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ORIGINAL PAPER

Self-adaptive algorithms for solving split feasibility problem with multiple output sets

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Abstract

In this paper, we study the split feasibility problem with multiple output sets in Hilbert spaces. For solving the aforementioned problem, we propose two new self-adaptive relaxed CQ algorithms which involve computing of projections onto half-spaces instead of computing onto the closed convex sets, and it does not require calculating the operator norm. We establish a weak and a strong convergence theorems for the proposed algorithms. We apply the new results to solve some other problems. Finally, we present some numerical examples to show the efficiency and accuracy of our algorithm compared to some existing results. Our results extend and improve some existing methods in the literature.

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1 Introduction

Let H_1 and H_2 be two real Hilbert spaces. Let C and Q be nonempty, closed, and convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a nonzero bounded linear operator and let $A^* : H_2 \to H_1$ be its adjoint. The `split feasibility problem (SFP)` is formulated to find a point $x^* \in H_1$ satisfying

$$x^* \in C \text{ such that } Ax^* \in Q. \tag{1}$$

The SFP was first introduced in 1994 by Censor and Elfving [1] in finite-dimensional Hilbert spaces for modeling certain inverse problems and has received a great attention since then. This is because the SFP can be used to model several inverse problems arising from, for example, phase retrievals and in medical image reconstruction [1, 2], intensity-modulated radiation therapy (IMRT) [3–5], gene regulatory network inference [6], just to mention but few, for more details one can, see, e.g., [7–14] and the references therein. In the span of the last twenty five years, focusing on real world applications, several iterative methods for solving the SFP (1) have been introduced and analyzed. Among them, Byrne [2, 9] introduced the first applicable and most celebrated method called the well-known `CQ-algorithm` as follows: for $x_0 \in H_1$;

$$x_{n+1} := P_C(x_n - \tau_n A^*(I - P_Q)Ax_n)), \tag{2}$$

where P_C and P_Q are the metric projections onto C and Q, respectively, and the stepsize $\tau_n \in \left(0, \frac{2}{\|A\|^2}\right)$ where $\|A\|^2$ is the spectral radius of the matrix A^*A.

The CQ algorithm proposed by Byrne [2, 9], requires the computation of metric projection onto the sets C and Q (in some cases, it is impossible or is too expensive to exactly compute the metric projection). In addition, the determination of the stepsize depends on the operator norm which computation (or at least estimate) is not easy task. In practical applications, the sets C and Q are usually the level sets of convex functions which are given by

$$C := \{x \in H_1 : c(x) \leq 0\} \text{ and } Q = \{y \in H_2 : q(y) \leq 0\}, \tag{3}$$

where $c : H_1 \to \mathbb{R}$ and $q : H_2 \to \mathbb{R}$ are convex and subdifferentiable functions on H_1 and H_2, respectively, and that subdifferentials $\partial c(x)$ and $\partial q(y)$ of c and q, respectively, are bounded operators (i.e., bounded on bounded sets).

Later, in 2004, Yang [12] generalized the CQ method to the so-called `relaxed CQ algorithm`, needing computation of the metric projection onto (relaxed sets) `half-spaces` C_n and Q_n, where

$$C_n := \{x \in H_1 : c(x_n) \leq \langle \xi_n, x_n - x\rangle\}, \tag{4}$$

where $\xi_n \in \partial c(x_n)$ and

$$Q_n := \{y \in H_2 : q(Ax_n) \leq \langle \eta_n, Ax_n - y \rangle\}, \tag{5}$$

where $\eta_n \in \partial q(Ax_n)$. It is easy to see that $C \subseteq C_n$ and $Q \subseteq Q_n$ for all $n \geq 1$. Moreover, it is known that projections onto half-spaces C_n and Q_n have closed forms. In what follows, define

$$f_n(x_n) := \frac{1}{2}\|(I - P_{Q_n})Ax_n\|^2, \tag{6}$$

where Q_n is given as in (5). f_n is a convex and differentiable function with its gradient ∇f_n defined by

$$\nabla f_n(x_n) := A^*(I - P_{Q_n})Ax_n. \tag{7}$$

More precisely, Yang [12] introduced the following relaxed CQ algorithm for solving the SFP (1) in a finite-dimensional Hilbert space: for $x_0 \in H_1$;

$$x_{n+1} := P_{C_n}(x_n - \tau_n \nabla f_n(x_n)), \tag{8}$$

where $\tau_n \in \left(0, \frac{2}{\|A\|^2}\right)$. Since P_{C_n} and P_{Q_n} are easily calculated, this method appears to be very practical. However, to compute the norm of A turns out to be complicated and costly. To overcome this difficulty, in 2012, López et al. [15] introduced a relaxed CQ algorithm for solving the SFP (1) with a new adaptive way of determining the stepsize sequence τ_n defined as follows:

$$\tau_n := \frac{\rho_n f_n(x_n)}{\|\nabla f_n(x_n)\|^2}, \tag{9}$$

where $\rho_n \in (0, 4), \forall n \geq 1$ such that $\liminf_{n\to\infty} \rho_n(4 - \rho_n) > 0$. It was proved that the sequence $\{x_n\}$ generated by (8) with τ_n defined by (9) converges weakly to a solution of the SFP (1). That is, their algorithm has only weak convergence in the framework of infinite-dimensional Hilbert spaces. But, in the infinite-dimensional spaces norm (strong) convergence is more desirable than the weak convergence for solving our problems. In this regard, many authors proposed algorithms that generate a sequence $\{x_n\}$, converges strongly to a point in the solution set of the SFP (1), see, e.g., [15–19]. In particular, López et al. [15] proposed a `Halpern's iterative` scheme for solving the SFP (1) in the setting of infinite-dimensional Hilbert spaces as follows: for $u, x_0 \in H_1$;

$$x_{n+1} := \alpha_n u + (1 - \alpha_n)P_{C_n}\left(x_n - \tau_n \nabla f_n(x_n)\right), \forall n \geq 1, \tag{10}$$

where $\{\alpha_n\} \subset (0, 1)$, and $\nabla f_n(x_n)$ and τ_n are given by (7) and (9), respectively. In 2013, He et al. [16] also introduced a new relaxed CQ algorithm for solving the SFP (1) such that strong convergence is guaranteed in infinite-dimensional Hilbert space. Their algorithm generates a sequence $\{x_n\}$ by the following manner: for $u, x_0 \in H_1$;

$$x_{n+1} := P_{C_n}\left(\alpha_n u + (1 - \alpha_n)\big(x_n - \tau_n \nabla g_n(x_n)\big)\right), \tag{11}$$

where C_n and τ_n are given as in (4) and (9), respectively, and $\{\alpha_n\} \subset (0, 1)$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = +\infty$. Under some standard conditions, it was shown that the sequence $\{x_n\}$ generated by (10) and (11) converges strongly to $p^* = P_{\Omega}(u) \in \Omega = \{p \in H_1 : p \in C \text{ such that } Ap \in Q\}$ of the SFP (1). Both schemes (10) and (11) do not require any prior knowledge of the operator norm and compute the projections onto the half-spaces C_n and Q_n (which have closed-form), and thus both are easily implementable.

Some generalizations of the SFP have also been studied by many authors. We mention, for instance, the multiple-sets SFP (MSSFP) [3, 20–34], the split common fixed point problem (SFPP) [35, 36], the split variational inequality problem (SVIP) [37], and the split common null point problem (SCNPP) [38–42].

Very recently, Reich et al. [43] considered and studied the following split feasibility problem with multiple output sets in real Hilbert spaces.

Let H, H_i, $i = 1, 2, \ldots, N$, be real Hilbert spaces and let $A_i : H \to H_i$, $i = 1, 2, \ldots, N$, be bounded linear operators. Let C and Q_i, $i = 1, 2, \ldots, N$, be nonempty, closed, and convex subsets of H and H_i, $i = 1, 2, \ldots, N$, respectively. Given H, H_i, and A_i as above, the `split feasibility problem with multiple output sets` (`SFPMOS`, for short) is to find an element p^* such that

$$p^* \in \Omega := C \cap \left(\cap_{i=1}^{N} A_i^{-1}(Q_i)\right) \neq \emptyset. \tag{12}$$

That is $p^* \in C$ and $A_i p^* \in Q_i$ for each $i = 1, 2, \ldots, N$.

In 2020, Reich et al. [43] introduced the following two methods for solving the SFPMOS (12).

For any given points, $x_0, y_0 \in H$, $\{x_n\}$, and $\{y_n\}$ are sequences generated by

$$x_{n+1} := P_C\left(x_n - \lambda_n \sum_{i=1}^{N} A_i^*(I - P_{Q_i})A_i x_n\right), \tag{13}$$

$$y_{n+1} := \alpha_n f(y_n) + (1 - \alpha_n)P_C\left(y_n - \lambda_n \sum_{i=1}^{N} A_i^*(I - P_{Q_i})A_i y_n\right), \tag{14}$$

where $f : C \to C$ is a strict contraction mapping of H into itself with the contraction constant $\theta \in [0, 1)$, $\lambda_n \subset (0, \infty)$ and $\{\alpha_n\} \subset (0, 1)$. It was proved that if the sequence $\{\lambda_n\}$ satisfies the condition:

$$0 < a \leq \lambda_n \leq b < \frac{2}{N \max_{i=1,2,\ldots,N}\{\|A_i\|^2\}}$$

for all $n \geq 1$, then the sequence $\{x_n\}$ generated by (13) converges weakly to a solution point $p^* \in \Omega$ of the SFPMOS (12). Furthermore, if the sequence $\{\alpha_n\}$ satisfies the conditions:

$$\lim_{n\to\infty} \alpha_n = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \alpha_n = \infty,$$

then the sequence $\{y_n\}$ generated by (14) converges strongly to a solution point $p^* \in \Omega$ of the SFPMOS (12), which is a unique solution of the variational inequality

$$\langle (I - f)p^*, x - p^* \rangle \geq 0 \; \forall x \in \Omega.$$

An important observation here is that the iterative methods (Scheme (13) and Scheme (14)) introduced by Reich et al. [43] requires to compute the metric projections on to the sets C and Q_i. Moreover, it needs to compute the operator norm. Due to this reason, the following question naturally arises. *Question:* Can we design two new iterative algorithms (a weakly convergent and strongly convergent methods, different from Scheme (13) and Scheme (14)) for solving the SFPMOS (12) which mainly involves a self-adaptive step-size and requires to compute the projections onto half-spaces so that the algorithm is easily implementable?.

We have a positive answer for the above question which is motivated by the iterative schemes (13) and (14) proposed by Reich et al. [43] for solving the SFPMOS (12), the Halpern's-type iterative schemes (10) and (11) proposed by López et al. [15] and He et al. [16], respectively, to solve the SFP (1). In this paper, we propose two new self-adaptive relaxed CQ algorithms for solving the SFPMOS (12) in infinite-dimensional Hilbert spaces.

In the next section, we recall some necessary tools which are used in establishing our main results. In Section 3, we propose self-adaptive relaxed CQ algorithms for solving the SFPMOS (12), and we establish and analyze weak and strong convergence theorems for the proposed algorithms. In the same section, we also present some newly derived results for solving the SFP (1). In Section 4, we present the application of our methods to solve the generalized split feasibility problem (another generalization of the SFP). Finally, in the last section, we provide several numerical examples to illustrate the implementation of our algorithms compared to some existing results.

2 Preliminaries

In this section, we recall some definitions and basic results which are needed in the sequel. Let H be a real Hilbert space with the inner product $\langle ., . \rangle$, and induced norm $\|.\|$. Let I stands for the identity operator on H. Let the symbols "$\rightharpoonup$" and "$\rightarrow$", denote the weak and strong convergence, respectively. For any sequence $\{x_n\} \subset H$, $\omega_w(x_n) = \{x \in H : \exists \{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \rightharpoonup x\}$ denotes the weak w-limit set of $\{x_n\}$.

Definition 1 ([44]) Let C be a nonempty closed convex subset of H. Let $T : C \rightarrow H$ be a given operator. Then, T is called

(1) `Lipschitz continuous` with constant $\lambda > 0$ on C if

$$\|Tx - Ty\| \leq \lambda \|x - y\|, \forall x, y \in C; \tag{15}$$

(2) `nonexpansive` on C if

$$\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in C; \tag{16}$$

(3) `firmly nonexpansive` on C if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - ||(I - T)x - (I - T)y\|^2, \forall x, y \in C, \tag{17}$$

which is equivalent to

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle, \forall x, y \in C; \tag{18}$$

(4) `averaged` if there exists a number $\lambda \in (0, 1)$ and a nonexpansive operator $F : C \to H$ such that

$$T = \lambda F + (1 - \lambda)I, \text{ where } I \text{ is the identity operator.} \tag{19}$$

In this case, we say that T is λ-*averaged*.

