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ORIGINAL PAPER

Analysis of fractal dimension of mixed Riemann-Liouville integral

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Abstract
In this article, we provide a rigorous study on the fractal dimension of the graph of the mixed Riemann-Liouville fractional integral for various choices of continuous functions on a rectangular region. We estimate bounds for the box dimension and the Hausdorff dimension of the graph of the mixed Riemann-Liouville fractional integral of the functions which belong to the class of continuous functions and the class of Hölder continuous functions. We also show that the box dimension of the graph of the mixed Riemann-Liouville fractional integral of two-dimensional continuous functions is also two. Furthermore, we give the construction of unbounded variational continuous functions. Later, we prove that the box dimension and the Hausdorff dimension of the graph of the mixed Riemann-Liouville fractional integral of unbounded variational continuous functions are two. Moreover, we illustrate our results by using some examples.

Keywords Box dimension · Hausdorff dimension · Riemann-Liouville fractional integral · Hölder condition · Bounded variation

Mathematics Subject Classification (2010) 26A33 · 28A80 · 28A78 · 26A30

1 Introduction

Fractional calculus (FC) and fractal geometry (FG) have become rapidly growing fields in theory as well as applications. In the past, mathematics was primarily concerned with sets and functions on which classical calculus methods could be

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applied, and the study of irregular and non-smooth sets or functions has been ignored. Although irregular sets are much better at representing certain natural phenomena than the figures of classical geometry do. FG provides a broad context for studying such irregular sets. Since the last few decades, several researchers have been fascinated by the graph of a function, its Hausdorff dimension, and box dimension. The study of dimensions of graphs began with Weierstrass type functions. Readers may encourage to see [5, 11, 21], for the details of Hausdorff dimension and the box dimension of Weierstrass type functions. We refer the books [2] and [8] on FG, for more details. FC deals with the concept of non-integer order differentiation and integration and it is as old as classical calculus. In FC, generally, fractional derivatives are represented in terms of fractional integrals, for instance, we refer to [17, 18, 20]. Since random fractals are better examples of irregular functions and for analyzing such functions, FC is the best mathematical operator. We have also seen by using some examples and graphs that it smoothen the irregular functions. Nowadays researchers are very much interested in the fractal dimension of graph of fractional integrals and derivatives. A connection between FC and fractal dimension can be seen in [13–16, 19, 22, 24–26]. In the smoothness analysis of any irregular function, the box dimension plays an important role. Now, we will look over some of the available results on fractional calculus and fractal dimension. A linear relation between the order of the fractional integral of Riemann-Liouville (R-L) type and fractal dimension such as box dimension, K-dimension, Packing dimension is given in [25]. Liang [16] investigated the box dimension of the graph of the fractional integral of R-L type corresponding to a function having box dimension one. We know that in the study of rectifiable curves and integrals, the bounded variation property of any function plays a significant role. An important result on box dimension of a function which is of bounded variation and continuous is given in [14]. In [14], Liang proved that if $f \in C([0,1])$ and of bounded variation on $[0,1]$, then $\dim_B Gr(f,[0,1]) = 1$, and $\dim_B Gr(\mathcal{I}^{\nu} f,[0,1]) = 1$, where

$$\mathcal{I}^{\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_0^x (x-s)^{\nu-1} f(s)ds,$$

is the fractional integral of R-L type. Now, we are interested in the notions of bounded variation for several variables and we will see that how these notions play an important role in the study of fractal dimension of the graph of the fractional integral of mixed R-L type. Clarkson and Adams introduced the new notions of bounded variation such as Hahn, Peirpont, and Arzelá in [6] and related properties are given in [1]. Using the bounded variation property in Arzelá sense, Verma and Viswanathan established the results for the fractional integral of mixed R-L type in [23]. Additionally, they proved that if $f \in C([a,b]\times[c,d])$ and f is of bounded variation in sense of Arzelá on $[a,b]\times[c,d]$, then $\dim_B Gr(f,[a,b]\times[c,d]) = 2$, and $\dim_B Gr(\mathcal{I}^{\gamma} f,[a,b]\times[c,d]) = 2$, where

$$\mathcal{I}^{\gamma} f(x,y) = \frac{1}{\Gamma(\gamma_1)\Gamma(\gamma_2)} \int_a^x \int_c^y (x-u)^{\gamma_1-1}(y-v)^{\gamma_2-1} f(u,v)dudv,$$

with $\gamma = (\gamma_1,\gamma_2)$; $\gamma_1 > 0, \gamma_2 > 0$, is the fractional integral of mixed R-L type. Although some examples can be found of two-dimensional continuous functions which are not of bounded variation in [23]. Barnsley [3] developed the theory of fractal interpolation functions (FIFs). Ruan et al. [19] introduced a new method to compute the box dimension of linear fractal interpolation functions (FIFs) and by using this they have established a relation between the order of fractional integral of R-L type and box dimension of two linear FIFs. This is also motivation of this work and we try to estimate box dimension of bivariate case of fractional integral in other aspects. Feng [9] studied some properties of the variation and oscillation of bivariate continuous functions. Also, he investigated the Minkowski dimension of the fractal interpolation surface (FIS). Feng and Sun introduced a new construction method of FIS by considering arbitrary interpolation nodes in [10], and they estimated the box dimension of FIS. The generalized notion of FIS and its exact box dimension can be seen in [4]. We proved that the fractional integral of mixed R-L type of FIS is again FIS in [7].

From the above discussion, it is natural to arise the following questions:

(i) What is the bounds of the box dimension and the Hausdorff dimension of the graph of $\mathcal{I}^{\gamma} f$ when $f \in C(I \times J)$, where $C(I \times J)$ denotes the set of all continuous functions on $I \times J$.
(ii) What is the bounds of the box dimension and the Hausdorff dimension of the graph of $\mathcal{I}^{\gamma} f$ when $f \in H^{\mu}(I \times J)$, where $H^{\mu}(I \times J)$ denotes the set of all μ-Hölder continuous functions on $I \times J$.
(iii) What is the box dimension and the Hausdorff dimension of the graph of $\mathcal{I}^{\gamma} f$ when f is unbounded variational continuous function.
(iv) What is the box dimension of the graph of $\mathcal{I}^{\gamma} f$ when f is two-dimensional continuous function.

Above Questions (i), (ii) and (iii) are based on analytical aspects in the sense that we are using fundamental properties of function f. Question (iv) is based on dimensional aspects in the sense that we are using the dimension of function f to compute the dimension of the graph of $\mathcal{I}^{\gamma} f$. In this work, we investigate the above-mentioned points and give a detailed analysis. Moreover, we also analyze both analytical and dimensional aspects. To the best of our knowledge, not much literature is available on the fractional integral of mixed R-L type. Our results are new for the fractional integral of mixed R-L type and improve the exiting results as well.

This article is arranged as follows: Definitions of the mixed R-L fractional integral, box dimension, Hausdorff dimension and other basic terminologies are given in Section 2. In Sections 3 and 4, we provide bounds for the box dimension and the Hausdorff dimension of the graph of the fractional integral of mixed R-L type of various choice of functions. In Section 5, we estimate the box dimension of the graph of the fractional integral of mixed R-L type of a continuous function having box dimension two. Section 6 is devoted to the construction of unbounded variational continuous function and the fractal dimensions of its fractional integral of mixed R-L type. Section 7 is devoted to the graphical representation of the fractional integral of mixed R-L type. In Section 8, some open problems are formulated.

2 Preliminaries

Let us recall basic definitions and other terminologies which act as a prelude to this article.

Definition 2.1 [20] Let a function f which is defined on a closed rectangle $[a, b] \times [c, d]$ and $a \geq 0$, $c \geq 0$. Assuming that the following integral exists, mixed Riemann-Liouville fractional integral of f is defined by

$$\mathcal{I}^{\gamma} f(x, y) = \frac{1}{\Gamma(\gamma_1)\Gamma(\gamma_2)} \int_a^x \int_c^y (x-u)^{\gamma_1 - 1}(y-v)^{\gamma_2 - 1} f(u, v) du dv,$$

where $\gamma = (\gamma_1, \gamma_2)$ with $\gamma_1 > 0, \gamma_2 > 0$.

Definition 2.2 [8] Let $E \neq \emptyset$ be a bounded subset of $\mathbb{R}^n$. Let the smallest number of sets which can cover E is denoted by $N_\delta(E)$ having diameter at most δ. Then

$$\underline{\dim}_B(E) = \underline{\lim}_{\delta \to 0} \frac{\log N_\delta(E)}{-\log \delta} \quad \text{(Lower box dimension)} \tag{2.1}$$

and

$$\overline{\dim}_B(E) = \overline{\lim}_{\delta \to 0} \frac{\log N_\delta(E)}{-\log \delta} \quad \text{(Upper box dimension).} \tag{2.2}$$

If $\underline{\dim}_B(E) = \overline{\dim}_B(E)$, the common value is called the box dimension of E. That is,

$$\dim_B(E) = \lim_{\delta \to 0} \frac{\log N_\delta(E)}{-\log \delta}.$$

For the definition of the fractal dimensions, the reader may follow [8].

Definition 2.3 For a function $f : A =: [a, b] \times [c, d] \to \mathbb{R}$, the maximum range of f over A is defined by

$$R_f[A] := \sup_{(t_1, t_2), (x, y) \in A} |f(t_1, t_2) - (x, y)|.$$

Lemma 2.4 [23] *Let $f \in C(I \times J)$ and*

$$|f(z_1, t_1) - f(z_2, t_2)| \leq C \|(z_1, t_1) - (z_2, t_2)\|_2^{\mu}, \quad \forall (z_1, t_1), (z_2, t_2) \in I \times J, \tag{2.3}$$

for $C > 0$ and $0 \leq \mu \leq 1$. Then $2 \leq \dim_H Gr(f, I \times J) \leq \overline{\dim}_B Gr(f, I \times J) \leq 3 - \mu$. This remains true if (2.3) (Hölder condition) holds with $\|(z_1, t_1) - (z_2, t_2)\|_2 < \delta$ for some $\delta > 0$.
If $\mu = 1$, then f is called Lipschitz continuous.

For $0 < \mu < 1$, we can rewrite Lemma 2.4 as below:

Lemma 2.5 *For $0 < \mu < 1$ and $C > 0$, let*

$$H^{\mu}(I \times J) = \{f(x, y) : |f(x + k_1, y + k_2) - f(x, y)| \leq C \|(k_1, k_2)\|_2^{\mu}, \\ \forall\, (x + k_1, y + k_2), (x, y) \in I \times J\}.$$

If $f \in C(I \times J)$ *and belongs to* $H^{\mu}(I \times J)$*, then*

$$2 \leq \dim_H Gr(f, I \times J) \leq \overline{\dim}_B Gr(f, I \times J) \leq 3 - \mu.$$

The reader may refer [1] for the definition of bounded variation in Arzelá sense.

