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ORIGINAL PAPER

Hybrid coupling of finite element and boundary element methods using Nitsche's method and the Calderon projection

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Abstract
In this paper, we discuss a hybridised method for FEM-BEM coupling. The coupling from both sides use a Nitsche-type approach to couple to the trace variable. This leads to a formulation that is robust and flexible with respect to approximation spaces and can easily be combined as a building block with other hybridised methods. Energy error norm estimates and the convergence of Jacobi iterations are proved and the performance of the method is illustrated on some computational examples.

Keywords FEM-BEM coupling · Nitsche's method · Hybridised methods

1 Introduction

The coupling of finite element (FEM) and boundary element (BEM) methods is the most widely used approach for solving multi-physical problems on an unbounded domain. It allows to take advantage of both methods. On the one hand, the BEM reduces the dimension of the problem by using the boundary integral equation, hence it is commonly used in exterior unbounded domains. On the other hand, the

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FEM is known for its robustness and universal applicability even for problem of inhomogeneous or non-linear nature.

The first coupled procedure was introduced by Zienkiewicz, Kelly and Bettess [38]. It was analysed by Brezzi, Johnson and Nédélec [5], [4] and [24] for problem in unbounded domains. It is often referred to as the Johnson-Nédélec coupling. Extension for higher order equations was considered by Wendland [37]. The convergence analysis requires compactness of the double layer potential that can be obtained on smooth boundaries. Furthermore, even for a symmetric discretisation scheme, the coupling method produces a system of equations with a non-symmetric coefficient matrix.

In order to avoid these disadvantages, a symmetric coupling of FEM and BEM was devised by Costabel [14] and Han [22]. The independence of the compactness condition was obtained by using both equations of the Calderón system, contrary to the previously introduced methods that employ only one of the two equations of the Calderón system. Some years later, Sayas [32] showed that the weaker assumption of a Lipschitz coupling interface is sufficient for the Johnson-Nédélec coupling. His analysis has since been simplified by Steinbach [35].

More recent developments have focused on the coupling of BEM with mixed FEM. In [6] and [27] the authors analysed symmetric coupling of BEM and mixed FEM that uses Raviart-Thomas elements. Further work on coupling BEM with mixed FEM with such elements as Brezzi-Douglas-Marini or Brezzi-Douglas-Marini-Fortin was proposed by Carstensen and Funken [9].

Gatica, Heuer and Sayas [21] and [20] introduced the first coupling of BEM and discontinuous Galerkin (DG) methods, in order to exploit the possibility to easily use high order approximation in the latter. Another coupling of interior penalty DG methods with BEM was presented by Of, Rodin, Steinbach and Taus [29]. A general approach using the unified hybridisation technique was presented by Cockburn and Sayas [13]. The class of FEM considered includes the mixed, the DG and the hybridisable discontinuous Galerkin (HDG) methods. Further collaboration of these authors with Gúzman led to a new convergence result published in [12].

In this paper, we present the coupling of FEM and BEM using weak imposition of coupling conditions. Nitsche's method [28] is widely used in the context of FEM for imposing boundary conditions. In addition, methods based on Nitsche's approach have been successfully utilised for BEM domain decomposition problems in [19] and [10], and more recently for weakly imposing boundary conditions for BEM in [2]. Merging these two approaches for FEM and BEM we weakly impose both coupling conditions in a hybridised formulation on the boundary. The hybridisation is made by introducing a trace variable and imposing the coupling in the form of a Nitsche-type Dirichlet condition on the two systems. The use of Nitsche's method allows us to use the Dirichlet trace as the hybrid variable, ensuring continuity by a consistent penalty term. The test function partner of the trace variable then acts to ensure continuity of fluxes. The global system can be constructed using arbitrary approximation orders for the two sub systems and the trace variable and the sub problem can be solved independently. The stability of the method poses no constraint on the approximation spaces and mesh refinement does not require special treatment as in the case of Johnson-Nédélec coupling. This means that the two systems can have independent meshes, that both must be integrated only against the trace variable. We here consider the standard continuous FEM, but the formulation is by and large agnostic to the choice of FEM used in the bulk and the method can be applied with discontinuous FEM as well, such as DG, HDG [11], or HHO [16] using a hybridised coupling on the interior domain boundary. In the case of using discontinuous FEM, our formulation can be interpreted as a hybridised interior penalty formulation of the class of methods discussed in [13]. Finally, we note that, thanks to the use of Nitsche-type mortaring, the method proposed herein can be used in the framework for unfitted hybridised methods introduced in [7]. In that case a surface mesh is required for the definition of the BEM method, but the FEM approximation on the interior domain can be computed on an unfitted bulk discretisation.

As many existing approaches of coupling FEM and BEM, we use Finite Element Tearing and Interconnecting (FETI) and Boundary Element Tearing and Interconnecting (BETI) type of methods [25] to solve the reduced system for the hybrid variable. FETI is formulated using a Schur complement formulation, while BETI is usually formulated in terms of Steklov-Poincaré operators. Although Nitsche's method is an established framework for domain decomposition for finite elements methods such as FETI, it was not recognised by BETI community. In this paper, we demonstrate how the hybrid Nitsche approach can be integrated into the BETI framework.

The rest of the paper is organised as follows. We introduce the model problem in this section. In Section 2, we present on continuous level the symmetric coupling of BEM and FEM formulation known from [14] and [22]. The discrete formulation including weakly imposed coupling condition is introduced and analysed in Section 3. Although the formulation obtained is not symmetric, we comment of how symmetry can be obtained for the associated Steklov-Poincaré operator. We discuss iterative domain decomposition in the model case of a simple relaxed Jacobi algorithm in Section 4 and prove its convergence. In Section 5, we present some numerical results, and we conclude with some remarks in Section 6.

1.1 Model problem

Let us consider the unbounded domain $\Omega = \mathbb{R}^3$. We divide Ω into a bounded internal part Ω^- and an unbounded external part Ω^+ with common Lipschitz boundary Γ, with n the outer unit normal vector of the domain Ω^- on Γ. We let $\partial_n u := \frac{\partial u}{\partial_n}$ denote the outward normal derivative, $f \in L^2(\Omega)$ be a function with support in Ω^- and introduce a function $\varepsilon \in L^\infty(\Omega)$, $\epsilon \geq 0$. Then, we can formulate our model problem as follows

$$\begin{cases} -\Delta u^- + \varepsilon u^- &= f \quad \text{in } \Omega^-, \\ -\Delta u^+ &= 0 \quad \text{in } \Omega^+, \\ u^- &= u^+ \quad \text{on } \Gamma, \\ \partial_n u^- &= \partial_n u^+ \quad \text{on } \Gamma, \\ |u^+| &\rightarrow 0 \quad \text{while } |x| \rightarrow \infty. \end{cases} \tag{1}$$

where $u^i = u|_{\Omega^i}$, $i \in \{-, +\}$. The function ε is introduced to make the interior problem heterogeneous and hence unsuitable for treatment using the boundary element method alone.

Remark 1 It is straightforward to extend the discussion to the case with a smoothly varying diffusion coefficient in Ω^- which has a jump over Γ.

Let us begin by introducing the variational formulation of the coupled system.

2 Variational formulation

Let $\langle \cdot, \cdot \rangle_\Gamma$ denote the $L^2(\Gamma)$-inner product that can be extended to a duality pairing on $H^{-\frac{1}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma)$.

We start with the variational formulation of the internal problem. Applying integration by parts for first equation of (1) for every $v \in H_0^1(\Omega^-)$ we have

$$\int_{\Omega^-} \nabla u \cdot \nabla v \, dx + \int_{\Omega^-} \varepsilon u v \, dx - \langle \partial_n u, \ v \rangle_\Gamma = \int_{\Omega^-} f v \, dx. \tag{2}$$

We define the Green's function for the Laplace operator in $\mathbb{R}^3$ as follows

$$G(x, y) := \tfrac{1}{4\pi |x-y|}.$$

In this paper, we focus on the problem in $\mathbb{R}^3$. A similar analysis can be used for problems in $\mathbb{R}^2$, in which case this definition should be replaced by $G(x, y) := \frac{\log |x-y|}{2\pi}$. Following the standard approach (see, e.g., [34, Chapter 6]), we introduce single layer and double layer operators $\mathcal{V} : H^{-\frac{1}{2}}(\Gamma) \to H^1(\Omega^+)$ and $\mathcal{K} : H^{\frac{1}{2}}(\Gamma) \to H^1(\Omega^+)$ respectively as

$$\begin{aligned} (\mathcal{V}\varphi)(x) &:= \textstyle\int_\Gamma G(x, y)\varphi(y) \, dy \quad \text{for } \varphi \in H^{-\frac{1}{2}}(\Gamma), \\ (\mathcal{K}v)(x) &:= \textstyle\int_\Gamma \frac{\partial G(x,y)}{\partial n_y} v(y) \, dy \quad \text{for } v \in H^{\frac{1}{2}}(\Gamma), \end{aligned}$$

where $x \in \Omega^+ \setminus \Gamma$ and n_y is an outer unit normal vector (for Ω^i, $i \in \{-, +\}$) in the point y.

Following [34, Chapter 1], we define the Dirichlet and Neumann traces

$$\begin{aligned} \gamma_D^i : H^1(\Omega^i) &\to H^{\frac{1}{2}}(\Gamma) \quad \gamma_D^i f(x) &&:= \lim_{\Omega^i \ni y \to x \in \Gamma} f(y), \\ \gamma_N^i : H^1(\Delta, \Omega^i) &\to H^{-\frac{1}{2}}(\Gamma) \quad \gamma_N^i f(x) &&:= \lim_{\Omega^i \ni y \to x \in \Gamma} n_x \cdot \nabla f(y), \end{aligned}$$

where $H^1(\Delta, \Omega^i) := \left\{ v \in H^1(\Omega^i) : \Delta v \in L^2(\Omega^i) \right\}$, for $i \in \{-, +\}$, and n_x is an outer (for Ω^-) normal vector to Γ in the point x. The following results will be useful in what follows.