Definition 2 ([44]) Let $C \subset H$ be a nonempty, closed and convex set. For every element $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$ such that

$$\|x - P_C(x)\| = \min\{\|x - y\| : y \in C\}. \tag{20}$$

The operator P_C (mapping $P_C : H \to C$) is called a `metric projection` of H onto C and it has the following well-known properties.

Lemma 1 *([44, 45]) Let $C \subset H$ be a nonempty, closed and convex set. Then, the following assertions hold for any $x, y \in H$ and $z \in C$:*

(1) $\langle x - P_C(x), z - P_C(x)\rangle \leq 0$;
(2) $\|P_C(x) - P_C(y)\| \leq \|x - y\|$;
(3) $\|P_C(x) - P_C(y)\|^2 \leq \langle P_C(x) - P_C(y), x - y\rangle$;
(4) $\|P_C(x) - z\|^2 \leq \|x - z\|^2 - \|x - P_C(x)\|^2$.

We see from Lemma 1 that the metric projection mapping is firmly nonexpansive and nonexpansive. Moreover, it is not hard to show that $I - P_C$ is also firmly nonexpansive and nonexpansive.

Lemma 2 *For all $x, y \in H$ and for all $\alpha \in \mathbb{R}$, we have*

(1) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$;
(2) $\|x + y\|^2 = \|x\|^2 + \|y\|^2 + 2\langle x, y\rangle$;
(3) $\langle x, y\rangle = \frac{1}{2}\|x\|^2 + \frac{1}{2}\|y\|^2 - \frac{1}{2}\|x - y\|^2$;
(4) $\|\alpha x + (1 - \alpha)y\|^2 = \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2$.

Fejér-monotone sequences are very useful in the analysis of optimization iterative algorithms.

Definition 3 ([44]) Let C be a nonempty subset of H and let $\{x_n\}$ be a sequence in H. Then, $\{x_n\}$ is Fejér monotone with respect to C if

$$\|x_{n+1} - z\| \leq \|x_n - z\|, \ \forall z \in C.$$

It is easy to see that a Fejér monotone sequence $\{x_n\}$ is bounded and the limit $\lim_{n\to\infty} \|x_n - z\|$ exists.

Lemma 3 *(Demiclosedness principle of nonexpansive mappings* [44]*) Let C be a closed convex subset of H, $T : C \to C$ be a nonexpansive mapping with nonempty fixed point sets. If $\{x_n\}$ is a sequence in C converging weakly to x and $\{(I-T)x_n\}$ converges strongly to y, then $(I-T)x = y$. In particular, if $y = 0$, then $x = Tx$.*

Lemma 4 *(*[44, 46, 47]*) Let C be a nonempty, closed, and convex subset of a real Hilbert space H and let $\{x_n\}$ be a sequence in H satisfying the properties:*

(1) $\lim_{n\to\infty} \|x_n - x^*\|$ exists for every $x^* \in C$;
(2) $\omega_w(x_n) \subset C$.

Then, there exists a point $\hat{x} \in C$ such that $\{x_n\}$ converges weakly to $\hat{x}$.

Definition 4 Let $f : H \to \mathbb{R}$ be a function and $\lambda \in [0, 1]$. Then,

(1) f is `convex` if

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y), \forall x, y \in H.$$

(2) A vector $\xi \in H$ is a `subgradient` of f at a point x if

$$f(y) \ge f(x) + \langle \xi, y - x\rangle, \ \forall y \in H.$$

(3) The set of all subgradients of a convex function $f : H \to \mathbb{R}$ at $x \in H$, denoted by $\partial f(x)$, is called the `subdifferential` of f, and is defined by

$$\partial f(x) = \{\xi \in H : f(y) \ge f(x) + \langle \xi, y - x\rangle, \ \text{for each } y \in H\}.$$

(4) If $\partial f(x) \neq \emptyset$, f is said to be `subdifferentiable` at x. If the function f is continuously differentiable then $\partial f(x) = \{\nabla f(x)\}$. The convex function is subdifferentiable everywhere [44].

(5) f is called `weakly lower semicontinuous` at x_0 if for a sequence $\{x_n\}$ weakly converging to x_0 one has

$$f(x_0) \le \liminf_{n\to\infty} f(x_n).$$

A function which is weakly lower semicontinuous at each point of H is called `weakly lower semicontinuous` on H.

Lemma 5 *(*[48]*) Let H_1 and H_2 be real Hilbert spaces and $f : H_1 \to \mathbb{R}$ is given by $f(x) = \frac{1}{2}\|(I - P_Q)Ax\|^2$ where Q is a nonempty, closed convex subset of H_2 and $A : H_1 \to H_2$ be a bounded linear operator. Then, the following assertions hold:*

(1) f is convex and differentiable;
(2) f is weakly lower semicontinuous on H_1;
(3) $\nabla f(x) = A^*(I - P_Q)Ax$, for $x \in H_1$;
(4) ∇f is $\|A\|^2$-Lipschitz, i.e., $\|\nabla f(x) - \nabla f(y)\| \le \|A\|^2\|x - y\|, \forall x, y \in H_1$.

Lemma 6 *(*[49]*) Let $\{\Lambda_n\}$ be a sequence of real numbers that does not decrease at infinity. Also consider the sequence of integers $\{\varphi(n)\}_{n\ge n_0}$ defined by*

$$\varphi(n) = \max\{m \in \mathbb{N} : m \le n, \Lambda_m \le \Lambda_{m+1}\}.$$

Then, $\{\varphi(n)\}_{n\geq n_0}$ *is a nondecreasing sequence verifying* $\lim_{n\to\infty}\varphi(n)=\infty$*, and for all* $n\geq n_0$*, the following two estimates hold:*

$$\Lambda_{\varphi(n)}\leq\Lambda_{\varphi(n)+1} \text{ and } \Lambda_n\leq\Lambda_{\varphi(n)+1}.$$

Lemma 7 *([50]) Let* $\{s_n\}$ *be a sequence of nonnegative real numbers satisfying the following relation:*

$$s_{n+1}\leq(1-\varrho_n)s_n+\varrho_n\mu_n+\theta_n, n\geq 1,$$

where $\{\varrho_n\}$*,* $\{\mu_n\}$ *and* $\{\theta_n\}$ *satisfying the conditions:*

(1) $\{\varrho_n\}\subset[0,1]$, $\sum_{n=1}^{\infty}\varrho_n=\infty$;
(2) $\limsup_{n\to\infty}\mu_n\leq 0$;
(3) $\theta_n\geq 0$, $\sum_{n=1}^{\infty}\theta_n<\infty$.

Then, $\lim_{n\to\infty}s_n=0$.

3 The iterative algorithms for solving SFPMOS

In this section, we propose new self-adaptive relaxed iterative methods for solving the SFPMOS (12) in the infinite-dimensional Hilbert spaces, and we prove a weak and strong convergence theorems of the proposed methods.

The relaxed projection methods use metric projections onto half-spaces instead of projections onto the original closed convex sets. In what follows, we consider a general case of the SFPMOS (12), where the nonempty, closed and convex sets C and $Q_i(i=1,2,\ldots,N)$ are given by level sets of convex functions defined as follows:

$$C:=\{x\in H:c(x)\leq 0\} \text{ and } Q_i:=\{y\in H_i:q_i(y)\leq 0\} \tag{21}$$

where, $c:H\to\mathbb{R}$ and $q_i:H_i\to\mathbb{R}$, $i=1,2,\ldots,N$ are lower semicontinuous convex functions. We assume that both c and each q_i are subdifferentiable on H and H_i, respectively, with subdifferential ∂c and ∂q_i, respectively. Moreover, assume that for any $x\in H$ a subgradient $\xi\in\partial c(x)$ can be calculated, and for any $y\in H_i$ and for each $i\in\{1,2,\ldots,N\}$, a subgradient $\eta_i\in\partial q_i(y)$ can be calculated. Again, assume that both ∂c and $\partial q_i(i=1,2,\ldots,N)$ are bounded operators (i.e., bounded on bounded sets). The subdifferentials ∂c and ∂q_i are defined by

$$\partial c(x):=\{\xi\in H:c(z)\geq c(x)+\langle\xi,z-x\rangle,\ \forall z\in H\}$$

for all $x\in C$ and

$$\partial q_i(y):=\{\eta_i\in H_i:q_i(u)\geq q_i(y)+\langle\eta_i,u-y\rangle,\ \forall u\in H_i\}$$

for all $y\in Q_i$, $i=1,2,\ldots,N$.

In this situation, the projections onto C and Q_i are not easily implemented in general. To avoid this difficulty, we introduce a relaxed projection gradient methods, in which the projections onto the half-spaces are adopted in stead of the projections onto C and Q_i. In particular for $n\in\mathbb{N}$, we define the relaxed sets (half-spaces) C_n

and $Q_i^n (i = 1, 2, \ldots, N)$ of C and Q_i, respectively, at x_n as follows:

$$C_n := \{x \in H : c(x_n) \leq \langle \xi_n, x_n - x \rangle\}, \tag{22}$$

where $\xi_n \in \partial c(x_n)$ is subgradient of c at x_n and

$$Q_i^n := \{y \in H_i : q_i(A_i x_n) \leq \langle \eta_i^n, A_i x_n - y \rangle\}, \tag{23}$$

where $\eta_i^n \in \partial q_i(A_i x_n)$. By the definition of the subgradient, it is easy to see that $C \subseteq C_n$ and $Q_i \subseteq Q_i^n$ (see [51]), and the metric projections onto C_n and Q_i^n can be directly calculated (since the projections onto C_n and Q_i^n have closed-form expressions), for example, for $\xi_n \in \partial c(x_n)$

$$P_{C_n}(x_n) = \begin{cases} x_n - \frac{c(x_n)}{\|\xi_n\|^2}\xi_n, & \text{if} \quad \xi_n \neq 0, \\ x_n, & \text{otherwise} . \end{cases}$$

Now, we present the following easily implementable algorithms.

3.1 Weak convergence theorems

In this subsection, we propose a new self-adaptive relaxed iterative method for solving the SFPMOS (12) in the infinite-dimensional Hilbert spaces, and we prove a weak convergence theorem of the proposed method.

Algorithm 1 Weakly convergent self-adaptive CQ algorithm for solving SFPMOPS.

Choose a constant $\beta > 0$ and three sequences $\{\rho_1^n\}, \{\rho_2^n\} \subset (0, 1)$ and $\{\vartheta_i\}_{i=1}^N > 0$. Select an arbitrary starting point $x_0 \in H$, and set $n = 0$. Given the current iterate $x_n \in H$. Compute the next iterate x_{n+1} via the rule

$$x_{n+1} = x_n - \rho_1^n \left(I - P_{C_n}\right) x_n - \tau_n \sum_{i=1}^N \vartheta_i A_i^* \left(I - P_{Q_i^n}\right) A_i x_n$$

where the step-size τ_n is updated self-adaptively as

$$\tau_n := \frac{\rho_2^n \sum_{i=1}^N \vartheta_i \left\| \left(I - P_{Q_i^n}\right) A_i x_n \right\|^2}{\bar{\tau}_n^2} \tag{24}$$

where

$$\bar{\tau}_n := \max \left\{ \left\| \sum_{i=1}^N \vartheta_i A_i^* \left(I - P_{Q_i^n}\right) A_i x_n \right\|, \beta \right\},$$

and C_n and Q_i^n are the half-spaces given as in (22) and (23), respectively.

