Numerical Algorithms (2023) 92:1679–1705
https://doi.org/10.1007/s11075-022-01360-5

ORIGINAL PAPER

Unconditional error estimates of linearized BDF2-Galerkin FEMs for nonlinear coupled Schrödinger-Helmholtz equations

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Received: 22 April 2021 / Accepted: 15 June 2022 / Published online: 5 August 2022

Abstract

In this paper, we present two linearized BDF2 Galerkin FEMs for the nonlinear and coupled Schrödinger-Helmholtz equations. Different from the standard linearized second-order Crank-Nicolson methodology, we employ backward differential concept to obtain second-order temporal accuracy at the time step t_n (instead of the time instant $t_{n+1/2}$) and apply semi-implicit or explicit treatment of nonlinear terms to formulate the decoupled schemes. We prove optimal error estimates for r-order FEM without any grid-ratio condition through a so-called temporal-spatial error splitting technique, and some sharp estimations to cope with the nonlinear terms. Finally, we provide two numerical experiments to illustrate the theoretical analysis and the efficiency of the proposed methods. Here, h is the spatial subdivision parameter, and τ is the time step.

Keywords Schrödinger-Helmholtz equations · Galerkin FEMs · Linearized BDF2 scheme · Optimal error estimates

Mathematics Subject Classification (2010) 65N15 · 65N30

1 Introduction

Consider the following initial-boundary value problem of Schrödinger-Helmholtz equations

$$\begin{cases} \mathrm{i}u_t + \Delta u + \phi f(|u|)u = 0, & (X,t) \in \Omega \times (0,T], \\ \alpha\phi - \beta^2\Delta\phi = f(|u|)|u|^2, & (X,t) \in \Omega \times (0,T], \\ u = \phi = 0, & (X,t) \in \partial\Omega \times [0,T], \\ u(\boldsymbol{x},0) = u_0(X), & X \in \Omega, \end{cases} \tag{1.1}$$

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in which $X = (x, y)$, $T < +\infty$ and Ω is a convex bounded domain in $\mathbb{R}^d (d = 2, 3)$ with the boundary $\partial\Omega$. $\mathrm{i} = \sqrt{-1}$, α, β are real nonnegative constants with $\alpha+\beta \neq 0$. $f : \mathbb{R}^+ \to \mathbb{R}$ and $u_0 : \Omega \to \mathbb{C}$ are given functions. The complex-valued function u stands for the single particle wave function, the real-valued function $\phi(X, t)$ denotes the potential. The system (1.1) models many different physical phenomena in optics, quantum mechanics, and plasma physics, and so forth. When $\alpha = 0$, the system (1.1) reduces to the Schrödinger-Poisson model [1–5]. And when $\beta = 0$, the system (1.1) degenerate to a generalized nonlinear Schrödinger equation [6, 7]. Besides, we refer [8, 9] for other Schrödinger type equations such as the Schrödinger-Poisson-Slater model. We can see that the model (1.1) conserves the total mass

$$\mathcal{M}(t) = \int_\Omega |u(X, t)|^2 \mathrm{d}X \equiv \mathcal{M}(0), t > 0, \tag{1.2}$$

and the total energy

$$\mathcal{E}(t) = \int_\Omega |\nabla u(X, t)|^2 - \frac{\phi(X, t)|u(X, t)|^{\sigma+1}}{\sigma + 1} \mathrm{d}X \equiv \mathcal{E}(0), t > 0, \tag{1.3}$$

when $\alpha = 1$, $f(s) = s^{\sigma-1}$, $\sigma \geq 1$.

A series of mathematical studies have been devoted for diverse Schrödinger type equations. For example, the existence and uniqueness of solution to the Schrödinger-Poisson type equations in $\mathbb{R}^d (d = 2, 3)$ were investigated in [10, 11]. And in [12], a type of Schrödinger-Helmholtz system as a regularization of the generalized nonlinear Schrödinger equation was introduced, local and global existence of a unique solution of the system was studied. Along the numerical front, various numerical methods for Schrödinger type equations also have been proposed including finite difference methods [13–18], spectral or pseudo-spectral methods [19–22], and FEMs [23–29]. Especially, the linearized backward Euler Galerkin FEMs and Crank-Nicolson Galerkin FEMs were studied for Schrödinger-Helmholtz system in [30] and [31], respectively. Both of them derived optimal L^2 error estimates for r-order FEMs without any grid-ratio restriction condition. Due to some pollution arising from the approximation used for the nonlinear terms $\phi f(|u|)u$ and $f(|u|)|u|^2$, only the error estimate at the time instant $t_{n+1/2}$ instead of the time division node t_n for the potential ϕ was derived in [31], because Schrödinger part u and the Helmholtz part ϕ are solved at different time step levels so as to decouple the strongly nonlinear and coupled of Schrödinger-Helmholtz equations.

Generally speaking, we can derive error estimates at the time instant t_n of (linearized) Crank-Nicolson scheme for many different nonlinear PDEs (see [32–34]). But for some strong nonlinearity and coupled problems, it is not an easy thing to establish the linearized decoupled high accurate (in time) numerical schemes and obtain the error estimates at the time division node t_n. To decouple the schemes, a nature strategy is to solve the coupled problems at different time step levels, see, e.g., [35] for the time-dependent nonlinear thermistor equations, and [36] for Cahn-Hilliard equation, where they achieved the error estimates at the time instant $t_{n+1/2}$ for the electric potential ϕ in [35] and the chemical potential μ in [36], respectively. This phenomenon also produced in [37] for flux $\vec{p}$, where a Crank-Nicolson mixed FEM was used for the nonlinear Sobolev equation. Whether we have the accuracy

$O(\tau^2)$ at the time division node t_n for numerical solutions involved in a strongly nonlinear and coupled system such as Schrödinger-Helmholtz (1.1).

In this paper, we shall give an affirmative answer to this question for Schrödinger-Helmholtz (1.1). Based on the second-order BDF temporal approximation framework, we present two BDF2 schemes to solve (1.1) instead of the Crank-Nicolson formula, because the BDF type scheme has the following striking advantages: (i) It is multi-step methods and unconditionally stable [38, 39]; (ii) It can achieve high-order accuracy without increasing the computation significantly [40]; (iii) This kind of scheme treats and approximates every term at time step t_n (instead of the time instant $t_{n+1/2}$). To overcome the strong nonlinearity and coupling in $\phi f(|u|)u$, we adopt semi-implicit or explicit treatment of $\phi f(|u|)u$ to develop the decoupled schemes. Different from [30, 31], we solve Schrödinger equation for numerical solution u_h^n firstly, and then to solve Helmholtz equation for numerical solution ϕ_h^n at the same time level t_n for our first scheme (see (2.3)–(2.6)). We achieve optimal error estimates without any grid-ratio restriction condition by use of the approach of [41, 42] to split the error into two parts, i.e., the temporal error and the spatial error. Besides, some fine tricks are also applied to deal with the nonlinear terms. The novelty with respect to previous works is that our scheme decoupled the strong nonlinearity of (1.1), and we obtain second-order temporal accuracy at the time step t_n (instead of the time instant $t_{n+1/2}$) without time step constraint.

The outline of the article is arranged in the following way. In Section 2, two linearized BDF schemes are developed, and the mass and energy conservative laws are proved. In Section 3, a corresponding time-discrete system is proposed and the temporal error with order $O(\tau^2)$ is deduced. In Section 4, optimal spatial error estimates with order $O(h^2)$ are derived for low-order elements ($r = 1$), and uniform boundedness of numerical solutions in L^∞-norm are established, which lead to unconditional optimal L^2 error estimates of the r-order ($r \geq 1$) Galerkin FEMs in Section 5. Two numerical examples are given to confirm our theoretical analysis in Section 6 and a conclusion is presented in Section 7, respectively.

2 Linearized BDF Galerkin FEMs

In this section, we will construct two linearized BDF2 Galerkin FEMs for (1.1). For this purpose, we use the classical Sobolev spaces $W^{s,p}(\Omega)$ and their associated norms and semi-norms. We denote $H^s(\Omega) = W^{s,2}(\Omega)$ with the corresponding norm $\|v\|_s = \|v\|_{s,2,\Omega}$, seminorm $|v|_s = |u|_{s,2,\Omega}$ and $\|v\|_0 = \|u\|_{0,2,\Omega}$ defined by

$$\|v\|_s^2 = \sum_{|\alpha|\leq s} \|D^\alpha v\|_0^2, \quad |v|_s^2 = \sum_{|\alpha|=s} \|D^\alpha v\|_0^2, \quad \|v\|_0^2 = \int_\Omega |v|^2 \mathrm{d}X.$$

In addition, for any two complex functions $u, v \in L^2(\Omega)$, the inner product is defined by

$$(u, v) = \int_\Omega u v^* \mathrm{d}X,$$

in which v^* denotes the conjugate of v. Besides, for a Banach space $\mathcal{X}$ with the norm $\|\cdot\|_{\mathcal{X}}$ the function space $L^s(0,T;\mathcal{X})$ consists of all strongly measurable functions $f:[0,T]\to\mathcal{X}$ with

$$\|f\|_{L^s(0,T;\mathcal{X})} = \begin{cases} (\int_0^T \|f(t)\|_{\mathcal{X}}^s \mathrm{d}t)^{\frac{1}{s}}, & 1\le s<\infty, \\ ess\,\sup_{t\in[0,T]} \|f(t)\|_{\mathcal{X}}, & s=\infty. \end{cases}$$

Following the classical finite element theory [43], we define $\mathcal{T}_h = \{\mathcal{K}\}$ to be a quasi-uniform partition of Ω into triangles or rectangles (in $\mathbb{R}^2$) or hexahedrons (in $\mathbb{R}^3$) with mesh size $h = \max_{\forall \mathcal{K}\in\mathcal{T}_h}\{\mathrm{diam}\mathcal{K}\}$ and $0<h<1$. We introduce the finite element space functions defined as (see, e.g., [44])

$$V_{h0} := \{v_h \in C(\overline{\Omega});\, v_h|_{\mathcal{K}} \in \mathcal{P}_r(\mathcal{K}) \text{ or } \mathcal{Q}_r(\mathcal{K}) \text{ and } v_h = 0 \text{ on } \partial\Omega,\ \forall \mathcal{K}\in\mathcal{T}_h\},$$

where $\mathcal{P}_r(\mathcal{K})$ denotes the polynomial space of degree $\le r (r\ge 1)$ and $\mathcal{Q}_r(\mathcal{K})$ denotes the polynomial space of degree $\le r$ in each variable.

To proceed, we define the Ritz projection operator $R_h : H_0^1(\Omega)\to V_{h0}$ by [43, 44]:

$$(\nabla(v-R_hv),\nabla v_h) = 0,\quad v_h\in V_{h0},$$

which satisfies

$$\|v-R_hv\|_0 + h\|\nabla(v-R_hv)\|_0 \le Ch^{r+1}|v|_{r+1}, \forall v\in H^{r+1}(\Omega). \tag{2.1}$$

Here and later, C with or without superscripts and subscripts, denotes a generic positive constant, not necessarily the same at different occurrences, which is always dependent on the solution and the given data but independent of h and τ.

For a given positive integer N, let $\{t_n : t_n = n\tau; 0\le n\le N\}$ be a uniform partition of $[0,T]$ with the time step $\tau := T/N$, nodes $t_n := n\tau$ for $n = 0,1,2,\ldots,N$, and intermediate nodes $t_{n+1/2} := t_n + \frac{\tau}{2}$. Let

$$\sigma^n := \sigma(X,t_n),\ \ \bar{\sigma}^n := \frac{1}{2}(\sigma^n+\sigma^{n-1}),\ \ \bar{\partial}_\tau\sigma^n := \frac{1}{\tau}(\sigma^n-\sigma^{n-1}),\ \ 1\le n\le N,$$

and

$$D_\tau\sigma^n := \frac{1}{\tau}(\frac{3}{2}\sigma^n - 2\sigma^{n-1} + \frac{1}{2}\sigma^{n-2}),\ \widehat{\sigma}^n := 2\sigma^{n-1}-\sigma^{n-2},\ 2\le n\le N.$$

For the time derivative D_τ, the inner product $(D_\tau\sigma^n,\sigma^n)$ can be written by

$$(D_\tau\sigma^n,\sigma^n) = \frac{1}{4\tau}\Big(P(\sigma^n) - P(\sigma^{n-1}) + Q(\sigma^n)\Big). \tag{2.2}$$

Here and later $P(\sigma^n) := \|\sigma^n\|_0^2 + \|2\sigma^n - \sigma^{n-1}\|_0^2$, $Q(\sigma^n) := \|\sigma^n - 2\sigma^{n-1} + \sigma^{n-2}\|_0^2$.

