the recursion relations (2.5) plus (4.6). For convenience, we unify (2.5) plus (4.6) particularly in Section 4.2-4.3 into the following form

$$c_j^{(n)} = \frac{c_{j-1}^{(n+1)} - c_{j-1}^{(n)}}{h} + \frac{u_0^{(n)}}{c_0^{(n)}} \sum_{i=0}^{j-1} c_i^{(n)} c_{j-1-i}^{(n+1)}, \quad j = 0, 1, 2, 3, \dots \tag{4.9}$$

with the convention  $\sum_{i=p}^q \{ \} = 0$  for  $p > q$ . Noting the formulas (4.4) and (4.5) of Shanks transformation, it is easy to see that the generalized Shanks transformation  $\tilde{\varepsilon}_k$  will be reduced to the Shanks transformation if taking  $u_0^{(n)} = 0$ .

It is noted that, almost all the sequence transformations have the form  $f/\mathcal{D}f$  with  $\mathcal{D} = \sum_i \partial/\partial S_{n+i}$  (see, e.g. [14]). This fact also applies to the generalized Shanks transformation (4.8) described in the following theorem.

**Theorem 4.2.1** Let  $f(S_n, S_{n+1}, \dots, S_{n+2k}) = H_{k+1,-1}^{(n)}$  and  $\mathcal{D} = \sum_i \partial/\partial S_{n+i}$ , then,  $\mathcal{D}(H_{k+1,-1}^{(n)}) = H_{k,1}^{(n)}$ , and the generalized Shanks transformation (4.8) has the form as

$$\tilde{\varepsilon}_k(S_n) = \frac{f(S_n, S_{n+1}, \dots, S_{n+2k})}{\mathcal{D}f(S_n, S_{n+1}, \dots, S_{n+2k})}.$$

**Proof** First of all, from the convention  $c_{-1}^{(n)} = S_n$  and the recursion relation (4.9) for  $c_j^{(n)}$ , it is easy to find that

$$c_0^{(n)} = \frac{\Delta S_n}{h}, \quad c_1^{(n)} = \frac{\Delta S_{n+1} - \Delta S_n}{h^2} + u_0^{(n)} \frac{\Delta S_{n+1}}{h}.$$

Recursively, one can obtain that the  $c_j^{(n)}$  defined by (4.9) are functions in terms of  $\Delta S_{n+i}$ ,  $i = 0, 1, \dots, j$ , namely,

$$c_j^{(n)} = \varphi(\Delta S_n, \Delta S_{n+1}, \dots, \Delta S_{n+j}), \quad j = 0, 1, 2, \dots,$$

where  $\Delta$  is the forward difference operator defined by (4.1a).

Now, we apply the operator  $\mathcal{D}$  to  $H_{k+1,-1}^{(n)}$ . Since

$$\mathcal{D}(c_{-1}^{(n)}) = 1, \quad \mathcal{D}(\Delta S_{n+i}) = 0, \quad i = 0, 1, \dots,$$

and from the property of determinant derivative, it finally holds

$$\mathcal{D}(H_{k+1,-1}^{(n)}) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ c_0^{(n)} & c_1^{(n)} & \dots & c_k^{(n)} \\ c_1^{(n)} & c_2^{(n)} & \dots & c_{k+1}^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ c_{k-1}^{(n)} & c_k^{(n)} & \dots & c_{2k-1}^{(n)} \end{vmatrix} = H_{k,1}^{(n)}.$$

Therefore, by means of the definition (4.8) of the generalized Shanks transformation, we obtain  $\tilde{e}_k(S_n) = f/\mathcal{D}f$ .

In order to avoid direct calculations of determinants, now we design an algorithm to implement the generalized Shanks transformation.

**Theorem 4.2.2** Given an initial sequence  $\{S_n\}$ , and a sequence of parameters  $u_0^{(n)}$  for  $n = 0, 1, 2, \dots$ , the generalized  $\varepsilon$ -algorithm reads,

$$\tilde{\varepsilon}_{k+1}^{(n)} - \tilde{\varepsilon}_{k-1}^{(n+1)} = \frac{h\beta^{(n)}}{\tilde{\varepsilon}_k^{(n+1)} - \tilde{\varepsilon}_k^{(n)} - h\theta_{k+1}\alpha^{(n)}} - h\theta_k\alpha^{(n)}, \quad k = 0, 1, 2, \dots, \tag{4.10a}$$

$$\tilde{\varepsilon}_{-1}^{(n)} = 0, \quad \tilde{\varepsilon}_0^{(n)} = c_{-1}^{(n)} = S_n, \tag{4.10b}$$

where

$$\theta_k = \frac{1 + (-1)^k}{2},$$

and  $\alpha^{(n)}$  and  $\beta^{(n)}$  are defined as

$$\alpha^{(n)} = \frac{u_0^{(n)}}{c_0^{(n)}}, \quad \beta^{(n)} = 1 + hu_0^{(n)}.$$

It enjoys solution

$$\tilde{\varepsilon}_{2k}^{(n)} = \frac{H_{k+1,-1}^{(n)}}{H_{k,1}^{(n)}}, \quad \tilde{\varepsilon}_{2k+1}^{(n)} = \frac{H_{k,2}^{(n)}}{H_{k+1,0}^{(n)}}, \tag{4.11}$$

with the Hankel determinant expressions (2.3) and the elements  $c_j^{(n)}$  satisfying (4.9).

From (4.11), we can see that, the generalized Shanks transformation (4.8) of  $\tilde{\epsilon}_k(S_n)$  can be computed via  $\tilde{\epsilon}_{2k}^{(n)}$ , namely,

$$\tilde{\epsilon}_k(S_n) = \tilde{\epsilon}_{2k}^{(n)}.$$

But for  $\tilde{\epsilon}_{2k+1}^{(n)}$ , similar to Wynn's  $\epsilon$ -algorithm, it only acts as an auxiliary quantity.

The generalized  $\epsilon$ -algorithm is an extension of Wynn's  $\epsilon$ -algorithm, and it is connected with the generalized dLV (2.1). We illustrate these in the following remarks.

**Remark 4.2.3** If  $u_0^{(n)} = 0$ , then  $\alpha^{(n)} = 0$  and  $\beta^{(n)} = 1$ , the generalized  $\epsilon$ -algorithm (4.2.2) reduces to

$$\begin{aligned} \tilde{\epsilon}_{k+1}^{(n)} - \tilde{\epsilon}_{k-1}^{(n+1)} &= \frac{h}{\tilde{\epsilon}_k^{(n+1)} - \tilde{\epsilon}_k^{(n)}}, \\ \tilde{\epsilon}_{-1}^{(n)} = 0, \quad \tilde{\epsilon}_0^{(n)} &= c_{-1}^{(n)} = S_n, \quad k, n = 0, 1, 2, \dots, \end{aligned}$$

which is Wynn's  $\epsilon$ -algorithm (4.1) if taking the step size  $h = 1$ . Correspondingly, the solution (4.11) with (4.9) of the generalized  $\epsilon$ -algorithm reduces to that for Wynn's  $\epsilon$ -algorithm.

**Remark 4.2.4** The generalized  $\epsilon$ -algorithm (4.2.2) is connected with the generalized dLV equation via the following Miura transformation

$$\frac{u_{k+1}^{(n)}}{u_k^{(n+1)}} = \frac{(\theta_{k+1}\mu^{(n)} + \theta_k)\tilde{\epsilon}_{k+1}^{(n+1)} - \tilde{\epsilon}_{k+1}^{(n)} - \theta_k h \alpha^{(n)}}{(\theta_{k+1}\mu^{(n+1)} + \theta_k)\tilde{\epsilon}_{k-1}^{(n+2)} - \tilde{\epsilon}_{k-1}^{(n+1)} - \theta_k h \alpha^{(n+1)}}, \quad k = 0, 1, 2, \dots,$$

where

$$\mu^{(n)} = \frac{\beta^{(n)} + \alpha^{(n)}c_{-1}^{(n)}}{1 + \alpha^{(n)}c_{-1}^{(n+1)}},$$

and  $\alpha^{(n)}, \beta^{(n)}, \theta_k$  are defined in Theorem 4.2.2.

By employing the determinant expressions (2.2) of  $u_j^{(n)}$  and (4.11) of  $\tilde{\epsilon}_k^{(n)}$ , this Miura transformation can be easily checked with the help of Corollary A.3.3.

### 4.3 Numerical experiments

This section endeavors to investigate effect of acceleration convergence of the generalized  $\epsilon$ -algorithm (4.2.2). We will apply the generalized  $\epsilon$ -algorithm (4.2.2) to some linearly, logarithmically convergent sequences and a divergent series, and compare its numerical effects with those of Wynn's  $\epsilon$ -algorithm (4.1).

Note that, the generalized  $\epsilon$ -algorithm (4.2.2) relies on a sequence of parameters  $\{u_0^{(n)}\}$ , in addition to the initial sequence  $\{S_n\}$ . Namely, all the quantities  $\tilde{\epsilon}_k^{(n)}$  are determined uniquely and recursively via the generalized  $\epsilon$ -algorithm (4.2.2), if fixing  $\{S_n\}$  and  $\{u_0^{(n)}\}$ .

Now, we provide a way of choice of  $u_0^{(n)}$  by taking advantage of Brezinski’s  $\theta$ -algorithm. The  $\theta$ -algorithm [7, 14, 50],

$$\begin{aligned} \vartheta_{2k+1}^{(n)} &= \vartheta_{2k-1}^{(n+1)} + \frac{1}{\vartheta_{2k}^{(n+1)} - \vartheta_{2k}^{(n)}}, \\ \vartheta_{2k+2}^{(n)} &= \vartheta_{2k}^{(n+1)} + \frac{(\vartheta_{2k}^{(n+2)} - \vartheta_{2k}^{(n+1)}) (\vartheta_{2k+1}^{(n+2)} - \vartheta_{2k+1}^{(n+1)})}{\vartheta_{2k+1}^{(n+2)} - 2\vartheta_{2k+1}^{(n+1)} + \vartheta_{2k+1}^{(n)}}, \\ \vartheta_{-1}^{(n)} &= 0, \quad \vartheta_0^{(n)} = S_n. \quad k, n = 0, 1, \dots, \end{aligned}$$

proposed by Brezinski in 1971, is a very versatile algorithm. It is not only able to accelerate linearly convergent sequence and sum certain divergent series efficiently, but is also capable of accelerating the convergence of many logarithmically convergent sequences and series. As in the case of Wynn’s  $\varepsilon$ -algorithm and the generalized  $\varepsilon$ -algorithm, only the elements  $\vartheta_{2k}^{(n)}$  with even subscripts provide approximations to the limit of the sequence  $S_n$ , and  $\vartheta_{2k+1}^{(n)}$  with odd subscripts only acts as an auxiliary quantity. In particular, among these useful elements  $\vartheta_{2k}^{(n)}$ , the simplest element is  $\vartheta_2^{(n)}$ , which possesses the following form by implementing the above  $\theta$ -algorithm, recursively,

$$\vartheta_2^{(n)} = S_{n+1} - \frac{(\Delta S_n)(\Delta S_{n+1})(\Delta^2 S_{n+1})}{(\Delta S_{n+2})(\Delta^2 S_n) - (\Delta S_n)(\Delta^2 S_{n+1})}.$$