Theorem 1 *Assume that the SFPMOS (12) is consistent (i.e., $\Omega \neq \emptyset$). Suppose the sequences $\{\rho_1^n\}$ and $\{\rho_2^n\}$ in Algorithm 1 are in $(0, 1)$ such that $0 < a_1 \leq \rho_1^n \leq b_1 < 1$ and $0 < a_2 \leq \rho_2^n \leq b_2 < 1$, respectively. Then, the sequence $\{x_n\}$ generated by Algorithm 1 converges weakly to a solution $p^* \in \Omega$ of the SFPMOS (12).*

Proof For convenience, we set the following notations first (for $i = 1, 2, \ldots, N$)

$$f^n_{C_n} := \left(I - P_{C_n}\right) x_n, \quad f^n_{Q^n_i} := \left(I - P_{Q^n_i}\right) A_i x_n. \tag{25}$$

Consequently, the step-size τ_n given by (24) can be written as

$$\tau_n := \frac{\rho^n_2 \sum_{i=1}^N \vartheta_i \| f^n_{Q^n_i} \|^2}{\bar{\tau}^2_n} \tag{26}$$

where

$$\bar{\tau}_n := \max\left\{ \| \sum_{i=1}^N \vartheta_i A^*_i f^n_{Q^n_i} \|, \ \beta \right\}.$$

Then, the iterative sequence $\{x_n\}$ in Algorithm 1 can be rewritten as follows:

$$x_{n+1} = x_n - \rho^n_1 f^n_{C_n} - \tau_n \sum_{i=1}^N \vartheta_i A^*_i f^n_{Q^n_i}. \tag{27}$$

Let $p^* \in \Omega$ (Ω is the solution set of the SFPMOS (12)). By (27), we have

$$\begin{aligned}
\|x_{n+1} - p^*\|^2 &= \|x_n - \rho^n_1 f^n_{C_n} - \tau_n \sum_{i=1}^N \vartheta_i A^*_i f^n_{Q^n_i} - p^*\|^2 \\
&= \|x_n - p^*\|^2 - 2\left\langle x_n - p^*, \rho^n_1 f^n_{C_n} + \tau_n \sum_{i=1}^N \vartheta_i A^*_i f^n_{Q^n_i} \right\rangle \\
&+ \ \|\rho^n_1 f^n_{C_n} + \tau_n \sum_{i=1}^N \vartheta_i A^*_i f^n_{Q^n_i} \|^2 \\
&= \|x_n - p^*\|^2 - 2\left\langle x_n - p^*, \rho^n_1 f^n_{C_n} \right\rangle - 2\left\langle x_n - p^*, \tau_n \sum_{i=1}^N \vartheta_i A^*_i f^n_{Q^n_i} \right\rangle \\
&+ \ \|\rho^n_1 f^n_{C_n}\|^2 + \|\tau_n \sum_{i=1}^N \vartheta_i A^*_i f^n_{Q^n_i}\|^2 + 2\left\langle \rho^n_1 f^n_{C_n}, \tau_n \sum_{i=1}^N \vartheta_i A^*_i f^n_{Q^n_i} \right\rangle \\
&\leq \|x_n - p^*\|^2 - 2\left\langle x_n - p^*, \rho^n_1 f^n_{C_n} \right\rangle - 2\left\langle x_n - p^*, \tau_n \sum_{i=1}^N \vartheta_i A^*_i f^n_{Q^n_i} \right\rangle \\
&+ \ \|\rho^n_1 f^n_{C_n}\|^2 + \|\tau_n \sum_{i=1}^N \vartheta_i A^*_i f^n_{Q^n_i}\|^2 + 2\|\rho^n_1 f^n_{C_n}\| \|\tau_n \sum_{i=1}^N \vartheta_i A^*_i f^n_{Q^n_i}\| \\
&\leq \|x_n - p^*\|^2 - 2\left\langle x_n - p^*, \rho^n_1 f^n_{C_n} \right\rangle - 2\left\langle x_n - p^*, \tau_n \sum_{i=1}^N \vartheta_i A^*_i f^n_{Q^n_i} \right\rangle \\
&+ \ 2\|\rho^n_1 f^n_{C_n}\|^2 + 2\tau^2_n \| \sum_{i=1}^N \vartheta_i A^*_i f^n_{Q^n_i}\|^2.
\end{aligned} \tag{28}$$

Using Lemma 1 (1), we obtain the following two estimations.

$$\begin{aligned}\left\langle x_n - p^*, \rho_1^n f_{C_n}^n \right\rangle &= \rho_1^n \left\langle x_n - p^*, f_{C_n}^n \right\rangle \\ &= \rho_1^n (\langle x_n - P_{C_n}(x_n), f_{C_n}^n \rangle + \langle P_{C_n}(x_n) - p^*, f_{C_n}^n \rangle) \\ &= \rho_1^n (\langle f_{C_n}^n, f_{C_n}^n \rangle + \langle P_{C_n}(x_n) - p^*, f_{C_n}^n \rangle) \\ &\geq \rho_1^n \| f_{C_n}^n \|^2. \end{aligned} \tag{29}$$

$$\begin{aligned}\left\langle x_n - p^*, \tau_n \sum_{i=1}^N \vartheta_i A_i^* f_{Q_i^n}^n \right\rangle &= \tau_n \sum_{i=1}^N \vartheta_i \left\langle x_n - p^*, A_i^* f_{Q_i^n}^n \right\rangle \\ &= \tau_n \sum_{i=1}^N \vartheta_i \left\langle A_i x_n - A_i p^*, f_{Q_i^n}^n \right\rangle \\ &= \tau_n \sum_{i=1}^N \vartheta_i \left(\left\langle f_{Q_i^n}^n, f_{Q_i^n}^n \right\rangle + \left\langle P_{Q_i^n}(A_i x_n) - A_i p^*, f_{Q_i^n}^n \right\rangle \right) \\ &\geq \tau_n \sum_{i=1}^N \vartheta_i \left\| f_{Q_i^n}^n \right\|^2. \end{aligned} \tag{30}$$

Substituting (29) and (30) into (28) and since $\| \sum_{i=1}^N \vartheta_i A_i^* f_{Q_i^n}^n \| \leq \bar{\tau}_n$, we obtain that

$$\begin{aligned}\|x_{n+1} - p^*\|^2 &\leq \|x_n - p^*\|^2 - 2\rho_1^n \|f_{C_n}^n\|^2 - 2\tau_n \sum_{i=1}^N \vartheta_i \left\| f_{Q_i^n}^n \right\|^2 \\ &\quad +2\|\rho_1^n f_{C_n}^n\|^2 + 2\tau_n^2 \| \sum_{i=1}^N \vartheta_i A_i^* f_{Q_i^n}^n \|^2 \\ &\leq \|x_n - p^*\|^2 - 2\rho_1^n \left(1 - \rho_1^n\right) \|f_{C_n}^n\|^2 - 2\tau_n \sum_{i=1}^N \vartheta_i \left\| f_{Q_i^n}^n \right\|^2 + 2\tau_n^2 \bar{\tau}_n^2 \\ &= \|x_n - p^*\|^2 - 2\rho_1^n \left(1 - \rho_1^n\right) \|f_{C_n}^n\|^2 \\ &\quad -2 \left(\frac{\rho_2^n \sum_{i=1}^N \vartheta_i \left\| f_{Q_i^n}^n \right\|^2}{\bar{\tau}_n^2} \right) \sum_{i=1}^N \vartheta_i \left\| f_{Q_i^n}^n \right\|^2 + 2 \left(\frac{\rho_2^n \sum_{i=1}^N \vartheta_i \left\| f_{Q_i^n}^n \right\|^2}{\bar{\tau}_n^2} \right)^2 \bar{\tau}_n^2 \\ &= \|x_n - p^*\|^2 - 2\rho_1^n \left(1 - \rho_1^n\right) \|f_{C_n}^n\|^2 - 2\rho_2^n \left(1 - \rho_2^n\right) \frac{\left(\sum_{i=1}^N \vartheta_i \left\| f_{Q_i^n}^n \right\|^2 \right)^2}{\bar{\tau}_n^2}. \end{aligned} \tag{31}$$

Since $0 < a_1 \leq \rho_1^n \leq b_1 < 1$ and $0 < a_2 \leq \rho_2^n \leq b_2 < 1$, we have from (31) that

$$\|x_{n+1} - p^*\|^2 \leq \|x_n - p^*\|^2.$$

Therefore, the sequence $\{x_n\}$ is Fejér-monotone with respect to Ω. As a consequence, $\lim_{n\to\infty} \|x_n - p^*\|$ exists. That is, $\{x_n\}$ is bounded, and hence the sequence $\{A_i x_n\}_{i=1}^N$ is also bounded.

Noticing that $\rho_1^n \in [a_1, b_1] \subset (0, 1)$, we can obtain from (31) that

$$\begin{aligned} 2a_1(1-b_1)\|f_{C_n}^n\|^2 &\le 2\rho_1^n\left(1-\rho_1^n\right)\|f_{C_n}^n\|^2 \\ &\le \|x_n - p^*\|^2 - \|x_{n+1} - p^*\|^2. \end{aligned} \tag{32}$$

Since $\{x_n\}$ is bounded and $f_{C_n}^n$ is 1-Lipschitz continuous, there exists a real number $R > 0$ such that $\|f_{C_n}^n\|^2 \le R$. Thus, we can obtain from (32) that

$$\lim_{n\to\infty} \|f_{C_n}^n\|^2 = 0. \tag{33}$$

Hence, we obtain from (33)

$$\lim_{n\to\infty} \|f_{C_n}^n\| = 0. \tag{34}$$

Noticing that $\rho_2^n \in [a_2, b_2] \subset (0, 1)$, we can obtain from (31) that

$$\begin{aligned} 2a_2(1-b_2)\frac{\left(\sum_{i=1}^N \vartheta_i \left\|f_{Q_i^n}^n\right\|^2\right)^2}{\bar{\tau}_n^2} &\le 2\rho_2^n\left(1-\rho_2^n\right)\frac{\left(\sum_{i=1}^N \vartheta_i \left\|f_{Q_i^n}^n\right\|^2\right)^2}{\bar{\tau}_n^2} \\ &\le \|x_n - p^*\|^2 - \|x_{n+1} - p^*\|^2. \end{aligned} \tag{35}$$

Letting $n \to \infty$ on both sides of (35), we have

$$\lim_{n\to\infty} \frac{\left(\sum_{i=1}^N \vartheta_i \left\|f_{Q_i^n}^n\right\|^2\right)^2}{\bar{\tau}_n^2} = 0. \tag{36}$$

Since the iterative sequence $\{x_n\}$ is bounded and by the Lipschitz continuity of $f_{Q_i^n}^n$, the sequence $\left\{\left\|\sum_{i=1}^N \vartheta_i A_i^* f_{Q_i^n}^n\right\|\right\}_{n=1}^\infty$ is bounded and so the sequence $\{\bar{\tau}_n\}$ is bounded too. Therefore, we can get from (36) that

$$\lim_{n\to\infty} \|f_{Q_i^n}^n\| = 0 \ \text{ for } i = 1, 2, \ldots, N. \tag{37}$$

Next, we will prove that $\omega_w(x_n) \subset \Omega$. For each $i = 1, 2, \ldots, N$, since ∂q_i is bounded on bounded sets, there exists a constant $\gamma > 0$ such that $\|\eta_i^n\| \le \gamma$, where $\eta_i^n \in \partial q_i(A_i x_n)$. Then, for $i = 1, 2, \ldots, N$, notice that $P_{Q_i^n}(A_i x_n) \in Q_i^n$, we have

$$\begin{aligned} q_i(A_i x_n) &\le \langle \eta_i^n, A_i x_n - P_{Q_i^n}(A_i x_n)\rangle \\ &\le \|\eta_i^n\|\|A_i x_n - P_{Q_i^n}(A_i x_n)\| \\ &\le \gamma\left\|\left(I - P_{Q_i^n}\right)A_i x_n\right\|. \end{aligned} \tag{38}$$

By (37), we have for any $i = 1, 2, \ldots, N$, that

$$\limsup_{n\to\infty} q_i(A_i x_n) \le 0. \tag{39}$$

Let $\hat{p} \in \omega_w(x_n)$, there exists a subsequence $\{x_{n_m}\} \subset \{x_n\}$ such that $x_{n_m} \rightharpoonup \hat{p}$ as $m \to \infty$. By the weak lower semicontinuity of the function q_i and (39), we get

$$q_i(A_i \hat{p}) \le \liminf_{m\to\infty} q_i(A_i x_{n_m}) \le 0, \tag{40}$$

which means that $A_i \hat{p} \in Q_i$ for $i = 1, 2, \ldots, N$.

Since ∂c is bounded, there exists a constant $\delta > 0$ such that $\|\xi_n\| \leq \delta$, where $\xi_n \in \partial c(x_n)$. Then, notice that $P_{C_n}(x_n) \in C_n$, we have

$$\begin{aligned} c(x_n) &\leq \langle \xi_n, x_n - P_{C_n}(x_n) \rangle \\ &\leq \|\xi_n\| \|x_n - P_{C_n}(x_n)\| \\ &\leq \delta \| \left(I - P_{C_n}\right) x_n \|. \end{aligned} \tag{41}$$

By (34), we have that

$$\limsup_{n\to\infty} c(x_n) \leq 0. \tag{42}$$

By the weak lower semicontinuity of the convex function c and (42), we obtain

$$c(\hat{p}) \leq \liminf_{m\to\infty} c(x_{n_m}) \leq 0. \tag{43}$$

Consequently, $\hat{p} \in C$. Therefore, $\hat{p} \in \Omega$.