With these notations, we define the linearized BDF2 Galerkin FEM to (1.1) as: to find $(u_h^n,\phi_h^n)\in\widetilde{V}_{h0}\times V_{h0}$ for all $n\ge 2$ such that

$$\begin{cases} \mathrm{i}(D_\tau u_h^n, v_h) - (\nabla u_h^n,\nabla v_h) + \big(\widehat{\phi}_h^n f(|\widehat{u}_h^n|)u_h^n, v_h\big) = 0,\ v_h\in\widetilde{V}_{h0}, & \text{(2.3a)} \\ \alpha(\phi_h^n,\varphi_h) + \beta^2(\nabla\phi_h^n,\nabla\varphi_h) = \big(f(|u_h^n|)|u_h^n|^2,\varphi_h\big),\ \varphi_h\in V_{h0}, & \text{(2.3b)} \end{cases}$$

where $\widetilde{V}_{h0} := V_{h0} \oplus \mathrm{i}V_{h0}$. To ensure the second-order accuracy for the temporal direction, we adopt a predictor corrector method to compute (u_h^1, ϕ_h^1):

$$\begin{cases} \mathrm{i}(\bar{\partial}_\tau u_h^1, v_h) - (\nabla \overline{u}_h^1, \nabla v_h) + \left(\frac{\phi_h^{1,0}+\phi_h^0}{2} f\left(|\frac{u_h^{1,0}+u_h^0}{2}|\right)\overline{u}_h^1, v_h\right) = 0, & (2.4a) \\ \alpha(\overline{\phi}_h^1, \varphi_h) + \beta^2(\nabla\overline{\phi}_h^1, \nabla\varphi_h) = \left(f(|\overline{u}_h^1|)|\overline{u}_h^1|^2, \varphi_h\right), & (2.4b) \end{cases}$$

in which $(u_h^{1,0}, \phi_h^{1,0})$ is computed by

$$\begin{cases} \mathrm{i}\left(\frac{u_h^{1,0}-u_h^0}{\tau}, v_h\right) - \left(\frac{\nabla(u_h^{1,0}+u_h^0)}{2}, \nabla v_h\right) + \left(\phi_h^0 f(|u_h^0|)\frac{u_h^0+u_h^{1,0}}{2}, v_h\right) = 0, & (2.5a) \\ \alpha\left(\frac{\phi_h^{1,0}+\phi_h^0}{2}, \varphi_h\right) + \beta^2\left(\frac{\nabla(\phi_h^{1,0}+\phi_h^0)}{2}, \nabla\varphi_h\right) = \left(f\left(|\frac{u_h^{1,0}+u_h^0}{2}|\right)|\frac{u_h^{1,0}+u_h^0}{2}|^2, \varphi_h\right), & (2.5b) \end{cases}$$

with the initial value $u_h^0 = R_h u_0$ and ϕ_h^0 satisfies

$$\alpha(\phi_h^0, \varphi_h) + \beta^2(\nabla\phi_h^0, \nabla\varphi_h) = \left(f(|u_0|)|u_0|^2, \varphi_h\right), \tag{2.6}$$

for all $(v_h, \varphi_h) \in \widetilde{V}_{h0} \times V_{h0}$.

Theorem 1 *The discrete scheme* (2.4)–(2.5) *is mass conservative in the recursive sense*

$$\|u_h^1\|_0^2 = \|u_h^{1,0}\|_0^2 = \|u_h^0\|_0^2, \quad \mathcal{M}^{n+1} = \mathcal{M}^n, \quad n \geq 1, \tag{2.7}$$

where

$$\mathcal{M}^n = \|u_h^n\|_0^2 + \|2u_h^n - u_h^{n-1}\|_0^2 + \mathcal{G}^n, \tag{2.8}$$

with

$$\mathcal{G}^1 = 0, \quad \mathcal{G}^n - \mathcal{G}^{n-1} = \|u_h^n - 2u_h^{n-1} + u_h^{n-2}\|_0^2, \quad n \geq 2. \tag{2.9}$$

Proof Taking $v_h = u_h^1 + u_h^0$ and $v_h = u_h^{1,0} + u_h^0$ in (2.4a) and (2.5a), respectively, and choosing the imaginary parts of the resulting equations, we derive

$$\|u_h^1\|_0^2 = \|u_h^{1,0}\|_0^2 = \|u_h^0\|_0^2. \tag{2.10}$$

Then, taking $v_h = u_h^n$ in (2.3a) and extracting the imaginary part, we get

$$\frac{1}{4\tau}\left(\|u_h^n\|_0^2 - \|u_h^{n-1}\|_0^2 + \|2u_h^n - u_h^{n-1}\|_0^2 - \|2u_h^{n-1} - u_h^{n-2}\|_0^2 + \|u_h^n - 2u_h^{n-1} + u_h^{n-2}\|_0^2\right) = 0 \tag{2.11}$$

which leads to

$$\|u_h^n\|_0^2 + \|2u_h^n - u_h^{n-1}\|_0^2 + \|u_h^n - 2u_h^{n-1} + u_h^{n-2}\|_0^2 = \|u_h^{n-1}\|_0^2 + \|2u_h^{n-1} - u_h^{n-2}\|_0^2. \tag{2.12}$$

Therefore, the proof is completed. □

Theorem 2 *The discrete scheme (2.4)–(2.5) is energy conservative in the recursive sense*

$$\|\nabla u_h^{1,0}\|_0^2 - \int_\Omega \phi_h^0 f(|u_h^0|)u_h^{1,0}\mathrm{d}X = \|\nabla u_h^0\|_0^2 - \int_\Omega \phi_h^0 f(|u_h^0|)u_h^0\mathrm{d}X, \tag{2.13}$$

$$\|\nabla u_h^1\|_0^2 - \int_\Omega \frac{\phi_h^{1,0}+\phi_h^0}{2} f(|u_h^0|)u_h^1\mathrm{d}X = \|\nabla u_h^0\|_0^2 - \int_\Omega \frac{\phi_h^{1,0}+\phi_h^0}{2} f(|u_h^0|)u_h^0\mathrm{d}X, \tag{2.14}$$

$$\mathcal{E}^{n+1} = \mathcal{E}^n, \quad n \geq 1, \tag{2.15}$$

where

$$\mathcal{E}^n = \|\nabla u_h^n\|_0^2 + \|2\nabla u_h^n - \nabla u_h^{n-1}\|_0^2 + \mathcal{F}^n, \tag{2.16}$$

with

$$\begin{aligned}\mathcal{F}^1 = 0, \quad \mathcal{F}^n - \mathcal{F}^{n-1} &= \|\nabla u_h^n - 2\nabla u_h^{n-1} + \nabla u_h^{n-2}\|_0^2 \\ &-4\tau Re\left(\widehat{\phi}_h^n f(|\widehat{u}_h^n|)u_h^n, D_\tau u_h^n\right), \quad n \geq 2.\end{aligned} \tag{2.17}$$

Proof Taking $v_h = \bar{\partial}_\tau u_h^1$ and $v_h = \frac{u_h^{1,0} - u_h^0}{\tau}$ in (2.4a) and (2.5a), respectively, and choosing the real parts of the resulting equations, we can obtain (2.14) and (2.13), respectively. Finally, taking $v_h = D_\tau u_h^n$ in (2.3a) and extracting the real part, we have

$$\begin{aligned}&\frac{1}{4\tau}\Big(\|\nabla u_h^n\|_0^2 - \|\nabla u_h^{n-1}\|_0^2 + \|2\nabla u_h^n - \nabla u_h^{n-1}\|_0^2 - \|2\nabla u_h^{n-1} \\ &-\nabla u_h^{n-2}\|_0^2 + \|\nabla u_h^n - 2\nabla u_h^{n-1} + \nabla u_h^{n-2}\|_0^2\Big) \\ &-\mathrm{Re}\big(\widehat{\phi}_h^n f(|\widehat{u}_h^n|)u_h^n, D_\tau u_h^n\big) = 0\end{aligned} \tag{2.18}$$

Thus, we can see that

$$\begin{aligned}\|\nabla u_h^n\|_0^2 + \|2\nabla u_h^n - \nabla u_h^{n-1}\|_0^2 + \|\nabla u_h^n - 2\nabla u_h^{n-1} + \nabla u_h^{n-2}\|_0^2 - 4\tau\mathrm{Re}\left(\widehat{\phi}_h^n f(|\widehat{u}_h^n|)u_h^n, D_\tau u_h^n\right) \\ = \|\nabla u_h^{n-1}\|_0^2 + \|2\nabla u_h^{n-1} - \nabla u_h^{n-2}\|_0^2,\end{aligned} \tag{2.19}$$

which leads to the desired results (2.13)–(2.15). □

The scheme (2.3)–(2.6) can be seen as a semi-decoupled scheme because one only needs to solve a linear system for u_h^n firstly, and then for ϕ_h^n at each time step. This is different from that in [31] where one needs to solve $\phi_h^{n-1/2}$ firstly, and then to solve for u_h^n. However, by use of an explicit treatment of the nonlinear term of (2.3b), it allow us to define the following fully decoupled linearized BDF2 scheme: to seek $(u_h^n, \phi_h^n) \in \widetilde{V}_{h0} \times V_{h0}$ for all $n \geq 2$ such that

$$\begin{cases} \mathrm{i}(D_\tau u_h^n, v_h) - (\nabla u_h^n, \nabla v_h) + \left(\widehat{\phi}_h^n f(|\widehat{u}_h^n|)u_h^n, v_h\right) = 0, \ v_h \in \widetilde{V}_{h0}, & \text{(2.20a)} \\ \alpha(\phi_h^n, \varphi_h) + \beta^2(\nabla\phi_h^n, \nabla\varphi_h) = \left(f(|\widehat{u}_h^n|)|\widehat{u}_h^n|^2, \varphi_h\right), \ \varphi_h \in V_{h0}, & \text{(2.20b)} \end{cases}$$

with (u_h^1, ϕ_h^1) and $(u_h^{1,0}, \phi_h^{1,0})$ is computed by (2.4) and (2.5), respectively, the initial value $u_h^0 = R_h u_0$ and ϕ_h^0 satisfies (2.6). We point out that we can solve the two (2.20a)–(2.20b) for u_h^n and $\phi_h^n (n \geq 2)$ in parallel at each time step.

Similar to Theorem 1 and Theorem 2, we also have the following conservative laws for scheme (2.20).