In order to seek for a choice of the parameter  $u_0^{(n)}$  in the generalized  $\varepsilon$ -algorithm (4.2.2), we now impose the assumption  $\tilde{\varepsilon}_2^{(n)} = \vartheta_2^{(n)}$ . Then, using the determinant expressions (4.11) of  $\tilde{\varepsilon}_k^{(n)}$  and the recursion relation (4.9) of the determinant elements  $c_j^{(n)}$  by noting  $c_{-1}^{(n)} = S_n$ , one can easily obtain the following result by direct calculation of determinant,

$$u_0^{(n)} = \frac{(\Delta S_n)(\Delta S_{n+2}) - (\Delta S_{n+1})^2}{(S_{n+2} - S_n)\Delta S_{n+1} - 2(\Delta S_n)(\Delta S_{n+2})}. \tag{4.12}$$

Under the choice of  $u_0^{(n)}$  in (4.12), it is easy to find that, given  $N$  terms of initial data of the sequence  $\{S_n\}$ , it will produce  $u_0^{(n)}$  with  $(N - 3)$  terms. Then, the number obtained by the generalized sequence transformation  $\tilde{\varepsilon}_{2k}^{(n)}$  with  $k = 1, 2, \dots, \lceil (N - 4)/2 \rceil$  is  $(N - 2k - 2)$ . Here, the notation  $\lceil \cdot \rceil$  means the ceiling function.

It is well known that, Wynn’s  $\varepsilon$ -algorithm can efficiently accelerate linearly convergent sequences and is also able to sum even wildly divergent series. To investigate the acceleration effect of the generalized  $\varepsilon$ -algorithm (4.2.2), we will apply it to some linearly, logarithmically convergent sequences and a certain divergent series, and compare their numerical effects with those of Wynn’s  $\varepsilon$ -algorithm in the following.

**Example 4.1** We consider the linearly convergent sequence

$$S_n = \sum_{k=0}^n \frac{(-1)^k}{k + 1}, \quad n = 0, 1, 2, \dots,$$

which converges to  $S = \log 2$ .

**Table 1** Numerical results of Example 4.1

$n$	$ S_n - S $	$ \epsilon_2^{(n)} - S $	$ \epsilon_4^{(n)} - S $	$ \epsilon_8^{(n)} - S $	$ \epsilon_{12}^{(n)} - S $	$ \epsilon_{18}^{(n)} - S $
0	$3.07 \times 10^{-1}$	$6.85 \times 10^{-3}$	$1.86 \times 10^{-4}$	$1.52 \times 10^{-7}$	$1.28 \times 10^{-10}$	$3.33 \times 10^{-15}$
1	$1.93 \times 10^{-1}$	$2.67 \times 10^{-3}$	$5.77 \times 10^{-5}$	$3.81 \times 10^{-8}$	$2.91 \times 10^{-11}$	$5.55 \times 10^{-16}$
2	$1.40 \times 10^{-1}$	$1.30 \times 10^{-3}$	$2.22 \times 10^{-5}$	$1.14 \times 10^{-8}$	$7.62 \times 10^{-12}$	
5	$7.65 \times 10^{-2}$	$2.90 \times 10^{-4}$	$2.68 \times 10^{-6}$	$6.26 \times 10^{-10}$	$2.60 \times 10^{-13}$	
7	$5.86 \times 10^{-2}$	$1.44 \times 10^{-4}$	$9.48 \times 10^{-7}$	$1.36 \times 10^{-10}$	$4.04 \times 10^{-14}$	
11	$3.99 \times 10^{-2}$	$5.05 \times 10^{-5}$	$1.92 \times 10^{-7}$	$1.17 \times 10^{-11}$		
15	$3.03 \times 10^{-2}$	$2.32 \times 10^{-5}$	$5.71 \times 10^{-8}$			
17	$2.70 \times 10^{-2}$	$1.68 \times 10^{-5}$				
19	$2.44 \times 10^{-2}$					

  

$n$	$ S_n - S $	$ \tilde{\epsilon}_2^{(n)} - S $	$ \tilde{\epsilon}_6^{(n)} - S $	$ \tilde{\epsilon}_{10}^{(n)} - S $	$ \tilde{\epsilon}_{12}^{(n)} - S $	$ \tilde{\epsilon}_{16}^{(n)} - S $
0	$3.07 \times 10^{-1}$	$1.30 \times 10^{-3}$	$2.15 \times 10^{-7}$	$1.35 \times 10^{-10}$	$1.67 \times 10^{-11}$	$3.22 \times 10^{-15}$
1	$1.93 \times 10^{-1}$	$4.39 \times 10^{-4}$	$8.12 \times 10^{-8}$	$1.61 \times 10^{-11}$	$2.76 \times 10^{-13}$	$8.88 \times 10^{-16}$
2	$1.40 \times 10^{-1}$	$1.86 \times 10^{-4}$	$3.00 \times 10^{-8}$	$8.84 \times 10^{-12}$	$1.31 \times 10^{-13}$	
5	$7.65 \times 10^{-2}$	$2.96 \times 10^{-5}$	$2.33 \times 10^{-9}$	$5.61 \times 10^{-13}$	$9.99 \times 10^{-15}$	
7	$5.86 \times 10^{-2}$	$1.23 \times 10^{-5}$	$5.89 \times 10^{-10}$	$1.06 \times 10^{-13}$		
11	$3.99 \times 10^{-2}$	$3.22 \times 10^{-6}$	$6.42 \times 10^{-11}$			
15	$3.03 \times 10^{-2}$	$1.18 \times 10^{-6}$				
17	$2.70 \times 10^{-2}$					
19	$2.44 \times 10^{-2}$					

Numerical results of applying Wynn’s  $\epsilon$ -algorithm (4.1) and the generalized  $\epsilon$ -algorithm (4.2.2) to the sequence  $\{S_n\}$  of Example 4.1 are listed in Table 1, where it should be noted that, the generalized Shanks transformation  $\tilde{\epsilon}_{2k}^{(n)}$  is compared with the Shanks transformation  $\epsilon_{2k+2}^{(n)}$  with  $k = 1, 3, 5, 8$ , based on the same number of used data of sequence  $\{S_n\}$ . This table shows that, the generalized  $\epsilon$ -algorithm (4.2.2) efficiently accelerates this linearly convergent sequence as well. The numerical effect of  $\epsilon_{2k+2}^{(n)}$  is a little bit better than the one of  $\tilde{\epsilon}_{2k}^{(n)}$ . Under the same initial data of sequence  $\{S_n\}$ , applying the generalized  $\epsilon$ -algorithm (4.2.2) and Wynn’s  $\epsilon$ -algorithm finally reach to the same level of the error order of approximation, but the former is iterated twice lesser than the latter.

**Example 4.2** We consider the linearly convergent sequence

$$S_n = \sum_{k=0}^n \frac{(0.8)^{k+1}}{k+1}, \quad n = 0, 1, 2, \dots,$$

which converges to  $S = \log 5$ .

The corresponding numerical results of Example 4.2 are placed in Table 2. We choose the same number of sequence  $\{S_n\}$  as the one in Example 4.1, and compare the generalized Shanks transformation  $\tilde{\epsilon}_{2k}^{(n)}$  with the Shanks transformation  $\epsilon_{2k+2}^{(n)}$ ,  $k = 1, 3, 5, 8$ . From this table we can see that, in general, for  $k = 3, 5, 8$ , the effects of  $\tilde{\epsilon}_{2k}^{(n)}$  are slightly better than those of  $\epsilon_{2k+2}^{(n)}$ . The final error order of approximation of applying

**Table 2** Numerical results of Example 4.2

$n$	$ S_n - S $	$ \epsilon_2^{(n)} - S $	$ \epsilon_4^{(n)} - S $	$ \epsilon_8^{(n)} - S $	$ \epsilon_{12}^{(n)} - S $	$ \epsilon_{18}^{(n)} - S $
0	$8.09 \times 10^{-1}$	$1.24 \times 10^{-1}$	$1.86 \times 10^{-2}$	$4.05 \times 10^{-4}$	$8.70 \times 10^{-6}$	$2.60 \times 10^{-8}$
1	$4.89 \times 10^{-1}$	$6.28 \times 10^{-2}$	$8.69 \times 10^{-3}$	$1.75 \times 10^{-4}$	$3.64 \times 10^{-6}$	$1.30 \times 10^{-8}$
2	$3.19 \times 10^{-1}$	$3.43 \times 10^{-2}$	$4.34 \times 10^{-3}$	$8.02 \times 10^{-5}$	$1.59 \times 10^{-6}$	
5	$1.07 \times 10^{-1}$	$7.28 \times 10^{-3}$	$7.10 \times 10^{-4}$	$9.79 \times 10^{-6}$	$1.65 \times 10^{-7}$	
7	$5.62 \times 10^{-2}$	$2.95 \times 10^{-3}$	$2.46 \times 10^{-4}$	$2.78 \times 10^{-6}$	$4.16 \times 10^{-8}$	
11	$1.70 \times 10^{-2}$	$5.82 \times 10^{-4}$	$3.65 \times 10^{-5}$	$2.84 \times 10^{-7}$		
15	$5.55 \times 10^{-3}$	$1.34 \times 10^{-4}$	$6.56 \times 10^{-6}$			
17	$3.23 \times 10^{-3}$	$6.66 \times 10^{-5}$				
19	$1.89 \times 10^{-3}$					

  

$n$	$ S_n - S $	$ \tilde{\epsilon}_2^{(n)} - S $	$ \tilde{\epsilon}_6^{(n)} - S $	$ \tilde{\epsilon}_{10}^{(n)} - S $	$ \tilde{\epsilon}_{12}^{(n)} - S $	$ \tilde{\epsilon}_{16}^{(n)} - S $
0	$8.09 \times 10^{-1}$	$7.95 \times 10^{-2}$	$2.54 \times 10^{-4}$	$1.41 \times 10^{-6}$	$1.48 \times 10^{-6}$	$6.55 \times 10^{-9}$
1	$4.89 \times 10^{-1}$	$3.57 \times 10^{-2}$	$1.71 \times 10^{-4}$	$1.25 \times 10^{-6}$	$3.11 \times 10^{-8}$	$6.50 \times 10^{-9}$
2	$3.19 \times 10^{-1}$	$1.77 \times 10^{-2}$	$2.27 \times 10^{-4}$	$5.94 \times 10^{-7}$	$3.46 \times 10^{-8}$	
5	$1.07 \times 10^{-1}$	$3.00 \times 10^{-3}$	$3.65 \times 10^{-7}$	$3.67 \times 10^{-8}$	$6.17 \times 10^{-9}$	
7	$5.62 \times 10^{-2}$	$1.08 \times 10^{-3}$	$1.00 \times 10^{-6}$	$1.50 \times 10^{-8}$		
11	$1.70 \times 10^{-2}$	$1.77 \times 10^{-4}$	$1.99 \times 10^{-7}$			
15	$5.55 \times 10^{-3}$	$3.49 \times 10^{-5}$				
17	$3.23 \times 10^{-3}$					
19	$1.89 \times 10^{-3}$					

the generalized  $\epsilon$ -algorithm (4.2.2) is higher than applying Wynn’s  $\epsilon$ -algorithm (4.1), beyond the advantage of twice iterations lesser than the latter.