Theorem 2.1 [1] *(Necessary and sufficient condition)*
A function $g : [a, b] \times [c, d] \rightarrow \mathbb{R}$ *is said to be of bounded variation in the sense of Arzelá if it can be written in the difference of two bounded functions* g_1 *and* g_2 *satisfying the inequities*

$$\Delta_{10} g_i(x, y) \geq 0, \quad \Delta_{01} g_i(x, y) \geq 0, \quad i = 1, 2,$$

where $\Delta_{10} g(x_i, y_j) = g(x_{i+1}, y_j) - g(x_i, y_j)$, $\Delta_{01} g(x_i, y_j) = g(x_i, y_{j+1}) - g(x_i, y_j)$.

Following notations are also used in this article: $Gr(f)$ represents the graph of f. $I \times J = [a, b] \times [c, d]$. C is absolute constant and it may have different values even in the same line at different occurrence. Sometimes, we use the abbreviation "the fractional integral of mixed R-L type" in the place of "the mixed Riemann-Liouville fractional integral".

3 Fractal dimensions of $\mathcal{I}^{\gamma} f(x, y)$ with $f(x, y) \in C(I \times J)$

In this section, we establish the bounds for the fractal dimensions of the fractional integral of mixed R-L type corresponding to the continuous functions.

Theorem 3.1 *For* $0 < a < b < \infty$, $0 < c < d < \infty$ *and* $0 < \gamma_1, \gamma_2 < 1$. *If* $f : [a, b] \times [c, d] \rightarrow \mathbb{R}$ *is continuous, then*

$$\dim_H Gr(\mathcal{I}^{\gamma} f, I \times J) \leq \overline{\dim}_B Gr(\mathcal{I}^{\gamma} f, I \times J) \leq 3 - \min\{\gamma_1, \gamma_2\}.$$

Proof Let $0 < a \leq x < x + k_1 \leq b$; $0 < c \leq y < y + k_2 \leq d$. Then

$$\begin{aligned} &(\mathcal{I}^{\gamma} f)(x + k_1, y + k_2) - (\mathcal{I}^{\gamma} f)(x, y) \\ &= \frac{1}{\Gamma(\gamma_1)\Gamma(\gamma_2)} \int_a^{x+k_1} \int_c^{y+k_2} (x + k_1 - u)^{\gamma_1 - 1} (y + k_2 - v)^{\gamma_2 - 1} f(u, v) du dv \\ &- \frac{1}{\Gamma(\gamma_1)\Gamma(\gamma_2)} \int_a^{x} \int_c^{y} (x - u)^{\gamma_1 - 1} (y - v)^{\gamma_2 - 1} f(u, v) du dv = L_1 + L_2 + L_3 + L_4, \end{aligned}$$

where

$$L_1 = \frac{1}{\Gamma(\gamma_1)\Gamma(\gamma_2)} \int_a^x \int_c^y \left[(x + k_1 - u)^{\gamma_1 - 1} (y + k_2 - v)^{\gamma_2 - 1} - (x - u)^{\gamma_1 - 1} (y - v)^{\gamma_2 - 1} \right] f(u, v) du dv$$
$$L_2 = \frac{1}{\Gamma(\gamma_1)\Gamma(\gamma_2)} \int_a^x \int_y^{y+k_2} (x + k_1 - u)^{\gamma_1 - 1} (y + k_2 - v)^{\gamma_2 - 1} f(u, v) du dv$$
$$L_3 = \frac{1}{\Gamma(\gamma_1)\Gamma(\gamma_2)} \int_x^{x+k_1} \int_c^y (x + k_1 - u)^{\gamma_1 - 1} (y + k_2 - v)^{\gamma_2 - 1} f(u, v) du dv$$
$$L_4 = \frac{1}{\Gamma(\gamma_1)\Gamma(\gamma_2)} \int_x^{x+k_1} \int_y^{y+k_2} (x + k_1 - u)^{\gamma_1 - 1} (y + k_2 - v)^{\gamma_2 - 1} f(u, v) du dv.$$

Because of continuity of f on $[a, b] \times [c, d]$, there exists M such that $|f(t_1, t_2)| \leq M \; \forall (t_1, t_2) \in [a, b] \times [c, d]$.
Now, we estimate the bound for L_1 as below:

$$|L_1| \leq \frac{1}{\Gamma(\gamma_1)\Gamma(\gamma_2)} \int_a^x \int_c^y \left[(x - u)^{\gamma_1 - 1} (y - v)^{\gamma_2 - 1} - (x + k_1 - u)^{\gamma_1 - 1} (y + k_2 - v)^{\gamma_2 - 1} \right] |$$
$$f(u, v)| du dv \leq \frac{M}{\Gamma(\gamma_1)\Gamma(\gamma_2)} \int_a^x \int_c^y \left[(x - u)^{\gamma_1 - 1} (y - v)^{\gamma_2 - 1} - (x + k_1 - u)^{\gamma_1 - 1} (y + k_2 - v)^{\gamma_2 - 1} \right]$$
$$du dv = \frac{M}{\Gamma(\gamma_1)\Gamma(\gamma_2)} \int_a^x \int_c^y \Big[(x - u)^{\gamma_1 - 1} (y - v)^{\gamma_2 - 1} - (x + k_1 - u)^{\gamma_1 - 1} (y - v)^{\gamma_2 - 1}$$
$$+ (x + k_1 - u)^{\gamma_1 - 1} (y - v)^{\gamma_2 - 1} - (x + k_1 - u)^{\gamma_1 - 1} (y + k_2 - v)^{\gamma_2 - 1} \Big]$$
$$du dv = \frac{M}{\Gamma(\gamma_1)\Gamma(\gamma_2)} \Big[\int_a^x \int_c^y (y - v)^{\gamma_2 - 1} \left[(x - u)^{\gamma_1 - 1} - (x + k_1 - u)^{\gamma_1 - 1} \right] du dv$$
$$+ \int_a^x \int_c^y (x + k_1 - u)^{\gamma_1 - 1} \left[(y - v)^{\gamma_2 - 1} - (y + k_2 - v)^{\gamma_2 - 1} \right] du dv \Big].$$

Let J_1 and J_2 defined as follows and by using Bernoulli's inequality $(1 + u)^{r'} \leq 1 + r'u$ for $0 \leq r' \leq 1$ and $u \geq -1$, we obtain

$$\begin{aligned} J_1 &= \int_a^x \left[(x - u)^{\gamma_1 - 1} - (x + k_1 - u)^{\gamma_1 - 1} \right] du \\ &= \frac{1}{\gamma_1} \left[(x + k_1 - x)^{\gamma_1} - (x + k_1 - a)^{\gamma_1} + (x - a)^{\gamma_1} \right] \\ &= \frac{1}{\gamma_1} \left[(k_1^{\gamma_1} - (x + k_1 - a)^{\gamma_1} + (x - a)^{\gamma_1} \right] \\ &\leq \frac{k_1^{\gamma_1}}{\gamma_1}. \end{aligned}$$

$$\begin{aligned} J_2 &= \int_c^y \left[(y - v)^{\gamma_2 - 1} - (y + k_2 - v)^{\gamma_2 - 1} \right] dv \\ &= \frac{1}{\gamma_2} \left[(y + k_2 - y)^{\gamma_2} - (y + k_2 - c)^{\gamma_2} + (y - c)^{\gamma_2} \right] \\ &= \frac{1}{\gamma_2} \left[(k_2^{\gamma_2} - (y + k_2 - c)^{\gamma_2} + (y - c)^{\gamma_2} \right] \\ &\leq \frac{k_2^{\gamma_2}}{\gamma_2}. \end{aligned}$$

By using the values of J_1 and J_2, we get

$$\begin{aligned} |L_1| &\leq \frac{M}{\Gamma(\gamma_1)\Gamma(\gamma_2)}\left[\frac{k_1^{\gamma_1}}{\gamma_1}\int_c^y (y-v)^{\gamma_2-1}dv + \frac{k_2}{\gamma_2}\int_a^x (x+k_1-u)^{\gamma_1-1}du\right] \\ &\leq \frac{M}{\Gamma(\gamma_1)\Gamma(\gamma_2)}\left[\frac{k_1^{\gamma_1}}{\gamma_1\gamma_2}(d-c)^{\gamma_2} + \frac{k_2^{\gamma_2}}{\gamma_1\gamma_2}(b-a)^{\gamma_1}\right]. \end{aligned}$$

Therefore for a suitable constant C, we obtain

$$|L_1| \leq C(k_1^{\gamma_1} + k_2^{\gamma_2}).$$

Now, we estimate L_2 as follows:

$$\begin{aligned} |L_2| &\leq \frac{1}{\Gamma(\gamma_1)\Gamma(\gamma_2)}\int_a^x\int_y^{y+k_2}(x+k_1-u)^{\gamma_1-1}(y+k_2-v)^{\gamma_2-1}|f(u,v)|dudv \\ &\leq \frac{M}{\Gamma(\gamma_1)\Gamma(\gamma_2)}\int_a^x\int_y^{y+k_2}(x+k_1-u)^{\gamma_1-1}(y+k_2-v)^{\gamma_2-1}dudv \\ &\leq \frac{(b-a)^{\gamma_1}k_2^{\gamma_2}}{\gamma_1\gamma_2}. \end{aligned}$$

For suitable C, we get

$$|L_2| \leq Ck_2^{\gamma_2}.$$

Similarly

$$|L_3| \leq Ck_1^{\gamma_1}.$$

In similar way, we estimate L_4

$$\begin{aligned} |L_4| &\leq \frac{1}{\Gamma(\gamma_1)\Gamma(\gamma_2)}\int_x^{x+k_1}\int_y^{y+k_2}(x+k_1-u)^{\gamma_1-1}(y+k_2-v)^{\gamma_2-1}|f(u,v)|dudv \\ &\leq \frac{M}{\Gamma(\gamma_1)\Gamma(\gamma_2)}\int_x^{x+k_1}\int_y^{y+k_2}(x+k_1-u)^{\gamma_1-1}(y+k_2-v)^{\gamma_2-1}dudv \\ &= \frac{k_1^{\gamma_1}k_2^{\gamma_2}}{\gamma_1\gamma_2}. \end{aligned}$$

For suitable C, we have

$$|L_4| \leq Ck_1^{\gamma_1}k_2^{\gamma_2}.$$

Say $\alpha = \min\{\gamma_1, \gamma_2\}$. For suitable C and sufficiently small positive constants k_1, k_2, α, we get

$$|(\mathcal{I}^\gamma f)(x+k_1, y+k_2) - (\mathcal{I}^\gamma f)(x,y)| \leq |L_1| + |L_2| + |L_3| + |L_4| \leq C(k_1^{\gamma_1} + k_2^{\gamma_2}) \leq C(k_1^\alpha + k_2^\alpha).$$

Since k_1 and k_2 are sufficiently small, we have $k_1 \leq \sqrt{k_1^2 + k_2^2}$ and $k_2 \leq \sqrt{k_1^2 + k_2^2}$. Consequently, we get

$$|(\mathcal{I}^\gamma f)(x+k_1, y+k_2) - (\mathcal{I}^\gamma f)(x,y)| \leq C\|(x+k_1, y+k_2) - (x,y)\|_2^\alpha.$$

The proof follows from Lemma 2.4. □

Now, we prove a very important property of such integrals which is called the semigroup property. This property will be used to prove the further results.