Theorem 3 *The discrete scheme* (2.20) *conserve the following conservative laws in the recursive sense*

$$\widetilde{\mathcal{M}}^{n+1} = \widetilde{\mathcal{M}}^n, \tag{2.21}$$

$$\widetilde{\mathcal{E}}^{n+1} = \widetilde{\mathcal{E}}^n, \quad n \geq 1, \tag{2.22}$$

where

$$\widetilde{\mathcal{M}}^n = \|u_h^n\|_0^2 + \|2u_h^n - u_h^{n-1}\|_0^2 + \widetilde{\mathcal{G}}^n, \tag{2.23}$$

$$\widetilde{\mathcal{E}}^n = \|\nabla u_h^n\|_0^2 + \|2\nabla u_h^n - \nabla u_h^{n-1}\|_0^2 + \widetilde{\mathcal{F}}^n, \tag{2.24}$$

with

$$\widetilde{\mathcal{G}}^1 = 0, \quad \widetilde{\mathcal{G}}^n - \widetilde{\mathcal{G}}^{n-1} = \|u_h^n - 2u_h^{n-1} + u_h^{n-2}\|_0^2 \tag{2.25}$$

$$\widetilde{\mathcal{F}}^1 = 0, \quad \widetilde{\mathcal{F}}^n - \widetilde{\mathcal{F}}^{n-1} = \|\nabla u_h^n - 2\nabla u_h^{n-1} + \nabla u_h^{n-2}\|_0^2 - 4\tau \mathrm{Re}\big(\widehat{\phi}_h^n f(|\widehat{u}_h^n|)u_h^n, D_\tau u_h^n\big), \quad n \geq 2. \tag{2.26}$$

In this paper, we only give out the error estimates for the linearized scheme (2.3)–(2.6). The analysis of the second linearized scheme (2.20) can be derived analogously, which will be confirmed numerically in Section 6. Like [31], we assume that $f : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous, i.e., for any $\gamma_1, \gamma_2 \in [-K^*, K^*]$,

$$|f(\gamma_1) - f(\gamma_2)| \leq L_{K^*}|\gamma_1 - \gamma_2|, \tag{2.27}$$

where L_{K^*} is the Lipschitz constant dependent on K^*. Besides, we assume that the solution to the problem (1.1) exists and satisfies

$$\begin{aligned} &\|u\|_{L^\infty(0,T;H^{r+1})} + \|u_t\|_{L^2(0,T;H^{r+1})} + \|u_{tt}\|_{L^2(0,T;H^2)} + \|u_{ttt}\|_{L^2(0,T;L^2)} \\ &+ \|\phi\|_{L^\infty(0,T;H^{r+1})} + \|\phi_{tt}\|_{L^2(0,T;H^2)} + \|u_0\|_{H^{r+1}} \leq M, \end{aligned} \tag{2.28}$$

where M is a positive constant depends only on Ω.

In our analysis, we need the following lemma which can be found in [45] for the details.

Lemma 1 *Assume $g \in H^m(\Omega)$ and $\partial\Omega$ is C^{m+2} for any nonnegative integer m. Suppose ψ is the unique solution of the boundary value problem*

$$\alpha\psi - \beta^2\Delta\psi = g, \text{ in } \Omega,$$
$$\psi = 0, \ on \ \partial\Omega.$$

Then $\psi \in H^{m+2}(\Omega)$ satisfies

$$\|\psi\|_{m+2} \leq \widetilde{C}\|g\|_m \tag{2.29}$$

where $\widetilde{C}$ depending on m, Ω and α, β. Especially, (2.29) *holds for convex domains when $m = 0$.*

3 Error estimates for temporal discretization

In this section, we first introduce a time-discrete system, then estimate the error functions $u^n - U^n$ and $\phi^n - \Phi^n$, as well as the boundedness of the time-discrete solutions in some norms.

When $n \geq 2$, we introduce the following auxiliary equations:

$$\begin{cases} \mathrm{i}D_\tau U^n + \Delta U^n + \widehat{\Phi}^n f(|\widehat{U}^n|)U^n = 0, & (3.1\mathrm{a}) \\ \alpha \Phi^n - \beta^2 \Delta \Phi^n = f(|U^n|)|U^n|^2, & (3.1\mathrm{b}) \\ U^n|_{\partial\Omega} = \Phi^n|_{\partial\Omega} = 0, & (3.1\mathrm{c}) \end{cases}$$

Similar to (2.4)–(2.5), we compute (U^1, Φ^1) and $(U^{1,0}, \Phi^{1,0})$ by

$$\begin{cases} \mathrm{i}\bar{\partial}_\tau U^1 + \Delta \overline{U}^1 + \frac{\Phi^{1,0}+\Phi^0}{2} f(|\frac{U^{1,0}+U^0}{2}|)\overline{U}^1 = 0, & (3.2\mathrm{a}) \\ \alpha \overline{\Phi}^1 - \beta^2 \Delta \overline{\Phi}^1 = f(|\overline{U}^1|)|\overline{U}^1|^2, & (3.2\mathrm{b}) \\ U^1|_{\partial\Omega} = \Phi^1|_{\partial\Omega} = 0, & (3.2\mathrm{c}) \end{cases}$$

and

$$\begin{cases} \mathrm{i}\frac{U^{1,0}-U^0}{\tau} + \frac{\Delta(U^{1,0}+U^0)}{2} + \Phi^0 f(|U^0|)\frac{U^{1,0}+U^0}{2} = 0, & (3.3\mathrm{a}) \\ \alpha \frac{\Phi^{1,0}+\Phi^0}{2} - \beta^2 \frac{\Delta(\Phi^{1,0}+\Phi^0)}{2} = f(|\frac{U^{1,0}+U^0}{2}|)|\frac{U^{1,0}+U^0}{2}|^2, & (3.3\mathrm{b}) \\ U^{1,0}|_{\partial\Omega} = \Phi^{1,0}|_{\partial\Omega} = 0, & (3.3\mathrm{c}) \end{cases}$$

respectively, where $U^0 = u_0$ in Ω and $\Phi^0 = \phi_0$ satisfies

$$\alpha\phi_0 - \beta^2 \Delta\phi_0 = f(|u_0|)|u_0|^2. \tag{3.4}$$

In what follows, we analyze the error functions $u^n - U^n$ and $\phi^n - \Phi^n$, respectively. To this end, under the regularity assumption (2.28), we define

$$K_0 := 1 + \max_{0\leq n\leq N}\{\|u^n\|_{0,\infty} + \|\phi^n\|_{0,\infty}\},$$

and let

$$e_u^{1,0} = u^1 - U^{1,0},\ e_\phi^{1,0} = \phi^1 - \Phi^{1,0},\ e_u^n = u^n - U^n,\ e_\phi^n = \phi^n - \Phi^n,\ \ 0 \leq n \leq N.$$

From (2.3)–(2.5) and (3.1)–(3.3), we have the error equations for $n \geq 2$:

$$\begin{cases} \mathrm{i}D_\tau e_u^n + \Delta e_u^n + (\widehat{\phi}^n f(|\widehat{u}^n|)u^n - \widehat{\Phi}^n f(|\widehat{U}^n|)U^n) = R_1^n, \\ \alpha e_\phi^n - \beta^2 \Delta e_\phi^n = f(|u^n|)|u^n|^2 - f(|U^n|)|U^n|^2, \end{cases}$$

and

$$\begin{cases} \mathrm{i}\bar{\partial}_\tau e_u^1 + \Delta \bar{e}_u^1 + \left(\overline{\phi}^1 f(|\overline{u}^1|)\overline{u}^1 - \frac{\Phi^{1,0}+\Phi^0}{2} f(|\frac{U^{1,0}+U^0}{2}|)\overline{U}^1\right) = S_1 + S_2, & (3.6\mathrm{a}) \\ \alpha \bar{e}_\phi^1 - \beta^2 \Delta \bar{e}_\phi^1 = f(|\overline{u}^1|)|\overline{u}^1|^2 - f(|\overline{U}^1|)|\overline{U}^1|^2 + S_3 + S_4, & (3.6\mathrm{b}) \end{cases}$$

$$\begin{cases} \mathrm{i}\frac{e_u^{1,0}}{\tau} + \frac{\Delta e_u^{1,0}}{2} + \Phi^0 f(|U^0|)\frac{e_u^{1,0}}{2} = S_1 + S_5, & (3.7\mathrm{a}) \\ \alpha \frac{e_\phi^{1,0}}{2} - \beta^2 \frac{\Delta e_\phi^{1,0}}{2} = f(|\overline{u}^1|)|\overline{u}^1|^2 - f(|\frac{U^{1,0}+U^0}{2}|)|\frac{U^{1,0}+U^0}{2}|^2 + S_3 + S_4, & (3.7\mathrm{b}) \end{cases}$$

where

$$R_1^n = \mathrm{i}(D_\tau u^n - u_t^n) + \widehat{\phi}^n f(|\widehat{u}^n|)u^n - \phi^n f(|u^n|)u^n,$$

and

$$\begin{aligned}
&S_1 = \mathrm{i}(\bar{\partial}_\tau u^1 - u_t^{1/2}) + \Delta(\overline{u}^1 - u^{1/2}), \quad S_2 = \overline{\phi}^1 f(|\overline{u}^1|)\overline{u}^1 - \phi^{1/2} f(|u^{1/2}|)u^{1/2},\\
&S_3 = \alpha(\overline{\phi}^1 - \phi^{1/2}) - \beta^2\Delta(\overline{\phi}^1 - \phi^{1/2}), \quad S_4 = f(|u^{1/2}|)|u^{1/2}|^2 - f(|\overline{u}^1|)|\overline{u}^1|^2,\\
&S_5 = \phi^0 f(|u^0|)\overline{u}^1 - \phi^{1/2} f(|u^{1/2}|)u^{1/2}.
\end{aligned}$$

By Taylor expansion formula, we have

$$\|S_i\|_0 + \tau\|S_5\|_0 + \|R_1^n\|_0 \le C\tau^2, \quad i = 1, ..., 4. \tag{3.8}$$

Theorem 4 *Suppose that the system (1.1) has unique solution (u, ϕ) satisfying (2.28). Then there exists $\tau_0 > 0$ such that when $\tau \le \tau_0$, the time-discrete system (3.1)–(3.3) is uniquely solvable for $n = 1, ..., N$, satisfying*

$$\|e_u^{1,0}\|_0 + \tau^{1/2}\|\nabla e_u^{1,0}\|_0 + \tau\|\Delta e_u^{1,0}\|_0 + \|e_\phi^{1,0}\|_2 + \|e_u^n\|_0 + \tau\|\Delta e_u^n\|_0 + \|e_\phi^n\|_2 \le C_0\tau^2, \tag{3.9}$$

and

$$\max\{\|\Phi^{1,0}\|_{0,\infty},\ \|U^{1,0}\|_{0,\infty},\ \|\Phi^n\|_{0,\infty},\ \|U^n\|_{0,\infty}\} \le K_0, \tag{3.10}$$

$$\|U^n\|_2 + \|\Phi^n\|_2 + \|\bar{\partial}_\tau U^n\|_2 + \|D_\tau U^n\|_2 \le C^*. \tag{3.11}$$

Proof Since the system (3.1)–(3.3) are linear elliptic equations, the classical theory of elliptic PDEs ensure that the solution of system (3.1)–(3.3) is unique solvable. In what follows, we prove (3.9)–(3.10) by mathematical induction. For the initial time step, multiplying (3.7a) by $(e_u^{1,0})^*$, and integrating it over Ω, then taking the imaginary parts, we find

$$\frac{1}{\tau}\|e_u^{1,0}\|_0^2 = \mathrm{Im}(S_1 + S_5, e_u^{1,0}) \le C(\|S_1\|_0 + \|S_5\|_0)\|e_u^{1,0}\|_0,$$

which together with (3.8) shows that

$$\|e_u^{1,0}\|_0 \le C_1\tau^2. \tag{3.12}$$

Similarly, multiplying (3.7a) by $(\Delta e_u^{1,0})^*$, integrating it over Ω and summing the real and imaginary parts together, we can see that

$$\begin{aligned}
\frac{1}{\tau}\|\nabla e_u^{1,0}\|_0^2 + \frac{1}{2}\|\Delta e_u^{1,0}\|_0^2 &= \mathrm{Im}\Big(\Phi^0 f(|U^0|)\frac{e_u^{1,0}}{2}, \Delta e_u^{1,0}\Big) - \mathrm{Im}(S_1 + S_5, \Delta e_u^{1,0})\\
&\quad -\mathrm{Re}\Big(\Phi^0 f(|U^0|)\frac{e_u^{1,0}}{2}, \Delta e_u^{1,0}\Big) + \mathrm{Re}(S_1 + S_5, \Delta e_u^{1,0})\\
&\le C_{K_0}\|e_u^{1,0}\|_0^2 + C(\|S_1\|_0^2 + \|S_5\|_0^2) + \frac{1}{4}\|\Delta e_u^{1,0}\|_0^2\\
&\le C\|e_u^{1,0}\|_0^2 + C\tau^2 + \frac{1}{4}\|\Delta e_u^{1,0}\|_0^2,
\end{aligned}$$

which together with (3.12), one has

$$\|\nabla e_u^{1,0}\|_0 + \tau^{1/2}\|\Delta e_u^{1,0}\|_0 \le C_2\tau^{3/2}. \tag{3.13}$$