**Example 4.3** We consider the partial sums

$$S_n = \sum_{k=0}^n (-1)^k k! z^{-k}, \quad n = 0, 1, 2, \dots, \tag{4.13}$$

of the factorially divergent asymptotic series with  $z = 3$ .

This corresponds to an exponential integral

$$E_1(z) = \int_z^\infty \frac{e^{-t}}{t} dt$$

satisfying

$$e^z E_1(z) = \int_0^\infty \frac{e^{-t}}{z+t} dt = \frac{1}{z} \int_0^\infty \frac{e^{-t}}{1+t/z} dt,$$

which admits the asymptotic expansion

$$ze^z E_1(z) \sim \sum_{k=0}^{\infty} \frac{(-1)^k k!}{z^k} = {}_2F_0(1, 1; -1/z), \quad z \rightarrow \infty. \quad (4.14)$$

If we replace  $z$  by  $1/z$ , we obtain the so-called series

$$\sum_{k=0}^{\infty} (-1)^k k! z^k = {}_2F_0(1, 1; -z), \quad z \rightarrow 0$$

which is the most simple prototype of a factorially divergent power series, and involves a factorially divergent generalized hypergeometric series  ${}_2F_0$  as  $z \rightarrow 0$ . It can be seen as the asymptotic expansion of the Euler integral

$$\mathcal{E}(z) = \int_0^{\infty} \frac{e^{-t}}{1+zt} dt.$$

We note that the Euler series has many physical applications, such as quantum physics [24], optics [5] etc.

It has been shown that sequence transformations are principal tools that can accomplish an efficient summation of the factorially divergent expansions of the type of the Euler series [6, 19]. For example, it was shown rigorously and explicitly in [6] that Wynn's epsilon algorithm or equivalently Padé approximants as well as some Levin-type transformations are able to sum the Euler series  $\mathcal{E}(z)$  provided that  $z \notin (-\infty, 0]$ . Since reliable programs for the exponential integral  $E_1(z)$  with  $z \in \mathbb{R}_+$  are available, (4.14) is well suited to test the ability of a sequence transformation of summing even wildly divergent series. In the following, we will take the partial sums (4.13) of the factorially divergent asymptotic series (4.14) as an example.

Now, we take advantage of the known result [6, 19] of  ${}_2F_0(1, 1; -1/z)|_{z=3} = ze^z E_1(z)|_{z=3} \approx 0.78625122076594$ , to investigate the ability of a sequence transformation of summing the divergent series  ${}_2F_0(1, 1; -1/z)$  for  $z = 3$ . It has been shown that, the subsequence  $\varepsilon_{2\lfloor n/2 \rfloor}^{(n-2\lfloor n/2 \rfloor)}$  of the Padé table of  ${}_2F_0(1, 1; -1/z)$  converges to the Euler integral for  $z = 3$ , (see [50, Eq. (4.3-6)-(4.3-7), Table 13-1] and [19]). Here, the notation  $\lfloor \cdot \rfloor$  means the floor function.

Let  $S = 0.78625122076594$ , and choose 22 terms of data of the sequence  $\{S_n\}$  given in Example 4.3. We compare summation results of using Wynn's  $\varepsilon$ -algorithm and the generalized  $\varepsilon$ -algorithm (4.2.2) respectively to  $\{S_n\}$  in Table 3. Similar to the above two examples, we compare  $\tilde{\varepsilon}_{2\lfloor m/2 \rfloor}^{(m-2\lfloor m/2 \rfloor)}$  with  $\varepsilon_{2\lfloor (m+2)/2 \rfloor}^{(m+2-2\lfloor (m+2)/2 \rfloor)}$  for  $m = 14, 15, \dots, 21$ . The results show that, compared to Wynn's  $\varepsilon$ -algorithm, the generalized  $\varepsilon$ -algorithm (4.2.2) behaves slightly more powerful, and the error order of approximation corresponding to the latter is improved at least an order of magnitude.

In summary, from the above three examples, we conclude that, as an extension of Wynn's  $\varepsilon$ -algorithm, the generalized  $\varepsilon$ -algorithm (4.2.2) under the choice of parameter  $u_0^{(n)}$  (4.12) enjoys similar numerical effects to the former when it comes to linearly convergent sequences or certain divergent series. For a given number of sequence  $\{S_n\}$ , the generalized  $\varepsilon$ -algorithm generally saves twice iterations to reach the same or even slightly higher error order of magnitude.



**Table 3** Numerical results of Example 4.3

$n$	$ S_n - S $	$ \varepsilon_{2\lfloor n/2 \rfloor}^{(n-2\lfloor n/2 \rfloor)} - S $	$ \tilde{\varepsilon}_{2\lfloor n/2 \rfloor}^{(n-2\lfloor n/2 \rfloor)} - S $
14	$1.50 \times 10^4$	$9.33 \times 10^{-7}$	$6.58 \times 10^{-9}$
15	$7.61 \times 10^4$	$7.01 \times 10^{-7}$	$6.67 \times 10^{-9}$
16	$4.10 \times 10^5$	$2.88 \times 10^{-7}$	$1.56 \times 10^{-9}$
17	$2.34 \times 10^6$	$2.19 \times 10^{-7}$	$1.65 \times 10^{-9}$
18	$1.42 \times 10^7$	$9.45 \times 10^{-8}$	$4.00 \times 10^{-10}$
19	$9.05 \times 10^7$	$7.29 \times 10^{-8}$	$4.42 \times 10^{-10}$
20	$6.07 \times 10^8$	$3.27 \times 10^{-8}$	
21	$4.28 \times 10^9$	$2.55 \times 10^{-8}$	

We end of this section by an example that shows an obviously better performance of the generalized  $\varepsilon$ -algorithm than Wynn’s  $\varepsilon$ -algorithm. Here, a different choice of  $u_0^{(n)}$  is considered.

**Example 4.4** We consider the sequence

$$S_n = S + 100 \prod_{i=0}^{n-2} \left( 1 - \frac{8}{9} \prod_{j=1}^i \frac{4j}{4j+3} \right) \left( 1 + \frac{1}{2^n} \right), \quad n = 1, 2, \dots,$$

which converges to  $S = 1$ .

This is a logarithmically convergent sequence because

$$\lim_{n \rightarrow \infty} \frac{S_{n+1} - S}{S_n - S} = 1.$$

Numerical results of applying Wynn’s  $\varepsilon$ -algorithm (4.1) and the generalized  $\varepsilon$ -algorithm (4.2.2) with  $u_0^{(n)} = \frac{3}{4n}$  to the sequence  $\{S_n\}$  of Example 4.4 are listed in Table 4, from which we see that our proposed algorithm (4.2.2) accelerates the convergence, while Wynn’s  $\varepsilon$ -algorithm does not. It is noted that, for given sequence  $\{S_n\}$  with number  $N$ , the number obtained by the generalized sequence transformation  $\tilde{\varepsilon}_{2k}^{(n)}$  with  $k = 1, 2, \dots, \lfloor (N - 2)/2 \rfloor$  is  $(N - 2k)$ .

### 5 Conclusion and final remarks

This paper extends the restricted fully discrete LV equation to a generalized case with a sequence of given constants  $\{u_0^{(n)}\}$ . This generalized dLV equation possesses a Hankel-type solution and can be connected with a discrete Riccati system. As the discrete step size  $h$  goes to zero, the generalized dLV equation with its solution, and the corresponding discrete Riccati system, both tend to the counterparts of the unrestricted semi-discrete LV equation given by [42]. In addition, we present a Lax pair of the

**Table 4** Numerical results of Example 4.4

$n$	$ S_n - S $	$ \varepsilon_2^{(n)} - S $	$ \varepsilon_8^{(n)} - S $	$ \varepsilon_{14}^{(n)} - S $	$ \varepsilon_{22}^{(n)} - S $	$ \varepsilon_{28}^{(n)} - S $
2	$1.39 \times 10$	2.48	$1.64 \times 10^{-1}$	$2.71 \times 10^{-2}$	$4.48 \times 10^{-3}$	$2.42 \times 10^{-3}$
5	2.50	$7.32 \times 10^{-1}$	$6.95 \times 10^{-2}$	$1.37 \times 10^{-2}$	$3.35 \times 10^{-3}$	
8	1.16	$3.43 \times 10^{-1}$	$3.73 \times 10^{-2}$	$7.95 \times 10^{-3}$	$2.63 \times 10^{-3}$	
12	$5.92 \times 10^{-1}$	$1.70 \times 10^{-1}$	$1.94 \times 10^{-2}$	$4.37 \times 10^{-3}$		
16	$3.59 \times 10^{-1}$	$1.01 \times 10^{-1}$	$1.15 \times 10^{-2}$	$2.42 \times 10^{-3}$		
22	$2.00 \times 10^{-1}$	$5.47 \times 10^{-2}$	$6.00 \times 10^{-3}$			
28	$1.26 \times 10^{-1}$	$3.34 \times 10^{-2}$				
30	$1.10 \times 10^{-1}$					
$n$	$ S_n - S $	$ \tilde{\varepsilon}_2^{(n)} - S $	$ \tilde{\varepsilon}_8^{(n)} - S $	$ \tilde{\varepsilon}_{14}^{(n)} - S $	$ \tilde{\varepsilon}_{22}^{(n)} - S $	$ \tilde{\varepsilon}_{28}^{(n)} - S $
2	$1.39 \times 10$	$2.05 \times 10^{-2}$	$1.27 \times 10^{-3}$	$1.64 \times 10^{-9}$	$1.63 \times 10^{-11}$	$5.55 \times 10^{-16}$
5	2.50	$1.16 \times 10^{-1}$	$4.59 \times 10^{-7}$	$3.79 \times 10^{-11}$	$9.79 \times 10^{-13}$	
8	1.16	$2.17 \times 10^{-2}$	$1.24 \times 10^{-8}$	$4.60 \times 10^{-12}$	$3.31 \times 10^{-14}$	
12	$5.92 \times 10^{-1}$	$1.56 \times 10^{-3}$	$3.81 \times 10^{-11}$	$2.11 \times 10^{-12}$		
16	$3.59 \times 10^{-1}$	$9.90 \times 10^{-5}$	$1.35 \times 10^{-13}$	$1.61 \times 10^{-13}$		
22	$2.00 \times 10^{-1}$	$1.49 \times 10^{-6}$	$2.07 \times 10^{-13}$			
28	$1.26 \times 10^{-1}$	$2.18 \times 10^{-8}$				
30	$1.10 \times 10^{-1}$					

generalized dLV equation in terms of symmetric OPs. We also show that the resulting equation is linked to a generalization of the famous Wynn’s  $\varepsilon$ -algorithm. Finally, we apply this generalized  $\varepsilon$ -algorithm to two linearly convergent sequences, one logarithmically convergent sequence as well as one divergent sequence, and compare the numerical effects between the generalized  $\varepsilon$ -algorithm and Wynn’s  $\varepsilon$ -algorithm.