Theorem 3.2 [***Semigroup property***] *Let* $\gamma_1 > 0, \gamma_1' > 0, \gamma_2 > 0, \gamma_2' > 0$ *and* $0 < a < b < \infty, 0 < c < d < \infty$. *Let* $f : [a, b] \times [c, d] \to \mathbb{R}$ *is an integrable function for which the fractional integral of mixed R-L type* $\mathcal{I}^{(\gamma_1,\gamma_2)} f$ *exists, then*

$$\mathcal{I}^{(\gamma_1,\gamma_2)}\mathcal{I}^{(\gamma_1',\gamma_2')} f = \mathcal{I}^{(\gamma_1+\gamma_1',\gamma_2+\gamma_2')} f.$$

Proof From the Dirichlet technique and Fubini's theorem , we have

$$(\mathcal{I}^{(\gamma_1,\gamma_2)}\mathcal{I}^{(\gamma_1',\gamma_2')} f)(x, y) = \frac{1}{\Gamma(\gamma_1)\Gamma(\gamma_2)\Gamma(\gamma_1')\Gamma(\gamma_2')} \int_a^x \int_c^y \left[\int_s^x \int_t^y (x-v)^{\gamma_1-1}(v-s)^{\gamma_1'-1} \right.$$
$$\left. .\ (y-w)^{\gamma_2-1}(w-t)^{\gamma_2'-1} dv dw\right] f(s, t) ds dt$$

With the change of variable $z_1 = \frac{v-s}{x-v}$, we have

$$\begin{aligned}\int_s^x (x-v)^{\gamma_1-1}(v-s)^{\gamma_1'-1} dv &= (x-s)^{\gamma_1+\gamma_1'-1}\int_0^1 (1-z_1)^{\gamma_1-1} z_1^{\gamma_1'-1} dz_1 \\ &= (x-s)^{\gamma_1+\gamma_1'-1}\frac{\Gamma(\gamma_1)\Gamma(\gamma_1')}{\Gamma(\gamma_1+\gamma_1')},\end{aligned}$$

according to the known formulae for the beta function [12, 17].

Similarly, we can estimate with $z_2 = \frac{w-t}{y-w}$,

$$\begin{aligned}\int_t^y (y-w)^{\gamma_2-1}(w-t)^{\gamma_2'-1} dw &= (y-t)^{\gamma_2+\gamma_2'-1}\int_0^1 (1-z_2)^{\gamma_2-1} z_2^{\gamma_2'-1} dz_2 \\ &= (y-t)^{\gamma_2+\gamma_2'-1}\frac{\Gamma(\gamma_2)\Gamma(\gamma_2')}{\Gamma(\gamma_2+\gamma_2')}.\end{aligned}$$

Consequently, we get

$$\begin{aligned}(\mathcal{I}^{(\gamma_1,\gamma_2)}\mathcal{I}^{(\gamma_1',\gamma_2')} f)(x, y) &= \frac{1}{\Gamma(\gamma_1+\gamma_1')\Gamma(\gamma_2+\gamma_2')}\int_a^x\int_c^y (x-s)^{\gamma_1+\gamma_1'-1}(y-t)^{\gamma_2+\gamma_2'-1} f(s,t) ds dt \\ &= (\mathcal{I}^{(\gamma_1+\gamma_1',\gamma_2+\gamma_2')} f)(x, y),\end{aligned}$$

which completes the proof. □

Theorem 3.3 *Let* $f : [a, b] \times [c, d] \to \mathbb{R}$ *is continuous and* $0 < a < b < \infty, 0 < c < d < \infty$.

(1) *If* $0 < \gamma_1, \gamma_2 < 1$, *then*

$$2 \le \dim_H Gr(\mathcal{I}^\gamma f, I \times J) \le \dim_B Gr(\mathcal{I}^\gamma f, I \times J) \le 3 - \min\{\gamma_1, \gamma_2\}.$$

(2) *If* $\gamma_1, \gamma_2 \ge 1$, *then*

$$\dim_H Gr(\mathcal{I}^\gamma f, I \times J) = \dim_B Gr(\mathcal{I}^\gamma f, I \times J) = 2.$$

The proof of the above theorem follows from Theorem 3.1, Theorem 3.2 and from the relation between fractal dimensions.

4 Fractal dimensions of $\mathcal{I}^{\gamma} f(x, y)$ with $f(x, y) \in H^{\mu}(I \times J)$

In this section, we establish the bounds for the fractal dimensions of the fractional integral of mixed R-L type corresponding to the μ-Hölder continuous functions.

Theorem 4.1 *Let $f(x, y)$ is continuous on $I \times J$ and $f(x, y) \in H^{\mu}(I \times J)$ with $f(0, 0) = (0, 0)$ and provided that the fractional integral of mixed R-L type of f exists. Then*

$$\dim_H Gr(\mathcal{I}^{\gamma} f, I \times J) \leq \overline{\dim}_B Gr(\mathcal{I}^{\gamma} f, I \times J) \leq 3-\mu, \quad 0 < \gamma_1, \gamma_2 < 1, \ 0 < \mu < 1.$$

Proof Let $0 \leq a \leq x < x + k_1 \leq b$, $0 \leq c \leq y < y + k_2 \leq d$ and $0 < \gamma_1, \gamma_2 < 1$. Then

$$\begin{aligned} &(\mathcal{I}^{\gamma} f)(x + k_1, y + k_2) - (\mathcal{I}^{\gamma} f)(x, y) \\ &= \frac{1}{\Gamma(\gamma_1)\Gamma(\gamma_2)} \int_a^{x+k_1} \int_c^{y+k_2} (x + k_1 - u)^{\gamma_1 - 1} (y + k_2 - v)^{\gamma_2 - 1} f(u, v) du dv \\ &- \frac{1}{\Gamma(\gamma_1)\Gamma(\gamma_2)} \int_a^{x} \int_c^{y} (x - u)^{\gamma_1 - 1} (y - v)^{\gamma_2 - 1} f(u, v) du dv = I_1 + I_2 + I_3 + I_4 - I_5, \end{aligned}$$

where

$$\begin{aligned} I_1 =& \frac{1}{\Gamma(\gamma_1)\Gamma(\gamma_2)} \int_a^{a+k_1} \int_c^{c+k_2} (x + k_1 - u)^{\gamma_1 - 1} (y + k_2 - v)^{\gamma_2 - 1} f(u, v) du dv \\ I_2 =& \frac{1}{\Gamma(\gamma_1)\Gamma(\gamma_2)} \int_a^{a+k_1} \int_{c+k_2}^{y+k_2} (x + k_1 - u)^{\gamma_1 - 1} (y + k_2 - v)^{\gamma_2 - 1} f(u, v) du dv \\ I_3 =& \frac{1}{\Gamma(\gamma_1)\Gamma(\gamma_2)} \int_{a+k_1}^{x+k_1} \int_c^{c+k_2} (x + k_1 - u)^{\gamma_1 - 1} (y + k_2 - v)^{\gamma_2 - 1} f(u, v) du dv \\ I_4 =& \frac{1}{\Gamma(\gamma_1)\Gamma(\gamma_2)} \int_{a+k_1}^{x+k_1} \int_{c+k_2}^{y+k_2} (x + k_1 - u)^{\gamma_1 - 1} (y + k_2 - v)^{\gamma_2 - 1} f(u, v) du dv \\ I_5 =& \frac{1}{\Gamma(\gamma_1)\Gamma(\gamma_2)} \int_a^{x} \int_c^{y} (x - u)^{\gamma_1 - 1} (y - v)^{\gamma_2 - 1} f(u, v) du dv. \end{aligned}$$

By change of variable in I_5, we have

$$\begin{aligned} I_5' =& \frac{1}{\Gamma(\gamma_1)\Gamma(\gamma_2)} \int_{a+k_1}^{x+k_1} \int_{c+k_2}^{y+k_2} (x + k_1 - u)^{\gamma_1 - 1} (y + k_2 - v)^{\gamma_2 - 1} f(u - k_1, v - k_2) du dv. \\ I_4 - I_5' = I_6 =& \frac{1}{\Gamma(\gamma_1)\Gamma(\gamma_2)} \int_{a+k_1}^{x+k_1} \int_{c+k_2}^{y+k_2} (x + k_1 - u)^{\gamma_1 - 1} (y + k_2 - v)^{\gamma_2 - 1} \\ &.[f(u - k_1, v - k_2) - f(u, v)] du dv. \\ |I_6| \leq& \frac{1}{\Gamma(\gamma_1)\Gamma(\gamma_2)} \int_{a+k_1}^{x+k_1} \int_{c+k_2}^{y+k_2} |(x + k_1 - u)^{\gamma_1 - 1} (y + k_2 - v)^{\gamma_2 - 1} \\ &.[f(u - k_1, v - k_2) - f(u, v)]| du dv. \end{aligned}$$

Since $f(x, y) \in H^{\mu}(I \times J)$ on $[a, b] \times [c, d]$, we have

$$\begin{aligned}|I_6| \leq& \frac{C\|(k_1, k_2)\|_2^{\mu}}{\Gamma(\gamma_1)\Gamma(\gamma_2)} \int_{a+k_1}^{x+k_1} \int_{c+k_2}^{y+k_2} |(x + k_1 - u)^{\gamma_1 - 1}(y + k_2 - v)^{\gamma_2 - 1}| dudv \\ =& \frac{C\|(k_1, k_2)\|_2^{\mu}}{\Gamma(\gamma_1 + 1)\Gamma(\gamma_2 + 1)}(x - a)^{\gamma_1}(y - c)^{\gamma_2}\end{aligned}$$

For $(x, y) \in [a, b] \times [c, d]$, we get

$$\begin{aligned}|I_6| \leq& \frac{C\|(k_1, k_2)\|_2^{\mu}}{\Gamma(\gamma_1 + 1)\Gamma(\gamma_2 + 1)}(b - a)^{\gamma_1}(d - c)^{\gamma_2}. \\ |I_6| \leq& C\|(k_1, k_2)\|_2^{\mu}, \quad \text{where } C = \frac{(b - a)^{\gamma_1}(d - c)^{\gamma_2}}{\Gamma(\gamma_1 + 1)\Gamma(\gamma_2 + 1)}.\end{aligned}$$