Combining (3.12) and (3.13), we derive

$$\|e_u^{1,0}\|_2 \le C_3\tau, \tag{3.14}$$

which implies that

$$\|U^{1,0}\|_{0,\infty} \le \|e_u^{1,0}\|_{0,\infty} + \|u^1\|_{0,\infty} \le CC_3\tau + \|u^1\|_{0,\infty} \le K_0, \tag{3.15}$$

when $\tau \le \tau_1 = 1/CC_3$. By use of Lemma 1 and (3.12), we can see that

$$\begin{aligned}
\|e_\phi^{1,0}\|_2 &\le C\Big\| f(|\overline{u}^1|)|\overline{u}^1|^2 - f\big(|\frac{U^{1,0}+U^0}{2}|\big)|\frac{U^{1,0}+U^0}{2}|^2 + S_3 + S_4\Big\|_0 \\
&\le C\Big\|\Big(f(|\overline{u}^1|) - f\big(|\frac{U^{1,0}+U^0}{2}|\big)\Big)|\overline{u}^1|^2\Big\|_0 \\
&\quad + C\Big\| f\big(|\frac{U^{1,0}+U^0}{2}|\big)\Big(|\overline{u}^1|^2 - |\frac{U^{1,0}+U^0}{2}|^2\Big)\Big\|_0 + C\|S_3+S_4\|_0 \\
&\le C_{K_0}\|e_u^{1,0}\|_0 + C\tau^2 \le C_4\tau^2.
\end{aligned} \tag{3.16}$$

Therefore

$$\|\Phi^{1,0}\|_{0,\infty} \le CC_4\tau^2 + \|\phi^1\|_{0,\infty} \le K_0, \tag{3.17}$$

when $\tau \le \tau_2 = 1/(CC_4)^{1/2}$.

Next, multiplying (3.6a) by $(e_u^1)^*$, and integrating it over Ω, then taking the imaginary parts, we obtain

$$\begin{aligned}
\frac{1}{\tau}\|e_u^1\|_0^2 &= -\mathrm{Im}\Big(\overline{\phi}^1 f(|\overline{u}^1|)\overline{u}^1 - \frac{\Phi^{1,0}+\Phi^0}{2} f\big(|\frac{U^{1,0}+U^0}{2}|\big)\overline{U}^1, e_u^1\Big) + \mathrm{Im}(S_1+S_2, e_u^1) \\
&= -\mathrm{Im}\Big(\big(\overline{\phi}^1 - \frac{\Phi^{1,0}+\Phi^0}{2}\big) f(|\overline{u}^1|)\overline{u}^1, e_u^1\Big) - \mathrm{Im}\Big(\frac{\Phi^{1,0}+\Phi^0}{2}\big(f(|\overline{u}^1|) - f(|\frac{U^{1,0}+U^0}{2}|)\big)\overline{u}^1, , e_u^1\Big) \\
&\quad -\mathrm{Im}\Big(\frac{\Phi^{1,0}+\Phi^0}{2} f\big(|\frac{U^{1,0}+U^0}{2}|\big)(\overline{u}^1 - \overline{U}^1), e_u^1\Big) + \mathrm{Im}(S_1+S_2, e_u^1) \\
&\le C_{K_0}(\|e_u^{1,0}\|_0^2 + \|e_\phi^{1,0}\|_0^2) + C\tau^4 + C\|e_u^1\|_0^2,
\end{aligned}$$

which together with (3.12) and (3.16), we conclude

$$\|e_u^1\|_0 \le C_5\tau^2, \tag{3.18}$$

when $\tau \le \tau_3 = 1/2C$.

Moreover, multiplying (3.6a) by $(\Delta e_u^1)^*$, integrating it over Ω and taking the real and imaginary parts, respectively. Then summing them together to get

$$\begin{aligned}
\frac{1}{\tau}\|\nabla e_u^1\|_0^2 + \frac{1}{2}\|\Delta e_u^1\|_0^2 &= \mathrm{Im}\Big(\overline{\phi}^1 f(|\overline{u}^1|)\overline{u}^1 - \tfrac{\Phi^{1,0}+\Phi^0}{2} f(|\tfrac{U^{1,0}+U^0}{2}|)\overline{U}^1, \Delta e_u^1\Big) - \mathrm{Im}(S_1+S_2, \Delta e_u^1) \\
&\quad -\mathrm{Re}\Big(\overline{\phi}^1 f(|\overline{u}^1|)\overline{u}^1 - \tfrac{\Phi^{1,0}+\Phi^0}{2} f(|\tfrac{U^{1,0}+U^0}{2}|)\overline{U}^1, \Delta e_u^1\Big) + \mathrm{Re}(S_1+S_2, \Delta e_u^1) \\
&\le C_{K_0}(\|e_u^{1,0}\|_0^2 + \|e_\phi^{1,0}\|_0^2 + \|e_u^1\|_0^2) + C\tau^4 + \tfrac{1}{4}\|\Delta e_u^1\|_0^2,
\end{aligned} \tag{3.19}$$

which together with (3.12), (3.16) and (3.18), we find that

$$\|\nabla e_u^1\|_0 + \tau^{1/2}\|\Delta e_u^1\|_0 \leq C_6\tau^2. \tag{3.20}$$

This yields

$$\|e_u^1\|_2 \leq C_7\tau^{3/2}, \tag{3.21}$$

and

$$\|U^1\|_{0,\infty} \leq CC_7\tau^{3/2} + \|u^1\|_{0,\infty} \leq K_0, \tag{3.22}$$

when $\tau \leq \tau_4 = 1/(CC_7)^{2/3}$. By use of Lemma 1 again, and (3.18), we obtain

$$\begin{aligned}
\|e_\phi^1\|_2 &\leq C\left\| f(|\overline{u}^1|)|\overline{u}^1|^2 - f(|\overline{U}^1|)|\overline{U}^1|^2 + S_3 + S_4\right\|_0 \\
&\leq C\left\|\big(f(|\overline{u}^1|) - f(|\overline{U}^1|)\big)|\overline{u}^1|^2\right\|_0 + C\left\| f(|\overline{U}^1|)\big(|\overline{u}^1|^2 - |\overline{U}^1|^2\big)\right\|_0 + C\|S_3 + S_4\|_0 \\
&\leq C_{K_0}\|e_u^1\|_0 + C\tau^2 \leq C_8\tau^2.
\end{aligned} \tag{3.23}$$

Therefore

$$\|\Phi^1\|_{0,\infty} \leq CC_8\tau^2 + \|\phi^1\|_{0,\infty} \leq K_0, \tag{3.24}$$

when $\tau \leq \tau_5 = 1/(CC_8)^{1/2}$.

By mathematical induction, we assume that (3.9)–(3.10) holds for $m \leq n-1$. Then there exists positive $\tau_6 = (CC_0)^{2/3}$, such that when $\tau \leq \tau_6$, we have

$$\|U^m\|_{0,\infty} + \|\Phi^m\|_{0,\infty} \leq CC_0\tau^{3/2} + \|u^m\|_{0,\infty} + \|\phi^m\|_{0,\infty} \leq K_0. \tag{3.25}$$

Now we prove (3.9)–(3.10) holds for $m = n$. Multiplying (3.5a) by $(e_u^n)^*$, and taking the imaginary part, it follows that

$$\begin{aligned}
\frac{1}{4\tau}\big(P(e_u^n) - P(e_u^{n-1}) + Q(e_u^n)\big) &\leq C\|\widehat{\phi}^n f(|\widehat{u}^n|)u^n - \widehat{\Phi}^n f(|\widehat{U}^n|)U^n\|_0\|e_u^n\|_0 + C\|R_1^n\|_0\|e_u^n\|_0 \\
&\leq C_M(\|\widehat{\phi}^n - \widehat{\Phi}^n\|_0 + \|\widehat{u}^n - \widehat{U}^n\|_0 + \|u^n - U^n\|_0 + \|R_1^n\|_0)\|e_u^n\|_0 \\
&\leq C\tau^4 + C(\|e_\phi^{n-1}\|_0^2 + \|e_\phi^{n-2}\|_0^2 + \|e_u^n\|_0^2 + \|e_u^{n-1}\|_0^2 + \|e_u^{n-2}\|_0^2) + C\|e_u^n\|_0^2,
\end{aligned}$$

and summing up from 2 to n, we get

$$\|e_u^n\|_0^2 \leq C\tau^4 + C\|e_u^1\|_0^2 + C\|e_u^2\|_0^2 + C\tau\sum_{i=1}^n(\|e_\phi^{i-1}\|_0^2 + \|e_u^i\|_0^2). \tag{3.26}$$

By use of the Gronwall's inequality, we have

$$\|e_u^n\|_0 \leq C_{10}\tau^2, \tag{3.27}$$

when $\tau \leq \tau_7$. The above estimate further shows that

$$\|D_\tau e_u^n\|_0 \leq C\tau, \tag{3.28}$$

which together with (3.27), and from (3.5a), we obtain

$$\|\Delta e_u^n\|_0 \leq \|D_\tau e_u^n\|_0 + \|\widehat{\phi}^n f(|\widehat{u}^n|)u^n - \widehat{\Phi}^n f(|\widehat{U}^n|)U^n\|_0 + \|R_1^n\|_0 \leq C_{K_0}\tau. \tag{3.29}$$

By using Gagliardo-Nirenberg inequality [31], we can see that

$$\|e_u^n\|_{0,\infty} \leq C\|e_u^n\|_2^{3/4}\|e_u^n\|_0^{1/4} + C\|e_u^n\|_0 \leq CC_{K_0}\tau^{5/4} \leq 1, \tag{3.30}$$

when $\tau \le \tau_8 = 1/(CC_{K_0})^{4/5}$. Therefore,

$$\|U^n\|_{0,\infty} \le \|e_u^n\|_{0,\infty} + \|u^n\|_{0,\infty} \le K_0. \tag{3.31}$$

By use of Lemma 1 and (3.27), we obtain from (3.5b) that

$$\begin{aligned}\|e_\phi^n\|_2 &\le C\left\| f(|u^n|)|u^n|^2 - f(|U^n|)|U^n|^2\right\|_0 \\ &\le C\left\|\big(f(|u^n|) - f(|U^n|)\big)|u^n|^2\right\|_0 + C\left\| f(|U^n|)\big(|u^n|^2 - |U^n|^2\big)\right\|_0 \\ &\le C_{K_0}\|e_u^n\|_0 \le C_{11}\tau^2, \end{aligned}\tag{3.32}$$

which further shows that

$$\|\Phi^n\|_{0,\infty} \le C\|e_\phi^n\|_2 + \|\phi^n\|_{0,\infty} \le CC_{11}\tau^2 \le K_0, \tag{3.33}$$

when $\tau \le \tau_9 = (CC_{11})^{1/2}$. Thus (3.9)–(3.10) holds for $m = n$. The induction is closed. Furthermore, we obtain

$$\begin{aligned}&\|U^n\|_2 \le \|u^n\|_2 + \|e_u^n\|_2 \le \|u^n\|_2 + C\|\Delta e_u^n\|_0 \le \frac{C^*}{4},\\ &\|\Phi^n\|_2 \le \|\phi^n\|_2 + \|e_\phi^n\|_2 \le \|\phi^n\|_2 + C\|\Delta e_\phi^n\|_0 \le \frac{C^*}{4},\\ &\|\bar\partial_\tau U^n\|_2 \le \|\bar\partial_\tau u^n\|_2 + \|\bar\partial_\tau e_u^n\|_2 \le \frac{C^*}{4},\\ &\|D_\tau U^n\|_2 \le \|D_\tau u^n\|_2 + \|D_\tau e_u^n\|_2 \le \frac{C^*}{4},\end{aligned}$$

for all $n = 1, 2, ..., N$. Taking $\tau_0 \le \min_{i=1}^{9} \tau_i$ and $C_0 \ge \max_{i=1}^{11} C_i$, the proof of Theorem 4 is completed. □

