Numerical Algorithms (2022) 90:563–576
https://doi.org/10.1007/s11075-021-01198-3

ORIGINAL PAPER

Existence results and numerical method for solving a fourth-order nonlinear integro-differential equation

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Received: 25 January 2021 / Accepted: 9 September 2021 / Published online: 17 September 2021

Abstract
In this paper, we consider a boundary value problem (BVP) for a fourth-order nonlinear integro-differential equation. By reducing the problem to an operator equation, we establish the existence and uniqueness of the solution and construct a numerical method for solving it. We prove that the method is of second-order accuracy and obtain an estimate for the total error. Some examples demonstrate the validity of the obtained theoretical results and the efficiency of the numerical method.

Keywords Fourth-order nonlinear integro-differential equation · Existence and uniqueness of solution · Iterative method · Total error

1 Introduction

Integro-differential equations are the mathematical models of many phenomena of physics, biology, hydromechanics, chemistry, etc. In general, it is impossible to find the exact solutions of the problems involving these equations, especially when they are nonlinear. Therefore, many analytical approximate methods and numerical methods have been developed for these equations (see, e.g., [1, 2, 11–18]).

Below, we mention some works concerning the solution methods for integro-differential equations. First, it is worthy to mention the recent work of Tahernezhad and Jalilian in 2020 [15]. In this work the authors consider the second-order linear

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problem

$$u''(x) + p(x)u'(x) + q(x)u(x) = f(x) + \int_a^b k(x,t)u(t)dt, \quad a < x < b,$$
$$u(a) = \alpha, \ u(b) = \beta,$$

where $p(x), q(x), k(x,t)$ are sufficiently smooth functions.

Using non-polynomial spline functions, namely, the exponential spline functions, the authors constructed the numerical solution of the problem and proved that the error of the approximate solution is $O(h^2)$, where h is the grid size on $[a, b]$. Before [15] there are interesting works of Chen et al. [2, 3], where the authors used a multiscale Galerkin method for constructing a approximate solution of the above second-order problem, for which the computed convergence rate is two.

Besides the researches evolving the second-order integro-differential equations, recently many authors have been interested in fourth-order integro-differential equations due to their wide applications. We first mention the work of Singh and Wazwaz [13]. In this work the authors developed a technique based on the Adomian decomposition method with the Green's function for constructing a series solution of the nonlinear Voltera equation associated with the Dirichlet boundary conditions

$$y^{(4)}(x) = g(x) + \int_0^x k(x,t)f(y(t))dt, \quad 0 < x < b, \tag{1}$$
$$y(0) = \alpha_1, \ y'(0) = \alpha_2, \ y(b) = \alpha_3, \ y'(b) = \alpha_4. \tag{2}$$

Under some conditions it was proved that the series solution converges as a geometric progression.

For the linear Fredholm IDE [1]

$$y^{(4)}(x) + \alpha y''(x) + \beta y(x) - \int_a^b K(x,t)y(t)dt = f(x), \quad a < x < b,$$

with the Navier boundary conditions, the difference method and the trapezium rule are used to design the corresponding linear system of algebraic equations. A new variant called the Modified Arithmetic Mean iterative method is proposed for solving the latter system, but the error estimate of the method is not obtained.
The boundary value problem for the nonlinear IDE

$$y^{(4)}(x) - \varepsilon y''(x) - \frac{2}{\pi}\Big(\int_0^\pi |y'(t)|^2 dt\Big) y''(x) = p(x), \quad 0 < x < \pi,$$
$$y(0) = 0, \ y(\pi) = 0, \ y''(0) = 0, \ y''(\pi) = 0$$

was considered in [10, 18], where the authors constructed approximate solutions by the iterative and spectral methods, respectively. Recently, Dang and Nguyen [6] studied the existence and uniqueness of solution and constructed iterative method for finding the solution for the IDE

$$u^{(4)}(x) - M\Big(\int_0^L |u'(t)|^2 dt\Big) u''(x) = f(x, u, u', u'', u'''), \quad 0 < x < L,$$
$$u(0) = 0, \ u(L) = 0, \ u''(0) = 0, \ u''(L) = 0,$$

where M is a continuous non-negative function.