As is described in the paper, the generalized  $\varepsilon$ -algorithm relies on the choice of  $u_0^{(n)}$ . It deserves to be investigated further to distinguish its roles in accelerating different types of sequences. Recall that the first three examples in Section 4.3 are based the choice of  $u_0^{(n)}$  in (4.12), while the last example involves a choice that  $u_0^{(n)} = \frac{3}{4n}$ , which exhibits an extremely good performance. In fact, we can obtain the so-called kernel of  $\tilde{\varepsilon}_2^{(n)}$ . More precisely, as for the sequence given by

$$S_n = S + \varrho \prod_{i=0}^{n-2} \left( 1 + \frac{\kappa}{\prod_{j=1}^i (1 + u_0^{(j)})} \right), \quad n = 1, 2, \dots,$$

where  $S, \varrho, \kappa$  are arbitrary constants with  $\varrho \neq 0, \kappa \neq 0$ , applying the generalized  $\varepsilon$ -algorithm to  $\{S_n\}$  will exactly give  $\tilde{\varepsilon}_2^{(n)} = S$ . The last example is constructed based on a perturbation of such a sequence.

Furthermore, we note that the generalized dLV equation obtained in this paper is a discrete analogue of the isospectral unrestricted semi-discrete LV equation, while Hankel-type solutions for the first two equations of the Volterra lattice hierarchy and the first two equations of its non-isospectral extension are addressed in [21]. A non-isospectral integrable equation means, the spectrum in its Lax pair is dependent on

time  $t$  instead of a constant. It is interesting to see whether a certain non-isospectral generalized dLV equation together with its solution exist and how to design it as an convergence acceleration algorithm, which we leave for future work.

### Appendix A: Technical details for the main results

This appendix is devoted to providing some linear and bilinear relations which serve for the proofs of the theorems in Sections 2–4. In Appendix A.1, some notations and auxiliary determinants are introduced. Appendix A.2 contains some linear relations, while Appendix A.3 includes some bilinear relations. Based on these relations, one can prove the main results in Sections 2–4.

#### A.1. Notations

For integers  $k \geq 1, j, r \geq -1, s, n \geq 0$ , and  $l = 0, 1$ , we first define two sets of sequences,  $\{a_s^{(n)}\}$  and  $\{d_r^{(n)}\}$  satisfying

$$a_s^{(n)} = c_s^{(n)} - \frac{h\alpha^{(n)}}{\beta^{(n)}} \sum_{i=0}^{s-1} a_i^{(n)} c_{s-i}^{(n)}$$

and

$$d_r^{(n)} = c_r^{(n)} - \frac{h\alpha^{(n)}}{\beta^{(n)}} \sum_{i=-1}^{r-1} d_i^{(n)} c_{r-i}^{(n)}$$

where  $\alpha^{(n)}$  and  $\beta^{(n)}$  are defined by (3.12). Then, in order to express determinants in this appendix conveniently, we introduce the following vectors,

$$A_{k,s}^{(n)} = \begin{pmatrix} a_s^{(n)} \\ a_{s+1}^{(n)} \\ \vdots \\ a_{s+k-1}^{(n)} \end{pmatrix}, \quad B_{k,j}^{(n+1)} = \begin{pmatrix} \frac{\beta^{(n)} c_j^{(n+1)}}{h} \\ \frac{\beta^{(n)} c_{j+1}^{(n+1)}}{h} + \alpha^{(n)} c_1^{(n)} c_j^{(n+1)} \\ \vdots \\ \frac{\beta^{(n)} c_{j+k-1}^{(n+1)}}{h} + \alpha^{(n)} \sum_{i=1}^{k-1} c_i^{(n)} c_{j+k-1-i}^{(n+1)} \end{pmatrix}, \quad (A.1a)$$

$$C_{k,j}^{(n)} = \begin{pmatrix} c_j^{(n)} \\ c_{j+1}^{(n)} \\ \vdots \\ c_{j+k-1}^{(n)} \end{pmatrix}, \quad D_{k,j}^{(n)} = \begin{pmatrix} d_j^{(n)} \\ d_{j+1}^{(n)} \\ \vdots \\ d_{j+k-1}^{(n)} \end{pmatrix}, \quad \xi_{k,s}^{(n+1)} = \begin{pmatrix} \sum_{i=1}^s c_i^{(n)} c_{s-i}^{(n+1)} \\ \sum_{i=2}^{s+1} c_i^{(n)} c_{s+1-i}^{(n+1)} \\ \vdots \\ \sum_{i=k}^{s+k-1} c_i^{(n)} c_{s+k-1-i}^{(n+1)} \end{pmatrix}. \quad (A.1b)$$

with the convention  $\xi_{k,0}^{(n+1)} = \mathbf{0}$ .

Use these vectors, we define some auxiliary determinants as follows,

$$\begin{aligned}
 F_{k,j,s}^{(n+1)} &= |A_{k,s}^{(n)}, C_{k,j}^{(n+1)}, \dots, C_{k,j+k-3}^{(n+1)}, C_{k,j+k-2}^{(n+1)}|, \\
 \Gamma_{k,j,r}^{(n+1)} &= |D_{k,r}^{(n)}, C_{k,j}^{(n+1)}, \dots, C_{k,j+k-3}^{(n+1)}, C_{k,j+k-2}^{(n+1)}|, \\
 E_{k,j,r}^{(n+1)} &= |C_{k,r}^{(n)}, B_{k,j}^{(n+1)}, \dots, B_{k,j+k-3}^{(n+1)}, B_{k,j+k-2}^{(n+1)}|, \\
 \tilde{E}_{k,s,r}^{(n+1)} &= |B_{k,r}^{(n+1)}, B_{k,s}^{(n+1)}, \dots, B_{k,s+k-3}^{(n+1)}, B_{k,s+k-2}^{(n+1)}|, \\
 K_{k,j,1}^{(n+1)} &= \begin{vmatrix} 1 & -\alpha^{(n)}C_{j+1}^{(n+1)} & -\alpha^{(n)}C_{j+1}^{(n+1)} & \dots & -\alpha^{(n)}C_{j+k-2}^{(n+1)} \\ C_{k-1,1}^{(n)} & B_{k-1,j+1}^{(n+1)} & B_{k-1,j+2}^{(n+1)} & \dots & B_{k-1,j+k-1}^{(n+1)} \end{vmatrix}, \\
 G_{2k,l}^{(n+1)}(x) &= \begin{vmatrix} C_{k,l}^{(n)} & B_{k,l}^{(n+1)} & \dots & B_{k,l+k-2}^{(n+1)} & B_{k,l+k-1}^{(n+1)} \\ 1 & y_{l,2} & \dots & y_{l,2k-2} & y_{l,2k} \end{vmatrix}, \\
 \tilde{G}_{2k,l}^{(n+1)}(x) &= \begin{vmatrix} B_{k-1,l}^{(n+1)} & B_{k-1,l+1}^{(n+1)} & \dots & B_{k-1,l+k-1}^{(n+1)} \\ y_{l,2} & y_{l,4} & \dots & y_{l,2k} \end{vmatrix},
 \end{aligned}$$

with the conventions

$$F_{0,j,s}^{(n+1)} = 0, \quad F_{1,j,s}^{(n+1)} = a_s^{(n)}, \quad \Gamma_{0,j,r}^{(n+1)} = 0, \quad \Gamma_{1,j,r}^{(n+1)} = a_r^{(n)}, \tag{A.2a}$$

$$E_{1,j,r}^{(n+1)} = c_r^{(n)}, \quad \tilde{E}_{0,s,r}^{(n+1)} = 1, \quad \tilde{E}_{1,s,r}^{(n+1)} = B_{1,r}^{(n+1)}, \tag{A.2b}$$

$$K_{1,j,1}^{(n+1)} = 1, \quad G_{0,l}^{(n+1)}(x) = 1, \quad \tilde{G}_{2,l}^{(n+1)}(x) = y_{l,2}, \tag{A.2c}$$

where

$$y_{l,2k} = (x^2 + \frac{1}{h})x^{2k-2} - \alpha^{(n)} \sum_{i=1}^{k-1} c_{k-1-i}^{(n+1)} x^{2i}. \tag{A.3}$$

Using these notations, it is easy to see that the Hankel determinant  $H_{k,j}^{(n)}$  can be expressed in terms of vector  $C_{k,j}^{(n)}$  as follows

$$H_{k,j}^{(n)} = |C_{k,j}^{(n)}, C_{k,j+1}^{(n)}, \dots, C_{k,j+k-1}^{(n)}|.$$

Now, let us present the relations among the vectors defined in (A.1). First, we rewrite (2.5) as follows

$$c_j^{(n)} = \frac{\beta^{(n)}c_{j-1}^{(n+1)}}{h} - \frac{c_{j-1}^{(n)}}{h} + \alpha^{(n)} \sum_{i=1}^{j-1} c_i^n c_{j-1-i}^{(n+1)}, \quad j = 1, 2, 3, \dots \tag{A.4}$$

Then, we have the following lemma.

**Lemma A.1.1** Assume  $\{c_j^{(n)}\}$  defined by the recursion relation (A.4) and (4.6), then

$$C_{k,0}^{(n)} = \frac{hB_{k-1}^{(n+1)} - C_{k-1}^{(n)}}{h\gamma^{(n)}}, \tag{A.5a}$$

$$C_{kj}^{(n)} = B_{kj-1}^{(n+1)} + \alpha^{(n)} \xi_{kj-1}^{(n+1)} - \frac{1}{h} C_{kj-1}^{(n)}, \quad j = 1, 2, \dots, \tag{A.5b}$$

where  $\gamma^{(n)}$  is defined by

$$\gamma^{(n)} = 1 + \alpha^{(n)} c_{-1}^{(n+1)}.$$

**Proof** The relation (A.5b) is obtained directly by applying (A.4) to  $C_{kj}^{(n)}$ . For (A.5a), noting the definition of recursion relation (4.6) for  $c_0^{(n)}$ , and using (A.4) to  $C_{k,0}^{(n)}$ , we first have

$$\gamma^{(n)} C_{k,0}^{(n)} = \alpha^{(n)} c_{-1}^{(n+1)} C_{k,0}^{(n)} + \left( \begin{array}{c} \frac{c_{-1}^{(n+1)}}{h} \\ \frac{\beta^{(n)} c_0^{(n+1)}}{h} \\ \frac{\beta^{(n)} c_1^{(n+1)}}{h} + \alpha^{(n)} c_1^{(n)} c_0^{(n+1)} \\ \vdots \\ \frac{\beta^{(n)} c_{k-2}^{(n+1)}}{h} + \alpha^{(n)} \sum_{i=1}^{k-2} c_i^{(n)} c_{k-2-i}^{(n+1)} \end{array} \right) - \frac{1}{h} C_{k,-1}^{(n)}.$$

Then, it is easy to find that the summation of the first two terms in the above right hand of equation identifies with  $B_{k,-1}^{(n+1)}$ , by noting its definition in (A.1a). Thus, (A.5a) gets proved.