4 Error estimates for spatial discretization

In this section, we obtain a τ-independent error estimate for $U^n - u_h^n$ and $\Phi^n - \phi_h^n$. To do so, we split the errors as follows

$$\begin{aligned}&U^{1,0} - u_h^{1,0} = U^{1,0} - R_hU^{1,0} + R_hU^{1,0} - u_h^{1,0} := \eta^{1,0} + \xi^{1,0},\\ &\Phi^{1,0} - \phi_h^{1,0} = \Phi^{1,0} - R_h\Phi^{1,0} + R_h\Phi^{1,0} - \phi_h^{1,0} := \rho^{1,0} + \theta^{1,0},\\ &U^n - u_h^n = U^n - R_hU^n + R_hU^n - u_h^n := \eta^n + \xi^n,\\ &\Phi^n - \phi_h^n = \Phi^n - R_h\Phi^n + R_h\Phi^n - \phi_h^n := \rho^n + \theta^n,\ n = 1, 2, ..., N.\end{aligned}$$

Theorem 5 *Let* (U^n, Φ^n) *and* (u_h^n, ϕ_h^n) *be the solutions of* (3.1)–(3.3) *and* (2.3)–(2.6) *respectively for* $n = 1, 2, ..., N$. *Then there exists* $\tau_0' > 0,\ h_0' > 0$*, such that when* $\tau \le \tau_0',\ h \le h_0'$,

$$\|\xi^n\|_0 + \|\theta^n\|_0 \le C_0'h^2. \tag{4.1}$$

Proof Since $\|R_h\Phi^n\|_{0,\infty} \le C\|\Phi^n\|_2$ and $\|R_hU^n\|_{0,\infty} \le C\|U^n\|_2$, and by use of (3.11), we can see that $\|R_h\Phi^n\|_{0,\infty}$ and $\|R_hU^n\|_{0,\infty}$ are uniformly bounded, thus we denote by $K_0' := 1 + \max_{n=0}^{N}\{\|R_h\Phi^n\|_{0,\infty} + \|R_hU^n\|_{0,\infty}\}$. First, we estimate the initial error. Since $u_h^0 = R_hu_0$, by using (2.1) and (3.11), we have

$$\|U^0 - u_h^0\|_0 = \|u_0 - R_hu_0\|_0 \le C_1'h^2\|u_0\|_2. \tag{4.2}$$

From (2.6) and (3.4), we obtain

$$\alpha(\Phi^0 - \phi_h^0, \varphi_h) + \beta^2\big(\nabla(\Phi^0 - \phi_h^0), \nabla\varphi_h\big) = 0. \tag{4.3}$$

When $\beta = 0$, (4.3) leads to

$$\|R_h\Phi^0 - \phi_h^0\|_0 \le \|\Phi^0 - R_h\Phi^0\|_0 \le C_2'h^2\|\Phi^0\|_2. \tag{4.4}$$

When $\beta \neq 0$, (4.3) yields to

$$\|\nabla(R_h\Phi^0 - \phi_h^0)\|_0 \le \|\nabla(\Phi^0 - R_h\Phi^0)\|_0 \le Ch\|\Phi^0\|_2. \tag{4.5}$$

By use of the Aubin-Nitsche techniques, we can derive

$$\|\Phi^0 - \phi_h^0\|_0 \le Ch^2\|\Phi^0\|_2, \tag{4.6}$$

which shows that

$$\|R_h\Phi^0 - \phi_h^0\|_0 \le \|\Phi^0 - \phi_h^0\|_0 + \|\Phi^0 - R_h\Phi^0\|_0 \le C_3'h^2\|\Phi^0\|_2. \tag{4.7}$$

Combining (4.2), (4.4) and (4.7), and employing the inverse inequality, we have

$$\|\phi_h^0\|_{0,\infty} \le \|R_h\Phi^0 - \phi_h^0\|_{0,\infty} + \|R_h\Phi^0\|_{0,\infty} \le Ch^{-d/2}C_3'h^2 + \|R_h\Phi^0\|_{0,\infty} \le K_0', \tag{4.8}$$
$$\|u_h^0\|_{0,\infty} \le \|R_hU^0 - u_h^0\|_{0,\infty} + \|R_hU^0\|_{0,\infty} \le Ch^{-d/2}C_1'h^2 + \|R_hU^0\|_{0,\infty} \le K_0', \tag{4.9}$$

when $h \le h_1' = \min\{\frac{1}{(CC_1')^{2/(4-d)}}, \frac{1}{(CC_3')^{2/(4-d)}}\}$.

For the first time step, from (2.5) and (3.3), we derive

$$\begin{cases} \mathrm{i}(\frac{\xi^{1,0}}{\tau}, v_h) - (\frac{\nabla\xi^{1,0}}{2}, \nabla v_h) = \mathrm{i}(\frac{\eta^{1,0}}{\tau}, v_h) - (\frac{U^0-u_h^0}{\tau}, v_h) \\ \qquad\qquad -\big(\Phi^0 f(|U^0|)\frac{U^{1,0}+U^0}{2} - \phi_h^0 f(|u_h^0|)\frac{u_h^{1,0}+u_h^0}{2}, v_h\big), & (4.10\text{a}) \\ \frac{\alpha}{2}(\theta^{1,0}, \varphi_h) + \frac{\beta^2}{2}(\nabla\theta^{1,0}, \nabla\varphi_h) = -\frac{\alpha}{2}(\rho^{1,0}, \varphi_h) - \frac{\alpha}{2}(\Phi^0 - \phi_h^0, \varphi_h) \\ \qquad\qquad +\Big(f(|\frac{U^{1,0}+U^0}{2}|)|\frac{U^{1,0}+U^0}{2}|^2 - f(|\frac{u_h^{1,0}+u_h^0}{2}|)|\frac{u_h^{1,0}+u_h^0}{2}|^2, \varphi_h\Big). & (4.10\text{b}) \end{cases}$$

Choosing $v_h = \xi^{1,0}$ in (4.10a), and taking the imaginary and real parts, then adding them together, one has

$$\begin{aligned} \frac{1}{\tau}\|\xi^{1,0}\|_0^2 + \frac{1}{2}\|\nabla\xi^{1,0}\|_0^2 \;=\; & \mathrm{Re}(\frac{\eta^{1,0}}{\tau}, \xi^{1,0}) - \mathrm{Im}(\frac{U^0 - u_h^0}{\tau}, \xi^{1,0}) - \mathrm{Im}(\frac{\eta^{1,0}}{\tau}, \xi^{1,0}) + \mathrm{Re}(\frac{U^0 - u_h^0}{\tau}, \xi^{1,0}) \\ & -\mathrm{Im}\Big(\Phi^0 f(|U^0|)\frac{U^{1,0} + U^0}{2} - \phi_h^0 f(|u_h^0|)\frac{u_h^{1,0} + u_h^0}{2}, \xi^{1,0}\Big) \\ & +\mathrm{Re}\Big(\Phi^0 f(|U^0|)\frac{U^{1,0} + U^0}{2} - \phi_h^0 f(|u_h^0|)\frac{u_h^{1,0} + u_h^0}{2}, \xi^{1,0}\Big). \end{aligned} \tag{4.11}$$

Since $\|\frac{U^{1,0}-U^0}{\tau}\|_2 \le \|\frac{U^{1,0}-u^1}{\tau}\|_2 + \|\frac{u^1-u^0}{\tau}\|_2 \le C$, we have

$$\Big|(\frac{\eta^{1,0}}{\tau}, \xi^{1,0}) - (\frac{U^0 - u_h^0}{\tau}, \xi^{1,0})\Big| = \Big(\frac{U^{1,0}-U^0}{\tau} - R_h(\frac{U^{1,0}-U^0}{\tau}), \xi^{1,0}\Big) \le Ch^4\|\frac{U^{1,0}-U^0}{\tau}\|_2^2 + C\|\xi^{1,0}\|_0^2, \tag{4.12}$$

and

$$\Big|\Big(\Phi^0 f(|U^0|)\frac{U^{1,0}+U^0}{2} - \phi_h^0 f(|u_h^0|)\frac{u_h^{1,0}+u_h^0}{2}, \xi^{1,0}\Big)\Big| \le C_{K_0'}\big(\|U^0 - u_h^0\|_0^2 + \|\Phi^0 - \phi_h^0\|_0^2 + \|U^{1,0} - u_h^{1,0}\|_0^2\big) + C\|\xi^{1,0}\|_0^2. \tag{4.13}$$

Thus, (4.11) leads to

$$\begin{aligned}
&\frac{1}{\tau}\|\xi^{1,0}\|_0^2 + \frac{1}{2}\|\nabla\xi^{1,0}\|_0^2 \\
\le\ & Ch^4\|\frac{U^{1,0}-U^0}{\tau}\|_2^2 + C_{K_0'}\big(\|U^0 - u_h^0\|_0^2 + \|\Phi^0 - \phi_h^0\|_0^2 + \|U^{1,0} - u_h^{1,0}\|_0^2\big) + C\|\xi^{1,0}\|_0^2 \\
\le\ & Ch^4 + C\|\xi^{1,0}\|_0^2,
\end{aligned} \tag{4.14}$$

which implies that

$$\|\xi^{1,0}\|_0 + \|\nabla\xi^{1,0}\|_0 \le C_4' h^2, \tag{4.15}$$

and

$$\|u_h^{1,0}\|_{0,\infty} \le Ch^{-d/2}C_4'h^2 + \|R_h U^{1,0}\|_{0,\infty} \le K_0', \tag{4.16}$$

when $\tau \le \tau_1' = 1/2C$, $h \le h_2' = \frac{1}{(CC_4')^{2/(4-d)}}$.

Moreover, taking $\varphi_h = \theta^{1,0}$ in (4,10b), we derive

$$\begin{aligned}
\frac{\alpha}{2}\|\theta^{1,0}\|_0^2 + \frac{\beta^2}{2}\|\nabla\theta^{1,0}\|_0^2 =\ & -\frac{\alpha}{2}(\rho^{1,0}, \theta^{1,0}) - \frac{\alpha}{2}(\Phi^0 - \phi_h^0, \theta^{1,0}) \\
& + \Big(f(|\frac{U^{1,0}+U^0}{2}|)|\frac{U^{1,0}+U^0}{2}|^2 - f(|\frac{u_h^{1,0}+u_h^0}{2}|)|\frac{u_h^{1,0}+u_h^0}{2}|^2, \theta^{1,0}\Big) \\
\le\ & Ch^4\|\Phi^{1,0}\|_2^2 + C\|\Phi^0 - \phi_h^0\|_0^2 \\
& + C_{K_0'}\big(\|U^0 - u_h^0\|_0^2 + \|\Phi^0 - \phi_h^0\|_0^2 + \|U^{1,0} - u_h^{1,0}\|_0^2\big) + \frac{\alpha}{4}\|\theta^{1,0}\|_0^2,
\end{aligned} \tag{4.17}$$

which together with (4.15)–(4.16), and by use of the Aubin-Nitsche techniques again, we have

$$\|\theta^{1,0}\|_0^2 \le C_5' h^2, \tag{4.18}$$

this further implies that

$$\|\phi_h^{1,0}\|_{0,\infty} \le Ch^{-d/2}C_5'h^2 + \|\Phi^{1,0}\|_{0,\infty} \le K_0', \tag{4.19}$$

when $h \le h_3' = \frac{1}{(CC_5')^{2/(4-d)}}$.