Very recently, Wang [17] considered the problem

$$\begin{aligned} y^{(4)}(x) &= f(x, y(x), \int_0^1 k(x,t)y(t)dt),\ 0 < x < 1, \\ y(0) &= 0,\ y(1) = 0,\ y''(0) = 0,\ y''(1) = 0. \end{aligned} \tag{3}$$

This problem can be seen as a generalization of the linear fourth-order problem

$$\begin{aligned} u^{(4)}(x) &+ Mu(x) - N\int_0^1 k(x,t)u(t)dt) = p(x),\ 0 < x < 1, \\ u(0) &= 0,\ u(1) = 0,\ u''(0) = 0,\ u''(1) = 0, \end{aligned}$$

where M, N are constants, $p \in C[0, 1]$. The latter problem arises from the models for suspension bridges [19, 20], quantum theory [21].

Using the monotone method and a maximum principle, Wang constructed the sequences of functions, which converge to the extremal solutions of the problem (3).

Motivated by the above facts, in this paper we consider an extension of the above problem, namely, the problem

$$\begin{aligned} u^{(4)}(x) &= f(x, u(x), u'(x), \int_0^1 k(x,t)u(t)dt), \\ u(0) &= 0,\ u(1) = 0,\ u''(0) = 0,\ u''(1) = 0, \end{aligned} \tag{4}$$

where the function $f(x, u, v, z)$ and $k(x, t)$ are assumed to be continuous. The presence of an extra u' in the right hand side function does not allow to use the argument in [17] to study the existence of solutions of the problem. Here, using the method developed in our previous papers [4–9] we establish the existence and uniqueness of the solution and propose an iterative method at both continuous and discrete levels for finding the solution. The second-order convergence of the method is proved. The theoretical results are illustrated by some examples.

2 Existence results

Using the methodology in [4–9] we introduce the operator A defined in the space of continuous functions $C[0, 1]$ by the formula

$$(A\varphi)(x) = f(x, u(x), u'(x), \int_0^1 k(x,t)u(t)dt), \tag{5}$$

where $u(x)$ is the solution of the boundary value problem

$$\begin{aligned} u'''' &= \varphi(x),\ 0 < x < 1, \\ u(0) &= u''(0) = u(1) = u''(1) = 0. \end{aligned} \tag{6}$$

It is easy to verify the following lemma.

Lemma 1 *If the function φ is a fixed point of the operator A, i.e., φ is the solution of the operator equation*

$$A\varphi = \varphi, \tag{7}$$

where A is defined by (5)–(6) then the function $u(x)$ determined from the BVP (6) is a solution of the BVP (4). Conversely, if the function $u(x)$ is the solution of the BVP (4) then the function

$$\varphi(x) = f(x, u(x), u'(x), \int_0^1 k(x,t)u(t)dt)$$

satisfies the operator (7).

Due to the above lemma we shall study the original BVP (4) via the operator (7). Before doing this we notice that the BVP (6) has a unique solution representable in the form

$$u(x) = \int_0^1 G_0(x,s)\varphi(s)ds, \quad 0 < t < 1, \tag{8}$$

where

$$G_0(x,s) = \frac{1}{6}\begin{cases} s(x-1)(x^2 - x + s^2), & 0 \le s \le x \le 1 \\ x(s-1)(s^2 - s + x^2), & 0 \le x \le s \le 1 \end{cases} \tag{9}$$

is the Green's function of the operator $u''''(t) = 0$ associated with the homogeneous boundary conditions $u(0) = u''(0) = u(1) = u''(1) = 0$.
Differentiating both sides of (8) gives

$$u'(x) \ = \textstyle\int_0^1 G_1(x,s)\varphi(s)ds, \tag{10}$$

where

$$G_1(x,s) = \frac{1}{6}\begin{cases} s(3x^2 - 6x + s^2 + 2), & 0 \le s \le x \le 1, \\ (s-1)(3x^2 - 2s + s^2), & 0 \le x \le s \le 1. \end{cases} \tag{11}$$