Lemma 1 (Trace theorem) *Let $i \in \{-, +\}$, then for $\Omega^i \subset \mathbb{R}^3$ the trace operator $\gamma_D^i : H^1(\Omega^i) \to H^{\frac{1}{2}}(\Gamma)$ is bounded for all $v \in H^1(\Omega^i)$*

$$\|\gamma_D^i v\|_{H^{\frac{1}{2}}(\Gamma)} \le C_T \|v\|_{H^1(\Omega^i)}. \tag{3}$$

Proof See [26, Theorem 3.37]. □

We use $\{\cdot\}_\Gamma$ to denote an average of the interior and exterior traces of a function. Then, applying the trace mappings yields to the single-layer, double-layer, and adjoint double-layer potentials and hypersingular boundary integral operator

$$\begin{aligned} V : H^{-\frac{1}{2}}(\Gamma) &\to H^{\frac{1}{2}}(\Gamma), V := \{\gamma_D \mathcal{V}\}_\Gamma, \\ K : H^{\frac{1}{2}}(\Gamma) &\to H^{\frac{1}{2}}(\Gamma), K := \{\gamma_D \mathcal{K}\}_\Gamma, \\ K' : H^{-\frac{1}{2}}(\Gamma) &\to H^{-\frac{1}{2}}(\Gamma), K' := \{\gamma_N \mathcal{V}\}_\Gamma, \\ W : H^{\frac{1}{2}}(\Gamma) &\to H^{-\frac{1}{2}}(\Gamma), W := \{-\gamma_N \mathcal{K}\}_\Gamma. \end{aligned}$$

For the solution u of the problem (1), we have the following boundary integral equations on Γ

$$\begin{pmatrix} \gamma_D^- u \\ \gamma_N^- u \end{pmatrix} = C^- \begin{pmatrix} \gamma_D^- u \\ \gamma_N^- u \end{pmatrix}, \ \begin{pmatrix} \gamma_D^+ u \\ \gamma_N^+ u \end{pmatrix} = C^+ \begin{pmatrix} \gamma_D^+ u \\ \gamma_N^+ u \end{pmatrix}, \tag{4}$$

where $C^\pm : H^{\frac{1}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma)$ denotes two Calderón projectors defined as follows

$$C^\pm := \begin{pmatrix} \frac{1}{2}Id \pm K & \mp V \\ \mp W & \frac{1}{2}Id \mp K' \end{pmatrix}.$$

From the relation (4) for external traces we can construct the following exterior Dirichlet-to-Neumann operator

$$DtN^+ := -W + (\tfrac{1}{2}Id - K') \circ V^{-1} \circ (K - \tfrac{1}{2}Id). \tag{5}$$

Obviously, it makes sense only if the inverse of the operator V exists.

Using Dirichlet-to-Neumann operator (5) we introduce a new variable $\lambda = \gamma_N^+ u = \partial_n u^+$ as

$$\lambda := \left(V^{-1} \circ (K - \tfrac{1}{2}Id)\right) \gamma_D^+ u. \tag{6}$$

The classical symmetric coupling that satisfies the transmission conditions of (1) is as follows

Find $u \in H^1(\Omega^-)$ *and* $\lambda \in H^{-\frac{1}{2}}(\Gamma)$ *such that for all* $v \in H^1(\Omega^-)$ *and* $\zeta \in H^{-\frac{1}{2}}(\Gamma)$

$$\begin{cases} \int_{\Omega^-} \nabla u \cdot \nabla v \, dx + \int_{\Omega^-} \varepsilon u v \, dx + \langle W u, \ v \rangle_\Gamma - \left\langle \left(\frac{1}{2}Id - K'\right)\lambda, \ v \right\rangle_\Gamma = \int_{\Omega^-} f v \, dx, \\ \qquad\qquad\qquad\qquad \left\langle \left(\frac{1}{2}Id - K\right) u, \ \zeta \right\rangle_\Gamma + \langle V\lambda, \ \zeta \rangle_\Gamma = 0. \end{cases} \tag{7}$$

For simplicity, we omit Dirichlet trace for integration over boundary.

2.1 Well-posedness of the continuous problem

The following results are well known (see [14] or [22]), but for reader's convenience we present them in the case of problem (1). Let us propose a more compact formulation of (7).

Find $u \in H^1(\Omega^-)$ *and* $\lambda \in H^{-\frac{1}{2}}(\Gamma)$ *such that for all* $v \in H^1(\Omega^-)$ *and* $\zeta \in H^{-\frac{1}{2}}(\Gamma)$

$$A\left((u,\lambda),(v,\zeta)\right) = \int_{\Omega^-} f v \, dx, \tag{8}$$

where

$$\begin{aligned} A\left((w,\lambda),(v,\zeta)\right) := {}& \textstyle\int_{\Omega^-} \nabla w \cdot \nabla v \, dx + \int_{\Omega^-} \varepsilon u v \, dx - \frac{1}{2}\langle \lambda,\ v\rangle_\Gamma + \frac{1}{2}\langle w,\ \zeta\rangle_\Gamma \\ &+ \langle W w,\ v\rangle_\Gamma + \left\langle K'\lambda,\ v\right\rangle_\Gamma - \langle K w,\ \zeta\rangle_\Gamma + \langle V\lambda,\ \zeta\rangle_\Gamma . \end{aligned}$$

For simplicity we introduce the space $\mathbb{V} := H^1(\Omega^-) \times H^{-\frac{1}{2}}(\Gamma)$ and the associated norm

$$\|(v,\varphi)\|_{\mathbb{V}}^2 := \|v\|_{H^1(\Omega^-)}^2 + \|\varphi\|_{H^{-\frac{1}{2}}(\Gamma)}^2 . \tag{9}$$

We begin by showing the boundedness of the bilinear form A.

Lemma 2 (**Boundedness**) *There exists constant* $\beta > 0$ *such that for all* $w, v \in H^1(\Omega^-)$ *and* $\lambda, \varphi \in H^{-\frac{1}{2}}(\Gamma)$

$$\left|A\left((w,\lambda),(v,\varphi)\right)\right| \le \beta \, \|(w,\lambda)\|_{\mathbb{V}} \, \|(v,\varphi)\|_{\mathbb{V}} . \tag{10}$$

Proof We use the Cauchy-Schwarz inequality, the duality pairing relation and continuity of boundary operators (see [34, Section 6.2–6.5]) to obtain

$$\begin{aligned} \left|A\left((w,\lambda),(v,\varphi)\right)\right| \le {}& \max\{1, \|\varepsilon\|_{L^\infty(\Omega)}\} \, \|w\|_{H^1(\Omega^-)} \, \|v\|_{H^1(\Omega^-)} \\ &+ \tfrac{1}{2}\|w\|_{H^{\frac{1}{2}}(\Gamma)} \|\varphi\|_{H^{-\frac{1}{2}}(\Gamma)} + \tfrac{1}{2}\|\lambda\|_{H^{-\frac{1}{2}}(\Gamma)} \|v\|_{H^{\frac{1}{2}}(\Gamma)} \\ &+ C_K \|w\|_{H^{\frac{1}{2}}(\Gamma)} \|\varphi\|_{H^{-\frac{1}{2}}(\Gamma)} + C_V \|\lambda\|_{H^{-\frac{1}{2}}(\Gamma)} \|\varphi\|_{H^{-\frac{1}{2}}(\Gamma)} \\ &+ C_{K'} \|\lambda\|_{H^{-\frac{1}{2}}(\Gamma)} \|v\|_{H^{\frac{1}{2}}(\Gamma)} + C_W \|w\|_{H^{\frac{1}{2}}(\Gamma)} \|v\|_{H^{\frac{1}{2}}(\Gamma)} . \end{aligned}$$

We finished by applying the trace inequality (3) to terms including $\|\cdot\|_{H^{\frac{1}{2}}(\Gamma)}$ norm,

$$\|v\|_{H^{\frac{1}{2}}(\Gamma)} + \|w\|_{H^{\frac{1}{2}}(\Gamma)} \le C_T \left(\|v\|_{H^1(\Omega^-)} + \|w\|_{H^1(\Omega^-)}\right).$$

□

The next step in proving the existence and uniqueness of the solution is coercivity of the bilinear form A.

Lemma 3 (**Coercivity**) *There exists constant* $\alpha > 0$ *such that for all* $v \in H^1(\Omega^-)$ *and* $\varphi \in H^{-\frac{1}{2}}(\Gamma)$

$$A\left((v,\varphi),(v,\varphi)\right) \ge \alpha \, \|(v,\varphi)\|_{\mathbb{V}}^2 . \tag{11}$$

Proof Consider the form $A\left((v,\varphi),(v,\varphi)\right)$. Because of the relation between the double layer potential and its adjoint, the associated terms cancel. Using coercivity of

V (see [34, Theorem 6.22]), and coercivity of W on functions with zero average (see [34, Theorem 6.24] we obtain

$$A\left((v,\varphi),(v,\varphi)\right) \geq c_\epsilon \left\|v\right\|^2_{H^1(\Omega^-)} + \alpha_V \left\|\varphi\right\|^2_{H^{-\frac{1}{2}}(\Gamma)} + \alpha_W \left\|v-\bar{v}\right\|^2_{H^{\frac{1}{2}}(\Gamma)}$$

where $\bar{v} = |\Gamma|^{-1}\int_\Gamma v \, \mathrm{d}s$ and $c_\varepsilon = \min(1,\varepsilon)$. This shows (11) when $\varepsilon > 0$. For the case $\epsilon = 0$ we need a Poincaré inequality of the form

$$c_P \|v\|_{H^1(\Omega)} \leq \|\nabla v\|_{L^2(\Omega)} + |g(v)| \tag{12}$$

where $g(v)$ is some functional that is non-zero for constant non-zero v [18, Lemma B.63]. We claim that this holds with $g(v) = \langle V\varphi(v), \varphi(v)\rangle_\Gamma$ where $\varphi(v)$ is defined by the second equation of (7). We immediately see that if this is true then there exists $\alpha > 0$ such that

$$A\left((v,\varphi),(v,\varphi)\right) \geq \alpha \left\|v\right\|^2_{H^1(\Omega^-)}$$