On the other hand, from (2.4) and (3.2), we obtain

$$\begin{cases}
\mathrm{i}(\bar\partial_\tau\xi^1, v_h) - (\nabla\overline{\xi}^1, \nabla v_h) = -\mathrm{i}(\bar\partial_\tau\eta^1, v_h) \\
\qquad\qquad -\Big(\frac{\Phi^{1,0}+\Phi^0}{2} f(|\frac{U^{1,0}+U^0}{2}|)\overline{U}^1 - \frac{\phi^{1,0}+\phi^0}{2} f(|\frac{u_h^{1,0}+u_h^0}{2}|)\overline{u}_h^1, v_h\Big), & \text{(4.20a)} \\
\alpha(\overline{\theta}^1, \varphi_h) + \beta^2(\nabla\overline{\theta}^1, \nabla\varphi_h) = -\alpha(\overline{\rho}^1, \varphi_h) + \Big(f(|\overline{U}^1|)|\overline{U}^1|^2 - f(|\overline{u}_h^1|)|\overline{u}_h^1|^2, \varphi_h\Big). & \text{(4.20b)}
\end{cases}$$

Likewise, we can see that

$$\|\xi^1\|_0 + \|\theta^1\|_0 \le C_6' h^2, \tag{4.21}$$

which implies that

$$\|u_h^1\|_{0,\infty} + \|\phi_h^1\|_{0,\infty} \le Ch^{-d/2}C_6'h^2 + \|R_hU^1\|_{0,\infty} + \|R_h\Phi^1\|_{0,\infty} \le K_0', \tag{4.22}$$

when $\tau \le \tau_2' = 1/2C,\ h \le h_4' = \frac{1}{(CC_6')^{2/(4-d)}}$.

We prove (4.1) by mathematical induction. By mathematical induction, we assume that the result (4.1) holds for $m \le n-1 (n \ge 2)$, then there exists $h_5' = \frac{1}{(CC_0')^{2/(4-d)}}$, when $h \le h_5'$ such that

$$\|u_h^m\|_{0,\infty} + \|\phi_h^m\|_{0,\infty} \le Ch^{-d/2}C_0'h^2 + \|R_hU^m\|_{0,\infty} + \|R_h\Phi^m\|_{0,\infty} \le K_0'. \tag{4.23}$$

Now we prove (4.1) holds for $m = n$. By (2.3) and (3.1), one can derive the following error equations

$$\begin{cases} l\mathrm{i}(D_\tau\xi^n, v_h) - (\nabla\xi^n, \nabla v_h) = -\mathrm{i}(D_\tau\eta^n, v_h) - \big(\widehat{\Phi}^n f(|\widehat{U}^n|)U^n - \widehat{\phi}_h^n f(|\widehat{u}_h^n|)u_h^n, v_h\big), & \text{(4.24a)}\\ \alpha(\theta^n, \varphi_h) + \beta^2(\nabla\theta^n, \nabla\varphi_h) = -\alpha(\rho^n, \varphi_h) + \big(f(|U^n|)|U^n|^2 - f(|u_h^n|)|u_h^n|^2, \varphi_h\big). & \text{(4.24b)} \end{cases}$$

Setting $v_h = \xi^n$ in (4.24a) and taking the imaginary part, we have

$$\begin{aligned} &\frac{1}{4\tau}\Big(P(\xi^n) - P(\xi^{n-1}) + Q(\xi^n)\Big) = -\mathrm{Re}(D_\tau\eta^n, \xi^n) - \mathrm{Im}\big(\widehat{\Phi}^n f(|\widehat{U}^n|)U^n - \widehat{\phi}_h^n f(|\widehat{u}_h^n|)u_h^n, \xi^n\big)\\ &\le Ch^2\|D_\tau U^n\|_2\|\xi^n\|_0 + C_{K_0'}\|\widehat{\Phi}^n - \widehat{\phi}_h^n\|_0\|U^n\|_{0,\infty}\|\xi^n\|_0\\ &+ C_{K_0'}\|\widehat{U}^n - \widehat{u}_h^n\|_0\|\xi^n\|_0 + C_{K_0'}\|U^n - u_h^n\|_0\|\xi^n\|_0\\ &\le Ch^4(\|D_\tau U^n\|_2^2 + \|\Phi^{n-1}\|_2^2 + \|\Phi^{n-2}\|_2^2 + \|U^n\|_2^2 + \|U^{n-1}\|_2^2 + \|U^{n-2}\|_2^2) + C\|\xi^n\|_0^2. \end{aligned}$$

Replacing n by i and summing up the equation from 2 to n, we obtain

$$\|\xi^n\|_0^2 \le Ch^4 + C\tau\sum_{i=1}^n(\|\theta^{i-1}\|_0^2 + \|\xi^i\|_0^2). \tag{4.25}$$

On the other hand, setting $\varphi_h = \theta^n$ in (4.24b), if $\beta \neq 0$, we derive

$$\|\nabla\theta^n\|_0 \le Ch^2 + C\|\xi^n\|_0. \tag{4.26}$$

This shows that

$$\|\theta^n\|_0 \le Ch^2 + C\|\xi^n\|_0. \tag{4.27}$$

If $\beta = 0$, we also obtain from (4.24b) that

$$\|\theta^n\|_0 \le Ch^2 + C\|\xi^n\|_0. \tag{4.28}$$

Substituting (4.28) into (4.25), then by use of the Gronwall's inequality, there exists positive constants τ_3', C_7', such that when $\tau \le \tau_3'$

$$\|\xi^n\|_0 \le C_7'h^2, \tag{4.29}$$

Then, from (4.28) and (4.29), we also have

$$\|\theta^n\|_0 \le Ch^2. \tag{4.30}$$

Further

$$\|u_h^n\|_{0,\infty} \le Ch^{-d/2}C_7'h^2 + \|R_hU^n\|_{0,\infty} \le K_0', \tag{4.31}$$

when $h \leq h'_6 = \frac{1}{(CC'_7)^{2/(4-d)}}$. Thus, (4.1) holds for $m = n$ if we take $C'_0 \geq \sum_{i=1}^{7} C'_i$, $\tau_0 \leq \min_{i=1}^{3} \tau'_i$ and $h_0 \leq \min_{i=1}^{6} h'_i$. The proof is completed. □

Remark 1 Clearly, one can see that the error estimate in Theorem 5 is optimal in L^2-norm for linear Galerkin FEM, and we can derive optimal H^1 error estimate

$$\|\nabla\xi^n\|_0 + \|\nabla\theta^n\|_0 \leq C'_0 h. \tag{4.32}$$

Furthermore, from the proof of Theorem 5, we can see that the following supercloseness result can be derived when $\beta \neq 0$

$$\|\nabla\theta^n\|_0 \leq C'_0 h^2. \tag{4.33}$$

5 Optimal error estimates for the fully discrete scheme

In this section, we will derive L^2 optimal error estimates for the r-order ($r \geq 1$) Galerkin FEM by using the results in the above sections.

From (2.1), (3.9), and (4.1), we have optimal error estimates for the linear Galerkin FEM ($r = 1$) as follows.

$$\|u^n - u_h^n\|_0 \leq \|e_u^n\|_0 + \|\eta^n\|_0 + \|\xi^n\|_0 \leq C_0\tau^2 + Ch^2\|U^n\|_2 + Ch^2 \leq C(h^2 + \tau^2). \tag{5.1}$$

Similarly, we derive

$$\|\phi^n - \phi_h^n\|_0 \leq C(h^2 + \tau^2), \tag{5.2}$$

and

$$\|\nabla(\phi^n - \phi_h^n)\|_0 \leq C(h + \tau^2), \quad \|\nabla(u^n - u_h^n)\|_0 \leq C(h + \tau^2). \tag{5.3}$$

For $r > 1$, the above estimates are not optimal for the r-order Galerkin FEM. However, we can derive the uniform bounds of the numerical solutions in L^∞-norm from Theorem 2 as:

$$\|u_h^n\|_{0,\infty} \leq Ch^{-d/2}\|\xi^n\|_0 + \|R_h U^n\|_{0,\infty} \leq K'_0, \tag{5.4}$$

$$\|\phi_h^n\|_{0,\infty} \leq Ch^{-d/2}\|\theta^n\|_0 + \|R_h \Phi^n\|_{0,\infty} \leq K'_0, \tag{5.5}$$

for $n = 0, 1, \cdots, N$ when $\tau \leq \tau'_0, h \leq h'_0$. By the above uniform bounds, we can obtain optimal L^2 error estimates given in the following Theorem.

Theorem 6 *Let (U^n, Φ^n) and (u_h^n, ϕ_h^n) be the solutions of* (1.1) *and* (2.3)–(2.6) *respectively for $n = 1, 2, ..., N$. Then there holds*

$$\|u^n - u_h^n\|_0 + \|\phi^n - \phi_h^n\|_0 \leq C(h^{r+1} + \tau^2). \tag{5.6}$$

Proof Let $\xi_u^n = R_h u^n - u_h^n, \theta_\phi^n = R_h\phi^n - \phi_h^n$. At the time step $t = t_{1/2}$, we can easily get $\|\xi_u^1\|_0 \leq C(h^{r+1} + \tau^2)$ and $\|\theta_\phi^1\|_0 \leq C(h^{r+1} + \tau^2)$. Thus, we only analyze the errors $u^n - u_h^n$ and $\phi^n - \phi_h^n$ for $1 \le n \le N$ in the following. From (1.1) and (2.3), we obtain

$$\begin{cases} \mathrm{i}(D_\tau \xi_u^n, v_h) - (\nabla \xi_u^n, \nabla v_h) = -\mathrm{i}(u_t^n - D_\tau R_h u^n, v_h) - \big(\phi^n f(|u^n|)u^n - \widehat{\phi}_h^n f(|\widehat{u}_h^n|)u_h^n, v_h\big), & (5.7a)\\ \alpha(\theta_\phi^n, \varphi_h) + \beta^2(\nabla \theta_\phi^n, \nabla \varphi_h) = -\alpha(\phi^n - R_h\phi^n, \varphi_h) + \big(f(|u^n|)|u^n|^2 - f(|u_h^n|)|u_h^n|^2, \varphi_h\big). & (5.7b)\end{cases}$$

Setting $v_h = \xi_u^n$ in (5.7a) and taking the imaginary part to obtain

$$\frac{1}{4\tau}\Big(P(\xi_u^n) - P(\xi_u^{n-1}) + Q(\xi_u^n)\Big) = -\mathrm{Re}\Big(u_t^n - D_\tau R_h u^n, \xi_u^n\Big) - \mathrm{Im}\big(\phi^n f(|u^n|)u^n - \widehat{\phi}_h^n f(|\widehat{u}_h^n|)u_h^n, \xi_u^n\big). \tag{5.8}$$

Replacing n by i and summing up the equation from 2 to n, and noting that

$$\begin{aligned}\sum\nolimits_{i=2}^n \tau\|u_t^i - D_\tau R_h u^i\|_0^2 &\le 2\sum\nolimits_{i=2}^n \tau\|u_t^i - D_\tau u^i\|_0^2 + 2\sum\nolimits_{i=2}^n \tau\|D_\tau u^i - D_\tau R_h u^i\|_0^2\\ &\le C\tau^4\|u_{ttt}\|_{L^2(0,T;L^2)}^2 + Ch^{2(r+1)}\|u_t\|_{L^2(0,T;H^{(r+1)})}^2,\end{aligned} \tag{5.9}$$

$$\begin{aligned}&\sum\nolimits_{i=2}^n \tau\|\phi^i f(|u^i|)u^i - \widehat{\phi}_h^i f(|\widehat{u}_h^i|)u_h^i\|_0^2\\ &\le 2\sum\nolimits_{i=2}^n \tau\|\phi^i f(|u^i|)u^i - \widehat{\phi}^i f(|\widehat{u}^i|)u^i\|_0^2 + 2\sum\nolimits_{i=2}^n \tau\|\widehat{\phi}^i f(|\widehat{u}^i|)u^i - \widehat{\phi}_h^i f(|\widehat{u}_h^i|)u_h^i\|_0^2\\ &\le C\tau^4\Big(\|u_{tt}\|_{L^2(0,T;L^2)}^2 + \|\phi_{tt}\|_{L^2(0,T;L^2)}^2\Big)\\ &+ Ch^{2(r+1)}\Big(\|u\|_{L^2(0,T;H^{r+1})}^2 + \|\phi\|_{L^2(0,T;H^{r+1})}^2\Big) + C\tau\sum\nolimits_{i=1}^{n-1}\|\theta_\phi^i\|_0^2 + C\tau\sum\nolimits_{i=1}^n\|\xi_u^i\|_0^2.\end{aligned} \tag{5.10}$$