Set

$$\begin{aligned} M_0 &= \max_{0\le x\le 1} \int_0^1 |G_0(x,s)|ds, \\ M_1 &= \max_{0\le x\le 1} \int_0^1 |G_1(x,s)|ds, \\ M_2 &= \max_{0\le x\le 1} \int_0^1 |k(x,s)|ds \end{aligned} \tag{12}$$

It is easy to obtain

$$M_0 = \frac{5}{384}, M_1 = \frac{1}{24}. \tag{13}$$

Now for any positive number M, we define the domain

$$\begin{aligned} \mathcal{D}_M = \{(x,u,v,z) \mid \ & 0 \le x \le 1, \ |u| \le M_0 M, \\ & |v| \le M_1 M, \ |z| \le M_0 M_2 M\}. \end{aligned} \tag{14}$$

As usual, we denote by $B[0, M]$ the closed ball centered at 0 with radius M in the space $C[0, 1]$, i.e.,

$$B[0, M] = \{u \in C[0, 1] \mid \|u\| \leq M\},$$

where $\|u\| = \max_{0 \leq x \leq 1} |u(x)|$.

Theorem 1 (Existence and uniqueness) *Suppose that the function $k(x, t)$ is continuous in the square $[0, 1] \times [0, 1]$ and there exist numbers $M > 0$, $L_0, L_1, L_2 \geq 0$ such that:*

(i) *The function $f(x, u, v, z)$ is continuous in the domain $\mathcal{D}_M$ and $|f(x, u, v, z)| \leq M$, $\forall (x, u, v, z) \in \mathcal{D}_M$.*
(ii) $|f(x_2, u_2, v_2, z_2) - f(x_1, u_1, v_1, z_1)| \leq L_0|u_2 - u_1| + L_1|v_2 - v_1| + L_2|z_2 - z_1|$, $\forall (x_i, u_i, v_i, z_i) \in \mathcal{D}_M,\ i = 1, 2.$
(iii) $q = L_0 M_0 + L_1 M_1 + L_2 M_0 M_2 < 1.$

Then the problem (4) *has a unique solution $u \in C^4[0, 1]$ satisfying $|u(x)| \leq M_0 M$, $|u'(x)| \leq M_1 M$ for any $0 \leq x \leq 1$.*

Proof Under the assumptions of the theorem we shall prove that the operator A is a contraction mapping in the closed ball $B[O, M]$. Then the operator (7) has a unique solution $u \in C^{(4)}[0, 1]$ and this implies the existence and uniqueness of solution of the BVP (4).

Indeed, take $\varphi \in B[O, M]$. Then the problem (6) has a unique solution of the form (8). From there and (12) we obtain $|u(x)| \leq M_0\|\varphi\|$ for all $x \in [0, 1]$. Analogously, we have $\|u'(x)\| \leq M_1\|\varphi\|$ for all $x \in [0, 1]$. Denote by K the integral operator defined by

$$(Ku)(x) = \int_0^1 k(x, t)u(t)dt.$$

Then from the last equation in (12) we have the estimate $|(Ku)(x)| \leq M_0 M_2\|\varphi\|$, $x \in [0, 1]$. Thus, if $\varphi \in B[O, M]$, i.e., $\|\varphi\| \leq M$ then for any $x \in [0, 1]$ we have

$$|u(x)| \leq M_0 M,\ \ |u'(x)| \leq M_1 M,\ \ |(Ku)(x)| \leq M_0 M_2 M.$$

Therefore, $(x, u(x), u'(x), (Ku)(x)) \in \mathcal{D}_M$. By the assumption (i) there is

$$|f(x, u(x), u'(x), (Ku)(x))| \leq M \quad \forall x \in [0, 1].$$

Hence, $|(A\varphi)(x)| \leq M,\ \forall x \in [0, 1]$ and $\|A\varphi\| \leq M$. It means that A maps $B[O, M]$ into itself.