We need to show that for constant $v \neq 0$ there holds $\langle V\varphi(v), \varphi(v)\rangle_\Gamma \neq 0$. Let $v|_\Gamma = u_0$ be a non-zero constant. Then, we need to study

$$\left\langle \left(\tfrac{1}{2}Id - K\right)u_0,\ \zeta\right\rangle_\Gamma + \langle V\varphi(v),\ \zeta\rangle_\Gamma = 0, \quad \forall \zeta \in H^{-\frac{1}{2}}(\Gamma).$$

We argue by contradiction. Assume that $\varphi(v) = 0$. Then, u_0 is the trace of a solution to the homogeneous Neumann problem in Ω^-. However by the first line of the left relation of (4) there holds for all such traces

$$\left\langle \left(\tfrac{1}{2}Id - K\right)u_0,\ \zeta\right\rangle_\Gamma = \langle Id\, u_0,\ \zeta\rangle_\Gamma\,, \quad \forall \zeta \in H^{-\frac{1}{2}}(\Gamma). \tag{13}$$

Hence the functional is defined by

$$\langle V\varphi(v),\ \zeta\rangle_\Gamma = -\langle Id\, u_0,\ \zeta\rangle_\Gamma\,, \quad \forall \zeta \in H^{-\frac{1}{2}}(\Gamma).$$

However since the operator on the left-hand side, defined by, V is injective it follows that $\varphi(v) \neq 0$, which leads to a contradiction. Hence $\langle V\varphi(v), \varphi(v)\rangle_\Gamma \neq 0$ for v constant. This concludes the proof. □

The existence and uniqueness of the solution of problem (8) is achieved by using the Lax-Milgram theorem.

3 The discrete problem

In the previous section, we focused on the classical symmetric formulation. Now, using this formulation we propose the weak penalty formulation of a couple problem. The discretisation is made by using finite element methods inside the domain and boundary integral methods outside the domain. We prove the well-posedness of the discrete problem and analyse the convergence of the proposed method.

We assume that Ω^- is a polyhedral domain. The boundary Γ may be decomposed in a set of n planar surfaces $\{\Gamma_j\}_{j=1}^n$. It will be convenient to use following broken

Sobolev spaces over the polyhedral boundary Γ. For $s > 1$ define

$$\widetilde{H}^s(\Gamma) := \left\{ v \in H^1(\Gamma) : \; v|_{\Gamma_j} \in H^s(\Gamma_j), \; 1 \leq j \leq n \right\}.$$

When $s > 1$ the norm on $\widetilde{H}^s(\Gamma)$, is defined as the broken norm over the faces of the polyhedral boundary Γ

$$\|v\|_{\widetilde{H}^s(\Gamma)} := \left(\sum_{j=1}^{n} \|v\|^2_{H^s(\Gamma_j)} \right)^{\frac{1}{2}}.$$

When $0 \leq s \leq 1$ the space $\widetilde{H}^s(\Gamma)$ coincides with the usual space $H^s(\Gamma)$ and their norms are the same (for more details see [31, Definition 4.1.48]).

Let $\mathcal{T}_h$ be a triangulation of $\overline{\Omega}^-$ made of tetrahedrons. For each of element $K \in \mathcal{T}_h$, $h_K := \operatorname{diam}(K)$, and $h := \max_{K \in \mathcal{T}_h} h_K$. Let $\mathcal{G}_i$, $i = 1, 2$ denote two different surface triangulation of the boundary Γ. For notation convenience we here assume that the trace mesh of $\mathcal{T}_h$ and the $\mathcal{G}_i$ all have similar local mesh size.

The following result will be useful in what follows.

Lemma 4 (**Trace inequality**) *There exists $C_{max} > 0$, independent of h_K, such that for all $K \in \mathcal{T}_h$ and polynomial function v in K the following discrete trace inequality holds*

$$h_K^{\frac{1}{2}} \|v\|_{L^2(\partial K)} \leq C_{max} \|v\|_{L^2(K)}. \tag{14}$$

Proof See [15, Lemma 1.46]. □

To discretise the problem (7) over the triangulation one can choose either continuous or discontinuous finite elements. For simplicity of the analysis we choose the following spaces

$$\begin{aligned}
V_h^j &:= \left\{ v_h \in C^0\left(\Omega^-\right) : \; v_h|_K \in \mathbb{P}_j(K) \;\; \forall\, K \in \mathcal{T}_h \right\}, \\
W_h^k &:= \left\{ w_h \in C^0(\Gamma) : \; w_h|_E \in \mathbb{P}_k(E) \;\; \forall\, E \in \mathcal{G}_1 \right\}, \\
\Lambda_h^l &:= \left\{ \lambda_h \in L^2(\Gamma) : \; \lambda_h|_E \in \mathbb{P}_l(E) \;\; \forall\, E \in \mathcal{G}_1 \right\}, \\
M_h^m &:= \left\{ \widetilde{v}_h \in L^2(\Gamma) : \; \widetilde{v}_h|_E \in \mathbb{P}_m(E) \;\; \forall\, E \in \mathcal{G}_2 \right\}.
\end{aligned}$$

Let us denote $\mathcal{V}_h := V_h^j \times W_h^k$ and $v_h = \left(v_h^-, v_h^+\right) \in \mathcal{V}_h$. Using the above spaces we propose the hybrid discrete formulation of the problem (7)

Find $(u_h, \lambda_h, \widetilde{u}_h) \in \mathcal{V}_h \times \Lambda_h^l \times M_h^m$ such that for all $(v_h, \varphi_h, \widetilde{v}_h) \in \mathcal{V}_h \times \Lambda_h^l \times M_h^m$

$$a_h\left(\left(u_h^-, \widetilde{u}_h\right), \left(v_h^-, \widetilde{v}_h\right)\right) + b_h\left(\left(u_h^+, \lambda_h, \widetilde{u}_h\right), \left(v_h^+, \varphi_h, \widetilde{v}_h\right)\right) = \int_{\Omega^-} f v_h^- dx, \tag{15}$$

where

$$\begin{aligned} a_h\left(\left(w_h, \widetilde{w}_h\right), \left(v_h, \widetilde{v}_h\right)\right) := &\quad \int_{\Omega^-} \nabla w_h : \nabla v_h dx + \int_{\Omega^-} \varepsilon w_h v_h dx \\ &- \left\langle \partial_n w_h,\ v_h - \widetilde{v}_h \right\rangle_\Gamma - \left\langle w_h - \widetilde{w}_h,\ \partial_n v_h \right\rangle_\Gamma \\ &+ \tfrac{\tau_F}{h} \left\langle w_h - \widetilde{w}_h,\ v_h - \widetilde{v}_h \right\rangle_\Gamma, \\ b_h\left(\left(w_h, \lambda_h, \widetilde{w}_h\right), \left(v_h, \varphi_h, \widetilde{v}_h\right)\right) := &\quad \left\langle \left(\tfrac{1}{2} Id - K\right) w_h,\ \varphi_h \right\rangle_\Gamma + \left\langle V\lambda_h,\ \varphi_h \right\rangle_\Gamma \\ &+ \left\langle W w_h,\ v_h \right\rangle_\Gamma - \left\langle \left(\tfrac{1}{2} Id - K'\right) \lambda_h,\ v_h \right\rangle_\Gamma \\ &+ \left\langle \lambda_h,\ v_h - \widetilde{v}_h \right\rangle_\Gamma - \left\langle w_h - \widetilde{w}_h,\ \varphi_h \right\rangle_\Gamma \\ &+ \tau_B \left\langle w_h - \widetilde{w}_h,\ v_h - \widetilde{v}_h \right\rangle_\Gamma . \end{aligned}$$

The formulation of bilinear form a_h is well known for example from [17]. As usual for symmetric Nitsche methods the stabilisation parameter $\tau_F > 0$ has to be chosen sufficiently large for stability.

Remark 2 (*Impedance boundary condition*) The hybrid weakly imposed Dirichlet and Neumann boundary conditions is related to an impedance boundary condition of the type

$$\widetilde{u} = -\gamma \frac{\partial u}{\partial n} + u,$$

with $\gamma = \frac{h}{\tau_F} = \tau_B$. This can be seen considering terms associated with $\widetilde{v}_h$ in above definition of the bilinear forms.

Remark 3 (*Relation to a standard Nitsche-type method without hybridisation*) To transform the hybridised method to a Nitsche-type method without hybridisation we proceed as follows. The trace variable $\widetilde{u}_h$ is eliminated by replacing it by a linear combination of u_h^+ and u_h^- (i.e. replace $\widetilde{u}_h$ with $\omega u_h^+ + (1-\omega)u_h^-$, $\omega \in [0, 1]$) and similarly replace the test function $\widetilde{v}_h$ replaced by the same linear combination of the test functions v_h^+ and v_h^- (i.e. replace $\widetilde{v}_h$ with $\omega v_h^+ + (1-\omega)v_h^-$) (see [8, Section 4.2]). The below analysis carries over verbatim to this case.

3.1 Symmetric formulation

Despite using the symmetric Nitsche method, our whole system is not symmetric. This is a consequence of the lack of symmetry of the boundary element method with weak imposition. We can use the Steklov-Poincaré operator to eliminate the flux variable, so that the non-symmetric method above is transformed into a symmetric reduced system as we show below.