### A.2. Linear relations

In Lemma A.2.1, we list some important linear relations, which serve to deduce Corollary A.2.2. The proof of Lemma A.2.1 is implemented by performing some row and column transformations on determinants based on the determinant properties. Due to the lengthy details, we place its proof in Appendix A.4.

**Lemma A.2.1** Assume  $\{c_j^{(n)}\}$  satisfy the recursion relations (2.5) and (4.6), then for integers  $k \geq 1, j, r \geq -1$ , and  $l = 0, 1$ , the following formulae hold

$$H_{k+1,-1}^{(n)} = \frac{1}{\gamma^{(n)}} E_{k+1,-1,-1}^{(n+1)}, \tag{A.6a}$$

$$H_{k,l}^{(n)} = E_{k,l,l}^{(n+1)}, \tag{A.6b}$$

$$H_{k,2}^{(n)} = K_{k+1,0,1}^{(n+1)} - \frac{1}{h} E_{k,2,1}^{(n+1)}, \tag{A.6c}$$

$$E_{k,j,-1}^{(n+1)} = \left( \frac{\beta^{(n)}}{h} \right)^{k-1} \Gamma_{k,j,-1}^{(n+1)}, \tag{A.6d}$$

$$E_{k,j,0}^{(n+1)} = \left(\frac{\beta^{(n)}}{h}\right)^{k-1} F_{k,j,0}^{(n+1)}, \quad (\text{A.6e})$$

$$E_{k,j,0}^{(n+1)} = \left(\frac{\beta^{(n)}}{h}\right)^{k-1} \Gamma_{k,j,0}^{(n+1)} + \frac{h\alpha^{(n)}c_{-1}^{(n)}}{\beta^{(n)}} E_{k,j,1}^{(n+1)}, \quad (\text{A.6f})$$

$$E_{k,j,0}^{(n+1)} = \frac{h\tilde{E}_{k,j,-1}^{(n+1)} - E_{k,j,-1}^{(n+1)}}{hy^{(n)}}, \quad (\text{A.6g})$$

$$E_{k,j,1}^{(n+1)} = \frac{(\beta^{(n)})^k}{h^{k-1}} F_{k,j,1}^{(n+1)}, \quad (\text{A.6h})$$

$$E_{k,j,1}^{(n+1)} = \tilde{E}_{k,j,0}^{(n+1)} - \frac{1}{h} E_{k,j,0}^{(n+1)}, \quad (\text{A.6i})$$

$$\tilde{E}_{k,r+1,r}^{(n+1)} = \left(\frac{\beta^{(n)}}{h}\right)^k H_{k,r}^{(n+1)}, \quad (\text{A.6j})$$

$$K_{k,j,1}^{(n+1)} = \frac{(\beta^{(n)})^k}{h^{k-1}} H_{k-1,j+1}^{(n+1)} - \frac{\alpha^{(n)}(\beta^{(n)})^{k-1}}{h^{k-2}} F_{k,j,0}^{(n+1)}, \quad (\text{A.6k})$$

$$G_{2k,l}^{(n+1)}(x) = \frac{H_{k,l}^{(n)} P_{2k+l}^{(n)}(x)}{x^l}, \quad (\text{A.6l})$$

$$\begin{aligned} \tilde{G}_{2k,l}^{(n+1)}(x) &= \frac{(\beta^{(n)})^{k-1} H_{k-1,l}^{(n+1)} \left( (hx^2+1)P_{2k-2+l}^{(n+1)}(x) - hx\alpha^{(n)}Q_{2k-3+l}^{(n+1)}(x) \right)}{h^k x^l}. \end{aligned} \quad (\text{A.6m})$$

Using Lemma A.2.1, it is not hard to derive the following corollary, whose detail is omitted.

**Corollary A.2.2** Under the recursion relations (2.5) and (4.6) satisfied by  $\{c_j^{(n)}\}$ , there hold the following important identities for  $k = 1, 2, \dots$ ,

$$H_{k+1,-1}^{(n)} = \frac{1}{\gamma^{(n)}} \left(\frac{\beta^{(n)}}{h}\right)^k \Gamma_{k+1,-1,-1}^{(n+1)}, \quad (\text{A.7a})$$

$$H_{k,0}^{(n)} = \frac{(\beta^{(n)})^{k-1} (\beta^{(n)} H_{k-1}^{(n+1)} - \Gamma_{k,0,-1}^{(n+1)})}{h^k \gamma^{(n)}}, \quad (\text{A.7b})$$

$$H_{k,0}^{(n)} = \left(\frac{\beta^{(n)}}{h}\right)^{k-1} F_{k,0,0}^{(n+1)}, \quad (\text{A.7c})$$

$$H_{k,1}^{(n)} = \frac{(\beta^{(n)})^k}{h^{k-1}} F_{k,1,1}^{(n+1)}, \quad (\text{A.7d})$$

$$H_{k,1}^{(n)} = \left(\frac{\beta^{(n)}}{h}\right)^k H_{k,0}^{(n+1)} - \frac{(\beta^{(n)})^{k-1}}{h^k} F_{k,1,0}^{(n+1)}, \tag{A.7e}$$

$$H_{k,1}^{(n)} = \frac{(\beta^{(n)})^k (\beta^{(n)} H_{k,0}^{(n+1)} - \Gamma_{k,1,0}^{(n+1)})}{h^k (\beta^{(n)} + \alpha^{(n)} c_{-1}^{(n)})}, \tag{A.7f}$$

$$H_{k,2}^{(n)} = \frac{(\beta^{(n)})^{k+1}}{h^k} H_{k,1}^{(n+1)} - h\alpha^{(n)} H_{k+1,0}^{(n)} - \left(\frac{\beta^{(n)}}{h}\right)^k F_{k,2,1}^{(n+1)}. \tag{A.7g}$$

Moreover, it is easy to check that relations (A.7e) and (A.7g) also hold for  $k = 0$ , by using the conventions (2.4) and (A.2a).

### A.3. Bilinear relations

Now, we apply the Jacobi identity to produce some bilinear relations, which play crucial roles in the main results. For any determinant  $D$ , the *Jacobi determinant identity* [1] reads

$$D \cdot D \begin{bmatrix} i_1 & i_2 \\ j_1 & j_2 \end{bmatrix} = D \begin{bmatrix} i_1 \\ j_1 \end{bmatrix} \cdot D \begin{bmatrix} i_2 \\ j_2 \end{bmatrix} - D \begin{bmatrix} i_1 \\ j_2 \end{bmatrix} \cdot D \begin{bmatrix} i_2 \\ j_1 \end{bmatrix}, \tag{A.8}$$

where

$$D \begin{bmatrix} i_1 & i_2 & \dots & i_k \\ j_1 & j_2 & \dots & j_k \end{bmatrix}, \quad i_1 < i_2 < \dots < i_k, \quad j_1 < j_2 < \dots < j_k,$$

denotes the determinant obtained from  $D$  by removing the rows at positions  $i_1, i_2, \dots, i_k$ , and the columns at positions  $j_1, j_2, \dots, j_k$ , in the respective matrix.

**Lemma A.3.1** For  $k = 0, 1, 2, \dots$ , and  $j = 0, 1$ , the following bilinear relations hold,

$$F_{k+2,0,0}^{(n+1)} H_{k,1}^{(n+1)} - F_{k+1,0,0}^{(n+1)} H_{k+1,1}^{(n+1)} + F_{k+1,1,1}^{(n+1)} H_{k+1,0}^{(n+1)} = 0, \tag{A.9a}$$

$$F_{k+1,1,0}^{(n+1)} H_{k,0}^{(n+1)} - H_{k+1,0}^{(n+1)} F_{k,1,0}^{(n+1)} - F_{k+1,0,0}^{(n+1)} H_{k,1}^{(n+1)} = 0, \tag{A.9b}$$

$$H_{k,1}^{(n+1)} F_{k,1,0}^{(n+1)} - F_{k,2,1}^{(n+1)} H_{k,0}^{(n+1)} - F_{k+1,0,0}^{(n+1)} H_{k-1,2}^{(n+1)} = 0, \tag{A.9c}$$

$$F_{k+1,1,0}^{(n+1)} H_{k,1}^{(n+1)} - H_{k+1,0}^{(n+1)} F_{k,2,1}^{(n+1)} - F_{k+1,0,0}^{(n+1)} H_{k,2}^{(n+1)} = 0, \tag{A.9d}$$

$$\Gamma_{k+2,-1,-1}^{(n+1)} H_{k,1}^{(n+1)} - H_{k+1,0}^{(n+1)} \Gamma_{k+1,0,-1}^{(n+1)} + \Gamma_{k+1,1,0}^{(n+1)} H_{k+1,-1}^{(n+1)} = 0, \tag{A.9e}$$

$$\Gamma_{k+1,0,-1}^{(n+1)} H_{k,0}^{(n+1)} - H_{k+1,-1}^{(n+1)} \Gamma_{k,1,0}^{(n+1)} - \Gamma_{k+1,-1,-1}^{(n+1)} H_{k,1}^{(n+1)} = 0, \tag{A.9f}$$

$$G_{2k+2,j}^{(n+1)}(x) \tilde{E}_{k,j+1j}^{(n+1)} - \tilde{G}_{2k+2,j}^{(n+1)}(x) E_{k+1,jj}^{(n+1)} + G_{2k,j}^{(n+1)}(x) \tilde{E}_{k+1,j+1j}^{(n+1)} = 0. \tag{A.9g}$$

**Proof** For  $k = 0$ , all these equations are easily checked. For  $k \geq 1$ , (A.9a) is obtained from the Jacobi identity (A.8) with  $D = F_{k+2,0,0}^{(n+1)}$ , and noticing

$$\begin{aligned} D \begin{bmatrix} 1 & k+2 \\ 1 & k+2 \end{bmatrix} &= H_{k,1}^{(n+1)}, & D \begin{bmatrix} 1 \\ 1 \end{bmatrix} &= H_{k+1,1}^{(n+1)}, \\ D \begin{bmatrix} k+2 \\ k+2 \end{bmatrix} &= F_{k+1,0,0}^{(n+1)}, & D \begin{bmatrix} 1 \\ k+2 \end{bmatrix} &= F_{k+1,1,1}^{(n+1)}, & D \begin{bmatrix} k+2 \\ 1 \end{bmatrix} &= H_{k+1,0}^{(n+1)}. \end{aligned}$$