From (5.8)–(5.10), we have

$$\|\xi_u^n\|_0^2 \le C\|\xi_u^1\|_0^2 + Ch^{2(r+1)} + C\tau^4 + C\tau\sum\nolimits_{i=1}^{n-1}\|\theta_\phi^i\|_0^2 + C\tau\sum\nolimits_{i=1}^n\|\xi_u^i\|_0^2. \tag{5.11}$$

On the other hand, for any $n \ge 1$, setting $\varphi_h = \theta_\phi^n$ in (5.7b), we derive

$$\begin{aligned}\alpha\|\theta_\phi^n\|_0^2 + \beta^2\|\nabla\theta_\phi^n\|_0^2 &= -\alpha(\phi^n - R_h\phi^n, \theta_\phi^n) + \big(f(|U^n|)|U^n|^2 - f(|u_h^n|)|u_h^n|^2, \theta_\phi^n\big)\\ &\le Ch^2\|\Phi^n\|_2\|\theta_\phi^n\|_0 + C_{K_0'}\big(h^{2(r+1)}\|u^n\|_2 + \|\xi_u^n\|_0\big)\|\theta_\phi^n\|_0.\end{aligned} \tag{5.12}$$

By the same technique used in the proof of estimates (4.3)–(4.7), we can derive

$$\|\theta_\phi^n\|_0 \le C(h^{r+1} + \|\xi_u^n\|_0). \tag{5.13}$$

Thus, with (5.11) and (5.13), there exists a positive τ'' such that when $\tau \le \tau''$, we have

$$\|\theta_\phi^n\|_0 + \|\xi_u^n\|_0 \le Ch^{r+1}. \tag{5.14}$$

Therefore, by (2.1) and the triangle inequality, we derive (5.6), which completes the proof. □

6 Numerical results

In this section, we present two numerical examples to confirm the efficiency and accuracy of the proposed numerical schemes. In our test, we choose linear and quadratic basis functions on triangular and rectangular finite elements to derive numerical solutions.

Example 6.1 We consider the following Schrödinger-Helmholtz equation [30, 31].

$$\begin{cases} \mathrm{i}u_t + \Delta u + u\phi = f_1, & (X,t) \in \Omega \times (0,T], \\ \alpha\phi - \beta^2\Delta\phi = |u|^2 + f_2, & (X,t) \in \Omega \times (0,T], \\ u(X,t) = 0, \phi(X,t) = 0, & (X,t) \in \ \partial\Omega \times (0,T], \\ u(X,0) = u_0(X), & X \in \Omega, \end{cases} \tag{6.1}$$

where $\Omega = (0,1)^2, \alpha = \beta = 1$ in (6.1) and the final time is chosen as $T = 2$ in the computations. f_1, f_2 and $u_0(X)$ are chosen correspondingly to the exact solutions

$$u(X,t) = e^{(\mathrm{i}+1)t}\sin(x)\sin(y)\sin(\pi x)\sin(\pi y),\ \ \phi(X,t) = e^{t+x+y}\sin(x)\sin(y)(1-x)(1-y).$$

Now, we solve the problem (6.1) by the linearized BDF2 schemes (2.3)–(2.6) and (2.20), respectively, with linear triangular element on triangular (P_1) and quadratic element on rectangular (Q_2) approximation. We choose $\tau = 5h$ for P_1 element and $\tau = h^{3/2}$ for Q_2 element, respectively, and divide the domain Ω into $M+1$ nodes in each direction for P_1 element with different $M = 5, 10, 20, 40$, and different mesh-grids $m \times n = 5\times5$, 10×10, 20×20 and 40×40 for Q_2 element, respectively. The numerical results are listed in Tables 1, 2, 3 and 4 at time $t = 0.5$, 1.0 and 2.0. It can be observed that the errors in L^2 norm are proportional to h^2 for P_1 element and h^3 for Q_2 element, which are consistent with the theoretical analysis. Additionally, we also observe that the semi-implicit or explicit treatment of the nonlinear terms in the (1.1) has little impact on the convergence of the whole scheme.

To show the unconditional convergence of the linearized BDF2 scheme (2.2)–(2.6) and (2.20) , respectively, we solve the problem (6.1) for each $\tau = \frac{1}{5}, \frac{1}{10}, \frac{1}{20}$,

Table 1 L^2 errors and convergence rates of the first scheme (2.3)–(2.6) with P_1 element (Example 6.1)

M	5	10	20	40	Average order
$\Vert u^n - u_h^n\Vert_0$					
$t = 0.5$	2.7025E-2	7.0807E-3	1.7584E-3	4.3760E-4	1.9828
$t = 1.0$	4.4423E-2	1.1850E-2	3.0098E-3	7.6140E-4	1.9555
$t = 2.0$	1.2392E-1	3.3841E-2	8.5075E-3	2.1054E-3	1.9597
$\Vert \phi^n - \phi_h^n\Vert_0$					
$t = 0.5$	1.6064E-2	4.2365E-3	1.0785E-3	2.7354E-4	1.9586
$t = 1.0$	2.7078E-2	7.2060E-3	1.8582E-3	4.8325E-4	1.9360
$t = 2.0$	8.1912E-2	2.3024E-2	6.3843E-3	1.8606E-3	1.8200

Table 2 L^2 errors and convergence rates of the first scheme (2.3)–(2.6) with Q_2 element (Example 6.1)

$m \times n$	5×5	10×10	20×20	40×40	Average order
$\|u^n - u_h^n\|_0$					
$t = 0.5$	1.5614E-3	1.0533E-4	1.3135E-5	1.6411E-6	3.2980
$t = 1.0$	1.3485E-3	1.7253E-4	2.1629E-5	2.7056E-6	2.9871
$t = 2.0$	3.6544E-3	4.6841E-4	5.8831E-5	7.3741E-6	2.9843
$\|\phi^n - \phi_h^n\|_0$					
$t = 0.5$	6.3762E-4	3.0897E-5	4.0601E-6	5.9652E-7	3.3540
$t = 1.0$	4.0502E-4	5.2415E-5	7.3872E-6	1.2577E-6	2.7770
$t = 2.0$	1.1958E-3	1.8103E-4	2. 3023E-5	2.7846E-6	2.9154

Table 3 L^2 errors and convergence rates of the second scheme (2.20) with P_1 element (Example 6.1)

M	5	10	20	40	Average order
$\|u^n - u_h^n\|_0$					
$t = 0.5$	2.7025E-2	7.0804E-3	1.7582E-3	4.3752E-4	1.9829
$t = 1.0$	4.4421E-2	1.1848E-2	3.0087E-3	7.6085E-4	1.9558
$t = 2.0$	1.2381E-1	3.3751E-2	8.4604E-3	2.0816E-3	1.9648
$\|\phi^n - \phi_h^n\|_0$					
$t = 0.5$	1.6030E-2	4.2129E-3	1.0663E-3	2.6742E-4	1.9685
$t = 1.0$	2.6893E-2	7.0808E-3	1.7923E-3	4.4958E-4	1.9675
$t = 2.0$	7.9177E-2	2.1191E-2	5.3613E-3	1.3337E-3	1.9639

Table 4 L^2 errors and convergence rates of the second scheme (2.20) with Q_2 element (Example 6.1)

$m \times n$	5×5	10×10	20×20	40×40	Average order
$\|u^n - u_h^n\|_0$					
$t = 0.5$	1.5612E-3	1.0532E-4	1.3135E-5	1.6411E-6	3.2979
$t = 1.0$	1.3485E-3	1.7253E-4	2.1628E-5	2.7053E-6	2.9871
$t = 2.0$	36494E-3	4.6854E-4	5.8931E-5	7.3621E-6	2.9844
$\|\phi^n - \phi_h^n\|_0$					
$t = 0.5$	6.1710E-4	3.0346E-5	3.7927E-6	4.7408E-7	3.4487
$t = 1.0$	4.0050E-4	5.0033E-5	6.2532E-6	7.8163E-6	3.0004
$t = 2.0$	1.2358E-3	1.8173E-4	2.3133E-5	2.7796E-6	2.9321

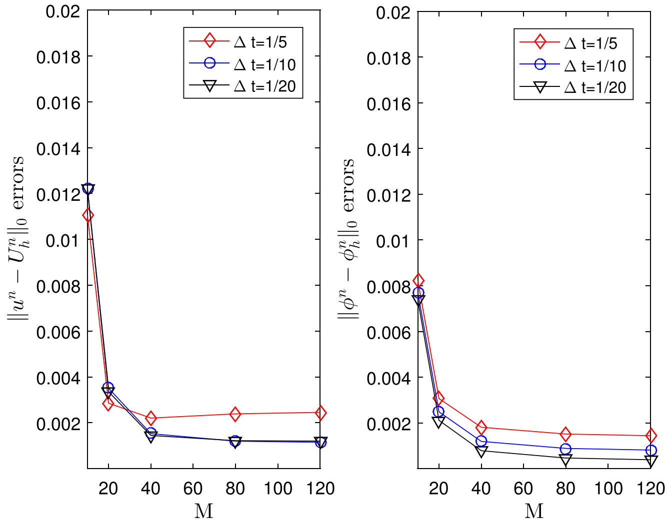

Fig. 1 L^2-norm errors of P_1 element computed by scheme (2.3)–(2.6) (Example 6.1)

with different mesh-grids at time $t = 1.0$. The numerical results are presented in Figs. 1, 2, 3 and 4 for P_1 and Q_1 elements. We can see that the numerical errors tend to be a constant as $\frac{\tau}{h} \to \infty$ for each fixed τ, which show that grid-ratio condition is unnecessary.

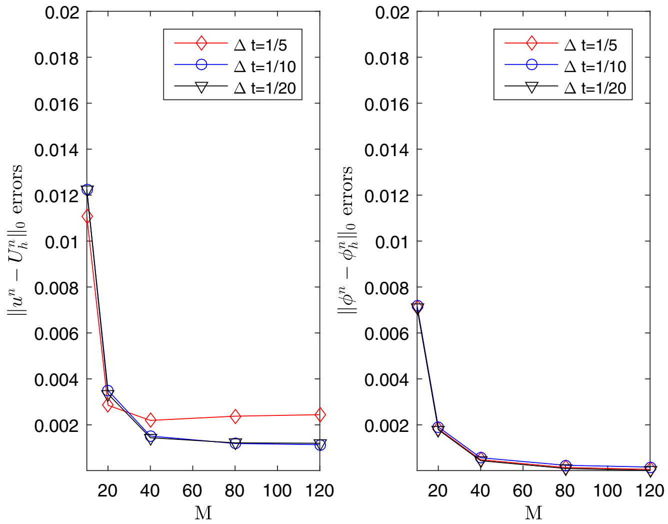

Fig. 2 L^2-norm errors of P_1 element computed by scheme (2.20) (Example 6.1)

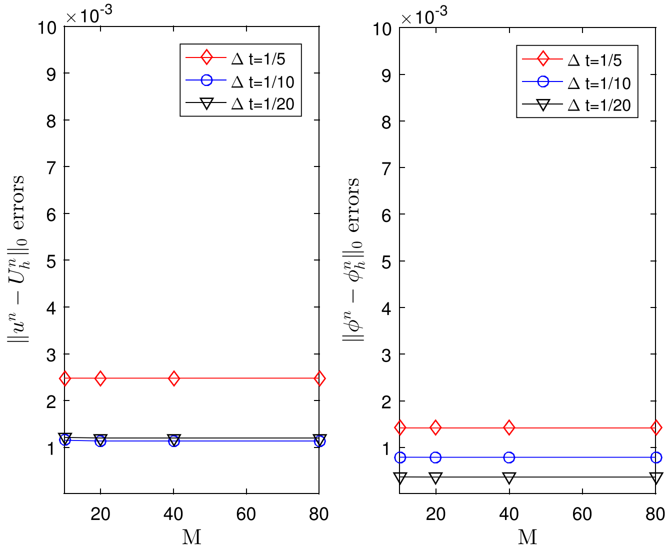

Fig. 3 L^2-norm errors of Q_2 element computed by scheme (2.3)–(2.6) (Example 6.1)