Notice that for any $p^* \in \Omega$, $\lim_{n\to\infty} \|x_n - p^*\|$ exists and $\omega_w(x_n) \subset \Omega$. Therefore, applying Lemma 4, we conclude that the iterative sequence $\{x_n\}$ converges weakly to a solution of the SFPMOS (12). This completes the proof. □

For $N = 1$, we note the following iterative method for solving the SFP (1).

Algorithm 2 Weakly convergent self-adaptive CQ algorithm for solving SFP.

Choose a constant $\beta > 0$ and two real sequences $\{\rho_1^n\}, \{\rho_2^n\} \subset (0, 1)$. Select an arbitrary starting point $x_0 \in H_1$, and set $n = 0$. Given the current iterate $x_n \in H_1$. Compute the next iterate x_{n+1} via the rule

$$x_{n+1} = x_n - \rho_1^n \left(I - P_{C_n}\right) x_n - \tau_n A^*(I - P_{Q_n})Ax_n$$

where the step-size τ_n is updated self-adaptively as

$$\tau_n := \frac{\rho_2^n \|(I - P_{Q_n})Ax_n\|^2}{\bar{\tau}_n^2} \tag{44}$$

where

$$\bar{\tau}_n := \max\left\{ \left\| A^* \left(I - P_{Q_n}\right) Ax_n \right\|, \ \beta \right\},$$

and C_n and Q_n are the half-spaces given as in (4) and (5), respectively.

As an immediate consequence of Theorem 1, we obtain the following corollary.

Corollary 1 *Assume that the SFP (1) is consistent. Suppose the sequences $\{\rho_1^n\}$ and $\{\rho_2^n\}$ in Algorithm 2 are in $(0, 1)$ such that $0 < a_1 \leq \rho_1^n \leq b_1 < 1$ and $0 < a_2 \leq \rho_2^n \leq b_2 < 1$, respectively. Then, the sequence $\{x_n\}$ generated by Algorithm 2 converges weakly to a solution $p^* \in \Omega = \{p \in H_1 : p \in C \text{ such that } Ap \in Q\}$.*

3.2 Strong convergence theorem

In this subsection, we propose a new iterative method for solving the SFPMOS (12) in the infinite-dimensional Hilbert spaces, and we prove a strong convergence theorem of the proposed method.

Algorithm 3 Strongly convergent self-adaptive CQ algorithm for solving SFP-MOPS.

Choose a constant $\beta > 0$ and sequences $\{\rho_1^n\}$, $\{\rho_2^n\}$, $\{\alpha_n\} \subset (0, 1)$ and $\{\vartheta_i\}_{i=1}^N > 0$. Let $u \in H$ be a fixed point, take an arbitrary starting point $x_0 \in H$, and set $n = 0$. Given the current iterate $x_n \in H$. Compute the next iterate x_{n+1} via the rule

$$x_{n+1} = \alpha_n u + (1-\alpha_n)\left(x_n - \rho_1^n\left(I - P_{C_n}\right)x_n - \tau_n \sum_{i=1}^N \vartheta_i A_i^*(I - P_{Q_i^n})A_i x_n\right),$$

where the step-size τ_n is updated self-adaptively as

$$\tau_n := \frac{\rho_2^n \sum_{i=1}^N \vartheta_i \left\|\left(I - P_{Q_i^n}\right)A_i x_n\right\|^2}{\bar{\tau}_n^2} \tag{45}$$

where

$$\bar{\tau}_n := \max\left\{\left\|\sum_{i=1}^N \vartheta_i A_i^*(I - P_{Q_i^n})A_i x_n\right\|, \beta\right\},$$

and C_n and Q_i^n are the half-spaces given as in (22) and (23), respectively.

Theorem 2 *Assume that the SFPMOS (12) is consistent (i.e., $\Omega \neq \emptyset$). Suppose the sequences $\{\rho_1^n\}$, $\{\rho_2^n\}$, and $\{\alpha_n\}$ in Algorithm 3 are in $(0, 1)$ such that $0 < a_1 \leq \rho_1^n \leq b_1 < 1$ and $0 < a_2 \leq \rho_2^n \leq b_2 < 1$, and $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^\infty \alpha_n = \infty$. Then, the sequence $\{x_n\}$ generated by Algorithm 3 converges strongly to the point $p^* \in \Omega$, where $p^* = P_\Omega u$.*

Proof For simplicity, the same as we did in the proof of Theorem 1, we introduce some notations first.

$$f_{C_n}^n := \left(I - P_{C_n}\right)x_n, \quad f_{Q_i^n}^n := \left(I - P_{Q_i^n}\right)A_i x_n \ \text{ for } i = 1, 2, \ldots, N,$$

$$y_n = x_n - \rho_1^n f_{C_n}^n - \tau_n \sum_{i=1}^N \vartheta_i A_i^* f_{Q_i^n}^n,$$

where τ_n is the stepsize given in the Algorithm 3 and can be defined as

$$\tau_n := \frac{\rho_2^n \sum_{i=1}^N \vartheta_i \|f_{Q_i^n}^n\|^2}{\bar{\tau}_n^2} \tag{46}$$

where

$$\bar{\tau}_n := \max\left\{\left\|\sum_{i=1}^{N}\vartheta_i A_i^* f_{Q_i^n}^n\right\|,\ \beta\right\}.$$

Then, the iterative sequence $\{x_n\}$ in Algorithm 3 can be rewritten as follows:

$$x_{n+1} = \alpha_n u + (1-\alpha_n)y_n. \tag{47}$$

Let $p^* \in \Omega$. Using Lemma 2 (1) and by (47), we have that

$$\begin{aligned}
\|x_{n+1}-p^*\|^2 &= \|\alpha_n u + (1-\alpha_n)y_n - p^*\|^2\\
&= \|\alpha_n u + (1-\alpha_n)y_n - p^* + \alpha_n p^* - \alpha_n p^*\|^2\\
&= \|\alpha_n(u-p^*) + (1-\alpha_n)(y_n-p^*)\|^2\\
&\le (1-\alpha_n)^2\|y_n-p^*\|^2 + 2\langle\alpha_n(u-p^*),\ x_{n+1}-p^*\rangle\\
&\le (1-\alpha_n)\|y_n-p^*\|^2 + 2\alpha_n\langle u-p^*,\ x_{n+1}-p^*\rangle.
\end{aligned} \tag{48}$$

From (31), we have

$$\|y_n-p^*\|^2 \le \|x_n-p^*\|^2 - 2\rho_1^n\left(1-\rho_1^n\right)\|f_{C_n}^n\|^2 - 2\rho_2^n\left(1-\rho_2^n\right)\frac{\left(\sum_{i=1}^N\vartheta_i\left\|f_{Q_i^n}^n\right\|^2\right)^2}{\bar{\tau}_n^2}. \tag{49}$$

From (48) and (49), we obtain

$$\begin{aligned}
&\|x_{n+1}-p^*\|^2 \le (1-\alpha_n)\|x_n-p^*\|^2 + 2\alpha_n\langle u-p^*, x_{n+1}-p^*\rangle -\\
&(1-\alpha_n)\left[2\rho_1^n\left(1-\rho_1^n\right)\|f_{C_n}^n\|^2 + 2\rho_2^n\left(1-\rho_2^n\right)\frac{\left(\sum_{i=1}^N\vartheta_i\|f_{Q_i^n}^n\|^2\right)^2}{\bar{\tau}_n^2}\right].
\end{aligned} \tag{50}$$

Now, we prove the sequence $\{x_n\}$ is bounded. Indeed, using the assumptions imposed on $\{\rho_1^n\}$, $\{\rho_2^n\}$ and $\{\alpha_n\}$, we have from (50) that

$$\begin{aligned}
\|x_{n+1}-p^*\|^2 &\le (1-\alpha_n)\|x_n-p^*\|^2\| + 2\alpha_n\langle u-p^*, x_{n+1}-p^*\rangle\\
&\le (1-\alpha_n)\|x_n-p^*\|^2\| + 4\alpha_n\|u-p^*\|^2 + \frac{1}{4}\alpha_n\|x_{n+1}-p^*\|^2.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\|x_{n+1}-p^*\|^2 &\le \frac{1-\alpha_n}{1-\frac{1}{4}\alpha_n}\|x_n-p^*\|^2 + \frac{\frac{3}{4}\alpha_n}{1-\frac{1}{4}\alpha_n}\frac{16}{3}\|u-p^*\|^2\\
&\le \max\left\{\|x_n-p^*\|, \frac{16}{3}\|u-p^*\|\right\}\\
&\quad\vdots\\
&\le \max\left\{\|x_0-p^*\|, \frac{16}{3}\|u-p^*\|\right\}.
\end{aligned} \tag{51}$$

This shows that the sequence $\{x_n\}$ is bounded, and $\{y_n\}$ and $\{A_i x_n\}_{i=1}^N$ as well.

Next, with no loss of generality, we may assume that there exist $\sigma_1, \sigma_2 > 0$ such that $2\rho_1^n \left(1-\rho_1^n\right)(1-\alpha_n) \geq \sigma_1$ and $2\rho_2^n \left(1-\rho_2^n\right)(1-\alpha_n) \geq \sigma_2$ for all n.

Setting $s_n = \|x_n - p^*\|^2$, we get from (50) that

$$s_{n+1} - (1-\alpha_n)s_n + \sigma_1 \|f_{C_n}^n\|^2 + \frac{\sigma_2 \left(\sum_{i=1}^N \vartheta_i \|f_{Q_i^n}^n\|^2\right)^2}{\bar{\tau}_n^2} \leq 2\alpha_n \langle u - p^*, x_{n+1} - p^* \rangle \leq 2\alpha_n \|u - p^*\| \|x_{n+1} - p^*\|. \quad (52)$$

Now, we prove $s_n \to 0$ by distinguishing two cases.

Case 1: Assume that $\{s_n\}$ is eventually decreasing. That is, there exists $k \geq 0$ such that $s_{n+1} < s_n$ holds for all $n \geq k$. In this case, $\{s_n\}$ must be convergent, and from (52) it follows that

$$\left(\sigma_1 \|f_{C_n}^n\|^2 + \frac{\sigma_2 \left(\sum_{i=1}^N \vartheta_i \|f_{Q_i^n}^n\|^2\right)^2}{\bar{\tau}_n^2}\right) \leq \alpha_n K + (s_n - s_{n+1}), \quad (53)$$

where $K > 0$ is a constant such that $2\|u - p^*\| \|x_{n+1} - p^*\| \leq K$ for all $n \in \mathbb{N}$. Since $\sigma_1, \sigma_2 > 0$, and $\alpha_n \to 0$ as $n \to \infty$, we have from (53) that

$$\lim_{n\to\infty} \|f_{C_n}^n\|^2 = 0 \Rightarrow \lim_{n\to\infty} \|f_{C_n}^n\| = 0 \Rightarrow \lim_{n\to\infty} \|\left(I - P_{C_n}\right) x_n\| = 0, \quad (54)$$

and

$$\lim_{n\to\infty} \frac{\left(\sum_{i=1}^N \vartheta_i \left\|f_{Q_i^n}^n\right\|^2\right)^2}{\bar{\tau}_n^2} = 0. \quad (55)$$

Next, we show that $\left\{f_{Q_i^n}^n\right\} \to 0$. To do so, it suffices to verify that $\left\{\| \sum_{i=1}^N \vartheta_i A_i^* f_{Q_i^n}^n \|^2\right\}$ is bounded. Since $p^* \in \Omega$, we note that $A_i^* \left(I - P_{Q_i^n}\right) A_i p^* = 0$. Hence, it follows from Lemma 5 that

$$\|A_i^* \left(I - P_{Q_i^n}\right) A_i x_n - A_i^* \left(I - P_{Q_i^n}\right) A_i p^*\| \leq \left(\max_{1\leq i\leq N} \|A_i\|^2\right) \|x_n - p^*\| \quad (56)$$

and since $\{x_n\}$ is bounded, for all $i = 1, 2, \ldots, N$, we have the sequence $\left\{\left\|A_i^* \left(I - P_{Q_i^n}\right) A_i x_n\right\|\right\}_{n=1}^\infty$ is bounded. This implies that the sequence $\left\{\left\|\sum_{i=1}^N \vartheta_i A_i^* f_{Q_i^n}^n\right\|\right\}_{n=1}^\infty$ is also bounded. Consequently, $\{\bar{\tau}_n\}$ is bounded too. Therefore, we can get from (55) that

$$\lim_{n\to\infty} \|f_{Q_i^n}^n\| = 0 \Rightarrow \lim_{n\to\infty} \|\left(I - P_{Q_i^n}\right) A_i x_n\| = 0 \quad (57)$$

for i=1, 2, ..., N.