(A.9b) follows from the Jacobi identity with

$$D = \begin{vmatrix} a_0^{(n)} & c_0^{(n+1)} & \cdots & c_{k-1}^{(n+1)} & c_k^{(n+1)} \\ a_1^{(n)} & c_1^{(n+1)} & \cdots & c_k^{(n+1)} & c_{k+1}^{(n+1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_k^{(n)} & c_k^{(n+1)} & \cdots & c_{2k-1}^{(n+1)} & c_{2k}^{(n+1)} \\ 0 & 0 & \cdots & 0 & 1 \end{vmatrix} = F_{k+1,0,0}^{(n+1)},$$

and

$$\begin{aligned} D \begin{bmatrix} k+1 & k+2 \\ 1 & 2 \end{bmatrix} &= H_{k,1}^{(n+1)}, & D \begin{bmatrix} k+1 \\ 1 \end{bmatrix} &= H_{k,0}^{(n+1)}, \\ D \begin{bmatrix} k+2 \\ 2 \end{bmatrix} &= F_{k+1,1,0}^{(n+1)}, & D \begin{bmatrix} k+1 \\ 2 \end{bmatrix} &= F_{k,1,0}^{(n+1)}, & D \begin{bmatrix} k+2 \\ 1 \end{bmatrix} &= H_{k+1,0}^{(n+1)}. \end{aligned}$$

(A.9c) is derived by employing the Jacobi identity with

$$D = \begin{vmatrix} 1 & 1 & 1 & 0 & \cdots & 0 \\ 0 & a_0^{(n)} & c_0^{(n+1)} & c_1^{(n+1)} & \cdots & c_{k-1}^{(n+1)} \\ 0 & a_1^{(n)} & c_1^{(n+1)} & c_2^{(n+1)} & \cdots & c_k^{(n+1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & a_k^{(n)} & c_k^{(n+1)} & c_{k+1}^{(n+1)} & \cdots & c_{2k-1}^{(n+1)} \end{vmatrix} = F_{k+1,0,0}^{(n+1)},$$

and



$$\begin{aligned}
 D \begin{bmatrix} 2 & k+2 \\ 1 & 2 \end{bmatrix} &= H_{k-1,2}^{(n+1)}, & D \begin{bmatrix} k+2 \\ 2 \end{bmatrix} &= H_{k,0}^{(n+1)}, & D \begin{bmatrix} 2 \\ 2 \end{bmatrix} &= H_{k,1}^{(n+1)}, \\
 D \begin{bmatrix} 2 \\ 1 \end{bmatrix} &= \begin{vmatrix} 1 & 1 & 0 & \dots & 0 \\ a_1^{(n)} & c_1^{(n+1)} & c_2^{(n+1)} & \dots & c_k^{(n+1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_k^{(n)} & c_k^{(n+1)} & c_{k+1}^{(n+1)} & \dots & c_{2k-1}^{(n+1)} \end{vmatrix} &= H_{k,1}^{(n+1)} - F_{k,2,1}^{(n+1)}, \\
 D \begin{bmatrix} k+2 \\ 1 \end{bmatrix} &= \begin{vmatrix} 1 & 1 & 0 & \dots & 0 \\ a_0^{(n)} & c_0^{(n+1)} & c_1^{(n+1)} & \dots & c_{k-1}^{(n+1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{k-1}^{(n)} & c_{k-1}^{(n+1)} & c_k^{(n+1)} & \dots & c_{2k-2}^{(n+1)} \end{vmatrix} &= H_{k,0}^{(n+1)} - F_{k,1,0}^{(n+1)},
 \end{aligned}$$

where we subtract the first columns of the determinants from the second ones in the calculations of  $D \begin{bmatrix} 2 \\ 1 \end{bmatrix}$  and  $D \begin{bmatrix} k+2 \\ 1 \end{bmatrix}$ , and use the determinant property.

To prove (A.9d), we apply the Jacobi identity to

$$D = \begin{vmatrix} 0 & 0 & 0 & \dots & 0 & 1 \\ a_0^{(n)} & c_0^{(n+1)} & c_1^{(n+1)} & \dots & c_{k-1}^{(n+1)} & c_k^{(n+1)} \\ a_1^{(n)} & c_1^{(n+1)} & c_2^{(n+1)} & \dots & c_k^{(n+1)} & c_{k+1}^{(n+1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_k^{(n)} & c_k^{(n+1)} & c_{k+1}^{(n+1)} & \dots & c_{2k-1}^{(n+1)} & c_{2k}^{(n+1)} \end{vmatrix} = (-1)^{k+3} F_{k+1,0,0}^{(n+1)},$$

with

$$\begin{aligned}
 D \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} &= H_{k,2}^{(n+1)}, & D \begin{bmatrix} 1 \\ 1 \end{bmatrix} &= H_{k+1,0}^{(n+1)}, & D \begin{bmatrix} 2 \\ 2 \end{bmatrix} &= (-1)^{k+2} F_{k,2,1}^{(n+1)}, \\
 D \begin{bmatrix} 1 \\ 2 \end{bmatrix} &= F_{k+1,1,0}^{(n+1)}, & D \begin{bmatrix} 2 \\ 1 \end{bmatrix} &= (-1)^{k+2} H_{k,1}^{(n+1)}.
 \end{aligned}$$

(A.9e) is obtained from the Jacobi identity with  $D = \Gamma_{k+2,-1,-1}^{(n+1)}$ , and

$$\begin{aligned}
 D \begin{bmatrix} 1 & k+2 \\ 1 & 2 \end{bmatrix} &= H_{k,1}^{(n+1)}, & D \begin{bmatrix} 1 \\ 1 \end{bmatrix} &= H_{k+1,0}^{(n+1)}, \\
 D \begin{bmatrix} k+2 \\ 2 \end{bmatrix} &= \Gamma_{k+1,0,-1}^{(n+1)}, & D \begin{bmatrix} 1 \\ 2 \end{bmatrix} &= \Gamma_{k+1,1,0}^{(n+1)}, & D \begin{bmatrix} k+2 \\ 1 \end{bmatrix} &= H_{k+1,-1}^{(n+1)}.
 \end{aligned}$$

(A.9f) is proved by applying the Jacobi identity to

$$D = \begin{vmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ d_{-1}^{(n)} & c_{-1}^{(n+1)} & c_0^{(n+1)} & \dots & c_{k-2}^{(n+1)} & c_{k-1}^{(n+1)} \\ d_0^{(n)} & c_0^{(n+1)} & c_1^{(n+1)} & \dots & c_{k-1}^{(n+1)} & c_k^{(n+1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ d_{k-1}^{(n)} & c_{k-1}^{(n+1)} & c_k^{(n+1)} & \dots & c_{2k-2}^{(n+1)} & c_{2k-1}^{(n+1)} \end{vmatrix} = -\Gamma_{k+1,0,-1}^{(n+1)},$$

with

$$\begin{aligned}
 D \begin{bmatrix} k+1 & k+2 \\ 1 & k+2 \end{bmatrix} &= \tilde{E}_{k,j+1,j}^{(n+1)}, & D \begin{bmatrix} k+1 \\ 1 \end{bmatrix} &= \tilde{G}_{2k+2,j}^{(n+1)}(x), & D \begin{bmatrix} k+2 \\ k+2 \end{bmatrix} &= E_{k+1,j,j}^{(n+1)}, \\
 D \begin{bmatrix} k+1 \\ k+2 \end{bmatrix} &= G_{2k,j}^{(n+1)}(x), & D \begin{bmatrix} k+2 \\ 1 \end{bmatrix} &= \tilde{E}_{k+1,j+1,j}^{(n+1)}.
 \end{aligned}$$

**Corollary A.3.2** If we let

$$\tau_{2k}^{(n)} = H_{k,0}^{(n)}, \quad \tau_{2k+1}^{(n)} = H_{k,1}^{(n)}, \quad k, n = 0, 1, \dots,$$

with the elements of  $H_{k,j}^{(n)}$  satisfying (2.5) as well as arbitrary constants  $c_0^{(n)} \neq 0$ , then  $\tau_k^{(n)}$  satisfy the bilinear relations

$$\beta^{(n)} \tau_{k+2}^{(n+1)} \tau_{k+1}^{(n)} - \tau_{k+2}^{(n)} \tau_{k+1}^{(n+1)} - h \tau_{k+3}^{(n)} \tau_k^{(n+1)} = 0, \quad k, n = 0, 1, \dots \tag{A.10}$$

**Proof** We prove (A.10) with respect to its odd and even parts, respectively.

For  $k = 2j + 1, j = 0, 1, 2, \dots$ , the relation (A.10) can be obtained from the identity (A.9a) and eliminating  $F_{k+2,0,0}^{(n+1)}, F_{k+1,0,0}^{(n+1)}$  and  $F_{k+1,1,1}^{(n+1)}$  via the relations (A.7c) and (A.7d) respectively.

For  $k = 2j, j = 0, 1, 2, \dots$ , the (A.10) immediately follows from the identities (A.9b) and eliminating  $F_{k+1,0,0}^{(n+1)}$  and  $F_{k,1,0}^{(n+1)}, F_{k+1,1,0}^{(n+1)}$  via the relations (A.7c) and (A.7e) respectively.

**Corollary A.3.3** For  $k = 0, 1, 2, \dots$ , the Hankel determinants  $H_{k,j}^{(n)}$  with the elements restricted by (2.5) and (4.6) satisfy the following bilinear relations

$$\mu^{(n)} H_{k+1,1}^{(n+1)} H_{k+1,-1}^{(n+1)} - \beta^{(n)} H_{k+1,0}^{(n+1)} H_{k+1,0}^{(n)} - H_{k+2,-1}^{(n)} H_{k,1}^{(n+1)} = 0, \tag{A.11a}$$

$$\mu^{(n)} H_{k+1,-1}^{(n+1)} H_{k,1}^{(n)} - H_{k+1,-1}^{(n)} H_{k,1}^{(n+1)} - h H_{k+1,0}^{(n)} H_{k,0}^{(n+1)} = 0, \tag{A.11b}$$

$$\begin{aligned}
 H_{k,2}^{(n)} H_{k,0}^{(n+1)} - \beta^{(n)} H_{k,1}^{(n+1)} H_{k,1}^{(n)} - H_{k+1,0}^{(n)} H_{k-1,2}^{(n+1)} \\
 + h \alpha^{(n)} H_{k,0}^{(n+1)} H_{k+1,0}^{(n)} = 0,
 \end{aligned} \tag{A.11c}$$

$$\begin{aligned}
 H_{k+1,0}^{(n+1)} H_{k,2}^{(n)} - H_{k+1,0}^{(n)} H_{k,2}^{(n+1)} - h H_{k+1,1}^{(n)} H_{k,1}^{(n+1)} \\
 + h \alpha^{(n)} H_{k+1,0}^{(n+1)} H_{k+1,0}^{(n)} = 0,
 \end{aligned} \tag{A.11d}$$

where

$$\alpha^{(n)} = \frac{u_0^{(n)}}{c_0^{(n)}}, \quad \beta^{(n)} = 1 + hu_0^{(n)}, \quad \mu^{(n)} = \frac{\beta^{(n)} + \alpha^{(n)} c_{-1}^{(n)}}{1 + \alpha^{(n)} c_{-1}^{(n+1)}}.$$

**Proof** The proof can be achieved by employing the determinant relations given in Lemma A.3.1 and linear relations of determinants presented in Corollary A.2.2.