The following equations are associated with bilinear form b_h from (15) reads

$$\begin{aligned} -\left\langle \left(\tfrac{1}{2} Id - K\right) w_h, \varphi_h \right\rangle_\Gamma - \left\langle V\lambda_h, \varphi_h \right\rangle_\Gamma &= -\left\langle w_h - \widetilde{w}_h, \varphi_h \right\rangle_\Gamma, \\ -\left\langle W w_h, v_h \right\rangle_\Gamma + \left\langle \left(\tfrac{1}{2} Id - K'\right) \lambda_h, v_h \right\rangle_\Gamma - \left\langle \lambda_h, v_h - \widetilde{v}_h \right\rangle_\Gamma &= \tau_B \left\langle w_h - \widetilde{w}_h, v_h - \widetilde{v}_h \right\rangle_\Gamma . \end{aligned}$$

Similar to the continuous formulation we use the Dirichlet-to-Neumann operator (5) to obtain

$$\lambda_h := \left(V^{-1} \circ (K - \tfrac{1}{2}Id)\right) w_h + V^{-1}\left(w_h - \widetilde{w}_h\right). \tag{16}$$

Injecting this relation into the second equation leads to the formulation of the new symmetric bilinear form

$$\begin{aligned} \widehat{b}_h\left(\left(w_h, \widetilde{w}_h\right), \left(v_h, \widetilde{v}_h\right)\right) := \quad & \langle W w_h,\ v_h\rangle_\Gamma - \left\langle \left(\tfrac{1}{2}Id - K'\right) V^{-1}(K - \tfrac{1}{2}Id) w_h,\ v_h \right\rangle_\Gamma \\ & - \left\langle \left(\tfrac{1}{2}Id - K'\right) V^{-1}\left(w_h - \widetilde{w}_h\right),\ v_h \right\rangle_\Gamma \\ & + \left\langle V^{-1}(K - \tfrac{1}{2}Id) w_h,\ v_h - \widetilde{v}_h \right\rangle_\Gamma \\ & + \left\langle V^{-1}\left(w_h - \widetilde{w}_h\right),\ v_h - \widetilde{v}_h \right\rangle_\Gamma + \tau_B \left\langle w_h - \widetilde{w}_h,\ v_h - \widetilde{v}_h \right\rangle_\Gamma . \end{aligned}$$

For clarity we carry out the analysis of the discrete problem using non-symmetric formulation.

3.2 Well-posedness of the discrete problem

Let us consider the following norms

$$\begin{aligned} \|(w_h, \widetilde{w}_h)\|^2_{\mathcal{F}_*} &:= \|w_h\|^2_{H^1(\Omega^-)} + \tfrac{\tau_F}{h}\|w_h - \widetilde{w}_h\|^2_{L^2(\Gamma)}, \\ \|(w_h, \widetilde{w}_h)\|^2_{\mathcal{F}} &:= \|(w_h, \widetilde{w}_h)\|^2_{\mathcal{F}_*} + h\|\partial_n w_h\|^2_{L^2(\Gamma)}, \end{aligned} \tag{17}$$

$$\begin{aligned} \|(w_h, \lambda_h, \widetilde{w}_h)\|^2_{\mathcal{B}_*} &:= \|w_h\|^2_{H^{\frac{1}{2}}(\Gamma)} + \|\lambda_h\|^2_{H^{-\frac{1}{2}}(\Gamma)} + \tau_B\|w_h - \widetilde{w}_h\|^2_{L^2(\Gamma)}, \\ \|(w_h, \lambda_h, \widetilde{w}_h)\|^2_{\mathcal{B}} &:= \|(w_h, \lambda_h, \widetilde{w}_h)\|^2_{\mathcal{B}_*} + h\|\lambda_h\|^2_{L^2(\Gamma)}. \end{aligned} \tag{18}$$

Lemma 5 (Equivalence of the norms) *For all* $(w_h, \lambda_h, \widetilde{w}_h) \in \mathcal{V}_h \times \Lambda_h^l \times M_h^m$ *there exist positive constants* $C_\mathcal{F}, C_\mathcal{B}$ *such that*

$$\left\|\left({w_h}^-, \widetilde{w}_h\right)\right\|_{\mathcal{F}_*} \le \left\|\left({w_h}^-, \widetilde{w}_h\right)\right\|_{\mathcal{F}} \le C_\mathcal{F}\left\|\left({w_h}^-, \widetilde{w}_h\right)\right\|_{\mathcal{F}_*}, \tag{19}$$

$$\left\|\left({w_h}^+, \lambda_h, \widetilde{w}_h\right)\right\|_{\mathcal{B}_*} \le \left\|\left({w_h}^+, \lambda_h, \widetilde{w}_h\right)\right\|_{\mathcal{B}} \le C_\mathcal{B}\left\|\left({w_h}^+, \lambda_h, \widetilde{w}_h\right)\right\|_{\mathcal{B}_*}. \tag{20}$$

Proof For (19) we use the trace inequality (14) and for (20) we use the inverse inequality $h^{\frac{1}{2}}\|\lambda_h\|_{L^2(\Gamma)} \le C\|\lambda_h\|_{H^{-\frac{1}{2}}(\Gamma)}$ (see [31, Remark 4.4.4]). □

After introducing the norms that we use we can start by showing the boundedness of the bilinear forms a_h and b_h.

Lemma 6 (Boundedness) *There exists positive constant* $\beta_\mathcal{F}$ *such that for all* $w, v \in H^{\frac{3}{2}+\delta}(\Omega^-)$, *for* $\delta > 0$, *and* $\widetilde{w}, \widetilde{v} \in L^2(\Gamma)$

$$\left|a_h\left((w, \widetilde{w}), (v, \widetilde{v})\right)\right| \le \beta_\mathcal{F}\|(w, \widetilde{w})\|_\mathcal{F}\|(v, \widetilde{v})\|_\mathcal{F}. \tag{21}$$

There exists positive constant $\beta_{\mathcal{B}}$ *such that for all* $w, v \in H^{\frac{1}{2}}(\Gamma)$, $\lambda, \varphi \in L^2(\Gamma)$ *and* $\widetilde{w}, \widetilde{v} \in L^2(\Gamma)$

$$\left|b_h\left((w, \lambda, \widetilde{w}), (v, \varphi, \widetilde{v})\right)\right| \leq \beta_{\mathcal{B}} \left\|(w, \lambda, \widetilde{w})\right\|_{\mathcal{B}} \left\|(v, \varphi, \widetilde{v})\right\|_{\mathcal{B}}. \tag{22}$$

Proof We use Cauchy-Schwarz inequality to obtain (21). In the case of (22), we use Cauchy-Schwarz inequality and Lemma 2. □

To show the discrete well-posedness of (15) we need the ellipticity of the bilinear forms a_h and b_h. Our formulation contains two stabilisation parameters τ_F and τ_B. The first parameter associated with finite element bilinear form must be chosen to allow us to use trace inequality (14). On the other hand, for τ_B for the bilinear form b_h positivity is the only constraint.

Lemma 7 (**Coercivity**) *Assume that* $\tau_B > 0$ *and positive constant* τ_F *is large enough. Then, there exists positive constant* α *such that for all* $(w_h, \lambda_h, \widetilde{w}_h) \in \mathcal{V}_h \times \Lambda_h^l \times M_h^m$

$$\begin{aligned} a_h\left(\left(w_h^-, \widetilde{w}_h\right), \left(w_h^-, \widetilde{w}_h\right)\right) + \quad & b_h\left(\left(w_h^+, \lambda_h, \widetilde{w}_h\right), \left({w_h}^+, \lambda_h, \widetilde{w}_h\right)\right) \\ & \geq \alpha\left(\left\|(w_h, \widetilde{w}_h)\right\|_{\mathcal{F}}^2 + \left\|(w_h, \lambda_h, \widetilde{w}_h)\right\|_{\mathcal{B}}^2\right). \end{aligned} \tag{23}$$

Proof Let us start with bilinear form a_h. First we assume that $\varepsilon \geq \varepsilon_{min} > 0$

$$\begin{aligned} a_h\left(\left(w_h^-, \widetilde{w}_h\right), \left(w_h^-, \widetilde{w}_h\right)\right) = \left|w_h^-\right|_{H^1(\Omega^-)}^2 + \left\|\varepsilon^{1/2} w_h^-\right\|_{\Omega^-}^2 - 2\left\langle \partial_n w_h^-, w_h^- - \widetilde{w}_h \right\rangle_\Gamma \\ + \tfrac{\tau_F}{h} \left\|w_h^- - \widetilde{w}_h\right\|_{L^2(\Gamma)}^2. \end{aligned}$$

Using Cauchy-Schwarz and trace inequality (14), followed by Young's inequality, we arrive at

$$\begin{aligned} a_h\left(\left(w_h^-, \widetilde{w}_h\right), \left(w_h^-, \widetilde{w}_h\right)\right) \geq \quad & \left|w_h^-\right|_{H^1(\Omega^-)}^2 + \varepsilon_{\min}\left\|w_h^-\right\|_{\Omega^-}^2 + \tfrac{\tau_F}{h}\left\|w_h^- - \widetilde{w}_h\right\|_{L^2(\Gamma)}^2 \\ & -2\left\|\partial_n w_h^-\right\|_{L^2(\Gamma)}\left\|w_h^- - \widetilde{w}_h\right\|_{L^2(\Gamma)} \\ \geq \quad & \left|w_h^-\right|_{H^1(\Omega^-)}^2 + \varepsilon_{\min}\left\|w_h^-\right\|_{\Omega^-}^2 + \tfrac{\tau_F}{h}\left\|w_h^- - \widetilde{w}_h\right\|_{L^2(\Gamma)}^2 \\ & -2\left|w_h^-\right|_{H^1(\Omega^-)}\left(C_{max} h^{-\frac{1}{2}}\left\|w_h^- - \widetilde{w}_h\right\|_{L^2(\Gamma)}\right) \\ \geq \quad & \tfrac{1}{2}\left|w_h^-\right|_{H^1(\Omega^-)}^2 + \varepsilon_{\min}\left\|w_h^-\right\|_{\Omega^-}^2 + \tfrac{\tau_F - 2C_{max}^2}{h}\left\|w_h^- - \widetilde{w}_h\right\|_{L^2(\Gamma)}^2. \end{aligned}$$

We finish by applying the equivalence of the norms (19) under the assumption that $\tau_F > 2C_{max}^2$.