Next, we verify that $\omega_w(x_n) \subset \Omega$. For each $i = 1, 2, \ldots, N$, since ∂q_i is bounded on bounded sets, there exists a constant $\delta > 0$ such that $\|\eta_i^n\| \leq \delta$ for all $n \geq 0$, where $\eta_i^n \in \partial q_i(A_i x_n)$. Then, from the fact that $P_{Q_i^n}(A_i x_n) \in Q_i^n$ and (23), it follows (for $i = 1, 2, \ldots, N$) that

$$\begin{aligned} q_i(A_i x_n) &\leq \langle \eta_i^n, A_i x_n - P_{Q_i^n}(A_i x_n)\rangle \\ &\leq \|\eta_i^n\| \|A_i x_n - P_{Q_i^n}(A_i x_n)\| \\ &\leq \delta \| \left(I - P_{Q_i^n}\right) A_i x_n \|. \end{aligned} \tag{58}$$

Let $\hat{p} \in \omega_w(x_n)$, there exists a subsequence $\{x_{n_m}\}$ of $\{x_n\}$ such that $x_{n_m} \rightharpoonup \hat{p}$ as $m \to \infty$. Then, the weak lower semicontinuity of q_i and (58) imply that

$$q_i(A_i \hat{p}) \leq \liminf_{m\to\infty} q_i(A_i x_{n_m}) \leq 0. \tag{59}$$

It turns out that $A_i \hat{p} \in Q_i$ for $i = 1, 2, \ldots, N$. Next, we turn to prove that $\hat{p} \in C$. Since ∂c is bounded, there exists a constant $\gamma > 0$ such that $\|\xi_n\| \leq \gamma$ for all $n \geq 0$, where $\xi_n \in \partial c(x_n)$. Then, from that trivial fact that $P_{C_n}(x_n) \in C_n$ and (22), it follows that

$$\begin{aligned} c(x_n) &\leq \langle \xi_n, x_n - P_{C_n}(x_n)\rangle \\ &\leq \|\xi_n\| \|x_n - P_{C_n}(x_n)\| \\ &\leq \gamma \| \left(I - P_{C_n}\right) x_n \|. \end{aligned} \tag{60}$$

The weak lower semicontinuity of c then implies that

$$c(\hat{p}) \leq \liminf_{m\to\infty} c(x_{n_m}) \leq 0. \tag{61}$$

Consequently, $\hat{p} \in C$. Therefore, $\hat{p} \in \Omega$. Hence, $\omega_w(x_n) \subset \Omega$.

Moreover, we have the following estimation

$$\begin{aligned} \|x_n - x_{n+1}\| &= \left\| x_n - \left[\alpha_n u + (1-\alpha_n)\left(x_n - \rho_1^n f_{C_n}^n - \tau_n \sum_{i=1}^N \vartheta_i A_i^* f_{Q_i^n}^n\right)\right]\right\| \\ &= \left\| \alpha_n(x_n - u) + (1-\alpha_n)\left(\rho_1^n f_{C_n}^n + \tau_n \sum_{i=1}^N \vartheta_i A_i^* f_{Q_i^n}^n\right)\right\| \\ &\leq \alpha_n\|x_n - u\| + (1-\alpha_n)\left(\rho_1^n \|f_{C_n}^n\| + \frac{\rho_2^n \sum_{i=1}^N \vartheta_i \|f_{Q_i^n}^n\|^2}{\bar{\tau}_n}\right), \end{aligned}$$

since $\{x_n\}$ is bounded and $\lim_{n\to\infty} \alpha_n = 0$, we have that $\lim_{n\to\infty} \alpha_n\|x_n - u\| = 0$. Noting that $\{\|A_i^*(I - P_{Q_i^n})A_i x_n\|\}_{n=1}^{\infty}$ is bounded, (57) together with the conditions $0 < a_1 \leq \rho_1^n \leq b_1 < 1$ and $0 < a_2 \leq \rho_2^n \leq b_2 < 1$, we have that

$$\lim_{n\to\infty}(1-\alpha_n)\left(\rho_1^n \|f_{C_n}^n\| + \frac{\rho_2^n \sum_{i=1}^N \vartheta_i \|f_{Q_i^n}^n\|^2}{\bar{\tau}_n}\right) = 0,$$

which implies that

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{62}$$

Furthermore, due to Lemma 1 (1), we get

$$\limsup_{n\to\infty}\langle u-p^*, x_n-p^*\rangle = \max_{z\in\omega_w(x_n)}\langle u-P_\Omega(u), z-P_\Omega(u)\rangle \le 0. \tag{63}$$

Taking into account of (52), we have

$$s_{n+1} \le (1-\alpha_n)s_n + 2\alpha_n\langle u-p^*, x_{n+1}-p^*\rangle. \tag{64}$$

Applying Lemma 7 to (64), we obtain $s_n = \|x_n - p^*\|^2 \to 0$.

Case 2: Assume that $\{s_n\}$ is not eventually decreasing. That is, we can find an integer n_0 such that $s_{n_0} \le s_{n_0+1}$. Now we define

$$M_n := \{n_0 \le m \le n : s_m \le s_{m+1}\}, n > n_0. \tag{65}$$

It is easy to see that M_n is nonempty and satisfies $M_n \subseteq M_{n+1}$. Let

$$\phi(n) := \max M_n, n > n_0. \tag{66}$$

It is clear that $\phi(n) \to \infty$ as $n \to \infty$ (otherwise, $\{s_n\}$ is eventually decreasing). It is also clear that $s_{\phi(n)} \le s_{\phi(n)+1}$ for all $n > n_0$. Moreover,

$$s_n \le s_{\phi(n)+1},\ \ n > n_0. \tag{67}$$

In fact, if $\phi(n) = n$, then (67) is trivial: if $\phi(n) < n$, from (66), there exists some $j \in \mathbb{N}$ such that $\phi(n) + j = n$, we deduce that

$$s_n = s_{\phi(n)+j} < \cdots < s_{\phi(n)+2} < s_{\phi(n)+1}, \tag{68}$$

and (67) holds again. Since $s_{\phi(n)} < s_{\phi(n)+1}$ for all $n > n_0$, it follows from (53) that

$$\left(\sigma_1\|f_{C_{\phi(n)}}^{\phi(n)}\|^2 + \frac{\sigma_2(\sum_{i=1}^N \vartheta_i\|f_{Q_i^{\phi(n)}}^{\phi(n)}\|^2)^2}{\bar{\tau}_{\phi(n)}^2}\right) \le \alpha_{\phi(n)}K \to 0, \tag{69}$$

so that

$$\lim_{n\to\infty}\|f_{C_{\phi(n)}}^{\phi(n)}\| = 0 \Rightarrow \lim_{n\to\infty}\|(I-P_{C_{\phi(n)}})x_{\phi(n)}\| = 0, \tag{70}$$

and

$$\lim_{n\to\infty}\frac{(\sum_{i=1}^N \vartheta_i\|f_{Q_i^{\phi(n)}}^{\phi(n)}\|^2)^2}{\bar{\tau}_{\phi(n)}^2} = 0. \tag{71}$$

Noting that $\left\{\left\|A_i^*(I-P_{Q_i^{\phi(n)}})A_ix_{\phi(n)}\right\|\right\}_{n=1}^\infty$ is bounded, for $i = 1, 2, \ldots, N$, we also have that

$$\lim_{n\to\infty}\left\|f_{Q_i^{\phi(n)}}^{\phi(n)}\right\| = 0 \Rightarrow \lim_{n\to\infty}\left\|(I-P_{Q_i^{\phi(n)}})A_ix_{\phi(n)}\right\| = 0. \tag{72}$$

By the same argument to the proof in Case 1, we have $\omega_w(x_{\phi(n)}) \subset \Omega$.

Furthermore, by the same argument to the proof in Case 1, from (62), we have that

$$\lim_{n\to\infty}\|x_{\phi(n)} - x_{\phi(n)+1}\| = 0. \tag{73}$$

Thus, one can deduce that

$$\begin{aligned}\limsup_{n\to\infty}\langle u-p^*, x_{\phi(n)+1}-p^*\rangle &= \limsup_{n\to\infty}\langle u-p^*, x_{\phi(n)}-p^*\rangle \\ &= \max_{z\in\omega_w(x_{\phi(n)})}\langle u-P_\Omega(u), z-P_\Omega(u)\rangle \\ &\leq 0. \end{aligned} \tag{74}$$

Since $s_{\phi(n)} \leq s_{\phi(n)+1}$, it follows from (52) that

$$s_{\phi(n)} \leq 2\langle u-p^*, x_{\phi(n)+1}-p^* >, \; n > n_0. \tag{75}$$

(74) and (75) together gives

$$\limsup_{n\to\infty} s_{\phi(n)} \leq 0. \tag{76}$$

Hence, $\lim_{n\to\infty} s_{\phi(n)} = 0$, which together with (73)

$$\begin{aligned}\sqrt{s_{\phi(n)+1}} &\leq \|x_{\phi(n)+1}-p^*\| \\ &= \|(x_{\phi(n)}-p^*)+(x_{\phi(n)+1}-x_{\phi(n)})\| \\ &\leq \|x_{\phi(n)}-p^*\|+\|x_{\phi(n)+1}-x_{\phi(n)}\| \\ &= \sqrt{s_{\phi(n)}}+\|x_{\phi(n)+1}-x_{\phi(n)}\| \to 0, \end{aligned} \tag{77}$$

which, together with (67), in turn implies that $s_n \to 0$, that is, $x_n \to p^*$. Therefore, the full iterative sequence $\{x_n\}$ converges strongly to the solution $p^* = P_\Omega(u)$ of SFPMOS (12). This completes the proof. □

Corollary 2 *Assume that the SFPMOS (12) is consistent (i.e., $\Omega \neq \emptyset$). Let $x_0 \in H$ be an arbitrary initial point, and set $n = 0$. Let $\{x_n\}$ be a sequence generated via the manner*

$$x_{n+1} = \alpha_n x_0 + (1-\alpha_n)\left(x_n - \rho_1^n\left(I-P_{C_n}\right)x_n - \tau_n\sum_{i=1}^N \vartheta_i A_i^*\left(I-P_{Q_i^n}\right)A_i x_n\right)$$

s where the step-size τ_n is updated self-adaptively as

$$\tau_n := \frac{\rho_2^n\sum_{i=1}^N \vartheta_i\|(I-P_{Q_i^n})A_i x_n\|^2}{\bar{\tau}_n^2} \tag{78}$$

where for a constant $\beta > 0$

$$\bar{\tau}_n := \max\left\{\|\sum_{i=1}^N \vartheta_i A_i^*\left(I-P_{Q_i^n}\right)A_i x_n\|, \; \beta\right\},$$

and C_n and Q_i^n are the half-spaces given as in (22) and (23), respectively. Suppose the parameters $\{\rho_1^n\}, \{\rho_2^n\}, \{\alpha_n\}$ are in $(0, 1)$ satisfying the conditions in Theorem 2, and $\{\vartheta_i\}_{i=1}^N > 0$. Then, the sequence $\{x_n\}$ converges strongly to the point $p^* \in \Omega$, where $p^* = P_\Omega(x_0)$.

Similarly as in Subsection 3.1, for $N = 1$, we again obtain the following strongly convergent result regarding the SFP (1).

Algorithm 4 Strongly convergent self-adaptive CQ algorithm for solving SFP.

Choose a constant $\beta > 0$ and three real sequences $\{\rho_1^n\}, \{\rho_2^n\}, \{\alpha_n\} \subset (0, 1)$. Let $u \in H_1$ be a fixed point, take an arbitrary starting point $x_0 \in H_1$, and set $n = 0$. Given the current iterate $x_n \in H_1$. Compute the next iterate x_{n+1} via the rule

$$x_{n+1} = \alpha_n u + (1-\alpha_n)(x_n - \rho_1^n \left(I - P_{C_n}\right) x_n - \tau_n A^* \left(I - P_{Q_n})x_n\right),$$

where the step-size τ_n is updated self-adaptively as

$$\tau_n := \frac{\rho_2^n \left\|(I - P_{Q_n})Ax_n\right\|^2}{\bar{\tau}_n^2} \tag{79}$$

where

$$\bar{\tau}_n := \max\left\{\|A^* \left(I - P_{Q_n}\right) Ax_n\|,\ \beta\right\},$$

and C_n and Q_n are the half-spaces given as in (4) and (5), respectively.