In particular, (A.11a) and (A.11b) are direct consequences of applying the relations (A.9e) and (A.9f) and replacing  $\Gamma_{k+1,-1,-1}^{(n+1)}, \Gamma_{k,0,-1}^{(n+1)}, \Gamma_{k,1,0}^{(n+1)}$ , in terms of  $H_{k,j}^{(n)}$  via (A.7a), (A.7b) and (A.7f).

Similarly, (A.11c) and (A.11d) can be obtained by the bilinear relations (A.9c), (A.9d) and replacing  $F_{k+1,0,0}^{(n+1)}, F_{k,1,0}^{(n+1)}, F_{k,2,1}^{(n+1)}$  in (A.9c) and (A.9d) via (A.7c), (A.7e) and (A.7g).

**A.4. Proof of Lemma A.2.1** For  $k = 1$ , all these relations (A.6a)–(A.6m) can be easily confirmed. We are going to prove that these relations also hold for  $k \geq 2$ .

First of all, with the help of determinant properties, (A.6g) and (A.6i) can be easily checked by substituting (A.5a) and (A.5b) into the first column of  $E_{k,j,0}^{(n+1)}$  and  $E_{k,j,1}^{(n+1)}$ , respectively, where the convention  $\xi_{k,0}^{(n+1)} = \mathbf{0}$  is used in the proof of (A.6i).

For the remaining relations, we will prove them by performing row or column transformation to a certain determinant and using the recursion relations (A.4) and (4.6) for  $\{c_j^{(n)}\}$ .

**(1) Column transformation.**

The proofs of (A.6a)–(A.6c) and (A.6l) are conducted by performing a series of column transformations on determinants.

We first consider (A.6b). To begin with, we deal with the last column of  $H_{k,l}^{(n)}$  by substituting the relation (A.5b) into it, which yields

$$H_{k,l}^{(n)} = |\mathbf{C}_{k,l}^{(n)}, \mathbf{C}_{k,l+1}^{(n)}, \dots, \mathbf{C}_{k,l+k-2}^{(n)}, (\mathbf{B}_{k,l+k-2}^{(n+1)} + \alpha^{(n)} \xi_{k,l+k-2}^{(n+1)} - \frac{1}{h} \mathbf{C}_{k,l+k-2}^{(n)})|.$$

Now, we perform some column transformations in order to eliminate the appeared vector  $\alpha^{(n)} \xi_{k,l+k-2}^{(n+1)}$  and  $\mathbf{C}_{k,l+k-2}^{(n)}/h$ . Concretely, we add the  $(k - 1)$ -th column of  $H_{k,l}^{(n)}$  multiplied by  $1/h$  to the  $k$ -th column, then  $\mathbf{C}_{k,l+k-2}^{(n)}/h$  is eliminated. Next, noticing  $l = 0, 1$ , adding the  $i$ -th column multiplied by  $-\alpha^{(n)} c_{k-1-i}^{(n+1)}$  to the  $k$ -th column for  $i = (2 - l), (3 - l), \dots, (k - 1)$ , it is not hard to check that  $\alpha^{(n)} \xi_{k,l+k-2}^{(n+1)}$  also disappears.

Then, we dispose the  $(k - 1)$ -th,  $(k - 2)$ -th, ..., 2nd columns of  $H_{k,l}^{(n)}$ , successively. For each of these columns, after replacing the column vector  $\mathbf{C}_{k,j}^{(n)}$  by (A.5b), we conduct similar column operations in order to eliminate the appeared vector  $\alpha^{(n)} \xi_{k,j-1}^{(n+1)}$  and  $\mathbf{C}_{k,j-1}^{(n)}/h$  in the corresponding column. Finally, it leads to

$$H_{k,l}^{(n)} = |\mathbf{C}_{k,l}^{(n)}, \mathbf{B}_{k,l}^{(n+1)}, \dots, \mathbf{B}_{k,l+k-3}^{(n+1)}, \mathbf{B}_{k,l+k-2}^{(n+1)}|,$$

which gives (A.6b).

Now, we turn to prove (A.6a). Based on the above calculations, we see that

$$H_{k+1,-1}^{(n)} = |\mathbf{C}_{k+1,-1}^{(n)}, \mathbf{C}_{k+1,0}^{(n)}, \mathbf{B}_{k+1,0}^{(n+1)}, \dots, \mathbf{B}_{k+1,k-3}^{(n+1)}, \mathbf{B}_{k+1,k-2}^{(n+1)}|.$$

Then using (A.5a) and adding the first column multiplied by  $1/(h\gamma^{(n)})$  to the second one, (A.6a) is immediately derived by eliminating  $\mathbf{C}_{k+1,-1}^{(n)}/(h\gamma^{(n)})$ .

To prove (A.6c), we first rewrite the determinant as

$$H_{k,2}^{(n)} = \begin{vmatrix} 1 & 0 & 0 & \dots & 0 \\ C_{k,1}^{(n)} & C_{k,2}^{(n+1)} & C_{k,3}^{(n+1)} & \dots & C_{k,k+1}^{(n+1)} \end{vmatrix}.$$

By following the similar column operations to the proof of (A.6b), we obtain

$$\begin{aligned} H_{k,2}^{(n)} &= \begin{vmatrix} 1 & \frac{1}{h} - \alpha^{(n)}c_0^{(n+1)} & -\alpha^{(n)}c_1^{(n+1)} & \dots & -\alpha^{(n)}c_{k-1}^{(n+1)} \\ C_{k,1}^{(n)} & B_{k,1}^{(n+1)} & B_{k,2}^{(n+1)} & \dots & B_{k,k}^{(n+1)} \end{vmatrix} \\ &= \begin{vmatrix} 1 & -\alpha^{(n)}c_0^{(n+1)} & -\alpha^{(n)}c_1^{(n+1)} & \dots & -\alpha^{(n)}c_{k-1}^{(n+1)} \\ C_{k,1}^{(n)} & B_{k,1}^{(n+1)} & B_{k,2}^{(n+1)} & \dots & B_{k,k}^{(n+1)} \end{vmatrix} \\ &\quad + \begin{vmatrix} 1 & \frac{1}{h} & -\alpha^{(n)}c_1^{(n+1)} & \dots & -\alpha^{(n)}c_{k-1}^{(n+1)} \\ C_{k,1}^{(n)} & 0 & B_{k,2}^{(n+1)} & \dots & B_{k,k}^{(n+1)} \end{vmatrix}, \end{aligned}$$

which obviously leads to (A.6c).

Now, we proceed to prove (A.6l). Noticing first the determinant expression (3.1) of  $P_k^{(n)}(x)$ , it is easy to see that

$$\mathcal{H}_{2k,l}^{(n)}(x) = \frac{H_{k,l}^{(n)} P_{2k+l}^{(n)}(x)}{x^l}, \quad l = 0, 1.$$

Thus, to prove (A.6l), we only need to prove  $\mathcal{H}_{2k,l}^{(n)}(x) = G_{2k,l}^{(n+1)}(x)$ . This can be proved by performing some column transformations to  $\mathcal{H}_{2k,l}^{(n)}(x)$  in a similar way to prove (A.6b). We omit the details.

**(2) Row transformation.**

The strategy to prove (A.6d)-(A.6f), (A.6h), (A.6j)-(A.6m) is based on a series of row transformations performed on determinants. The crucial observation is that, the column vectors with  $B_{k,j}^{(n+1)}$  can be converted to  $\beta^{(n)}C_{k,j}^{(n+1)}/h$  after performing appropriate row transformations so that the summation term of  $B_{k,j}^{(n+1)}$  is eliminated.

Moreover, it should be noted that, since the determinants  $\tilde{E}_{k,r+1,r}^{(n+1)}$ ,  $E_{k,j,r}^{(n+1)}$ ,  $K_{k,j,1}^{(n+1)}$ , and  $\tilde{G}_{2k,l}^{(n+1)}(x)$  have the majority columns in common in terms of  $B_{k,j}^{(n+1)}$ , we will conduct the same row operations to transform them. In order to illustrate these consistent row transformations performed on determinants uniformly and clearly, we now take the following determinant as an example.

Noting that the types of the columns of the determinants in question mainly involve vectors  $C_{k,s}^{(n)}$ ,  $B_{k,j}^{(n+1)}$  with  $s = -1, 0, 1$  and  $j \geq -1$ , we set

$$M_{k+2j,-1}^{(n)} = |C_{k+2,-1}^{(n)}, C_{k+2,0}^{(n)}, B_{k+2,j}^{(n+1)}, \dots, B_{k+2,j+k-2}^{(n+1)}, B_{k+2,j+k-1}^{(n+1)}|, \quad j = -1, 0, 1, \dots$$

The row transformations will be conducted on  $M_{k+2j,-1}^{(n)}$  from the second row until to the last row, successively. More precisely, for fixed  $p = 2, 3, \dots, k + 2$ , we add the  $i$ -th row multiplied by  $-h\alpha^{(n)}c_{p-i}^{(n)}/\beta^{(n)}$  with  $i = 1, 2, 3, \dots, p - 1$  to the  $p$ -th row of  $M_{k+2j,-1}^{(n)}$ , which leads to

$$M_{k+2,j,-1}^{(n)} = \begin{vmatrix} d_{-1}^{(n)} & a_0^{(n)} & \frac{\beta^{(n)} c_j^{(n+1)}}{h} & \dots & \frac{\beta^{(n)} c_{j+k-1}^{(n+1)}}{h} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d_{p-2}^{(n)} & a_{p-1}^{(n)} & \frac{\beta^{(n)} c_{j+p-1}^{(n+1)}}{h} & \dots & \frac{\beta^{(n)} c_{j+p+k-2}^{(n+1)}}{h} \\ c_{p-1}^{(n)} & c_p^{(n)} & \frac{\beta^{(n)} c_{j+p}^{(n+1)}}{h} + \alpha^{(n)} \sum_{i=1}^p c_i^{(n)} c_{j+p-i}^{(n+1)} & \dots & \frac{\beta^{(n)} c_{j+p+k-1}^{(n+1)}}{h} + \alpha^{(n)} \sum_{i=1}^p c_i^{(n)} c_{j+p+k-1-i}^{(n+1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_k^{(n)} & c_{k+1}^{(n)} & \frac{\beta^{(n)} c_{j+k+1}^{(n+1)}}{h} + \alpha^{(n)} \sum_{i=1}^{k+1} c_i^{(n)} c_{j+k+1-i}^{(n+1)} & \dots & \frac{\beta^{(n)} c_{j+2k}^{(n+1)}}{h} + \alpha^{(n)} \sum_{i=1}^{k+1} c_i^{(n)} c_{j+2k-i}^{(n+1)} \end{vmatrix}.$$

Obviously, for  $p = k + 2$ , we have

$$M_{k+2,j,-1}^{(n)} = \left( \frac{\beta^{(n)}}{h} \right)^k |D_{k+2,-1}^{(n)}, A_{k+2,0}^{(n)}, C_{k+2,j}^{(n+1)}, \dots, C_{k+2,j+k-1}^{(n+1)}|. \tag{A.5}$$

Then, (A.6d), (A.6e) and (A.6j) follow immediately since the involved columns of the determinants  $E_{k,j,-1}^{(n+1)}$ ,  $E_{k,j,0}^{(n+1)}$  and  $\tilde{E}_{k,r+1,r}^{(n+1)}$  are only part of the determinant  $M_{k+2,j,-1}^{(n)}$ .