Now for the bound of I_1, we apply similar steps as done above.

$$\begin{aligned}|I_1| \leq& \frac{1}{\Gamma(\gamma_1)\Gamma(\gamma_2)} \int_{a}^{a+k_1} \int_{c}^{c+k_2} |(x + k_1 - u)^{\gamma_1 - 1}(y + k_2 - v)^{\gamma_2 - 1}||f(u, v) - f(0, 0)| dudv \\ \leq& \frac{C\|(k_1, k_2)\|_2^{\mu}}{\Gamma(\gamma_1)\Gamma(\gamma_2)} \int_{a}^{a+k_1} \int_{c}^{c+k_2} |(x + k_1 - u)^{\gamma_1 - 1}(y + k_2 - v)^{\gamma_2 - 1}| dudv \\ \leq& \frac{C\|(k_1, k_2)\|_2^{\mu}}{\Gamma(\gamma_1)\Gamma(\gamma_2)} \int_{a}^{a+k_1} \int_{c}^{c+k_2} |(a + k_1 - u)^{\gamma_1 - 1}(c + k_2 - v)^{\gamma_2 - 1}| dudv \\ =& \frac{C\|(k_1, k_2)\|_2^{\mu}}{\Gamma(\gamma_1 + 1)\Gamma(\gamma_2 + 1)} k_1^{\gamma_1} k_2^{\gamma_2}.\end{aligned}$$

So, we have

$$|I_1| \leq C\|(k_1, k_2)\|_2^{\mu}, \quad \text{where } C = \frac{k_1^{\gamma_1} k_2^{\gamma_2}}{\Gamma(\gamma_1 + 1)\Gamma(\gamma_2 + 1)}.$$

In similar way, we obtain the bounds for I_2 and I_3 as follows

$$\begin{aligned}|I_2| \leq& C\|(k_1, k_2)\|_2^{\mu}, \quad \text{where } C = \frac{k_1^{\gamma_1}(d - c)^{\gamma_2}}{\Gamma(\gamma_1 + 1)\Gamma(\gamma_2 + 1)}, \\ |I_3| \leq& C\|(k_1, k_2)\|_2^{\mu}, \quad \text{where } C = \frac{(b - a)^{\gamma_1} k_2^{\gamma_2}}{\Gamma(\gamma_1 + 1)\Gamma(\gamma_2 + 1)}.\end{aligned}$$

Consequently, we get for a suitable constant C

$$\begin{aligned}|(\mathcal{I}^{\gamma} f)(x + k_1, y + k_2) - (\mathcal{I}^{\gamma} f)(x, y)| \leq& |I_1| + |I_2| + |I_3| + |I_4| + |I_5| \\ \leq& C\|(k_1, k_2)\|_2^{\mu}.\end{aligned}$$

In view of Lemma 2.5 the proof follows. □

Remark 4.1 If $f(x, y)$ is any fractal function having box dimension $3 - \mu$, then upper box dimension of the fractional integral of mixed R-L type corresponding to $f(x, y)$ is non-increasing. Since,

$$\dim_B Gr(f, I \times J) = 3 - \mu.$$

We have

$$\overline{\dim}_B Gr(\mathcal{I}^{\gamma} f, I \times J) \leq 3 - \mu.$$

That is

$$\overline{\dim}_B Gr(\mathcal{I}^{\gamma} f, I \times J) \leq \dim_B Gr(f, I \times J) = 3 - \mu.$$

Theorem 4.2 *Let $f(x, y)$ be a continuous function defined on $[a, b] \times [c, d]$ with $f(0, 0) = (0, 0)$ and satisfies Lipschitz condition, then for $0 < \gamma_1, \gamma_2 < 1$,*

$$\dim_H Gr(\mathcal{I}^{\gamma} f, I \times J) = \dim_B Gr(\mathcal{I}^{\gamma} f, I \times J) = 2.$$

In view of Lemma 2.4 and Theorem 4.1 the proof of the Theorem 4.2 follows.

5 Fractal dimension of $\mathcal{I}^{\gamma} f(x, y)$ of 2-dimensional continuous functions

So far we have estimated the bounds for fractal dimensions. Now a question arises: For which class of continuous functions the box dimension of the fractional integral of mixed R-L type is equal to two. The answer of this question is given below. First, we give the following Lemma 5.1 which acts as a prelude for the main Theorem 5.1 and then we corroborate our result with the help of existing results.

Lemma 5.1 *Let $f : [0, 1] \times [0, 1] \to \mathbb{R}$ is continuous and $0 < \delta < 1,\ \frac{1}{\delta} < m, n < 1 + \frac{1}{\delta}$ for some $m, n \in \mathbb{N}$. If the number of δ-cubes that intersect the graph $Gr(f)$ is denoted by $N_\delta(Gr(f))$, then*

$$\sum_{j=1}^{n} \sum_{i=1}^{m} \max\left\{\frac{R_f[A_{ij}]}{\delta}, 1\right\} \leq N_\delta(Gr(f)) \leq 2mn + \frac{1}{\delta} \sum_{j=1}^{n} \sum_{i=1}^{m} R_f[A_{ij}],$$

where A_{ij} is the (i, j)-th cell corresponding to the net under consideration.

Proof If $f(x, y)$ is continuous on $I \times J$, the number of cubes having side δ in the part above A_{ij} which intersect $Gr(f, I \times J)$ is at least

$$\max\left\{\frac{R_f[A_{ij}]}{\delta}, 1\right\}$$

and at most

$$2 + \frac{R_f[A_{ij}]}{\delta}.$$

By summing over all such parts we get the required result. □

Theorem 5.1 *Let a non-negative function $f(x, y) \in C([0, 1] \times [0, 1])$ and $0 < \gamma_1 < 1,\ 0 < \gamma_2 < 1$.*
If

$$\dim_B Gr(f, [0, 1] \times [0, 1]) = 2, \tag{5.1}$$

then, the box dimension of the fractional integral of mixed R-L type of $f(x, y)$ of order $\gamma = (\gamma_1, \gamma_2)$ exists and is equal to 2 on $[0, 1] \times [0, 1]$, as

$$\dim_B Gr(\mathcal{I}^\gamma f, [0, 1] \times [0, 1]) = 2. \tag{5.2}$$

Proof Since $f(x, y) \in C([0, 1] \times [0, 1])$, $\mathcal{I}^\gamma f(x, y)$ is also continuous on $[0, 1] \times [0, 1]$ (from Theorem 4.2 in [23]). From the definition of the box dimension, we can get

$$\underline{\dim}_B Gr(\mathcal{I}^\gamma f, [0, 1] \times [0, 1]) \geq 2. \tag{5.3}$$

Now, to prove (5.2), we have to prove the following inequality

$$\overline{\dim}_B Gr(\mathcal{I}^\gamma f, [0, 1] \times [0, 1]) \leq 2. \tag{5.4}$$

Suppose that $0 < \delta < \frac{1}{2}$, $\frac{1}{\delta} < m, n < 1 + \frac{1}{\delta}$ and $N_\delta(Gr(f))$ is the number of δ-cubes that intersect $Gr(f)$. From (5.1), it holds

$$\lim_{\delta \to 0} \frac{\log N_\delta(Gr(f))}{-\log \delta} = 2.$$

Let $N_\delta(Gr(\mathcal{I}^\gamma f))$ is the number of δ-cubes that intersect $Gr(\mathcal{I}^\gamma f)$. Thus, (5.4) can be written as

$$\overline{\lim_{\delta \to 0}} \frac{\log N_\delta(Gr(\mathcal{I}^\gamma f))}{-\log \delta} \leq 2. \tag{5.5}$$

Now, we are ready to prove (5.5).

Let $f(x, y) \in C([0, 1] \times [0, 1])$ and $0 < \gamma_1, \gamma_2 < 1$. If $k_1 > 0$, $k_2 > 0$ and $x + k_1 \leq 1$, $y + k_2 \leq 1$, then

$$\begin{aligned} &(\Gamma(\gamma_1)\Gamma(\gamma_2)) \left[(\mathcal{I}^\gamma f)(x + k_1, y + k_2) - (\mathcal{I}^\gamma f)(x, y)\right] \\ &= \int_0^{x+k_1} \int_0^{y+k_2} (x + k_1 - u)^{\gamma_1 - 1}(y + k_2 - v)^{\gamma_2 - 1} f(u, v)dudv \\ &\quad - \int_0^x \int_0^y (x - u)^{\gamma_1 - 1}(y - v)^{\gamma_2 - 1} f(u, v)dudv. \end{aligned}$$

By integral transform, let

$$\left(\frac{u}{x + k_1}\right) = s,$$

and

$$\left(\frac{v}{y + k_2}\right) = t.$$

Then

$$dudv = |J|dsdt,$$

where

$$J = \begin{bmatrix} \frac{\partial u}{\partial s} & \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial s} & \frac{\partial v}{\partial t} \end{bmatrix}$$

$$= (x + k_1)(y + k_2).$$

Thus, we have

$$\begin{aligned}
&(\Gamma(\gamma_1)\Gamma(\gamma_2))\left[(\mathcal{I}^{\gamma} f)(x + k_1, y + k_2) - (\mathcal{I}^{\gamma} f)(x, y)\right]\\
&= \int_0^1 \int_0^1 (x + k_1)^{\gamma_1}(1 - u)^{\gamma_1 - 1}(y + k_2)^{\gamma_2}(1 - v)^{\gamma_2 - 1} f\left((x + k_1)u, (y + k_2)v\right) dudv\\
&- \int_0^1 \int_0^1 x^{\gamma_1}(1 - u)^{\gamma_1 - 1} y^{\gamma_2}(1 - v)^{\gamma_2 - 1} f(xu, yv) dudv\\
&= \int_0^1 \int_0^1 (x + k_1)^{\gamma_1}(1 - u)^{\gamma_1 - 1}(y + k_2)^{\gamma_2}(1 - v)^{\gamma_2 - 1} f\left((x + k_1)u, (y + k_2)v\right) dudv\\
&- \int_0^1 \int_0^1 (x + k_1)^{\gamma_1}(1 - u)^{\gamma_1 - 1}(y + k_2)^{\gamma_2}(1 - v)^{\gamma_2 - 1} f(xu, yv) dudv\\
&+ \int_0^1 \int_0^1 (x + k_1)^{\gamma_1}(1 - u)^{\gamma_1 - 1}(y + k_2)^{\gamma_2}(1 - v)^{\gamma_2 - 1} f(xu, yv) dudv\\
&- \int_0^1 \int_0^1 x^{\gamma_1}(1 - u)^{\gamma_1 - 1} y^{\gamma_2}(1 - v)^{\gamma_2 - 1} f(xu, yv) dudv\\
&= \int_0^1 \int_0^1 (1 - u)^{\gamma_1 - 1}(1 - v)^{\gamma_2 - 1}(x + k_1)^{\gamma_1}(y + k_2)^{\gamma_2}\left[f\left((x + k_1)u, (y + k_2)v\right) - f(xu, yv)\right] dudv\\
&+ \int_0^1 \int_0^1 (1 - u)^{\gamma_1 - 1}(1 - v)^{\gamma_2 - 1} f(xu, yv)\left[(x + k_1)^{\gamma_1}(y + k_2)^{\gamma_2} - x^{\gamma_1} y^{\gamma_2}\right] dudv.
\end{aligned}$$