As an immediate consequence of Theorem 2, we obtain the following corollary.

Corollary 3 *Assume that the SFP* (1) *is consistent. Suppose the sequences* $\{\rho_1^n\}$, $\{\rho_2^n\}$ *and* $\{\alpha_n\}$ *in Algorithm 4 are in* $(0, 1)$ *such that* $0 < a_1 \leq \rho_1^n \leq b_1 < 1$ *and* $0 < a_2 \leq \rho_2^n \leq b_2 < 1$, *and* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$. *Then, the sequence* $\{x_n\}$ *generated by Algorithm 4 converges strongly to the point* $p^* = P_\Omega(u) \in \Omega = \{p \in H_1 : p \in C \text{ such that } Ap \in Q\}$.

4 Application to the generalized split feasibility problem

In this section, we present an application of Theorems 1 and 2 for solving generalized split feasibility problem (another generalization of the SFP) in Hilbert spaces. We recall the generalized split feasibility problem first.

In 2020, Reich and Tuyen [52] first introduced and studied the following generalized split feasibility problem (GSFP).

Let H_i, $i = 1, 2, \ldots, N$, be real Hilbert spaces and C_i, $i = 1, 2, \ldots, N$, be closed and convex subsets of H_i, respectively. Let $B_i : H_i \to H_{i+1}$, $i = 1, 2, \ldots, N-1$, be bounded linear operators such that

$$S := C_1 \cap B_1^{-1}(C_2) \cap \cdots \cap B_1^{-1}\left(B_2^{-1} \ldots \left(B_{N-1}^{-1}(C_N)\right)\right) \neq \emptyset. \tag{80}$$

Given H_i, C_i and A_i as above, the generalized SFP (GSFP)([52]) is to

$$\text{find an element } p^* \in S. \tag{81}$$

That is $p^* \in C_1,\ B_1 p^* \in C_2, \ldots, B_{N-1}B_{N-2}\ldots B_1 p^* \in C_N$. In [52], Reich and Tuyen proved a strong convergence theorem for a modification of the CQ method which solves the GSFP (81). For more details on the GSFP (81), one can read the paper [52].

Remark 1 ([43, Remark 1.1]) Letting $H = H_1, C = C_1, Q_i = C_{i+1}, 1 \leq i \leq N-1$, $A_1 = B_1, A_2 = B_2 B_1, \ldots$, and $A_{N-1} = B_{N-1} B_{N-2} B_{N-3} \ldots B_2 B_1$, then the SFPMOS (12) becomes the GSFP (81).

From Theorem 1 and Remark 1, we note the following theorem for solving the GSFP (81).

Theorem 3 *Let $H = H_1, C = C_1, Q_i = C_{i+1}, 1 \leq i \leq N-1$, $A_1 = B_1, A_2 = B_2 B_1, \ldots$, and $A_{N-1} = B_{N-1} B_{N-2} B_{N-3} \ldots B_2 B_1$. Assume that the GSFP (81) is consistent (i.e., $S \neq \emptyset$). Let $x_0 \in C_1$ be an arbitrary initial point and set $n = 0$. Let $\{x_n\}$ be the sequence generated by*

$$x_{n+1} = x_n - \rho_1^n \left(I - P_{C_1^n}\right) x_n - \tau_n \sum_{i=1}^{N-1} \vartheta_i A_i^* \left(I - P_{C_{i+1}^n}\right) A_i x_n \tag{82}$$

where C_1^n and C_{i+1}^n are half-spaces of C_1 and C_{i+1} (at the nth iterate), respectively,

$$\tau_n := \frac{\rho_2^n \sum_{i=1}^{N-1} \vartheta_i \left\| \left(I - P_{C_{i+1}^n}\right) A_i x_n \right\|^2}{\bar{\tau}_n^2}$$

where for a constant $\beta > 0$

$$\bar{\tau}_n := \max \left\{ \left\| \sum_{i=1}^{N-1} \vartheta_i A_i^* \left(I - P_{C_{i+1}^n}\right) A_i x_n \right\|, \ \beta \right\},$$

and the sequences $\{\rho_1^n\}$, $\{\rho_2^n\} \subset (0, 1)$ such that $0 < a_1 \leq \rho_1^n \leq b_1 < 1$ and $0 < a_2 \leq \rho_2^n \leq b_2 < 1$, and the parameter $\{\vartheta_i\}_{i=1}^N > 0$. Then, the sequence $\{x_n\}$ generated by the iterative scheme (82) converges weakly to a solution $p^* \in S$.

Again, using Theorem 2 and Remark 1, we note the following result to solve the GSFP (81).

Theorem 4 *Let $H = H_1, C = C_1, Q_i = C_{i+1}, 1 \leq i \leq N-1$, $A_1 = B_1, A_2 = B_2 B_1, \ldots$, and $A_{N-1} = B_{N-1} B_{N-2} B_{N-3} \ldots B_2 B_1$. Assume that the GSFP (81) is consistent (i.e., $S \neq \emptyset$). Let $u \in C_1$ be a fixed point and $x_0 \in C_1$ is an arbitrary initial point, and set $n = 0$. Let $\{x_n\}$ be the sequence generated by*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) \left(x_n - \rho_1^n \left(I - P_{C_1^n}\right) x_n - \tau_n \sum_{i=1}^{N-1} \vartheta_i A_i^* \left(I - P_{C_{i+1}^n}\right) A_i x_n \right) \tag{83}$$

where C_1^n and C_{i+1}^n are half-spaces of C_1 and C_{i+1}, respectively,

$$\tau_n := \frac{\rho_2^n \sum_{i=1}^{N-1} \vartheta_i \left\| \left(I - P_{C_{i+1}^n}\right) A_i x_n \right\|^2}{\bar{\tau}_n^2}$$

where for a constant $\beta > 0$ and

$$\bar{\tau}_n := \max\left\{\left\|\sum_{i=1}^{N-1} \vartheta_i A_i^*\left(I - P_{C_{i+1}^n}\right) A_i x_n\right\|,\ \beta\right\},$$

the sequences $\{\rho_1^n\}$, $\{\rho_2^n\}$, $\{\alpha_n\} \subset (0, 1)$ such that $0 < a_1 \le \rho_1^n \le b_1 < 1,\ 0 < a_2 \le \rho_2^n \le b_2 < 1,\ \lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$, and the parameter $\{\vartheta_i\}_{i=1}^N > 0$. Then, the sequence $\{x_n\}$ generated by the iterative scheme (83) converges strongly to the solution $p^* \in S$, where $p^* = P_S(u)$.

5 Numerical results

In this section, we present some numerical examples to illustrate the implementation and efficiency of our proposed methods compared to some existing results by solving some problems. The numerical results are completed on a standard FUJITSUNOTEBOOK laptop with 11th Gen Intel(R) Core(TM) i7-1165G7 @ 2.80GHz 2.80 GHz with memory 16GB. The code is implemented in MATLAB R2022a. In our numerical experiments, Iter. (n) stands for the number of iterations and CPU(s) for the Elapsed time-run in seconds.

Example 1 ([43]) Consider $H = \mathbb{R}^{10}$, $H_1 = \mathbb{R}^{20}$, $H_2 = \mathbb{R}^{30}$ and $H_3 = \mathbb{R}^{40}$. Find a point $p^* \in \mathbb{R}^{10}$ such that

$$p^* \in \Omega := C \cap A_1^{-1}(Q_1) \cap A_2^{-1}(Q_2) \cap A_3^{-1}(Q_3) \neq \emptyset, \tag{84}$$

where the sets C and Q_i, and the linear bounded operators A_i are defined by

$$\begin{aligned} C &= \left\{x \in \mathbb{R}^{10} : \|x - \mathbf{c}\|^2 \le \mathbf{r}^2\right\}, \\ Q_1 &= \left\{A_1 x \in \mathbb{R}^{20} : \|A_1 x - \mathbf{c}_1\|^2 \le \mathbf{r}_1^2\right\}, \\ Q_2 &= \left\{A_2 x \in \mathbb{R}^{30} : \|A_2 x - \mathbf{c}_2\|^2 \le \mathbf{r}_2^2\right\}, \\ Q_3 &= \left\{A_3 x \in \mathbb{R}^{40} : \|A_3 x - \mathbf{c}_3\|^2 \le \mathbf{r}_3^2\right\}. \end{aligned} \tag{85}$$

where $\mathbf{c} \in \mathbb{R}^{10}$, $\mathbf{c}_1 \in \mathbb{R}^{20}$, $\mathbf{c}_2 \in \mathbb{R}^{30}$, $\mathbf{c}_3 \in \mathbb{R}^{40}$, $\mathbf{r}, \mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3 \in \mathbb{R}$, and $A_1 : \mathbb{R}^{10} \to \mathbb{R}^{20}$, $A_2 : \mathbb{R}^{10} \to \mathbb{R}^{30}$, $A_3 : \mathbb{R}^{10} \to \mathbb{R}^{40}$. In this case, for any $x \in \mathbb{R}^{10}$ we have $c(x) = \|x - \mathbf{c}\|^2 - \mathbf{r}^2$ and $q_i(A_i x) = \|A_i x - \mathbf{c}_i\|^2 - \mathbf{r}_i^2$ for $i = 1, 2, 3$. According to (22) and (23), the half-spaces C_n and Q_i^n $(i = 1, 2, 3)$, respectively of the sets C and Q_i are determined at a point x_n and $A_i x_n$, respectively as follows:

$$\begin{aligned} C_n &= \{x \in \mathbb{R}^{10} : \|x_n - \mathbf{c}\|^2 - \mathbf{r}^2 \le 2\langle x_n - \mathbf{c}, x_n - x\rangle\}, \\ Q_1^n &= \{y \in \mathbb{R}^{20} : \|A_1 x_n - \mathbf{c}_1\|^2 - \mathbf{r}_1^2 \le 2\langle A_1 x_n - \mathbf{c}_1, A_1 x_n - y\rangle\}, \\ Q_2^n &= \{y \in \mathbb{R}^{30} : \|A_2 x_n - \mathbf{c}_2\|^2 - \mathbf{r}_2^2 \le 2\langle A_2 x_n - \mathbf{c}_2, A_2 x_n - y\rangle\}, \\ Q_3^n &= \{y \in \mathbb{R}^{40} : \|A_3 x_n - \mathbf{c}_3\|^2 - \mathbf{r}_3^2 \le 2\langle A_3 x_n - \mathbf{c}_3, A_3 x_n - y\rangle\}. \end{aligned} \tag{86}$$

Then, the metric projections onto the half-spaces C_n and Q_i^n $(i = 1, 2, 3)$, can be easily calculated. The elements of the representing matrices A_i are randomly generated in the closed interval $[-5, 5]$, the coordinates of the centers $\mathbf{c}, \mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3$ are randomly generated in the closed interval $[-1, 1]$, and the radii $\mathbf{r}, \mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ are randomly generated in the closed intervals $[10, 20]$, $[20, 40]$, $[30, 60]$ and $[40, 80]$, respectively. For simplicity, denote $e_1 = (1, 1, \ldots, 1)^T \in \mathbb{R}^{10}$.