In the case of the bilinear form b_h, by using the results from Lemma 3 we obtain, with $\bar{w}_h = |\Gamma|^{-1}\int_\Gamma w_h^+ \, \mathrm{d}s$,

$$\begin{aligned} b_h\left(\left(w_h^+, \lambda_h, \widetilde{w}_h\right), \left({w_h}^+, \lambda_h, \widetilde{w}_h\right)\right) \geq \alpha_V \left\|\lambda_h\right\|_{H^{-\frac{1}{2}}(\Gamma)}^2 + \alpha_W \left\|w_h^+ - \bar{w}_h\right\|_{H^{\frac{1}{2}}(\Gamma)}^2 \\ + \tau_B \left\|w_h^+ - \widetilde{w}_h\right\|_{L^2(\Gamma)}^2. \end{aligned}$$

Observe that when $\varepsilon > 0$ we may bound

$$\begin{aligned}
\|w_h^+\|_{H^{\frac{1}{2}}(\Gamma)} &\leq \|w_h^+ - \bar{w}_h\|_{H^{\frac{1}{2}}(\Gamma)} + \|\bar{w}_h\|_{L^2(\Gamma)} \\
&\leq \|w_h^+ - \bar{w}_h\|_{H^{\frac{1}{2}}(\Gamma)} + \|w_h^+ - \widetilde{w}_h\|_{L^2(\Gamma)} + \|w_h^- - \widetilde{w}_h\|_{L^2(\Gamma)} + \|w_h^-\|_{L^2(\Gamma)} \\
&\leq \|w_h^+ - \bar{w}_h\|_{H^{\frac{1}{2}}(\Gamma)} + \|w_h^+ - \widetilde{w}_h\|_{L^2(\Gamma)} + \|w_h^- - \widetilde{w}_h\|_{L^2(\Gamma)} + C\|w_h^-\|_{H^1(\Omega^-)}
\end{aligned}$$

where we applied the trace inequality (3) in the last step. The right-hand side is controlled by the lower bounds on a_h and b_h above. Once again, we finish by applying the equivalence of the norms (20).

In case $\varepsilon = 0$ we need to show that a Poincaré inequality holds, similar to (12), this time on the form

$$\begin{aligned}
c_P \|w_h^-\|_{H^1(\Omega^-)} \leq |w_h^-|_{H^1(\Omega^-)} + \tfrac{1}{h} \left\|w_h^- - \widetilde{w}_h\right\|_{L^2(\Gamma)} + \tfrac{1}{h} \left\|w_h^+ - \widetilde{w}_h\right\|_{L^2(\Gamma)} \\
+ \|\lambda_h\|_{H^{-\frac{1}{2}}(\Gamma)} + \left\|w_h^+ - \bar{w}_h\right\|_{H^{\frac{1}{2}}(\Gamma)}.
\end{aligned} \tag{24}$$

To this end, since coercivity holds up to a constant, we may assume that $w_h^-|_\Gamma = \widetilde{w}_h = w_h^+ = \bar{w}_h$ and proceed verbatim as in the continuous case, since in that case the continuous and discrete expressions corresponding to (13) are the same. ☐

The existence and uniqueness of the solution of problem (15) is achieved by using the Lax-Milgram theorem. In addition, the proposed method is consistent for sufficiently smooth exact solutions as the following result shows.

Lemma 8 (**Consistency**) *Let $\delta > 0$, $u^- \in H^{\frac{3}{2}+\delta}(\Omega^-)$, $u^+ \in H^{\frac{1}{2}}(\Gamma)$ and $\partial_n u^+ = \lambda \in L^2(\Gamma)$ be the solution of problem (1) and $\widetilde{u} = u^- = u^+$ on Γ. If $(u_h, \lambda_h, \widetilde{u}_h) \in \mathcal{V}_h \times \Lambda_h^l \times M_h^m$ solves (15) then, for all $(v_h, \varphi_h, \widetilde{v}_h) \in \mathcal{V}_h \times \Lambda_h^l \times M_h^m$ the following holds*

$$a_h\left(\left(u^- - u_h^-, \widetilde{u} - \widetilde{u}_h\right), \left(v_h^-, \widetilde{v}_h\right)\right) + b_h\left(\left(u^+ - u_h^+, \lambda - \lambda_h, \widetilde{u} - \widetilde{u}_h\right), \left(v_h^+, \varphi_h, \widetilde{v}_h\right)\right) = 0.$$

Proof Because of the transmission conditions from (1) we have $u = u^+ = u^-$ and $\partial_n u = \partial_n u^+ = \partial_n u^-$ on Γ

$$\begin{aligned}
a_h\left(\left(u - u_h^-, \widetilde{u} - \widetilde{u}_h\right), \left(v_h^-, \widetilde{v}_h\right)\right) &= \langle \partial_n u, \widetilde{v}_h\rangle_\Gamma - \left\langle u - \widetilde{u}, \partial_n v_h^-\right\rangle_\Gamma \\
&\quad + \tfrac{\tau_F}{h} \left\langle u - \widetilde{u}, v_h^- - \widetilde{v}_h\right\rangle_\Gamma, \\
b_h\left(\left(u - u_h^+, \lambda - \lambda_h, \widetilde{u} - \widetilde{u}_h\right), \left(v_h^+, \varphi_h, \widetilde{v}_h\right)\right) &= \ -\langle \lambda, \widetilde{v}_h\rangle_\Gamma - \langle u - \widetilde{u}, \varphi\rangle_\Gamma \\
&\quad + \tau_B \left\langle u - \widetilde{u}, v_h^+ - \widetilde{v}_h\right\rangle_\Gamma.
\end{aligned}$$

By adding above expressions and using the facts that $\lambda = \partial_n u$ and $\widetilde{u} = u$ on Γ, we obtain consistency. ☐

3.3 Error analysis

In this section we present the error estimates for the method. These estimates are proved using the following norm

$$\left\|\left(u, \lambda, \widetilde{u}\right)\right\|_{h} := \left\|\left(u^{-}, \widetilde{u}\right)\right\|_{\mathcal{F}} + \left\|\left(u^{+}, \lambda, \widetilde{u}\right)\right\|_{\mathcal{B}}. \tag{25}$$

The first step is the following version of Cea's lemma.

Proposition 1 (Cea's Lemma) *Let* $\delta > 0$, $u^{-} \in H^{\frac{3}{2}+\delta}(\Omega^{-})$, $u^{+} \in H^{\frac{1}{2}}(\Gamma)$ *and* $\partial_n u^{+} = \lambda \in L^2(\Gamma)$ *be the solution of problem (1),* $\widetilde{u} = u^{-} = u^{+}$ *on* Γ, *and let* $(u_h, \lambda_h, \widetilde{u}_h) \in \mathcal{V}_h \times \Lambda_h^l \times M_h^m$ *solve (15). Then, there exists* $C > 0$, *independent of* h, *such that*

$$\|(u-u_h, \lambda-\lambda_h, \widetilde{u}-\widetilde{u}_h)\|_h \leq C \inf_{(v_h,\varphi_h,\widetilde{v}_h)\in\mathcal{V}_h\times\Lambda_h^l\times M_h^m} \|(u-v_h, \lambda-\varphi_h, \widetilde{u}-\widetilde{v}_h)\|_h. \tag{26}$$

Proof Let us denote

$$\begin{aligned} A_h\left((w_h, \phi_h, \widetilde{w}_h), (v_h, \varphi_h, \widetilde{v}_h)\right) := \quad & a\left(\left(w_h^{-}, \widetilde{w}_h\right), \left(v_h^{-}, \widetilde{v}_h\right)\right) \\ & + b\left(\left(w_h^{+}, \phi_h, \widetilde{w}_h\right), \left(v_h^{+}, \varphi_h, \widetilde{v}_h\right)\right). \end{aligned}$$

By Lemma 7 there exists $\alpha > 0$, independent of h, such that for all $(v_h, \varphi_h, \widetilde{v}_h) \in \mathcal{V}_h \times \Lambda_h^l \times M_h^m$ there exists $(w_h, \phi_h, \widetilde{w}_h) \in \mathcal{V}_h \times \Lambda_h^l \times M_h^m$ with $\|(w_h, \phi_h, \widetilde{w}_h)\|_h = 1$, and

$$A_h\left((v_h, \varphi_h, \widetilde{v}_h), (w_h, \phi_h, \widetilde{w}_h)\right) \geq \alpha \|(v_h, \varphi_h, \widetilde{v}_h)\|_h. \tag{27}$$

Now using Lemma 6, we get continuity of A_h, there exists $\beta > 0$

$$|A_h\left((v, \varphi, \widetilde{v})\right), (w, \phi, \widetilde{w})| \leq \beta \|(v, \varphi, \widetilde{v})\|_h \|(w, \phi, \widetilde{w})\|_h. \tag{28}$$

Let $(v_h, \varphi_h, \widetilde{v}_h) \in \mathcal{V}_h \times \Lambda_h^l \times M_h^m$. Then, using the triangle inequality we see that

$$\begin{aligned} \|(u-u_h, \lambda-\lambda_h, \widetilde{u}-\widetilde{u}_h)\|_h \leq \quad & \|(u-v_h, \lambda-\varphi_h, \widetilde{u}-\widetilde{v}_h)\|_h \\ & + \|(v_h-u_h, \varphi_h-\lambda_h, \widetilde{v}_h-\widetilde{u}_h)\|_h. \end{aligned}$$

and it follows from (27), Lemma 8 and (28) that

$$\begin{aligned} \|(v_h-u_h, \varphi_h-\lambda_h, \widetilde{v}_h-\widetilde{u}_h)\|_h \leq \; & \tfrac{1}{\alpha} A_h\left((v_h-u, \varphi_h-\lambda, \widetilde{v}_h-\widetilde{u}), (w_h, \phi_h, \widetilde{w}_h)\right) \\ & + \tfrac{1}{\alpha} A_h\left((u-u_h, \lambda-\lambda_h, \widetilde{u}-\widetilde{u}_h), (w_h, \phi_h, \widetilde{w}_h)\right) \\ \leq \; & \tfrac{\beta}{\alpha} \|(v_h-u, \varphi_h-\lambda, \widetilde{v}_h-\widetilde{u})\|_h. \end{aligned}$$

Thus, we get (26) with $C := 1 + \frac{\beta}{\alpha}$. □

Lemma 9 (Energy norm estimates) *For* $s > \frac{3}{2}$, $r > 1$ *and* $p \geq \frac{1}{2}$, *let* $u^{-} \in H^{s}(\Omega^{-})$, $u^{+} \in \widetilde{H}^{r}(\Gamma)$ *and* $\partial_n u^{+} = \lambda \in \widetilde{H}^{p}(\Gamma)$ *be the solution of problem (1). On* Γ *there holds* $u^{-} = u^{+} = \widetilde{u}$ *on* Γ. *Let* $(u_h, \lambda_h, \widetilde{u}_h) \in \mathcal{V}_h \times \Lambda_h^l \times M_h^m$ *solve (15). If* *the mesh is quasi uniform, then there exists* $C > 0$, *independent of* h, *such that*

$$\|(u-u_h, \lambda-\lambda_h, \widetilde{u}-\widetilde{u}_h)\|_h \leq C\left(h^{\sigma-1}\|u\|_{H^\sigma(\Omega^-)} + h^{\phi-\frac{1}{2}}\|u\|_{\widetilde{H}^\phi(\Gamma)} + h^{\xi-\frac{1}{2}}\|u\|_{\widetilde{H}^\xi(\Gamma)} + h^{\psi+\frac{1}{2}}\|\lambda\|_{\widetilde{H}^\psi(\Gamma)}\right), \quad (29)$$

where $\sigma = \min\{j+1, s\}$, $\phi = \min\{k+1, r\}$, $\xi = \min\{m+1, r\}$ *and* $\psi = \min\{l+1, p\}$.