Example 6.2 Here, we consider a high-order Schrödinger-Poisson-Slater system:

$$
\begin{cases}
\mathrm{i}u_t + \Delta u + u\phi + |u|^4 u = g_1, & (X,t) \in \Omega \times (0,T], \\
-\Delta\phi = |u|^2 + g_2, & (X,t) \in \Omega \times (0,T], \\
u(X,t) = 0, \phi(X,t) = 0, & (X,t) \in \ \partial\Omega \times (0,T], \\
u(X,0) = u_0(X), & X \in \Omega,
\end{cases}
\tag{5.2}
$$

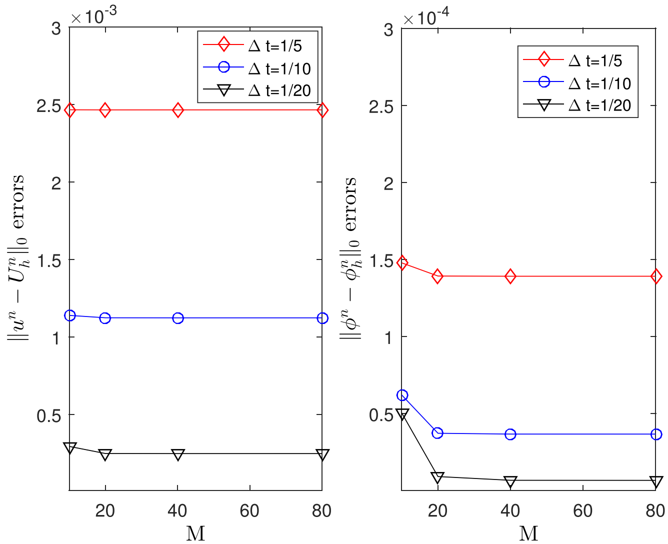

Fig. 4 L^2-norm errors of Q_2 element computed by scheme (2.20) (Example 6.1)

Table 5 L^2 errors and convergence rates of the first scheme (2.3)–(2.6) with P_1 element (Example 6.2)

M	5	10	20	40	Average order
$\|u^n - u_h^n\|_0$					
$t = 0.5$	1.4699E-2	3.9717E-3	1.0197E-3	2.6313E-4	1.9346
$t = 1.0$	3.3115E-2	8.6709E-3	2.2206E-3	5.5894E-4	1.9628
$\|\phi^n - \phi_h^n\|_0$					
$t = 0.5$	9.5764E-3	2.5072E-3	6.3929E-4	16324E-4	1.9581
$t = 1.0$	2.1599E-2	5.7514E-3	1.5089E-3	4.0493E-4	1.9123

Table 6 L^2 errors and convergence rates of the first scheme (2.3)–(2.6) with Q_2 element (Example 6.2)

$m \times n$	5×5	10×10	20×20	40×40	Average order
$\|u^n - u_h^n\|_0$					
$t = 0.5$	6.5287E-5	7.7743E-6	9.6503E-7	1.2048E-7	3.0273
$t = 1.0$	1.4095E-4	1.7650E-5	2.2034E-6	2.7532E-7	3.0000
$\|\phi^n - \phi_h^n\|_0$					
$t = 0.5$	1.2012E-5	1.5196E-6	1.9052E-7	2.3832E-8	2.9925
$t = 1.0$	2.6145E-5	3.3013E-6	4.1376E-7	5.1756E-8	2.9935

Table 7 L^2 errors and convergence rates of the second scheme (2.20) with P_1 element (Example 6.2)

M	5	10	20	40	Average order
$\|u^n - u_h^n\|_0$					
$t = 0.5$	1.4694E-2	3.9631E-3	1.0101E-3	2.5317E-4	1.9530
$t = 1.0$	3.3109E-2	8.6668E-3	2.2184E-3	5.5781E-4	1.9638
$\|\phi^n - \phi_h^n\|_0$					
$t = 0.5$	9.5429E-3	2.4866E-3	6.2822E-4	1.5748E-4	1.9737
$t = 1.0$	2.1300E-2	5.5668E-3	1.4089E-3	3.5334E-4	1.9712

Table 8 L^2 errors and convergence rates of the second scheme (2.20) with Q_2 element (Example 6.2)

$m \times n$	5×5	10×10	20×20	40×40	Average order
$\|u^n - u_h^n\|_0$					
$t = 0.5$	6.5276E-5	7.7741E-6	9.6502E-7	1.2048E-7	3.0272
$t = 1.0$	1.4095E-4	1.7649E-5	2.2034E-6	2.7532E-7	2.9999
$\|\phi^n - \phi_h^n\|_0$					
$t = 0.5$	1.2504E-5	1.5344E-6	1.9099E-7	2.3847E-8	3.0114
$t = 1.0$	2.6689E-5	3.3146E-6	4.1414E-7	5.1767E-8	3.0033

in which $\Omega = (0, 1)^2$, $T = 1$. g_1, g_2 and $u_0(X)$ are chosen correspondingly to the exact solutions

$$u(X, t) = 2e^{\mathrm{i}t+(x+y)/5}(1 + 3t^2)x(1 - x)y(1 - y),$$
$$\phi(X, t) = 5(1 + 3t^2 + \sin(t))\sin(\tfrac{x}{2})\sin(\tfrac{y}{2})(1 - x)(1 - y).$$

We solve this problem by the two linearized BDF2 schemes given in Section 2, with above P_1 and Q_2 elements. To show the convergence in L^2 norm, we adopt the same mesh generation as Example 6.1. The numerical results are presented in Tables 5, 6, 7 and 8 at time $t = 0.5$ and 1.0. We can see that the errors in L^2 norm are in line with the theoretical analysis. On the other hand, to verify unconditional stability of schemes, we also list numerical results at time $t = 1.0$. in Figs. 5, 6, 7

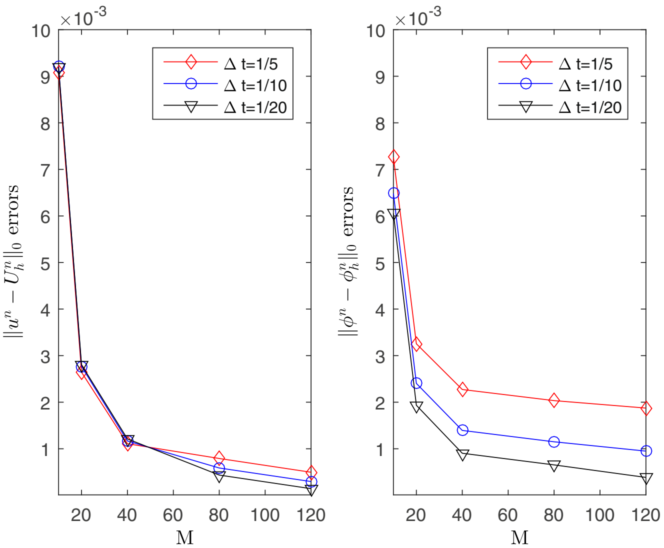

Fig. 5 L^2-norm errors of P_1 element computed by scheme (2.3)–(2.6) (Example 6.2)

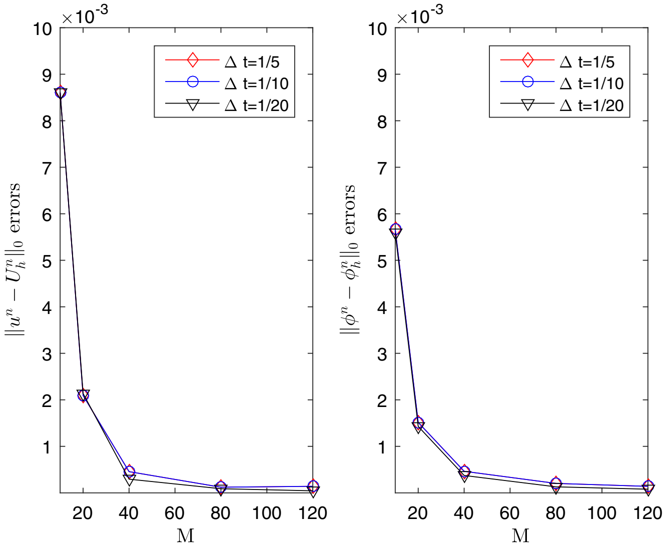

Fig. 6 L^2-norm errors of P_1 element computed by scheme (2.20) (Example 6.2)

and 8 for each $\tau = \frac{1}{5}, \frac{1}{10}, \frac{1}{20}$ with different mesh-grids. We can see that for a fixed τ, the errors in L^2 norm converge to a small constant when the mesh refine gradually, which show that the two proposed schemes are unconditionally stable and the grid-ratio condition is unnecessary.

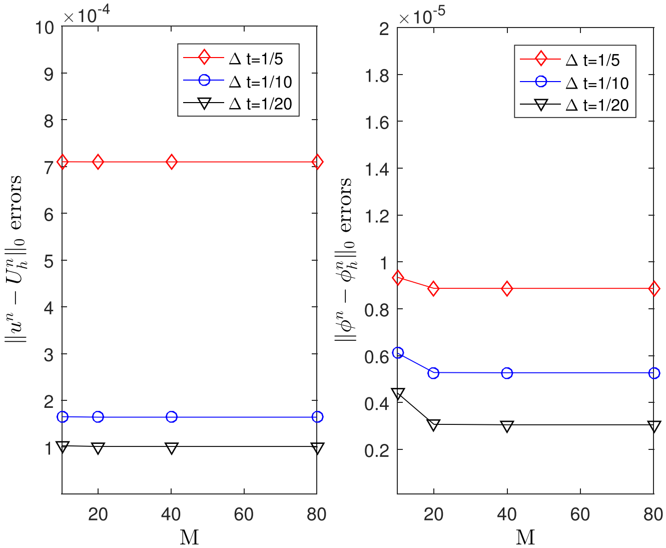

Fig. 7 L^2-norm errors of Q_2 element computed by scheme (2.3)–(2.6) (Example 6.2)

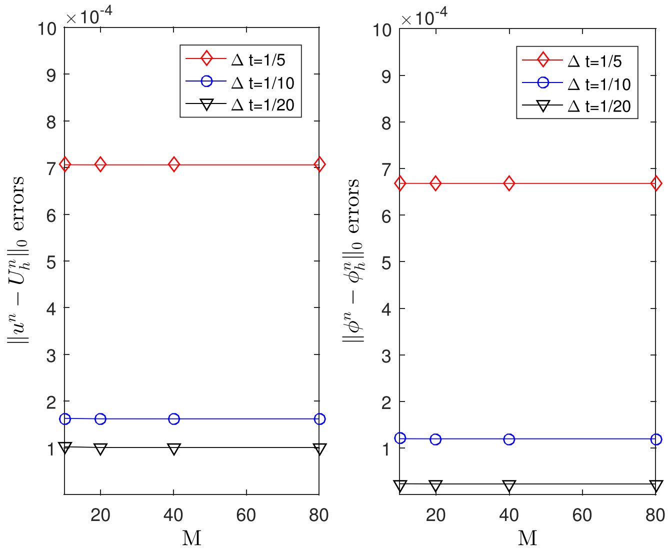

Fig. 8 L^2-norm errors of Q_2 element computed by scheme (2.20) (Example 6.2)

7 Conclusion

In this paper, we have presented two linearized BDF2 schemes with Galerkin finite elements approximation for the nonlinear Schrödinger-Helmholtz equations. Different from the existing second accurate (in time) numerical schemes for coupled equations, we derive optimal error estimates at the time step t_n (instead of the time instant $t_{n+1/2}$) for the proposed schemes without grid-ratio condition. At last, two numerical examples are provided to confirm the theoretical analysis. On the other hand, there are some interesting works on the variable-step BDF2 method for self-adaptive time stepping integrations for long-time simulations of phase field models, such as [46–48]. The analytic method in this paper can be extended to analyze other nonlinear physical models, such as the time-dependent nonlinear thermistor equations [35], Cahn-Hilliard equation [36], Keller-Segel system [49], and so forth.

Funding This work was supported by the National Natural Science Foundation of China (No. 12071443); the Key Scientific Research Projects of Henan Colleges and Universities (No. 20B110013).

Data availability Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Conflict of interest The authors declare no competing interests.

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