Next, take $\varphi_1, \varphi_2 \in B[O, M]$. Using the assumption (ii) and (iii) it is easy to obtain

$$\|A\varphi_2 - A\varphi_1\| \leq (L_0 M_0 + L_1 M_1 + L_2 M_0 M_2)\|\varphi_2 - \varphi_1\| = q\|\varphi_2 - \varphi_1\|.$$

Since $q < 1$ the operator A is a contraction in $B[O, M]$. This completes the proof of the theorem. □

Now, in order to study positive solutions of the BVP (4) we introduce the domain

$$\mathcal{D}_M^+ = \{(x,u,v,z) \mid 0 \le x \le 1,\ 0 \le u \le M_0 M, \\ |v| \le M_1 M,\ |z| \le M_0 M_2 M\}. \tag{15}$$

and denote

$$S_M = \{\varphi \in C[0,1], 0 \le \varphi(x) \le M\}.$$

Theorem 2 (Positivity of solution) *Suppose that the function $k(x,t)$ is continuous in the square $[0,1] \times [0,1]$ and there exist numbers $M > 0$, $L_0, L_1, L_2 \ge 0$ such that:*

(i) *The function $f(x,u,v,z)$ is continuous in the domain $\mathcal{D}_M^+$ and $0 \le f(x,u,v,z) \le M,\ \forall(x,u,v,z) \in \mathcal{D}_M^+$ and $f(x,0,0,0) \not\equiv 0$.*
(ii) $|f(x_2,u_2,v_2,z_2) - f(x_1,u_1,v_1,z_1)| \le L_0|u_2-u_1| + L_1|v_2-v_1| + L_2|z_2-z_1|$, $\forall(x_i,u_i,v_i,z_i) \in \mathcal{D}_M^+,\ i=1,2.$
(iii) $q = L_0 M_0 + L_1 M_1 + L_2 M_0 M_2 < 1.$

Then the problem (4) has a unique positive solution $u \in C^4[0,1]$ satisfying $0 \le u(x) \le M_0 M$, $|u'(x)| \le M_1 M$ for any $0 \le x \le 1$.

Proof Similarly to the proof of Theorem 1, where instead of $\mathcal{D}_M$ and $B[O;M]$ there stand $\mathcal{D}_M^+$ and S_M, we conclude that the problem has a non-negative solution. Due to the condition $f(x,0,0,0) \not\equiv 0$, this solution must be positive. □

3 Numerical method

In this section we suppose that all the conditions of Theorem 1 are satisfied. Then the problem (4) has a unique solution. For finding this solution consider the following iterative method:

1. Given

$$\varphi_0(x) = f(x,0,0,0). \tag{16}$$

2. Knowing $\varphi_m(x)$ $(m = 0,1,...)$ compute

$$\begin{aligned} u_m(x) &= \int_0^1 G_0(x,t)\varphi_m(t)dt, \\ v_m(x) &= \int_0^1 G_1(x,t)\varphi_m(t)dt, \\ z_m(x) &= \int_0^1 k(x,t)u_m(t)dt. \end{aligned} \tag{17}$$

3. Update

$$\varphi_{m+1}(x) = f(x,u_m(x),v_m(x),z_m(x)). \tag{18}$$

This iterative method indeed is the successive iterative method for finding the fixed point of operator A. Therefore, it converges with the rate of geometric progression and there holds the estimate

$$\|\varphi_m - \varphi\| \leq \frac{q^m}{1-q}\|\varphi_1 - \varphi_0\| = p_m d,$$

where φ is the fixed point of the operator A and

$$p_m = \tfrac{q^m}{1-q},\ d = \|\varphi_1 - \varphi_0\|. \tag{19}$$

This estimate implies the following result of the convergence of the iterative method (16)–(18).