For $0 < \delta < \frac{1}{2}, \frac{1}{\delta} < m, n < 1 + \frac{1}{\delta}$, let non-negative integers i and j such that $0 \leq i \leq m, 0 \leq j \leq n$. Then

$$(\Gamma(\gamma_1)\Gamma(\gamma_2)) R_{\mathcal{I}^{\gamma} f}[A_{ij}] = \sup_{(x+k_1, y+k_2), (x,y) \in A_{ij}} |(\mathcal{I}^{\gamma} f)(x + k_1, y + k_2) - (\mathcal{I}^{\gamma} f)(x, y)|,$$

where $A_{ij} = [i\delta, (i + 1)\delta] \times [j\delta, (j + 1)\delta]$.
Here,

$$\begin{aligned}
&= |(\mathcal{I}^{\gamma} f)(x + k_1, y + k_2) - (\mathcal{I}^{\gamma} f)(x, y)| \leq (x + k_1)^{\gamma_1}(y + k_2)^{\gamma_2} \Big|\\
&\int_0^1 \int_0^1 (1 - u)^{\gamma_1 - 1}(1 - v)^{\gamma_2 - 1}\left[f\left((x + k_1)u, (y + k_2)v\right) - f(xu, yv)\right] dudv\Big|\\
&+ \left((i + 1)^{\gamma_1}(j + 1)^{\gamma_2} - i^{\gamma_1} j^{\gamma_2}\right) \delta^{\gamma_1 + \gamma_2} \int_0^1 \int_0^1 (1 - u)^{\gamma_1 - 1}(1 - v)^{\gamma_2 - 1} f(xu, yv) dudv.
\end{aligned}$$

Let $i \geq 1$, $j \geq 1$. On the one hand

$$
\begin{aligned}
&\Big|\int_0^1\int_0^1 (1-u)^{\gamma_1-1}(1-v)^{\gamma_2-1}\left[f\left((x+k_1)u,(y+k_2)v\right)-f(xu,yv)\right]dudv\Big|\\
&=\Big|\int_0^{\frac{1}{i+1}}\int_0^{\frac{1}{j+1}} (1-u)^{\gamma_1-1}(1-v)^{\gamma_2-1}\left[f\left((x+k_1)u,(y+k_2)v\right)-f(xu,yv)\right]dudv\Big|\\
&+\sum_{p=1}^{j}\Big|\int_0^{\frac{1}{i+1}}\int_{\frac{p}{j+1}}^{\frac{p+1}{j+1}} (1-u)^{\gamma_1-1}(1-v)^{\gamma_2-1}\left[f\left((x+k_1)u,(y+k_2)v\right)-f(xu,yv)\right]dudv\Big|\\
&+\sum_{q=1}^{i}\Big|\int_{\frac{q}{i+1}}^{\frac{q+1}{i+1}}\int_0^{\frac{1}{j+1}} (1-u)^{\gamma_1-1}(1-v)^{\gamma_2-1}\left[f\left((x+k_1)u,(y+k_2)v\right)-f(xu,yv)\right]dudv\Big|\\
&+\sum_{q=1}^{i}\sum_{p=1}^{j}\Big|\int_{\frac{q}{i+1}}^{\frac{q+1}{i+1}}\int_{\frac{p}{j+1}}^{\frac{p+1}{j+1}} (1-u)^{\gamma_1-1}(1-v)^{\gamma_2-1}\left[f\left((x+k_1)u,(y+k_2)v\right)-f(xu,yv)\right]dudv\Big|\\
&\leq \frac{1}{(i+1)(j+1)}R_f\left[[0,\delta]\times[0,\delta]\right]\\
&+\sum_{p=1}^{j}\frac{1}{(i+1)(j+1)}\left(R_f\left[[0,\delta]\times[(p-1)\delta,p\delta]\right]+R_f\left[[0,\delta]\times[p\delta,(p+l)\delta]\right]\right)\\
&+\sum_{q=1}^{i}\frac{1}{(i+1)(j+1)}(R_f\left[[(q-1)\delta,q\delta]\times[0,\delta]\right]+R_f\left[[q\delta,(q+1)\delta]\times[0,\delta]\right])\\
&+\sum_{q=1}^{i}\sum_{p=1}^{j}\frac{1}{(i+1)(j+1)}(R_f\left[[(q-1)\delta,q\delta]\times[(p-1)\delta,p\delta]\right]+R_f\left[[(q-1)\delta,q\delta]\times[p\delta,(p+1)\delta]\right]\\
&+R_f\left[[q\delta,(q+1)\delta]\times[(p-1)\delta,p\delta]\right]+R_f\left[[q\delta,(q+1)\delta]\times[p\delta,(p+1)\delta]\right]).
\end{aligned}
$$

By using Bernoulli's inequality $(1+u)^{r'} \leq 1+r'u$ for $0 \leq r' \leq 1$ and $u \geq -1$, we can see that

$$
\int_0^{\frac{1}{i+1}}\int_0^{\frac{1}{j+1}}(1-u)^{\gamma_1-1}(1-v)^{\gamma_2-1}dudv \leq \frac{1}{(i+1)(j+1)}.
$$

On the other hand

$$
\begin{aligned}
\left((i+1)^{\gamma_1}(j+1)^{\gamma_2}-i^{\gamma_1}j^{\gamma_2}\right)\delta^{\gamma_1+\gamma_2}\int_0^1\int_0^1(1-s)^{\gamma_1-1}(1-t)^{\gamma_2-1}f(xs,yt)dsdt\\
\leq (i+1)^{\gamma_1}(j+1)^{\gamma_2}\delta^{\gamma_1+\gamma_2}\frac{\max_{0\leq(x,y)\leq 1}f(x,y)}{\gamma_1\gamma_2}.
\end{aligned}
$$

From Lemma 5.1, we have

$$N_\delta(Gr(\mathcal{I}^\gamma f)) \leq 2mn + \frac{1}{\delta}\sum_{j=1}^{n}\sum_{i=1}^{m} R_{\mathcal{I}^\gamma f}[A_{ij}] \leq 2mn + \frac{1}{\delta}\sum_{j=1}^{n}\sum_{i=1}^{m}\Bigg(\frac{1}{(i+1)(j+1)} R_f\,[[0,\delta]\times[0,\delta]]$$

$$+\sum_{p=1}^{j}\frac{1}{(i+1)(j+1)}\big(R_f\,[[0,\delta]\times[(p-1)\delta, p\delta]] + R_f\,[[0,\delta]\times[p\delta,(p+l)\delta]]\big)$$

$$+\sum_{q=1}^{i}\frac{1}{(i+1)(j+1)}(R_f\,[[(q-1)\delta, q\delta]\times[0,\delta]] + R_f\,[[q\delta,(q+1)\delta]\times[0,\delta]])$$

$$+\sum_{q=1}^{i}\sum_{p=1}^{j}\frac{1}{(i+1)(j+1)}\big(R_f\,[[(q-1)\delta, q\delta]\times[(p-1)\delta, p\delta]] + R_f\,[[(q-1)\delta, q\delta]\times$$

$$[p\delta,(p+1)\delta]] + R_f\,[[q\delta,(q+1)\delta]\times[(p-1)\delta, p\delta]] + R_f\,[[q\delta,(q+1)\delta]\times[p\delta,(p+1)\delta]]\big)\Bigg)$$

$$+\frac{1}{\delta}\sum_{j=1}^{n}\sum_{i=1}^{m}(i+1)^{\gamma_1}(j+1)^{\gamma_2}\delta^{\gamma_1+\gamma_2}\frac{\max_{0\leq(x,y)\leq 1} f(x,y)}{\gamma_1\gamma_2}$$

$$\leq \frac{1}{\delta}\Bigg(C + \sum_{j=1}^{n}\sum_{i=1}^{m}\Bigg(\frac{1}{(i+1)(j+1)} R_f\,[[0,\delta]\times[0,\delta]]$$

$$+\sum_{p=1}^{j}\frac{1}{(i+1)(j+1)}\big(R_f\,[[0,\delta]\times[(p-1)\delta, p\delta]] + R_f\,[[0,\delta]\times[p\delta,(p+l)\delta]]\big)$$

$$+\sum_{q=1}^{i}\frac{1}{(i+1)(j+1)}(R_f\,[[(q-1)\delta, q\delta]\times[0,\delta]] + R_f\,[[q\delta,(q+1)\delta]\times[0,\delta]])$$

$$+\sum_{q=1}^{i}\sum_{p=1}^{j}\frac{1}{(i+1)(j+1)}\big(R_f\,[[(q-1)\delta, q\delta]\times[(p-1)\delta, p\delta]] + R_f\,[[(q-1)\delta, q\delta]$$

$$\times[p\delta,(p+1)\delta]] + R_f\,[[q\delta,(q+1)\delta]\times[(p-1)\delta, p\delta]] + R_f\,[[q\delta,(q+1)\delta]\times[p\delta,(p+1)\delta]]\big)\Bigg)\Bigg)$$

$$\leq \frac{C}{\delta}\left(\sum_{j=0}^{n}\sum_{i=0}^{m}\frac{1}{(i+1)(j+1)}\right)\left(\sum_{j=0}^{n}\sum_{i=0}^{m} R_f[A_{ij}]\right) \leq \frac{C}{\delta}(\log m)(\log n)\sum_{j=0}^{n}\sum_{i=0}^{m} R_f[A_{ij}]$$

$$\leq C(\log m)(\log n)N_\delta(Gr(f)).$$

Therefore, we get

$$\frac{\log N_\delta(Gr(\mathcal{I}^\gamma f))}{-\log\delta} \leq \frac{\log\{C(\log m)(\log n)N_\delta(Gr(f))\}}{-\log\delta}$$
$$\leq \frac{\log C}{-\log\delta} + \frac{\log(\log m)}{-\log\delta} + \frac{\log(\log n)}{-\log\delta} + \frac{\log N_\delta(Gr(f))}{-\log\delta}.$$