Proof The result is a consequence of (26) and approximation. Applying triangle inequality and trace inequality [3, Theorem 1.6.6] followed by Young's inequality, we obtain

$$\begin{aligned}
\|(u-v_h, \widetilde{u}-\widetilde{v}_h)\|^2_{\mathcal{F}} :=& \quad \|u-v_h\|^2_{H^1(\Omega^-)} + \tfrac{\tau_F}{h}\|u-v_h-(\widetilde{u}-\widetilde{v}_h)\|^2_{L^2(\Gamma)} \\
& \quad + h\,\|\partial_n u - \partial_n v_h\|^2_{L^2(\Gamma)} \\
\leq& \quad \|u-v_h\|^2_{H^1(\Omega^-)} + \tfrac{\tau_F}{h}\|\widetilde{u}-\widetilde{v}_h\|^2_{L^2(\Gamma)} \\
& + C_1\left(\tfrac{\tau_F}{h^2}\|u-v_h\|^2_{L^2(\Omega^-)} + \tau_F \textstyle\sum_{K\in\mathcal{T}_h}\|u-v_h\|^2_{H^1(K)}\right) \\
& + C_2\left(\textstyle\sum_{K\in\mathcal{T}_h}\|u-v_h\|^2_{H^1(K)} + h^2\sum_{K\in\mathcal{T}_h}\|u-v_h\|^2_{H^2(K)}\right).
\end{aligned}$$

Using approximation results [18, Theorem 1.109] and [31, Theorem 4.3.19] for the second term containing hybrid variables, we claim

$$\inf_{(v_h,\widetilde{v}_h)\in\mathcal{V}_h\times M_h^m}\|(u-v_h, \widetilde{u}-\widetilde{v}_h)\|_{\mathcal{F}} \leq C_{\mathcal{F}}\left(h^{\sigma-1}\|u\|_{H^\sigma(\Omega^-)} + h^{\xi-\frac{1}{2}}\|u\|_{\widetilde{H}^\xi(\Gamma)}\right).$$

For the boundary part, by applying triangle inequality, we obtain

$$\begin{aligned}
\|(u-v_h, \lambda-\lambda_h, \widetilde{u}-\widetilde{v}_h)\|^2_{\mathcal{B}} :=& \ \|u-v_h\|^2_{H^{\frac{1}{2}}(\Gamma)} + \|\lambda-\lambda_h\|^2_{H^{-\frac{1}{2}}(\Gamma)} + h\,\|\lambda-\lambda_h\|^2_{L^2(\Gamma)} \\
& + \tau_B\,\|u-v_h-(\widetilde{u}-\widetilde{v}_h)\|^2_{L^2(\Gamma)} \\
\leq& \ \|u-v_h\|^2_{H^{\frac{1}{2}}(\Gamma)} + \|\lambda-\lambda_h\|^2_{H^{-\frac{1}{2}}(\Gamma)} + h\,\|\lambda-\lambda_h\|^2_{L^2(\Gamma)} \\
& + \tau_B\,\|u-v_h\|^2_{L^2(\Gamma)} + \tau_B\,\|\widetilde{u}-\widetilde{v}_h\|^2_{L^2(\Gamma)}.
\end{aligned}$$

Using approximation results [31, Theorems 4.3.19, 4.3.20 and 4.3.22] , we claim

$$\inf_{(v_h,\lambda_h,\widetilde{v}_h)\in\mathcal{V}_h\times\Lambda_h^l\times M_h^m}\|(u-v_h, \lambda-\lambda_h, \widetilde{u}-\widetilde{v}_h)\|_{\mathcal{B}} \leq C_{\mathcal{B}}\left(h^{\phi-\frac{1}{2}}\|u\|_{\widetilde{H}^\phi(\Gamma)} + h^{\xi}\|u\|_{\widetilde{H}^\xi(\Gamma)} + h^{\psi+\frac{1}{2}}\|\lambda\|_{\widetilde{H}^\psi(\Gamma)}\right).$$

We conclude the proof by applying Proposition 1. □

Remark 4 If M_h^m is conforming trace space for V_h^j and W_h^k, we can approximate terms containing hybrid variables $\widetilde{u}, \widetilde{v}_h$ with the H^σ-norm for interior domain and with the H^ϕ-norm for exterior domain. However, it does not improve the convergence rate.

4 Iterative solution

For the solution of the linear system we will iterate on the Schur complement for the trace variable, solving independently in the two sub domains. To justify this split approach we here show that a simple relaxed Jacobi iteration on the two systems will converge. The condition number of the Schur complement can be analysed using the arguments of [7, Section 4].

1. Given $\widetilde{u}^n$ solve for u^{n+1} and λ^{n+1} by solving the linear system
$$A_h[(u^{n+1}, \lambda^{n+1}, \widetilde{u}^n), (v, \varphi, 0)] = \int_{\Omega^-} f v \, dx.$$
2. Given u^{n+1} and λ^{n+1}, solve for the new trace variable $\widetilde{u}^{n+1}$, for $\sigma > 0$ and $\tau_h := \left(\frac{\tau_F}{h} + \tau_B\right)$,
$$A_h[(u^{n+1}, \lambda^{n+1}, \widetilde{u}^{n+1}), (0, 0, \widetilde{v})] + \sigma\tau_h \left\langle \widetilde{u}^{n+1} - \widetilde{u}^n, \widetilde{v} \right\rangle_\Gamma = 0.$$

To prove that the iterative algorithm converges we only need to show that if $f = 0$, u^{n+1}, $\widetilde{u}^{n+1}$ and λ^{n+1} all go to zero as $n \to \infty$.

We add and subtract $\widetilde{u}^{n+1}$ in the first equation and add the second to obtain

$$A_h[(u^{n+1}, \lambda^{n+1}, \widetilde{u}^{n+1}), (v, \varphi, \widetilde{v})] + \sigma\tau_h \left\langle \widetilde{u}^{n+1} - \widetilde{u}^n, \widetilde{v} \right\rangle_\Gamma = \left\langle (\widetilde{u}^{n+1} - \widetilde{u}^n), \partial_n v + \varphi - \tau_h((v^- + v^+) \right\rangle_\Gamma .$$

Test this equation with u^{n+1}, λ^{n+1}, $\widetilde{u}^{n+1}$ and use coercivity to obtain

$$\tfrac{1}{2}\sigma\tau_h \|\widetilde{u}^N\|^2_{L^2(\Gamma)} + \sum_{n=0}^{N-1} (\alpha \|(u^{n+1}, \lambda^{n+1}, \widetilde{u}^{n+1})\|_h^2 + \tfrac{1}{2}\sigma\tau_h \|\widetilde{u}^{n+1} - \widetilde{u}^n\|^2_{L^2(\Gamma)})$$
$$\leq \tfrac{1}{2}\sigma\tau_h \|\widetilde{u}^0\|^2_{L^2(\Gamma)} + \sum_{n=0}^{N-1} \left\langle (\widetilde{u}^{n+1} - \widetilde{u}^n), \partial_n u^{n+1} + \lambda^{n+1} - \tau_h((u^{n+1})^- + (u^{n+1})^+) \right\rangle_\Gamma .$$

Here we used the well-known formula

$$\sum_{n=0}^{N-1} \left\langle (\widetilde{u}^{n+1} - \widetilde{u}^n), \widetilde{u}^{n+1} \right\rangle_\Gamma = \tfrac{1}{2}\|\widetilde{u}^N\|^2_{L^2(\Gamma)} - \tfrac{1}{2}\|\widetilde{u}^0\|^2_{L^2(\Gamma)} + \tfrac{1}{2}\sum_{n=0}^{N-1} \|\widetilde{u}^{n+1} - \widetilde{u}^n\|^2_{L^2(\Gamma)}. \tag{30}$$

Considering the terms on the right-hand side and using trace inequality (14) we see that

$$\left\langle (\widetilde{u}^{n+1} - \widetilde{u}^n), \partial_n (u^{n+1})^- \right\rangle_\Gamma \leq \tfrac{1}{4}\sigma\tau_h \|\widetilde{u}^{n+1} - \widetilde{u}^n\|^2_{L^2(\Gamma)} + C_{max}(h\tau_h)^{-1}\sigma^{-1}\|u^{n+1}\|^2_{H^1(\Omega^-)}.$$

Using the duality pairing between $H^{\frac{1}{2}}$ and $H^{-\frac{1}{2}}$ followed by the global inverse inequality $\|\widetilde{u}^{n+1} - \widetilde{u}^n\|_{H^{\frac{1}{2}}(\Gamma)} \leq C_t h^{-\frac{1}{2}} \|\widetilde{u}^{n+1} - \widetilde{u}^n\|_{L^2(\Gamma)}$ (see [31, Theorem 4.4.3]) and Young's inequality we have

$$\left\langle (\widetilde{u}^{n+1} - \widetilde{u}^n), \lambda^{n+1} \right\rangle_\Gamma \leq \tfrac{1}{4}\sigma\tau_h \|\widetilde{u}^{n+1} - \widetilde{u}^n\|^2_{L^2(\Gamma)} + C_t^{-2}(h\tau_h)^{-1}\sigma^{-1}\|\lambda^{n+1}\|^2_{H^{-\frac{1}{2}}(\Gamma)}.$$