Theorem 3 *Under the conditions of Theorem 1 the iterative method* (16)–(18) *converges and for the approximate solution $u_k(t)$ there hold estimates*

$$\|u_m - u\| \leq M_0 p_m d,\ \|u'_m - u'\| \leq M_1 p_m d,$$

where u is the exact solution of the problem (4)*, p_m and d are defined by* (19)*.*

To numerically realize the above iterative method we construct a corresponding discrete iterative method. For this purpose cover the interval $[0, 1]$ by the uniform grid $\bar{\omega}_h = \{x_i = ih,\ h = 1/N, i = 0, 1, ..., N\}$ and denote by $\Phi_m(x), U_m(x), V_m(x), Z_m(x)$ the grid functions, which are defined on the grid $\bar{\omega}_h$ and approximate the functions $\varphi_m(x), u_m(x), v_m(x), z_m(x)$ on this grid.

Consider now the following discrete iterative method:

1. Given
$$\Phi_0(x_i) = f(x_i, 0, 0, 0),\ i = 0, ..., N. \tag{20}$$
2. Knowing $\Phi_m(x_i),\ m = 0, 1, ...;\ i = 0, ..., N$, compute approximately the definite integrals (17) by the trapezium formulas
$$U_m(x_i) = \sum_{j=0}^{N} h\rho_j G_0(x_i, x_j)\Phi_m(x_j),$$
$$V_m(x_i) = \sum_{j=0}^{N} h\rho_j G_1(x_i, x_j)\Phi_m(x_j),$$
$$Z_m(x_i) = \sum_{j=0}^{N} h\rho_j k(x_i, x_j)U_m(x_j),\ i = 0, ..., N, \tag{21}$$
where ρ_j is the weight of the trapezium formula, namely
$$\rho_j = \begin{cases} 1/2,\ j = 0, N \\ 1,\ j = 1, 2, ..., N-1. \end{cases}$$
3. Update
$$\Phi_{m+1}(x_i) = f(x_i, U_m(x_i), V_m(x_i), Z_m(x_i)). \tag{22}$$

In order to get the error estimates for the approximate solution for $u(t)$ and its derivatives on the grid we need some following auxiliary results.

Proposition 1 *Assume that the function $f(t,u,v,z)$ has all continuous partial derivatives up to second order in the domain $\mathcal{D}_M$ and the kernel function $k(x,t)$ also has all continuous partial derivatives up to second order in the square* $[0,1]\times[0,1]$. *Then for the functions $\varphi_m(x), u_m(x), v_m(x), z_m(x), m = 0,1,...,$ constructed by the iterative method* (16)–(18) *we have $\varphi_m(x) \in C^2[0,1]$, $u_m(x) \in C^6[0,1]$, $v_m(x) \in C^5[0,1]$, $z_m(x) \in C^2[0,1]$.*

Proof We prove the proposition by induction. For $k = 0$, by the assumption on the function f we have $\varphi_0(t) \in C^2[0,1]$ since $\varphi_0(x) = f(x,0,0,0)$. Taking into account

$$u_0(x) = \int_0^1 G_0(x,t)\varphi_0(t)dt$$

we deduce that the function $u_0(x)$ is the solution of the BVP

$$\begin{aligned} u_0^{(4)}(x) &= \varphi_0(x), \quad x \in (0,1), \\ u_0(0) &= u_0(1) = u_0''(0) = u_0''(1) = 0. \end{aligned}$$

Therefore, $u_0(x) \in C^6[0,1]$. It implies that $v_0(x) \in C^5[0,1]$ because $v_0(x) = u_0'(x)$. Since by assumptions $k(x,t)$ has all continuous derivatives up to second order, the function $z_0(x) = \int_0^1 k(x,t)u_0(t)dt$ belongs to $C^2[0,1]$.