In this example, we examine the convergence of the sequence $\{x_n\}$ which is defined by Algorithms 1 and 3 by solving problem (84) compared to the recently introduced iterative methods for solving the SFPMOS (12) given by Scheme (13), Scheme (14), and with the following viscosity approximation an optimization approach method proposed by Reich et al. [53] for solving the SFPMOS (12). For any given point $x_0 \in H$, $\{x_n\}$ is a sequence generated by the iterative method

$$x_{n+1} := \alpha_n f(x_n) + (1 - \alpha_n) P_C \left(x_n - \lambda_n \sum_{i \in I(x_n)} \gamma_{i,n} A_i^* \left(I - P_{Q_i} \right) A_i x_n \right), \quad (87)$$

where $f : C \to C$ is a strict contraction mapping of H into itself with the contraction constant $\theta \in [0, 1)$, $\{\alpha_n\} \subset (0, 1)$, $I(x_n) = \left\{ i : \left\| A_i x_n - P_{Q_i} A_i x_n \right\| = \max_{i=1,2,\ldots,N} \left\| A_i x_n - P_{Q_i} A_i x_n \right\| \right\}$, $\gamma_{i,n} \geq 0$ for all $i \in I(x_n)$ with $\sum_{i \in I(x_n)} \gamma_{i,n} = 1$, and for $\{\rho_n\} \subset [\bar{a}, \bar{a}] \subset (0, 2)$ $\{\lambda_n\} \subset [0, \infty)$ such that

$$\lambda_n = \begin{cases} \rho_n \frac{(\max_{i=1,2,\ldots,N} \|A_i x_n - P_{Q_i} A_i x_n\|)^2}{\|\sum_{i \in I(x_n)} \gamma_{i,n} A_i^* (I - P_{Q_i}) A_i x_n\|^2}, & \text{if } \|\sum_{i \in I(x_n)} \gamma_{i,n} A_i^* (I - P_{Q_i}) A_i x_n\| > 0, \\ 0, & \text{otherwise}. \end{cases} \quad (88)$$

For comparison purpose, we consider the values of the parameters appeared in the methods as follows. For Algorithms 1 and 3, we take $\beta = 0.05$, $\rho_1^n = \frac{1}{10^4 n + 1} = \rho_2^n$ and $\vartheta_i = \frac{i}{12}$, $i = 1, 2, 3$. For Algorithm 3, Scheme (14), and Scheme (87) $\alpha_n = \frac{1}{n+1}$. For Scheme (13) and Scheme (14), we take $\lambda_n = 0.00005$. For Scheme (14) and Scheme (87), we take $f(x) = 0.975x$. Moreover, for Scheme (87), we take $\gamma_{1,n} = \frac{1}{6}$, $\gamma_{2,n} = \frac{1}{3}$, $\gamma_{3,n} = \frac{1}{2}$ and $\rho_n = \frac{1}{10^4 n + 1}$. Using $E_n = \|x_{n+1} - x_n\|^2 < 10^{-8}$ as stopping criteria, for different choices of the fixed point u and the initial point x_0, the results of numerical experiments are reported in Table 1 and Fig. 1.

It can be observed from Table 1 and Fig. 1 that for each choices of (u, x_0), our proposed methods Algorithms 1 and 3 have better performance interms of the iteration numbers (Iter. (n)) and comparatively the CPU-run time in seconds (CPU(s)) than of the compared methods. More precisely, Algorithms 1 and 3 have less number of iterations and take small CPU-time to run than of the iterative methods given by Scheme (13), Scheme (14), and Scheme (87).

Example 2 Let $H_1 = H_2 = L_2([0, 2\pi])$ with the inner product $\langle . \rangle$ defined by

$$\langle x, y \rangle = \int_0^{2\pi} x(t) y(t) dt, \quad \forall x, y \in L_2([0, 2\pi])$$

and with the norm $\|.\|$ defined by

$$\|x\|_2 := \sqrt{\int_0^{2\pi} |x(t)|^2 dt}, \quad \forall x, y \in L^2([0, 2\pi]).$$

Table 1 Comparison of Algorithms 1 and 3 with Scheme (13), Scheme (14), and Scheme (87) for different choices of u and x_0

(u, x_0)		Algorithm 1	Algorithm 3	Scheme (13)	Scheme (14)	Scheme (87)
	Iter. (n)	21	397	580	709	611
$(10e_1, 15e_1)$	CPU(s)	0.000814	0.007049	0.009757	0.012382	0.007107
	E_n	9.7443e-09	9.9995e-09	9.9612e-09	9.9995e-09	9.9689e-09
	Iter. (n)	24	548	654	662	571
$(10e_1, \frac{1}{2}e_1)$	CPU(s)	0.000873	0.007506	0.009706	0.011476	0.008185
	E_n	9.5908e-09	9.9589e-09	9.9869e-09	9.9834e-09	9.9686e-09
	Iter. (n)	49	297	393	543	470
$(3e_1, \frac{1}{5}e_1)$	CPU(s)	0.010865	0.019628	0.028973	0.036331	0.019807
	E_n	9.7579e-09	9.9626e-09	9.7810e-09	9.9688e-09	9.9801e-09
	Iter. (n)	45	318	441	685	590
$(3e_1, -\frac{1}{5}e_1)$	CPU(s)	0.001141	0.006589	0.008854	0.011453	0.007842
	E_n	9.9649e-09	9.9562e-09	9.9854e-09	9.9735e-09	9.9709e-09
	Iter. (n)	53	180	349	611	528
$(e_1, -\frac{1}{50}e_1)$	CPU(s)	0.001559	0.004430	0.008021	0.011801	0.005567
	E_n	9.8328e-09	9.8907e-09	9.8595e-09	9.9901e-09	9.9765e-09
	Iter. (n)	43	178	272	470	409
$(e_1, -\frac{1}{1000}e_1)$	CPU(s)	0.002600	0.003973	0.006847	0.009095	0.006656
	E_n	9.7746e-09	9.8789e-09	9.8990e-09	9.9887e-09	9.9616e-09

Furthermore, we consider the following half-spaces

$$C := \left\{x \in L_2([0, 2\pi]) : \int_0^{2\pi} x(t)dt \le 1\right\} \text{ and } Q := \left\{y \in L_2([0, 2\pi]) : \int_0^{2\pi} |y(t) - sin(t)|^2 dt \le 16\right\}.$$

In addition, we consider a linear continuous operator $A : L_2([0, 2\pi]) \to L_2([0, 2\pi])$, where $(Ax)(t) = x(t)$. Then, $(A^*x)(t) = x(t)$ and $\|A\| = 1$. That is, A is an identity operator. The metric projection onto C and Q have an explicit formula [54]. We can also write the projections onto C and the projections onto Q as follows:

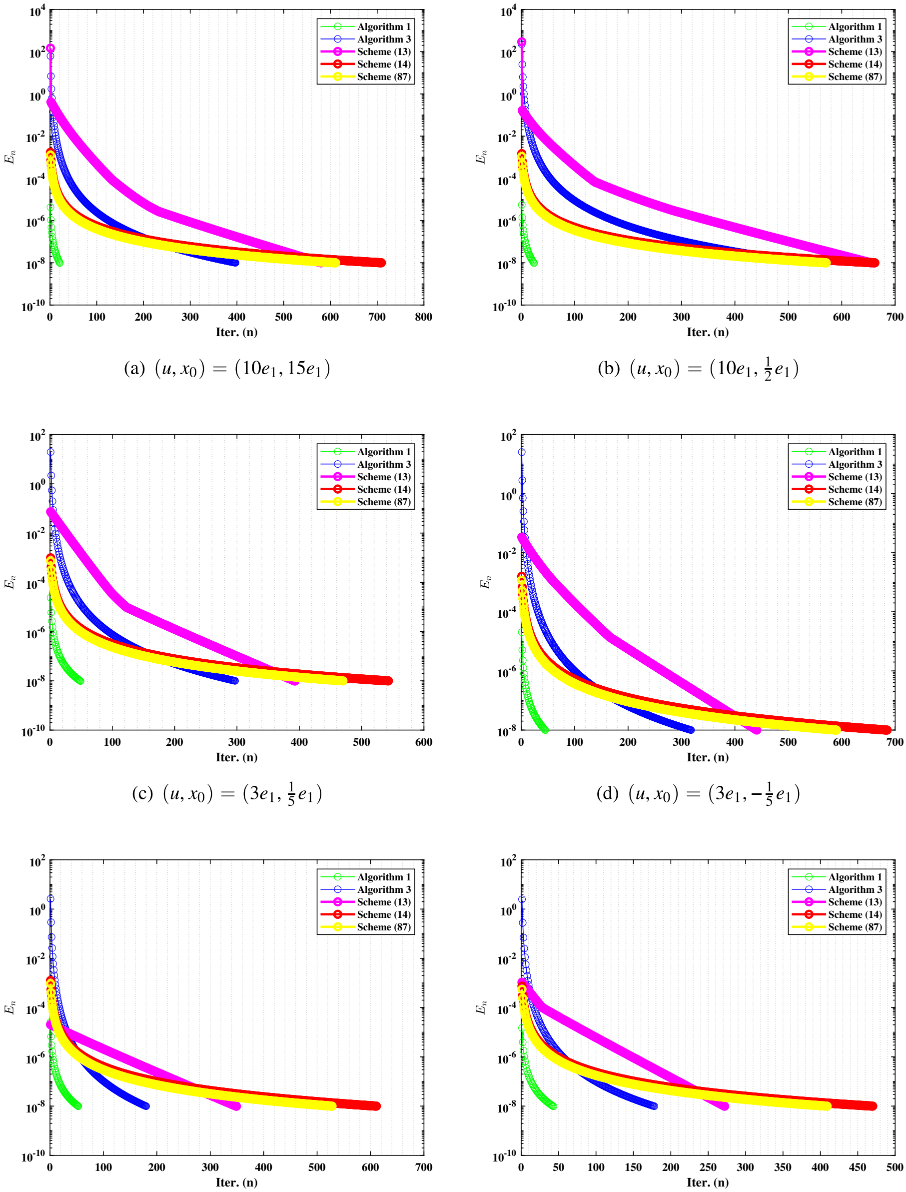
$$P_C(x(t)) = \begin{cases} x(t) + \frac{1-\int_0^{2\pi} x(t)dt}{4\pi^2}, & \text{if } \int_0^{2\pi} x(t)dt > 1, \\ x(t), & \text{if } \int_0^{2\pi} x(t)dt \le 1. \end{cases}$$

$$P_Q(y(t)) = \begin{cases} sin(t) + \frac{4(y(t)-sin(t))}{\sqrt{\int_0^{2\pi} |y(t)-sin(t)|^2 dt}}, & \text{if } \int_0^{2\pi} |y(t) - sin(t)|^2 dt > 16, \\ y(t), & \text{if } \int_0^{2\pi} |y(t) - sin(t)|^2 dt \le 16. \end{cases}$$

Now, we solve the following problem

$$\text{find } p^* \in C \text{ such that } Ap^* \in Q. \tag{89}$$

In this example, we examine the numerical behaviour of our proposed method: Algorithm 4 and compare it with the strongly convergent iterative algorithms given



(a) $(u, x_0) = (10e_1, 15e_1)$

(b) $(u, x_0) = (10e_1, \frac{1}{2}e_1)$

(c) $(u, x_0) = (3e_1, \frac{1}{5}e_1)$

(d) $(u, x_0) = (3e_1, -\frac{1}{5}e_1)$

(e) $(u, x_0) = (e_1, -\frac{1}{50}e_1)$

(f) $(u, x_0) = (e_1, -\frac{1}{1000}e_1)$

Fig. 1 Comparison of Algorithm 1 and Algorithm 3 with Scheme (13), Scheme (14), and Scheme (87) for different choices of u and x_0

by Scheme (10) and Scheme (11) by solving problem (89). For comparison purpose, we take the following data: For Algorithm 4, we take, $\beta = 0.05$, $\rho_1^n = \rho_2^n = \frac{n}{n+1}$ and $\alpha_n = \frac{1}{n+1}$. For Schemes (10) and (11), we take $\rho_n = \frac{n}{n+1}$ and $\alpha_n = \frac{1}{n+1}$.

Now, using $E_n = \|x_{n+1} - x_n\| < 10^{-4}$ as stopping criteria for all methods, for different choices of the fixed point u and the initial point x_0, the outcomes of the numerical experiments of the compared methods are reported in Table 2 and Fig. 2.