In order to prove the relations (A.6f), (A.6h) and (A.6k), we first conduct equivalent deformations on the determinants  $E_{k,j,0}^{(n+1)}$ ,  $E_{k,j,1}^{(n+1)}$  and  $K_{k,0,1}^{(n+1)}$ , before performing row transformations.

Note that

$$1 - \frac{hu_0^{(n)}}{\beta^{(n)}} = \frac{1}{\beta^{(n)}},$$

and

$$\begin{aligned} C_{k,0}^{(n)} &= (C_{k,0}^{(n)} - \frac{h\alpha^{(n)} d_{-1}^{(n)}}{\beta^{(n)}} C_{k,1}^{(n)}) + \frac{h\alpha^{(n)} d_{-1}^{(n)}}{\beta^{(n)}} C_{k,1}^{(n)}, \\ C_{k,1}^{(n)} &= \beta^{(n)} (1 - \frac{hu_0^{(n)}}{\beta^{(n)}}) C_{k,1}^{(n)}, \end{aligned}$$

from which, we immediately have

$$\begin{aligned} E_{k,j,0}^{(n+1)} &= \left| \left( C_{k,0}^{(n)} - \frac{h\alpha^{(n)} d_{-1}^{(n)}}{\beta^{(n)}} C_{k,1}^{(n)} \right), B_{k,j}^{(n+1)}, \dots, B_{k,j+k-3}^{(n+1)}, B_{k,j+k-2}^{(n+1)} \right| + \frac{h\alpha^{(n)} d_{-1}^{(n)}}{\beta^{(n)}} E_{k,j,1}^{(n+1)}, \\ E_{k,j,1}^{(n+1)} &= \beta^{(n)} \left| \left( 1 - \frac{hu_0^{(n)}}{\beta^{(n)}} \right) C_{k,1}^{(n)}, B_{k,j}^{(n+1)}, \dots, B_{k,j+k-3}^{(n+1)}, B_{k,j+k-2}^{(n+1)} \right|, \\ K_{k,j,1}^{(n+1)} &= \beta^{(n)} \left| \begin{array}{cccc} \frac{1}{\beta^{(n)}} & -\alpha^{(n)} C_j^{(n+1)} & -\alpha^{(n)} C_{j+1}^{(n+1)} & \dots & -\alpha^{(n)} C_{j+k-2}^{(n+1)} \\ \left( 1 - \frac{hu_0^{(n)}}{\beta^{(n)}} \right) C_{k-1,1}^{(n)} & B_{k-1,j+1}^{(n+1)} & B_{k-1,j+2}^{(n+1)} & \dots & B_{k-1,j+k-1}^{(n+1)} \end{array} \right|. \end{aligned}$$

Now, we operate the same row transformations as those on  $M_{k+2,j,-1}^{(n)}$  upon the above determinants. For the first two, we have

$$\begin{aligned} E_{k,j,0}^{(n+1)} &= |D_{k,0}^{(n)}, \frac{\beta^{(n)}}{h} C_{k,j}^{(n+1)}, \dots, \frac{\beta^{(n)}}{h} C_{k,j+k-2}^{(n+1)}| + \frac{h\alpha^{(n)} d_{-1}^{(n)}}{\beta^{(n)}} E_{k,j,1}^{(n+1)} \\ &= \left( \frac{\beta^{(n)}}{h} \right)^{k-1} \Gamma_{k,j,0}^{(n+1)} + \frac{h\alpha^{(n)} d_{-1}^{(n)}}{\beta^{(n)}} E_{k,j,1}^{(n+1)}, \\ E_{k,j,1}^{(n+1)} &= \beta^{(n)} |A_{k,1}^{(n)}, \frac{\beta^{(n)}}{h} C_{k,j}^{(n+1)}, \dots, \frac{\beta^{(n)}}{h} C_{k,j+k-2}^{(n+1)}| = \frac{(\beta^{(n)})^k}{h^{k-1}} F_{k,j,1}^{(n+1)}, \end{aligned}$$

which are nothing but (A.6f) and (A.6h). While for the third one, it is not hard to obtain the following result by conducting the row transformations from the third row to the last one,

$$\begin{aligned}
 K_{k,j,1}^{(n+1)} &= \beta^{(n)} \begin{vmatrix} \frac{1}{\beta^{(n)}} & -\alpha^{(n)} c_j^{(n+1)} & -\alpha^{(n)} c_{j+1}^{(n+1)} & \cdots & -\alpha^{(n)} c_{j+k-2}^{(n+1)} \\ A_{k-1,1}^{(n+1)} & \frac{\beta^{(n)}}{h} C_{k-1,j+1}^{(n+1)} & \frac{\beta^{(n)}}{h} C_{k-1,j+2}^{(n+1)} & \cdots & \frac{\beta^{(n)}}{h} C_{k-1,j+k-1}^{(n+1)} \end{vmatrix} \\
 &= -\frac{\alpha^{(n)}(\beta^{(n)})^{k-1}}{h^{k-2}} \begin{vmatrix} -\frac{1}{h\alpha^{(n)}} & c_j^{(n+1)} & c_{j+1}^{(n+1)} & \cdots & c_{j+k-2}^{(n+1)} \\ A_{k-1,1}^{(n+1)} & C_{k-1,j+1}^{(n+1)} & C_{k-1,j+2}^{(n+1)} & \cdots & C_{k-1,j+k-1}^{(n+1)} \end{vmatrix} \\
 &= -\frac{\alpha^{(n)}(\beta^{(n)})^{k-1}}{h^{k-2}} \begin{vmatrix} (A_{1,0}^{(n)} - \frac{\beta^{(n)}}{h\alpha^{(n)}}) & c_j^{(n+1)} & c_{j+1}^{(n+1)} & \cdots & c_{j+k-2}^{(n+1)} \\ A_{k-1,1}^{(n+1)} & C_{k-1,j+1}^{(n+1)} & C_{k-1,j+2}^{(n+1)} & \cdots & C_{k-1,j+k-1}^{(n+1)} \end{vmatrix} \\
 &= -\frac{\alpha^{(n)}(\beta^{(n)})^{k-1}}{h^{k-2}} \left( F_{k,j,0}^{(n+1)} - \frac{\beta^{(n)}}{h\alpha^{(n)}} H_{k-1,j+1}^{(n+1)} \right),
 \end{aligned}$$

which identifies with (A.6k) by observing the fact

$$-\frac{1}{h\alpha^{(n)}} = A_{1,0}^{(n)} - \frac{\beta^{(n)}}{h\alpha^{(n)}}.$$

Thus, (A.6f), (A.6h) and (A.6k) all get verified.

Now, we remain to prove (A.6m). For determinant  $\tilde{G}_{2k,l}^{(n+1)}(x)$ ,  $l = 0, 1$ , from the second row to the last second row, after conducting several row transformations the same as those on  $M_{k+2,j,-1}^{(n)}$ , and noticing the expressions (A.3), (3.1) and (3.10b) of  $y_{l,2k}$ ,  $P_k^{(n)}$  and  $Q_k^{(n)}$ , respectively, we obtain that

$$\begin{aligned}
 \tilde{G}_{2k,l}^{(n+1)}(x) &= \begin{vmatrix} \frac{\beta^{(n)}}{h} C_{k-1,l}^{(n+1)} & \frac{\beta^{(n)}}{h} C_{k-1,l+1}^{(n+1)} & \cdots & \frac{\beta^{(n)}}{h} C_{k-1,l+k-1}^{(n+1)} \\ y_{l,2} & y_{l,4} & \cdots & y_{l,2k} \end{vmatrix} \\
 &= \frac{(\beta^{(n)})^{k-1}(hx^2+1)}{h^k} \begin{vmatrix} C_{k-1,l}^{(n+1)} & C_{k-1,l+1}^{(n+1)} & \cdots & C_{k-1,l+k-1}^{(n+1)} \\ 1 & x^2 & \cdots & x^{2k-2} \end{vmatrix} \\
 &\quad - \alpha^{(n)} \left( \frac{\beta^{(n)}}{h} \right)^{k-1} \begin{vmatrix} C_{k-1,l}^{(n+1)} & C_{k-1,l+1}^{(n+1)} & \cdots & C_{k-1,l+k-1}^{(n+1)} \\ \sum_{i=1-l}^0 C_{-i}^{(n+1)} x^{2i} & \sum_{i=1-l}^1 C_{-i}^{(n+1)} x^{2i} & \cdots & \sum_{i=1-l}^{k-1} C_{-i}^{(n+1)} x^{2i} \end{vmatrix} \\
 &= \frac{(\beta^{(n)})^{k-1}(hx^2+1)H_{k-1,l}^{(n+1)}P_{2k-2+l}^{(n+1)}(x)}{h^k x^l} - \frac{\alpha^{(n)}(\beta^{(n)})^{k-1}H_{k-1,l}^{(n+1)}Q_{2k-3+l}^{(n+1)}(x)}{h^{k-1} x^{l-1}},
 \end{aligned}$$

which is exactly (A.6m).

Therefore, we complete the proof of Lemma A.2.1.

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## Declarations

**Conflict of interest** The authors declare no competing interests.

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## Authors and Affiliations

Xiao-Min Chen<sup>1</sup> · Xiang-Ke Chang<sup>2,3</sup>  · Yi He<sup>4,5</sup> · Xing-Biao Hu<sup>2,3</sup>

Xiao-Min Chen  
chenxm1016@bjut.edu.cn

Yi He  
heyj@wipm.ac.cn

Xing-Biao Hu  
hxb@lsec.cc.ac.cn

- <sup>1</sup> Institute of Applied Mathematics, Department of Mathematics, Faculty of Science, Beijing University of Technology, Beijing 100124, People's Republic of China
- <sup>2</sup> LSEC, ICMSEC, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, P.O.Box 2719, Beijing 100190, People's Republic of China
- <sup>3</sup> School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, People's Republic of China
- <sup>4</sup> Innovation Academy for Precision Measurement Science and Technology, Chinese Academy of Sciences, Wuhan 430071, People's Republic of China
- <sup>5</sup> Wuhan Institute of Physics and Mathematics, Chinese Academy of Sciences, Wuhan 430071, People's Republic of China