Now suppose $\varphi_m(x) \in C^2[0,1]$, $u_m(x) \in C^6[0,1]$, $v_m(x) \in C^5[0,1]$, $z_m(x) \in C^2[0,1]$. Then, because $\varphi_{m+1}(x) = f(x,u_m(x),v_m(x),z_m(x))$ and the functions f by the assumption has continuous derivative in all variables up to order 2, it follows that $\varphi_{m+1}(x) \in C^2[0,1]$. Repeating the same argument as for $\varphi_0(x)$ above we obtain that $u_{m+1}(x) \in C^6[0,1]$, $v_{m+1}(x) \in C^5[0,1]$, $z_{m+1}(x) \in C^2[0,1]$ Thus, the proposition is proved. □

Proposition 2 *For any function $\varphi(x) \in C^2[0,1]$ there holds the estimate*

$$\int_0^1 G_n(x_i,t)\varphi(t)dt = \sum_{j=0}^{N} h\rho_j G_n(x_i,t_j)\varphi(t_j) + O(h^2) \quad (n = 0,1). \tag{23}$$

Proof The above estimate is obvious in view of the error estimate of the compound trapezium formula because the functions $G_n(x_i,t)$ $(n = 0,1)$ are continuous at t_j and are polynomials in the intervals $[0,t_j]$ and $[t_j,1]$. □

Proposition 3 *Under the assumptions of Proposition 1, for any $m = 0,1,...$ there hold the estimates*

$$\|\Phi_m - \varphi_m\| = O(h^2), \quad \|U_m - u_m\| = O(h^2), \tag{24}$$

$$\|V_m - v_m\| = O(h^2), \ \|Z_m - z_m\| = O(h^2). \tag{25}$$

where $\|.\| = \|.\|_{\bar{\omega}_h}$ is the max-norm of function on the grid $\bar{\omega}_h$.

Proof We prove the proposition by induction. For $m = 0$ we have immediately $\|\Phi_0 - \varphi_0\| = 0$. Next, by the first equation in (17) and Proposition 2 we have

$$u_0(x_i) = \int_0^1 G_0(x_i, t)\varphi_0(t)dt = \sum_{j=0}^{N} h\rho_j G_0(x_i, t_j)\varphi_0(t_j) + O(h^2) \tag{26}$$

for any $i = 0, ..., N$. On the other hand, in view of the first equation in (21) we have

$$U_0(x_i) = \sum_{j=0}^{N} h\rho_j G_0(x_i, t_j)\Phi_0(t_j). \tag{27}$$

Therefore, $|U_0(t_i) - u_0(t_i)| = O(h^2)$ because $\Phi_0(t_j) = \varphi_0(t_j) = f(t_j, 0, 0, 0)$. Consequently, $\|U_0 - u_0\| = O(h^2)$.
Similarly, we have

$$\|V_0 - v_0\| = O(h^2). \tag{28}$$

Next, by the trapezium formula we have

$$z_0(x_i) = \int_0^1 k(x_i, t)u_0(t)dt = \sum_{j=0}^{N} h\rho_j k(x_i, t_j)u_0(t_j) + O(h^2),$$

while by the third equation in (21) we have

$$Z_0(x_i) = \sum_{j=0}^{N} h\rho_j k(x_i, t_j)U_0(t_j),\ i = 0, ..., N.$$

Therefore,

$$\begin{aligned}\Big|Z_0(x_i) - z_0(x_i)\Big| &= \Big|\sum_{j=0}^{N} h\rho_j k(x_i, t_j)(U_0(t_j) - u_0(t_j))\Big| + O(h^2)\\ &\le \sum_{j=0}^{N} h\rho_j |k(x_i, t_j)||U_0(t_j) - u_0(t_j)| + O(h^2)\\ &\le Ch^2 \sum_{j=0}^{N} h\rho_j |k(x_i, t_j)| + O(h^2)\\ &\le CC_1h^2 \sum_{j=0}^{N} h\rho_j + O(h^2) = O(h^2)\end{aligned}$$

because $|U_0(t_j) - u_0(t_j)| \le Ch^2,\ |k(x_i, t_j)| \le C_1$, where C, C_1 are some constants.