So, we obtain

$$\overline{\dim}_B Gr(\mathcal{I}^\gamma f, [0,1]\times[0,1]) = \overline{\lim_{\delta\to 0}}\frac{\log N_\delta(Gr(\mathcal{I}^\gamma f))}{-\log\delta}$$
$$\leq \overline{\lim_{\delta\to 0}}\left(\frac{\log C}{-\log\delta} + \frac{\log(\log m)}{-\log\delta} + \frac{\log(\log n)}{-\log\delta} + \frac{\log N_\delta(Gr(f))}{-\log\delta}\right)$$
$$\leq \overline{\lim_{\delta\to 0}}\frac{\log N_\delta(Gr(f))}{-\log\delta} = \lim_{\delta\to 0}\frac{\log N_\delta(Gr(f))}{-\log\delta} = 2.$$

Thus, Inequality (5.5) holds. By combining Inequalities (5.3) and (5.5), we get the desired result. □

Corollary 5.2 *Let $f : [0,1] \times [0,1] \to \mathbb{R}$ be a continuous function and $0 < \gamma_1 < 1,\ 0 < \gamma_2 < 1$. If f is of bounded variation in Arzelá sense, then*

$$\dim_B Gr(\mathcal{I}^{\gamma} f, [0,1] \times [0,1]) = 2.$$

Proof From Remark 3.13 in [23], if $f : [0,1] \times [0,1] \to \mathbb{R}$ is continuous and of bounded variation in Arzelá sense on $[0,1] \times [0,1]$, then

$$\dim_B Gr(f, [0,1] \times [0,1]) = 2.$$

Thus, by using Theorem 5.1, we get

$$\dim_B Gr(\mathcal{I}^{\gamma} f, [0,1] \times [0,1]) = 2.$$

This completes the proof. □

Remark 5.2 Thus, Theorem 4.5 of [23] follows from our Theorem 5.1. In [23], Verma and Viswanathan proved that the box dimension of the fractional integral of mixed R-L type of a continuous function which is of bounded variation in Arzelá sense on $[0,1] \times [0,1]$ is 2. Their results are more concerned with analytical aspects in the sense that they are using the notion of bounded variation. But in Theorem 5.1, we have proved that if a continuous function has box dimension two, then the box dimension of its fractional integral of mixed R-L type is also two. Here, we are using the dimension of function to compute the dimension of its fractional integral of mixed R-L type. So, our results are more concerned with dimensional aspects. From Theorem 5.1, we conclude that the fractional integral of mixed R-L type of any two-dimensional continuous function preserves its dimension.

Now, we are going to corroborate Theorem 5.1 by using existing results.

Lemma 5.3 [23] *Let a function $h : [c,d] \to \mathbb{R}$ be continuous. Consider a set as $H = \{(x, y, h(y)) : x \in [a,b], y \in [c,d]\}$ with $a < b$. Then it holds, $\overline{\dim}_B(H) \leq \overline{\dim}_B(Gr(h)) + 1$.*

Remark 5.4 Let $h_1 : [a,b] \to \mathbb{R}$ and $h_2 : [c,d] \to \mathbb{R}$ are two continuous maps. Now, define $g_1, g_2 : [a,b] \times [c,d] \to \mathbb{R}$ such that

$$g_1(x,y) = h_1(x) + h_2(y), \quad \text{and} \quad g_2(x,y) = h_1(x)h_2(y).$$

From Lemma 5.3, we get $\overline{\dim}_B Gr(g_1) \leq \overline{\dim}_B Gr(h_2) + 1$ and $\overline{\dim}_B Gr(g_2) \leq \overline{\dim}_B Gr(h_2) + 1$.

Remark 5.5 Let $g : [a,b] \to \mathbb{R}$ be a continuous function which box dimension is 1. We define a bivariate continuous function $f : [a,b] \times [c,d] \to \mathbb{R}$ such that

$f(x, y) = g(x)$. From definition 2.1, we have

$$\mathcal{I}^{\gamma} f(x, y) = \frac{1}{\Gamma(\gamma_1)\Gamma(\gamma_2)} \int_a^x \int_c^y (x-u)^{\gamma_1 - 1}(y-v)^{\gamma_2 - 1} f(u, v) du dv.$$

For $\gamma_2 = 1$, we get

$$\mathcal{I}^{\gamma} f(x, y) = \frac{1}{\Gamma(\gamma_1)} \int_a^x \int_c^y (x-u)^{\gamma_1 - 1} f(u, v) du dv.$$

By definition of f, we obtain

$$\mathcal{I}^{\gamma} f(x, y) = \frac{y-c}{\Gamma(\gamma_1)} \int_a^x (x-u)^{\gamma_1 - 1} g(u) du.$$

So, we have a relation between the fractional integral of R-L type of g, namely

$$\mathcal{I}^{\gamma_1} g(x) = \frac{1}{\Gamma(\gamma_1)} \int_a^x (x-u)^{\gamma_1 - 1} g(u) du,$$

and the fractional integral of mixed R-L type of f as

$$\mathcal{I}^{\gamma} f(x, y) = (y-c)\mathcal{I}^{\gamma_1} g(x).$$

Now, from remark 5.4, we know that $\overline{\dim}_B Gr(\mathcal{I}^{\gamma} f) \leq \overline{\dim}_B Gr(\mathcal{I}^{\gamma_1} g) + 1$. Since, $\dim_B Gr(g) = 1$, from Theorem 3.1 in [16], it follows that $\dim_B Gr(\mathcal{I}^{\gamma_1} g) = 1$, and hence $\dim_B Gr(\mathcal{I}^{\gamma} f) = 2$. This corroborates Theorem 5.1.

6 Fractal dimension of $\mathcal{I}^{\gamma} f(x, y)$ of unbounded variational continuous functions

To study the fractional integral of mixed R-L type of unbounded variational (UV) continuous functions with examples, we follow the construction from [24].

Construction of UV continuous functions Consider $[0, 1] \times [0, 1]$. Let $\lim_{n\to\infty} a_n = 1$, where (a_n) is the increasing sequence of real numbers in $[0, 1]$. For our construction, we take a sequence $(a_n)_{n\geq 0}$ by considering $a_0 = 0$ and $a_n = \frac{1}{2} + \frac{1}{2^2} + ... + \frac{1}{2^n}$, $n \in \mathbb{N}$. Let us define a continuous function $\Theta(x, y) = x(x - 0.5)y$ on $[0, 0.5] \times [0, 1]$ such that

$$\Theta(0, y) = \Theta(0.5, y), \quad \forall\, y \in [0, 1].$$

We shall refer Θ as generating function. Let Υ_n be map from $[a_{n-1}, a_n]$ onto $[0, 0.5]$ given by

$$\Upsilon_n(x, y) = 2^{(n-1)}(x - a_{n-1}).$$

Let us define $G_1(x, y) = \Theta(x, y)$ for $(x, y) \in [0, 0.5] \times [0, 1]$ and $n \geq 2$,

$$G_n(x, y) = \frac{1}{n}\Theta(\Upsilon_n(x), y) + \frac{n-1}{n}\Theta(0, y) \text{ for } (x, y) \in [a_{n-1}, a_n] \times [0, 1].$$

Now, we denote that $F_n(x, y)$ is the composed of $G_1(x, y), G_2(x, y), ..., G_n(x, y)$. Let

$$M(x, y) = \lim_{n\to\infty} F_n(x, y).$$

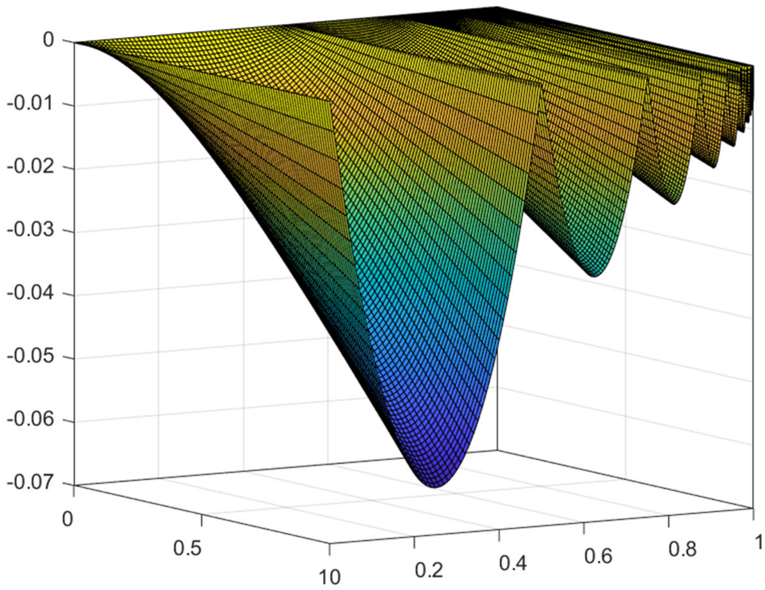

Fig. 1 $M(x, y)$

The graph of $M(x, y)$ is given in Fig. 1.

Theorem 6.1 *The function M is not of bounded variation on* $[0, 1] \times [0, 1]$.

Proof It can be seen that Θ is non-constant function along the line $y = y_0$ for some $y_0 \in [0, 1]$. For $C > 0$ and some $u_1, u_2 \in [a_0, a_1]$ with $u_1 < u_2$, we have

$$|\Theta(u_1, y_0) - \Theta(u_2, y_0)| \geq C.$$

Choose $w_1, w_2 \in [a_0, a_1]$, $w_1 < w_2$ such that

$$|G_1(w_1, y_0) - G_1(w_2, y_0)| = |\Theta(u_1, y_0) - \Theta(u_2, y_0)| \geq C.$$

We can choose $w_3, w_4 \in [a_1, a_2]$, $w_3 < w_4$ such that

$$|G_2(w_3, y_0) - G_2(w_4, y_0)| = \frac{1}{2}|\Theta(u_1, y_0) - \Theta(u_2, y_0)| \geq \frac{C}{2}.$$

We can see that for $i > 1$, $w_i = \Upsilon_{i-1}^{-1}(u_1)$ and $w_{i+1} = \Upsilon_{i-1}^{-1}(u_2)$. By proceeding in similar way, we get a collection $P' = \{w_i : w_1 < w_2 < w_3 < ... < w_{2n}\}$. Now, we take partition P of $[0, 1]$ such that $P' \subset P$. The variation of M along the line $y = y_0$ denoted by $V(M, [0, 1], y_0)$ is

$$V(M, [0, 1], y_0) \geq \sum_{i=1}^{2n} |M(w_{i+1}, y_0) - M(w_i, y_0)| \geq \sum_{i=1}^{n} |G_i(w_{i+1}, y_0) - G_i(w_i, y_0)| \geq \sum_{i=1}^{n} \frac{C}{i}.$$

Since $C > 0$ and $\sum_{i=1}^{n} \frac{1}{i} = \infty$, restriction of M, $M|_{y=y_0}$, is not of bounded variation on [0, 1] along the line $y = y_0$. So, $M|_{y=y_0}$ can not be written as difference of two increasing functions $g_1, g_2 : [0, 1] \to \mathbb{R}$ along the line $y = y_0$. That is, $M|_{y=y_0} = g_1 - g_2$ with $\Delta_{10} g_i(x, y_0) \geq 0$, $i = 1, 2$ does not hold. Now, by using Theorem 2.1, it is clear that the function M is not of bounded variation on $[0, 1] \times [0, 1]$ in Arzelá sense. □

Lemma 6.1 [23] *If* $f(x, y) \in C([0, 1] \times [0, 1])$ *and of bounded variation on* $[0, 1] \times [0, 1]$ *in Arzelá sense, then* $\mathcal{I}^{\gamma} f(x, y) \in C([0, 1] \times [0, 1])$ *and of bounded variation on* $[0, 1] \times [0, 1]$ *in Arzelá sense.*

The following theorem gives the box dimension and the Hausdorff dimension of $\mathcal{I}^{\gamma} M(x, y)$.