Finally

$$\begin{aligned}\left\langle(\widetilde{u}^{n+1}-\widetilde{u}^{n}),\tau_h((u^{n+1})^{-}+(u^{n+1})^{+})\right\rangle_\Gamma \leq \quad & \tfrac{1}{4}\sigma\tau_h\|\widetilde{u}^{n+1}-\widetilde{u}^{n}\|^2_{L^2(\Gamma)}\\ +\sigma^{-1}\big(\tau_h\|(u^{n+1})^{+}-\widetilde{u}^{n+1}\|^2_{L^2(\Gamma)} & +\tau_h\|(u^{n+1})^{-}-\widetilde{u}^{n+1}\|^2_{L^2(\Gamma)}\big)\\ & +2\tau_h\left\langle(\widetilde{u}^{n+1}-\widetilde{u}^{n}),\widetilde{u}^{n+1}\right\rangle_\Gamma .\end{aligned}$$

Using the once again the telescoping property (30) we see that for

$$\sigma > (h\tau_h)^{-1}\alpha^{-1}\max\left\{C_{max},C_t^{-2},h\tau_h\right\}+2,$$

the right-hand sides can all be absorbed in the left-hand side to yield

$$\sum_{n=0}^{N-1}\left(\widetilde{\alpha}\|(u^{n+1},\lambda^{n+1},\widetilde{u}^{n+1})\|^2_h+\tfrac{1}{4}\sigma\tau_h\|\widetilde{u}^{n+1}-\widetilde{u}^{n}\|^2_{L^2(\Gamma)}\right)\leq\tfrac{1}{2}(\sigma-2)\tau_h\|\widetilde{u}^{0}\|^2_{L^2(\Gamma)}.$$

It follows that as $N\to\infty$ u^{n+1}, $\widetilde{u}^{n+1}$ and λ^{n+1} all go to zero, since the sum of the left-hand side has to be bounded by the constant of the right-hand side.

5 Numerical experiments

Let $\boldsymbol{\phi} := [\phi_1,\dots,\phi_j]^T$ be the vector of canonical basis functions of the finite element space V_h^j, $\boldsymbol{\psi} := [\psi_1,\dots,\psi_k]^T$ — the vector of canonical basis functions of W_h^k, $\boldsymbol{\xi} := [\xi_1,\dots,\xi_l]^T$ — the vector of canonical basis functions of Λ_h^l and $\boldsymbol{\theta} := [\theta_1,\dots,\theta_m]^T$ be the vector of canonical basis functions of M_h^m. We define the following matrices and vector associated with the corresponding linear forms

$$\begin{aligned}A_{\alpha\beta}&=a_h((\phi_\alpha,0),(\phi_\beta,0))\ ,\ B^F_{\alpha\beta}=a_h((\phi_\alpha,0),(0,\theta_\beta)),\\ C^F_{\alpha\beta}&=a_h((0,\theta_\alpha),(\phi_\beta,0))\ ,\ D^F_{\alpha\beta}=a_h((0,\theta_\alpha),(0,\theta_\beta)),\\ \widetilde{W}_{\alpha\beta}&=b_h((\psi_\alpha,0,0),(\psi_\beta,0,0))\ ,\ \widetilde{K}_{\alpha\beta}=b_h((\psi_\alpha,0,0),(0,\xi_\beta,0))\\ \widetilde{K'}_{\alpha\beta}&=b_h((0,\xi_\alpha,0),(\psi_\beta,0,0))\ ,\ \widetilde{V}_{\alpha\beta}=b_h((0,\xi_\alpha,0),(0,\xi_\beta,0)),\\ B^D_{\alpha\beta}&=b_h((\psi_\alpha,0,0),(0,0,\theta_\beta))\ ,\ B^N_{\alpha\beta}=b_h((0,\xi_\alpha,0),(0,0,\theta_\beta)),\\ C^D_{\alpha\beta}&=b_h((0,0,\theta_\alpha),(\psi_\beta,0,0))\ ,\ C^N_{\alpha\beta}=b_h((0,0,\theta_\alpha),(0,\xi_\beta,0)),\\ D^B_{\alpha\beta}&=b_h((0,0,\theta_\alpha),(0,0,\theta_\beta))\ ,\ \boldsymbol{f}_\beta=\int_{\Omega^-}f\phi_\beta dx.\end{aligned}$$

In the following we present the full definition of these quantities.

$$A_{\alpha\beta} = \int_{\Omega^-} \nabla\phi_\alpha : \nabla\phi_\beta + \varepsilon\phi_\alpha\phi_\beta dx - \left\langle \partial_n\phi_\alpha, \phi_\beta \right\rangle_\Gamma - \left\langle \phi_\alpha,\ \partial_n\phi_\beta \right\rangle_\Gamma + \tfrac{\tau_F}{h} \left\langle \phi_\alpha,\ \phi_\beta \right\rangle_\Gamma ,$$

$$\widetilde{K'}_{\alpha\beta} = \left\langle \left(\tfrac{1}{2}Id - K'\right) \xi_\alpha,\ \psi_\beta \right\rangle_\Gamma \ ,\ \widetilde{V}_{\alpha\beta} = \left\langle V\xi_\alpha,\ \xi_\beta \right\rangle_\Gamma ,$$

$$\widetilde{W}_{\alpha\beta} = \left\langle (\tau_B Id + W)\, \psi_\alpha,\ \psi_\beta \right\rangle_\Gamma \ ,\ \widetilde{K}_{\alpha\beta} = -\left\langle \left(\tfrac{1}{2}Id + K\right) \psi_\alpha,\ \xi_\beta \right\rangle_\Gamma ,$$

$$B^F_{\alpha\beta} = \left\langle \theta_\alpha,\ \partial_n\phi_\beta \right\rangle_\Gamma - \tfrac{\tau_F}{h} \left\langle \theta_\alpha,\ \phi_\beta \right\rangle_\Gamma \ ,\ C^F_{\alpha\beta} = \left\langle \partial_n\phi_\alpha,\ \theta_\beta \right\rangle_\Gamma - \tfrac{\tau_F}{h} \left\langle \phi_\alpha,\ \theta_\beta \right\rangle_\Gamma ,$$

$$B^D_{\alpha\beta} = -\tau_B \left\langle \theta_\alpha,\ \psi_\beta \right\rangle_\Gamma \ ,\ C^D_{\alpha\beta} = -\tau_B \left\langle \psi_\alpha,\ \theta_\beta \right\rangle_\Gamma$$

$$B^N_{\alpha\beta} = \left\langle \theta_\alpha,\ \xi_\beta \right\rangle_\Gamma \ ,\ C^N_{\alpha\beta} = -\left\langle \xi_\alpha,\ \theta_\beta \right\rangle_\Gamma ,$$

$$D_{\alpha\beta} = \left(\tfrac{\tau_F}{h} + \tau_B\right) \left\langle \theta_\alpha,\ \theta_\beta \right\rangle_\Gamma \ ,\ \boldsymbol{f}_\beta = \int_{\Omega^-} f\phi_\beta dx.$$

Using the above definition the discrete problem (15) can be written in the following matrix form

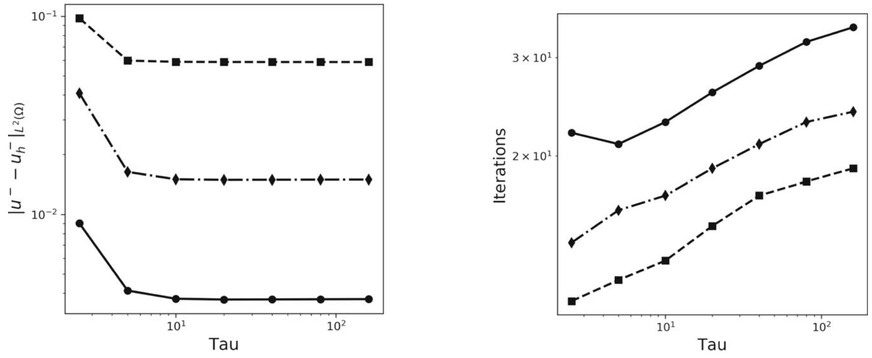
$$\begin{bmatrix} A & 0 & 0 & B^F \\ 0 & \widetilde{K} & \widetilde{V} & B^D \\ 0 & \widetilde{W} & \widetilde{K'} & B^N \\ C^F & C^D & C^N & D \end{bmatrix} \begin{bmatrix} \boldsymbol{\phi} \\ \boldsymbol{\psi} \\ \boldsymbol{\xi} \\ \boldsymbol{\theta} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

To solve this system we use the Schur complement and the Steklov-Poincaré operator to eliminate all variables except the hybrid one as follows

$$\left(D - C^F A^{-1} B^F - \begin{bmatrix} C^D & C^N \end{bmatrix} \begin{bmatrix} \widetilde{K} & \widetilde{V} \\ \widetilde{W} & \widetilde{K'} \end{bmatrix}^{-1} \begin{bmatrix} B^D \\ B^N \end{bmatrix} \right) \boldsymbol{\theta} = -C^F A^{-1} \boldsymbol{f} \tag{31}$$

In our experiment tests we consider $\varepsilon = 1$ and $\mathcal{V}_h \times \Lambda^l_h \times M^m_h$ with $j = k = m = 1$. The value l varies depending on the geometry of domains considered. We let the trace meshes $\mathcal{G}_1$ and $\mathcal{G}_2$ coincide with the trace mesh of $\mathcal{T}_h$ on Γ. As we know from our experiments as well as the one performed in [2], there is a flexibility with the choice of positive parameter τ_B, hence for simplicity we use $\tau_B = 1$.