Now suppose that (24) and (25) are valid for $m \ge 0$. We shall show that these estimates are valid for $m + 1$. By the Lipschitz condition of the function f and the estimates (24) and (25) it is easy to obtain the estimate

$$\|\Phi_{m+1} - \varphi_{m+1}\| = O(h^2).$$

Now from the first equation in (17) by Proposition 2 we have

$$u_{m+1}(x_i) = \int_0^1 G_0(x_i, t)\varphi_{m+1}(t)dt = \sum_{j=0}^{N} h\rho_j G_0(x_i, x_j)\varphi_{m+1}(x_j) + O(h^2).$$

On the other hand by the first formula in (21) we have

$$U_{m+1}(x_i) = \sum_{j=0}^{N} h\rho_j G_0(x_i, x_j)\Phi_{m+1}(x_j).$$

From this equality and the above estimates we obtain the estimate

$$\|U_{m+1} - u_{m+1}\| = O(h^2).$$

Similarly, we obtain

$$\|V_{m+1} - v_{m+1}\| = O(h^2), \ \|Z_{m+1} - z_{k+1}\| = O(h^2).$$

Thus, by induction we have proved the proposition. □

Now combining Proposition 3 and Theorem 3 results in the following theorem.

Theorem 4 *Assume that all the conditions of Theorem 1 and Proposition 1 are satisfied. Then, for the approximate solution of the problem* (4) *obtained by the discrete iterative method on the uniform grid with grid size h there hold the estimates*

$$\|U_m - u\| \le M_0 p_m d + O(h^2), \ \|V_m - u'\| \le M_2 p_m d + O(h^2). \tag{29}$$

Proof The first above estimate is easily obtained if representing

$$U_m(t_i) - u(t_i) = (u_m(t_i) - u(t_i)) + (U_m(t_i) - u_m(t_i))$$

and using the first estimate in Theorem 3 and the second estimate in (24). The remaining estimate is obtained in the same way. Thus, the theorem is proved. □

4 Examples

Example 1 Consider the problem (4) with

$$k(x,t) = e^x \sin(\pi t), \quad (x,t) \in [0,1] \times [0,1],$$
$$f\left(x, u(x), u'(x), \int_0^1 k(x,t)u(t)dt\right) = u^2(x)\int_0^1 k(x,t)u(t)dt + u(x)u'(x)$$
$$-\frac{1}{2}e^x \sin^2(\pi x) + \pi^4 \sin(\pi x) - \frac{\pi}{2}\sin(2\pi x).$$

In this case

$$f(x,u,v,z) = u^2 z + uv - \frac{1}{2}e^x \sin^2(\pi x) + \pi^4 \sin(\pi x) - \frac{\pi}{2}\sin(2\pi x)$$

and $M_2 = \dfrac{2e}{\pi}$. It is possible to verify that the function $u = \sin(\pi x)$ is the exact solution of the problem. In the domain $\mathcal{D}_M$ defined by

$$\mathcal{D}_M = \{(x, u, v, z) \mid \ 0 \le x \le 1, \ |u| \le M_0 M, |u'| \le M_1 M, |z| \le M_0 M_2 M\}$$

we have

$$|f(x, u, v, z)| \le M_0^3 M_2 M^3 + M_0 M_1 M^2 + \pi^4 + \frac{\pi}{2} + \frac{e}{2}.$$

It is possible to verify that for $M = 113$ all the conditions of Theorem 1 are satisfied with $L_0 = 12.2010$, $L_1 = 1.4714$, $L_2 = 2.1649$, $q = 0.2690$. Therefore, the problem has a unique solution $u(x)$ satisfying the estimates $|u(x)| \le 1.4714$, $|u'(x)| \le 4.7083$. These theoretical estimates are somewhat greater than the exact estimates $|u(x)| \le 1$, $|u'(x)| \le \pi$.