Theorem 6.2 *Let* $0 < \gamma_1 < 1, \ 0 < \gamma_2 < 1$. *Then* $\mathcal{I}^{\gamma} M(x, y)$ *is finite on* $[0, 1] \times [0, 1]$ *and*

$$\dim_H Gr(\mathcal{I}^{\gamma} M, [0, 1] \times [0, 1]) = \dim_B Gr(\mathcal{I}^{\gamma} M, [0, 1] \times [0, 1]) = 2.$$

Proof For $0 < \gamma_1 < 1, \ 0 < \gamma_2 < 1$, we have

$$\begin{aligned} |\mathcal{I}^{\gamma} M(x, y)| &= \left| \frac{1}{\Gamma(\gamma_1)\Gamma(\gamma_2)} \int_0^x \int_0^y (x-u)^{\gamma_1 - 1} (y-v)^{\gamma_2 - 1} M(u, v) du dv \right| \\ &\leq \frac{1}{\Gamma(\gamma_1 + 1)\Gamma(\gamma_2 + 1)} x^{\gamma_1} y^{\gamma_2} \max_{(x,y) \in [0,1] \times [0,1]} |M(x, y)| \leq \frac{1}{\Gamma(\gamma_1 + 1)\Gamma(\gamma_2 + 1)}. \end{aligned}$$

This shows that $\mathcal{I}^{\gamma} M(x, y)$ is finite on $[0, 1] \times [0, 1]$. Since $M(x, y)$ is a continuous function and of bounded variation on $[0, 1) \times [0, 1)$, then from Lemma 6.1 we know that $\mathcal{I}^{\gamma} M(x, y)$ is continuous and of bounded variation on $[0, 1) \times [0, 1)$ for $0 < \gamma_1 < 1, \ 0 < \gamma_2 < 1$.
Let $0 < \delta < 1$ and a positive constant C, when $(x, y) \in [0, 1-\delta) \times [0, 1-\delta)$, $\mathcal{I}^{\gamma} M(x, y)$ is of bounded variation. Let the smallest number of sets of diameter δ which can cover graph of $\mathcal{I}^{\gamma} M(x, y)$ is $\frac{C}{\delta^2}$. Now, when $(x, y) \in [1, 1-\delta) \times [1, 1-\delta)$, then the number of δ-cubes that intersect graph of $\mathcal{I}^{\gamma} M(x, y)$ is at most $\frac{1}{\delta^2}$.
Hence, the smallest number of sets of diameter δ which can cover graph of $\mathcal{I}^{\gamma} M(x, y)$ is at most $\frac{C+1}{\delta^2}$. Thus, we have

$$\overline{\dim}_B Gr(\mathcal{I}^{\gamma} M, [0, 1] \times [0, 1]) = \overline{\lim_{\delta \to 0}} \frac{\log N_\delta(Gr(\mathcal{I}^{\gamma} M, [0, 1] \times [0, 1]))}{-\log \delta} \leq \lim_{\delta \to 0} \frac{\log \frac{C+1}{\delta^2}}{-\log \delta} = 2.$$

From Definition 2.2 and Lemma 5.1, we know that

$$\underline{\dim}_B Gr(\mathcal{I}^{\gamma} M, [0, 1] \times [0, 1]) \geq 2.$$

This implies that

$$\dim_B Gr(\mathcal{I}^{\gamma} M, [0, 1] \times [0, 1]) = 2. \tag{6.1}$$

Also, we know that

$$2 \leq \dim_H Gr(\mathcal{I}^{\gamma} M, [0, 1] \times [0, 1]) \leq \dim_B Gr(\mathcal{I}^{\gamma} M, [0, 1] \times [0, 1]). \tag{6.2}$$

From (6.1) and (6.2), we get the required result. □

Remark 6.2 From [23], we know that if a function is continuous and of bounded variation in Arzelá sense, then its fractional integral of mixed R-L type is also continuous and of bounded variation in Arzelá sense and its box dimension and Hausdorff dimension is two. From Theorem 6.2, we conclude that the box dimension and the Hausdorff dimension of the fractional integral of mixed R-L type of unbounded variational continuous function is also two. So, M is such example which is of unbounded variational continuous function but the fractal dimension of its fractional integral of mixed R-L type is two.

7 Graph of $\mathcal{I}^{\gamma=(\gamma_1,\gamma_2)}M(x,y)$

In Section 6, we give the construction of $M(x, y)$. Now, we calculate its fractional integral of mixed R-L type and draw the following graphs.
If $(x, y) \in [0, \frac{1}{2}] \times [0, 1]$,

$$\mathcal{I}^{\gamma} M(x, y) = \left(\frac{2}{\Gamma(\gamma_1 + 3)} x^{\gamma_1+2} - \frac{1}{2\Gamma(\gamma_1 + 2)} x^{\gamma_1+1}\right) \left(\frac{1}{\Gamma(\gamma_2 + 2)} y^{\gamma_2+1}\right).$$