For our experiments we use two numerical softwares: FEniCS [1] and Bempp [33]. We use the solution of interior and exterior Dirichlet boundary value problems to construct a Schur complement system (31). The solution $\boldsymbol{\theta}$ on Γ of the eliminated system (31) is obtained using the nested conjugate gradient method (CG) [23]. Although one can use direct solvers to solve the interior and exterior Dirichlet boundary value problems, we here used preconditioned iterative solvers suitable for large scale applications. The interior Dirichlet boundary value problem that is a symmetric system associated with bilinear form a_h (15) is solved by using FEniCS and CG without and with algebraic multigrid preconditioner. The discrete exterior problem associated with bilinear form b_h (15) is not symmetric, however as we shown in Section 3.1, we can apply the Steklov-Poincaré operator to the flux variable and transform the equations into a symmetric system. For clarity of the code, we here simply used the generalised minimal residual method (GMRES) [30] without or with mass matrix preconditioner to solve in Bempp the external Dirichlet boundary value problem. The tolerance of the iterative solvers is chosen to be not greater the 10^{-8}.



(a) The error of the interior solution.

(b) Iteration taken by CG to solve the preconditioned system.

Fig. 1 The dependence of the errors and iteration count on the value of τ_F for $h \approx 2^{-2}$ (dashed line with circles), $h \approx 2^{-3}$ (dash-dotted line with diamonds), and $h \approx 2^{-4}$ (solid line with squares) for the problem on the unit sphere subdomain, with $j = k = m = l = 1$

A Jupyter notebook demonstrating the functionality used in this paper will be made available at www.bempp.com.

5.1 Choice of parameter τ_F

Thanks to Lemma 7 we know that the stabilisation parameter τ_F in the discrete problem (15) must be large enough to assure coercivity. We start with an experiment showing how the value of the parameter τ_F influences the convergence and the number of iterations. We consider Ω^- as a unit sphere with boundary Γ. We define

$$\begin{aligned} u^-(x,y,z) &= \tfrac{1}{2\pi}\sin\left(\pi(x^2+y^2+z^2)\right) + \tfrac{1}{2\pi}\cos\left(\pi(x^2+y^2+z^2)\right) + \tfrac{2\pi+1}{2\pi}, \\ u^+(x,y,z) &= \tfrac{1}{\sqrt{x^2+y^2+z^2}}. \end{aligned}$$

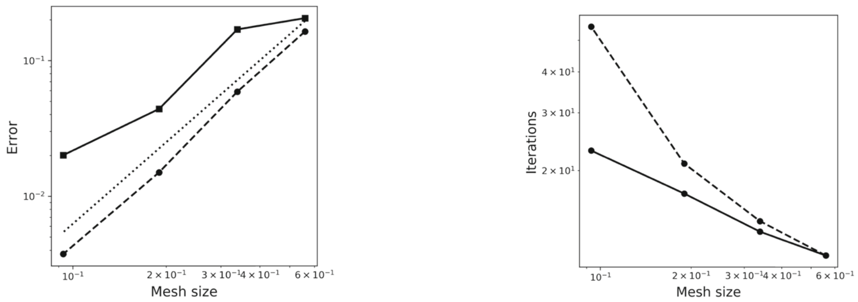
It is easy to check that for the unit sphere domain Ω^- the above elementary functions are the solution of our problem (1).

Figure 1 shows the error values for different values of τ and $j = k = m = l = 1$. In this case, Γ is smooth, and so $W_h^1 = \Lambda_h^1$. In Fig. 1a, we plot in log-log scale the error of the interior solution $\|u^- - u_h^-\|_{L^2(\Omega^-)}$ for $h \approx 2^{-2}$ (dashed line with circles), $h \approx 2^{-3}$ (dash-dotted line with diamonds), and $h \approx 2^{-4}$ (solid line with squares).

It can be seen from Fig. 1 that errors stop decreasing when τ_F is around 10. Furthermore, for $\tau_F > 10$ the iterations increase with growing τ_F, hence we fix $\tau_F = 10$ for the next experiments.

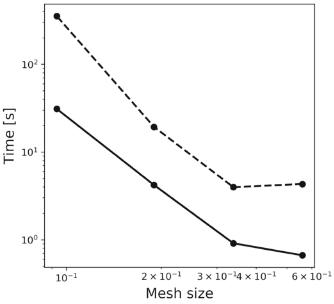
5.2 Spherical subdomain

Let Ω^- once again be the unit sphere, Γ its boundary and consider the same exact solution as above.



(a) Error of the interior (dashed line with circles) and exterior solutions (solid line with squares). The dotted line shows order 2 convergence.

(b) Iteration taken by CG to solve the non-preconditioned system (dashed line), compared with the preconditioned system (solid line)

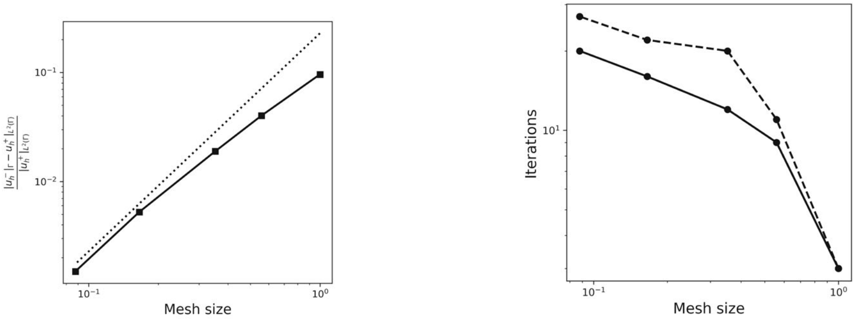


(c) Time to solve the non-preconditioned system (dashed line), compared with the preconditioned system (solid line) of the whole discrete problem (15) including solving interior and exterior systems.

Fig. 2 The convergence (left), CG iteration counts (middle) and solving time (right) for the problem on the unit sphere with $\tau_F = 10$ and $j = k = m = l = 1$

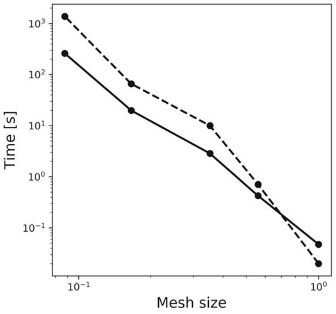
Figure 2 shows the convergence, CG iteration counts and solving time when $\tau_F = 10$ and $k = l = 1$. In Fig. 2a, we plot in log-log scale the error of the interior solution $\|u^- - u_h^-\|_{L^2(\Omega^-)}$ (dashed line with circles) and the error of the exterior solutions $\|u^+ - u_h^+\|_{L^2(\Gamma)} + \|\lambda^+ - \lambda_h^+\|_{L^2(\Gamma)}$ (solid line with squares).

In Fig. 2b, we plot in log-log scale the number of iterations taken by CG to solve the non-preconditioned system associated with exterior problem (dashed line), compared with the preconditioned system (solid line). In addition, Fig. 2c shows the time required by solvers of interior and exterior systems. The interior system is solved by CG with or without algebraic multigrid preconditioner and the exterior system is solved by GMRES with or without mass preconditioner. As we can see in Fig. 2b, the preconditioning reduces both the iteration count and the increase of iterations under refinement. In terms of CPU time needed by the solver, the preconditioned methods reduces the execution time by an order of magnitude compared to the method without preconditioning.



(a) Error between interior solution and exterior. The dashed line shows order 2 convergence.

(b) Iteration taken by CG to solve the non-preconditioned system (dashed line), compared with the preconditioned system (solid line)



(c) Time to solve the non-preconditioned system (dashed line), compared with the preconditioned system (solid line) of the whole discrete problem (15) including solving interior and exterior systems.

Fig. 3 The convergence (left), CG iteration counts (middle) and solving time (right) for the problem on the cube with $\tau_F = 10$ and $j = k = m = l + 1 = 1$

5.3 Cubical subdomain

Let $\Omega^- = (0, 1)^3$ be a cube and we solve the problem (1) with $f = 1$. We choose $\tau_F = 10$ and $j = k = m = l + 1 = 1$, where Λ_h^0 is the space of piecewise constants per element in the trace space. Since the domain has corners we must use an approximation of the flux variable that is discontinuous over corners on the boundary. Therefore we consider the approximation space consisting of functions that are piecewise constant per element, $l = 0$.

Figure 3a shows the convergence when $\tau_F = 10$ and $j = k = m = l + 1 = 1$. In this case, the exact solution is not known; thus, in Fig. 3a, we plot in log-log scale the error between interior solution and exterior $\frac{\|u_h^-|_\Gamma - u_h^+\|_{L^2(\Gamma)}}{\|u_h^+\|_{L^2(\Gamma)}}$ (solid line).

In Fig. 3, we plot as well in log-log scale the number of iterations and solving time taken by CG to solve the non-preconditioned system (dashed line), compared with the preconditioned system (solid line). Once again preconditioning brings improvement in terms of iteration counts and time taken to solve the problem. The reduction of the iteration count is less significant than in the previous case, however we can see similar improvement in terms of time reduction.

6 Conclusions

We have analysed and demonstrated the effectiveness of Nitsche-type methods for coupling finite element and boundary element formulations. Our approach gives flexibility to choose a continuous or discontinuous finite element space in the FEM solver, hence the interior problem can be solved essentially using any method that allows for the hybridised Nitsche method for interdomain coupling. We are also free to choose the trace variable minimising the coupling degrees of freedom. In this paper, we focus on the technical aspects and analysis to allow for flexibility within our framework. A work demonstrating the applicability for large problems using parallel approach is in preparation.

The method can be extended to other models such as the Helmholtz equations. In this case it is known [36] that the use of impedance interface conditions is advantageous and such an approach can be mimicked in the present framework by letting the stabilisation constant have non-zero imaginary part and depend on the wave number. Formulations of the presented FEM/BEM coupling method to the Helmholtz and Maxwell problems are currently in preparation. For these cases however more effective operator preconditioning techniques for exterior problem are essential, especially for high frequency problems. Despite that, we expect that their implementation will be similar to the presented Laplace case.

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Data availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations The authors declare no competing interests.

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