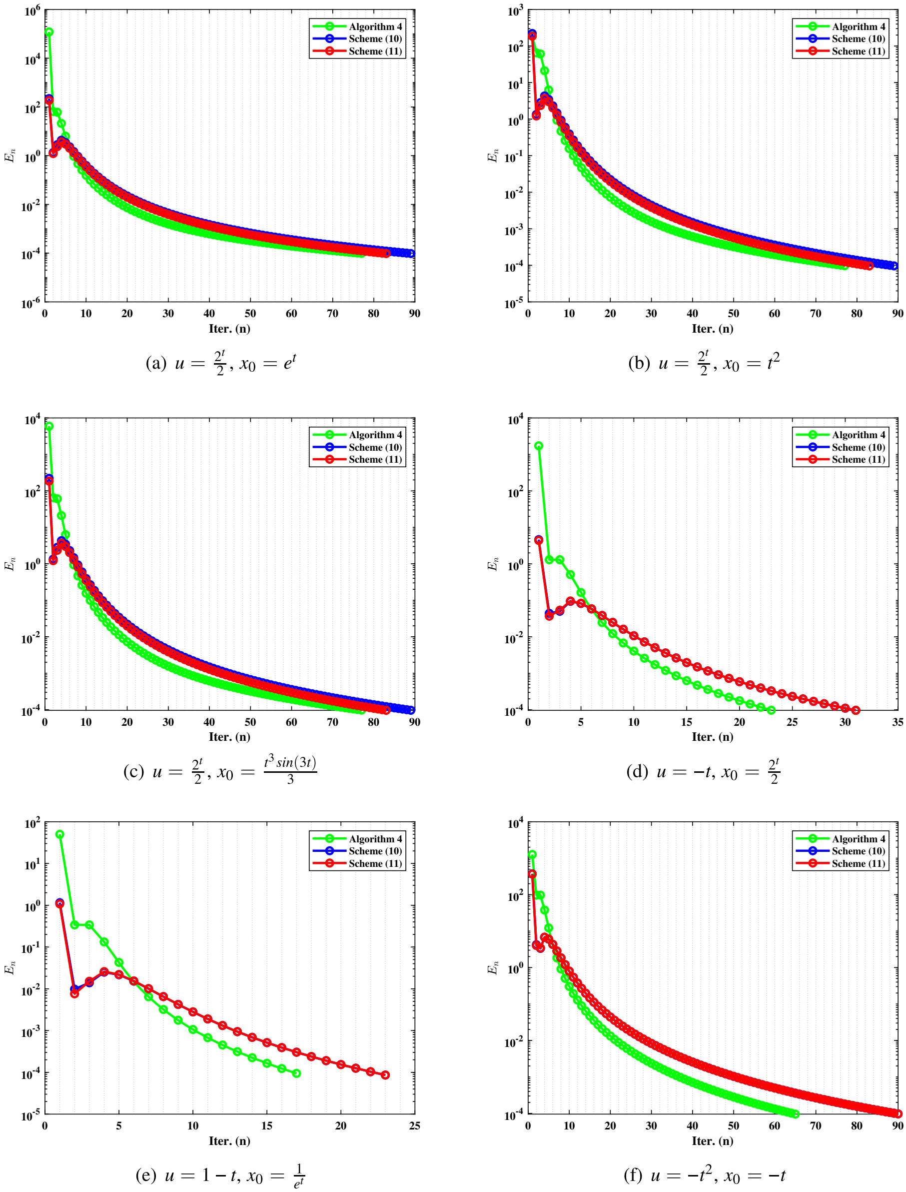
It can be observed from Table 2 and Fig. 2 that for each choices of u and x_0, Algorithm 4 is faster in terms of less number of iterations (Iter. (n)) and CPU-run time in seconds (CPU(s)) than the compared algorithms.

Example 3 The problem of computing sparse solutions (i.e., solutions where only a very small number of entries are nonzero) to linear inverse problems arises in a large number of application areas, for instance, in image restoration [55], channel equalization [56], echo cancellation [57], and stock market analysis [58]. The linear inverse problem consists of computing sparse solutions of a vector that has been digitized and has been degraded by an additive noise. Without loss of generality, for a vector $x \in H_1$ and an observed vector $y \in H_2$, a model including an additive noise can be written as

$$y = Ax + \eta,$$

Table 2 Comparison of Algorithm 4 with Scheme (10) and Scheme (11) for different choices of u and x_0

		Algorithm 4	Scheme (10)	Scheme (11)
$u = \frac{2^t}{2}, x_0 = e^t$	Iter. (n)	77	89	83
	CPU(s)	197.820341	258.683440	271.607658
	E_n	0.000097596	0.0000965296	0.000096184
$u = \frac{2^t}{2}, x_0 = t^2$	Iter. (n)	77	89	83
	CPU(s)	171.863335	195.831745	252.749699
	E_n	0.000097596	0.0000965297	0.000096184
$u = \frac{2^t}{2}, x_0 = \frac{t^3 sin(3t)}{3}$	Iter. (n)	77	89	83
	CPU(s)	154.680836	182.982568	290.567723
	E_n	0.000097596	0.0000965297	0.000096184
$u = -t, x_0 = \frac{2^t}{2}$	Iter. (n)	23	31	31
	CPU(s)	67.111117	97.401316	98.183831
	E_n	0.000097482	0.000096914	0.000096914
$u = 1 - t, x_0 = \frac{1}{e^t}$	Iter. (n)	17	23	23
	CPU(s)	33.258399	45.372658	45.323811
	E_n	0.000094479	0.000086201	0.0000862013
$u = -t^2, x_0 = -t$	Iter. (n)	65	90	90
	CPU(s)	128.514409	184.037896	297.695904
	E_n	0.000095464	0.000096829	0.000096829



(a) $u = \frac{2^t}{2}, x_0 = e^t$

(b) $u = \frac{2^t}{2}, x_0 = t^2$

(c) $u = \frac{2^t}{2}, x_0 = \frac{t^3 sin(3t)}{3}$

(d) $u = -t, x_0 = \frac{2^t}{2}$

(e) $u = 1 - t, x_0 = \frac{1}{e^t}$

(f) $u = -t^2, x_0 = -t$

Fig. 2 Comparison of Algorithm 4 with Scheme (10) and Scheme (11) for different choices of $u, \ x_0$

where A is a bounded linear operator between the two Hilbert spaces H_1 and H_2 and $\eta \in H_2$ denotes the additive noise.

Suppose that $H_1 = H_2 = L^2([0, 1])$ with norm $\|x\| := \left(\int_0^1 |x(t)|^2 dt\right)^{\frac{1}{2}}$ and inner product $\langle x, y\rangle := \int_0^1 x(t)y(t)dt, x, y \in L^2([0, 1])$. Define the Volterra integral operator $A : L^2([0,1]) \rightarrow L^2([0,1])$ by

$$Ax(t) := \int_0^t x(s)ds, \quad \forall x \in L^2([0,1]), t \in [0,1].$$

Then, A is bounded linear monotone and $\|A\| = \frac{2}{\pi}$ (see [59, Problem 188, p100; Solution 188, p300]). Using Algorithm 4, we develop an iterative algorithm to recover the solution of the linear equation $Ax = y - \eta$. Furthermore, we compare the performance of our proposed Algorithm 4 and Scheme (10).

We are interested in solutions $x^* \in \{x \in C : Ax \in Q\}$, where C is the cone of functions $x(t)$ that are negative for $t \in [0, 0.25]$ and positive for $t \in [0.25, 1]$ and $Q = [a(t), b(t)] := \{y(t) : a(t) \leq y(t) \leq b(t), 0 \leq t \leq 1\}$ is a box delimited by the functions $a(t)$ and $b(t)$. The metric projection P_Q can be computed by formula:

$$P_Q(y) := max\{a, min\{y, b\}\}.$$

For some problems, the solution is almost sparse. To ensure the existence of the solution of the consider problem, K-sparse vector $x^*(t)$ is generated randomly in C. Taking $y(t) = Ax^*(t)$ and $a(t) = y(t) - 0.01$, $b(t) = y(t) + 0.01$, we have $Q = [a(t), b(t)]$. We take $K = 30$, $K = 55$ and $K = 70$ (see Fig. 3). The problem of interest is to find $x \in C$ such that $Ax \in Q$.

We compare the behavior of Algorithm 4 and Scheme (10) for the same initial point $x_0 = e^{4t^2}$ and same fixed point $u = t^2$. Set $\rho_1^n = \frac{2n}{3n+1} = \rho_2^n$ and $\alpha_n = \frac{1}{n}$ in Algorithm 4 and $\rho_n = \frac{2n}{3n+1}$ and $\alpha_n = \frac{1}{n}$ in Scheme (10). In the implementation, we

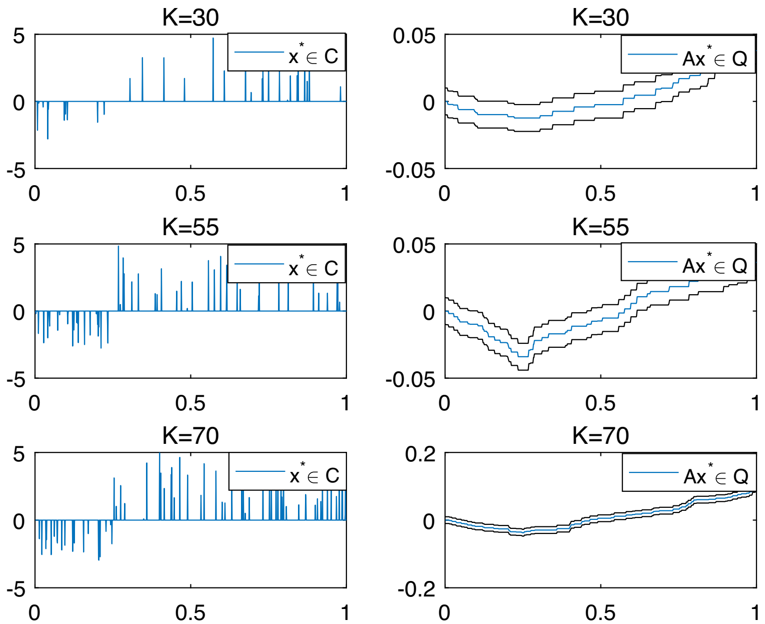


Fig. 3 $x^* \in C$ (left) and $Ax^* \in Q$ (right) for $K = 30, 55, 70$

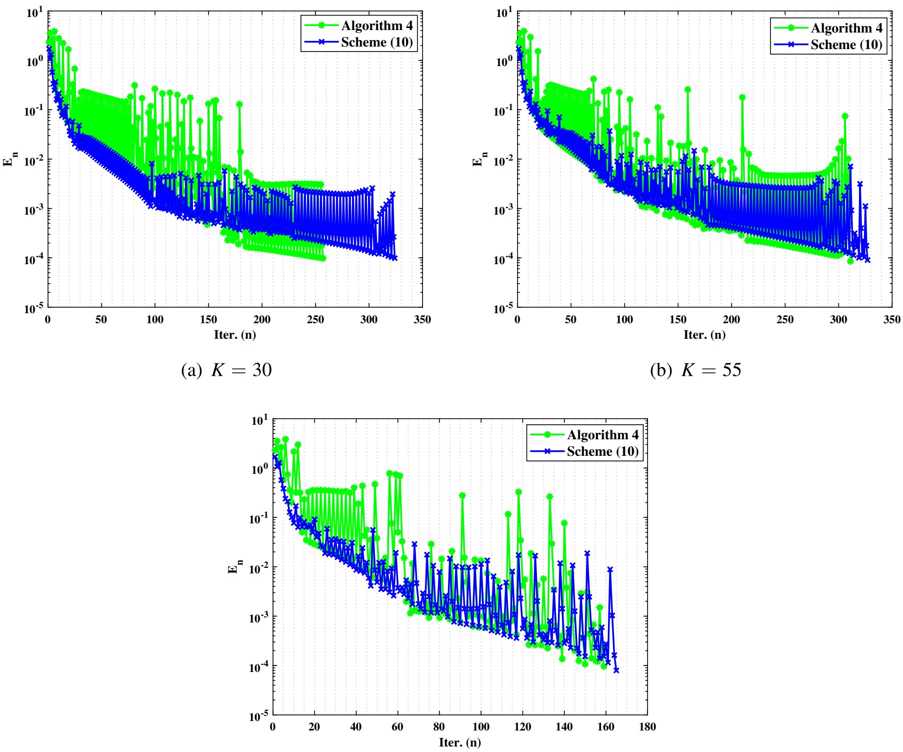
Table 3 Numerical results with different K-sparse

	Algorithm 4		Scheme (10)	
	Iter. (n)	CPU(s)	Iter. (n)	CPU(s)
$K = 30$	259	4.0322	324	9.0005
$K = 55$	311	5.0901	327	10.2111
$K = 70$	159	2.5093	165	5.1393

take $E_n < \varepsilon = 10^{-4}$ as the stopping criterion, where

$$E_n = \|x - P_C x\|^2 + \|Ax - P_Q Ax\|^2.$$

In Table 3, we present our numerical results with different K-sparse ($K = 30$, 55, 70). Table 3 shows the number of iterations and the time of execution in seconds (CPU(s)) of Algorithm 4 and Scheme (10). In Fig. 4, we report the behavior of Algorithm 4 and Scheme (10) for $K = 30, 55, 70$. Furthermore, Fig. 4 presents error value



(a) $K = 30$ (b) $K = 55$ (c) $K = 70$

Fig. 4 Number of iterations and error estimate for Algorithm 4 and Scheme (10) versus the iteration numbers. It can be seen that Algorithm 4 is significantly faster than Scheme (10). This shows the effectiveness of our proposed algorithms.

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Availability of data and material Not applicable.

Code availability Not available.

Declarations

Competing interests The authors declare no competing interests.

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