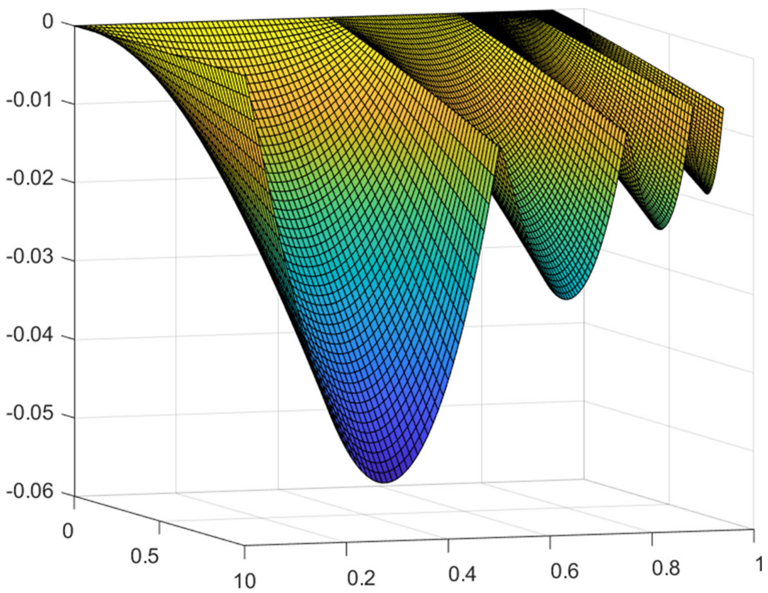

Fig. 2 $\mathcal{I}^{(0.1,0.1)} M(x, y)$

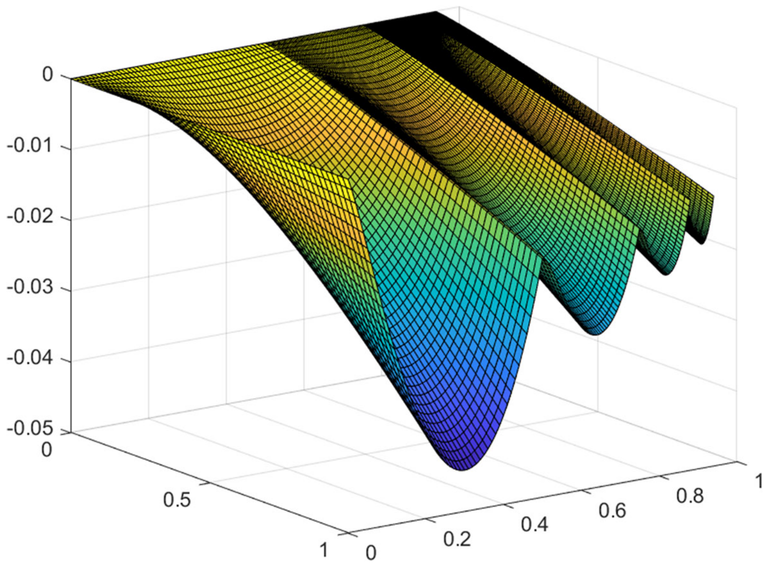

Fig. 3 $\mathcal{I}^{(0.2,0.2)}M(x, y)$

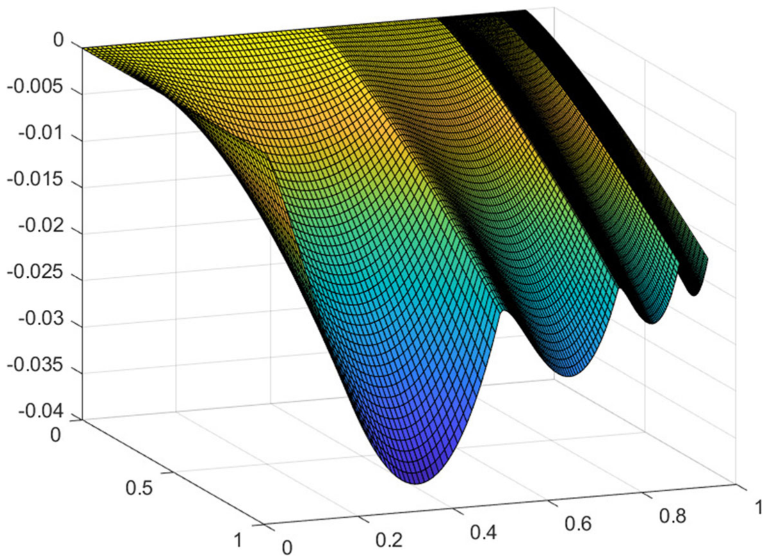

Fig. 4 $\mathcal{I}^{(0.3,0.3)}M(x, y)$

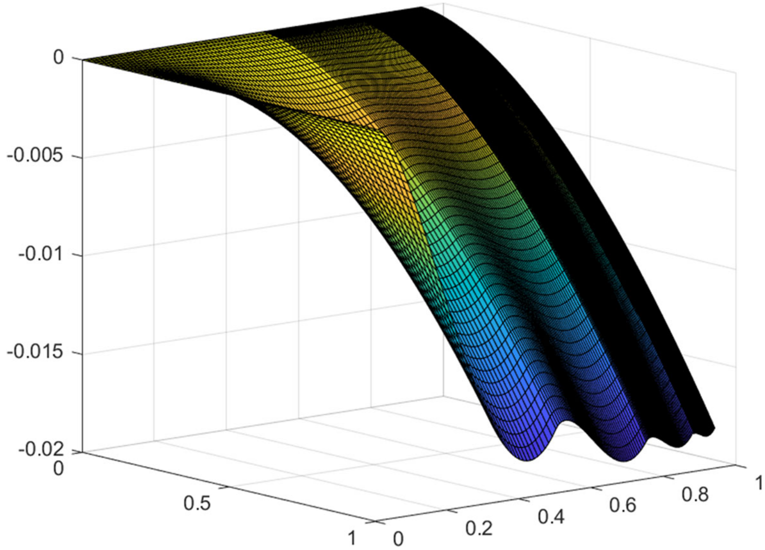

Fig. 5 $\mathcal{I}^{(0.7,0.7)}M(x, y)$

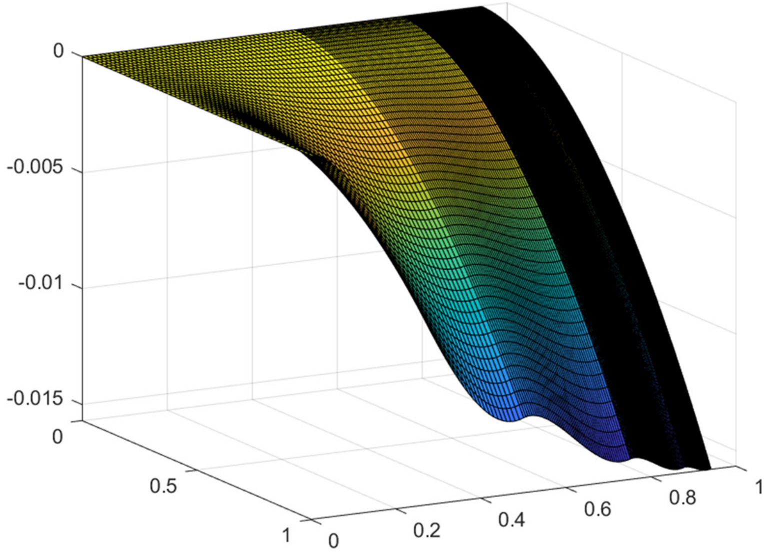

Fig. 6 $\mathcal{I}^{(0.9,0.9)}M(x, y)$

If $(x, y) \in [\frac{1}{2}, \frac{3}{4}] \times [0, 1]$,

$$\mathcal{I}^{\gamma} M(x, y) = \left(\frac{2}{\Gamma(\gamma_1+3)} x^{\gamma_1+2} - \frac{1}{2\Gamma(\gamma_1+2)} x^{\gamma_1+1} - \frac{1}{\Gamma(\gamma_1+2)} \left(x - \frac{1}{2}\right)^{\gamma_1+1} + \frac{2}{\Gamma(\gamma_1+3)} \left(x - \frac{1}{2}\right)^{\gamma_1+2} \right) \left(\frac{1}{\Gamma(\gamma_2+2)} y^{\gamma_2+1} \right).$$

If $(x, y) \in [\frac{3}{4}, \frac{7}{8}] \times [0, 1]$,

$$\mathcal{I}^{\gamma} M(x, y) = \left(\frac{2}{\Gamma(\gamma_1+3)} x^{\gamma_1+2} - \frac{1}{2\Gamma(\gamma_1+2)} x^{\gamma_1+1} - \frac{1}{\Gamma(\gamma_1+2)} \left(x - \frac{1}{2}\right)^{\gamma_1+1} + \frac{2}{\Gamma(\gamma_1+3)} \left(x - \frac{1}{2}\right)^{\gamma_1+2} - \frac{7}{6\Gamma(\gamma_1+2)} \left(x - \frac{3}{4}\right)^{\gamma_1+1} + \frac{20}{3\Gamma(\gamma_1+3)} \left(x - \frac{3}{4}\right)^{\gamma_1+2} \right) \left(\frac{1}{\Gamma(\gamma_2+2)} y^{\gamma_2+1} \right).$$

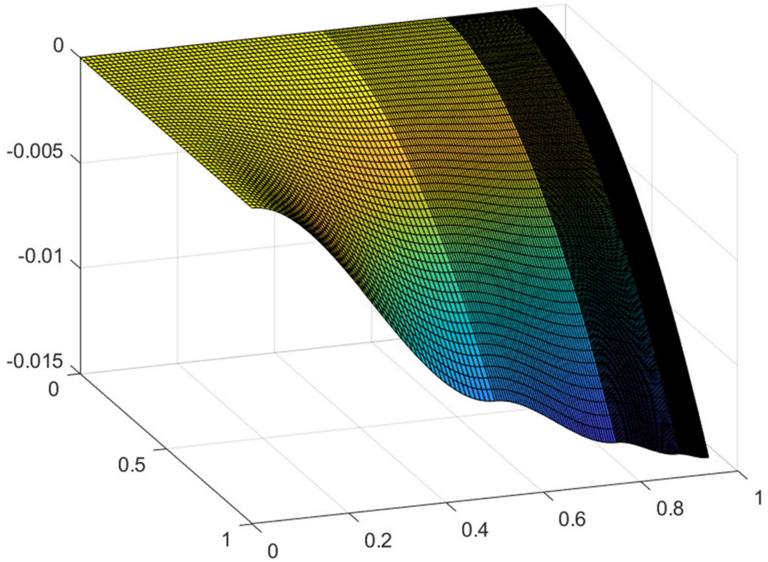

Fig. 7 $\mathcal{I}^{(1.0,1.0)} M(x, y)$

If $(x, y) \in [\frac{7}{8}, \frac{15}{16}] \times [0, 1]$,

$$\begin{aligned}\mathcal{I}^{\gamma} M(x, y) &= \Bigg(\frac{2}{\Gamma(\gamma_1+3)} x^{\gamma_1+2} - \frac{1}{2\Gamma(\gamma_1+2)} x^{\gamma_1+1} - \frac{1}{\Gamma(\gamma_1+2)}\left(x-\frac{1}{2}\right)^{\gamma_1+1} \\ &+ \frac{2}{\Gamma(\gamma_1+3)}\left(x-\frac{1}{2}\right)^{\gamma_1+2} - \frac{7}{6\Gamma(\gamma_1+2)}\left(x-\frac{3}{4}\right)^{\gamma_1+1} + \frac{20}{3\Gamma(\gamma_1+3)}\left(x-\frac{3}{4}\right)^{\gamma_1+2} \\ &- \frac{5}{3\Gamma(\gamma_1+2)}\left(x-\frac{7}{8}\right)^{\gamma_1+1} + \frac{64}{3\Gamma(\gamma_1+3)}\left(x-\frac{7}{8}\right)^{\gamma_1+2}\Bigg)\left(\frac{1}{\Gamma(\gamma_2+2)} y^{\gamma_2+1}\right).\end{aligned}$$

Similarly, we can calculate $\mathcal{I}^{\gamma} M(x, y)$ for other $G_n(x, y)$ on $(x, y) \in [a_{n-1}, a_n] \times [0, 1]$ but above values are sufficient for the smoothness analysis of the graphs (Figs. 2, 3, 4, 5, 6 and 7).

Remark 7.1 As we can notice from the graphs of the fractional integral of mixed R-L type that increments in γ will give more smooth surfaces, that is, the bigger the γ, the smoother the $Gr(\mathcal{I}^{\gamma} M, [0, 1] \times [0, 1])$. So, we can say that fractional integrals are the best mathematical operator to study the smoothness of irregular functions as we discussed in the introduction part.

8 Open problems

It will be interesting to explore the following problems:

(i) The box dimension of the graph of the fractional integral of mixed R-L type of fractal functions.
(ii) The box dimension of the graph of the fractional integral of mixed R-L type of a continuous function having the box dimension greater than two.

Conclusion Calculating fractal dimension of the graph of a function not simple even for real-valued functions. While in this article, we compute fractal dimension of the graph of the mixed Riemann-Liouville fractional integral. Various estimates, and in some special cases the exact value of the box dimensions and the Hausdorff dimension of the graph $Gr(\mathcal{I}^{\gamma} f)$ of $\mathcal{I}^{\gamma} f$ are given which depend on $\gamma = (\gamma_1, \gamma_2)$ and f. For a typical result, if f is μ-Hölder with $f(0, 0) = (0, 0)$, then $\dim_H Gr(\mathcal{I}^{\gamma} f) \leq \dim_B Gr(\mathcal{I}^{\gamma} f) \leq 3-\mu$ provided that $0 < \gamma_1, \gamma_2 < 1$. There are other results when f has the graph of dimension 2, and when f is of bounded variation. Particular emphasis is given to the cases where $\dim_B Gr(\mathcal{I}^{\gamma} f) = 2$; this is the minimum possible for a surface and this is the case where an exact value is found. There is an example of a function that is not of bounded variation but with the graph of the Riemann-Liouville integral function nevertheless equal to 2; these surfaces are illustrated for various choices of parameters.

Let us conclude this report with some remarks. Theorem 3.1 is more general having fewer restrictions. We can construct a continuous function (Weierstrass type, for instance, we refer to [24]) having 2.1-dimension, in this case, we can give bounds for fractal dimensions of the fractional integral of mixed R-L type by using Theorem 3.1

but we cannot use Theorem 5.1 for this case. Now, let f be a Hölder continuous function with Hölder exponent 0.5 and dim $f = 2.5$, in this case, we can use the Theorem 4.1 and also the Theorem 3.1 for the bounds of the fractal dimensions of the fractional integral of mixed R-L type by we cannot apply the Theorem 5.1 for this case. Similarly, we can illustrate other theorems. So, from the above discussion, we can conclude that each theorem has special attention and importance. This work is also an attempt to give geometric and physical interpretations to the fractional integrals by using fractal dimension as a suitable tool. Overall, this paper may be viewed as a contribution to link two fields-fractional calculus and fractal geometry. The paper should be of interest to a broad readership including those interested in fractional calculus and fractal geometry.

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Declarations

Conflict of interest The authors declare no competing interests.

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