Below we report the numerical results by the discrete iterative method (20)–(22) for the problem. In Tables 1 and 2 we use the notation $Error = \|U_m - u\|$, where u is the exact solution of the problem.

It is interesting to notice that if taking the stopping criterion $\|\Phi_m - \Phi_{m-1}\| \le 10^{-10}$ instead of $\|U_m - u\| \le h^2$ then we obtain better accuracy of the approximate solution with more iterations. See Table 2.

From Table 2 we see that the accuracy of the approximate solution is near $O(h^4)$ although by the proved theory it is only $O(h^2)$.

Example 2 (Example 4.2 in [17]) Consider the nonlinear fourth-order BVP

$$\begin{aligned} u^{(4)}(x) &= \sin(\pi x)[(2 - u^2(x))\textstyle\int_0^1 t u(t)dt + 1], x \in (0, 1) \\ u(0) &= 0, \ u(1) = 0, \ u''(0) = 0, \ u''(1) = 0. \end{aligned} \tag{30}$$

This is the problem (4) with

$$k(x, t) = \sin(\pi x)t, \quad (x, t) \in [0, 1] \times [0, 1],$$
$$f(x, u(x), u'(x), \textstyle\int_0^1 k(x, t)u(t)dt) = (2 - u^2(x))\int_0^1 \sin(\pi x) t u(t)dt + \sin(\pi x).$$

Table 1 The convergence in Example 1 for the stopping criterion $\|U_m - u\| \le h^2$

N	h^2	m	$Error$
50	4.0000e-04	2	1.4305e-04
100	1.0000e-04	3	2.8588e-06
150	4.4444e-05	3	2.8599e-06
200	2.5000e-05	3	2.8602e-06
300	1.1111e-05	3	2.8603e-06
400	6.2500e-06	3	2.8603e-06
500	4.0000e-06	3	2.8603e-06
800	1.5625e-06	4	5.7485e-08
1000	1.0000e-06	4	5.7486e-08

Table 2 The convergence in Example 1 for the stopping criterion $\|\Phi_m - \Phi_{m-1}\| \leq 10^{-10}$

N	h^2	m	$Error$
50	4.0000e-04	7	2.2152e-08
100	1.0000e-04	7	1.3831e-09
150	4.4444e-05	7	2.7279e-10
200	2.5000e-05	7	8.5995e-11
300	1.1111e-05	7	1.6618e-11
400	6.2500e-06	7	4.9447e-12
500	4.0000e-06	7	1.7567e-12
800	1.5625e-06	7	1.4588e-13
1000	1.0000e-06	7	3.3318e-13

So, $f(x, u, v, z) = (2 - u^2)z + \sin(\pi x)$.
It is easy to see that $M_2 = \max_{0 \leq x \leq 1} \int_0^1 |k(x, t)|dt = \frac{1}{2}$. Since M_0 and M_1 are given by (13) we define

$$\mathcal{D}_M = \{(x, u, v, z) \mid \ 0 \leq x \leq 1, \ |u| \leq \tfrac{5}{384}M, |v| \leq \tfrac{1}{24}M, |z| \leq \tfrac{5}{768}M\}. \quad (31)$$

It is possible to verify that for $M = 1.1$ all the assumptions of Theorem 1 are satisfied with $L_0 = 2.0515e - 04$, $L_1 = 0$, $L_2 = 2$, $q = 0.0130$. Therefore, the problem (30) has a unique solution satisfying $|u(x)| \leq 0.0143, \ |u'(x)| \leq 0.0458$.

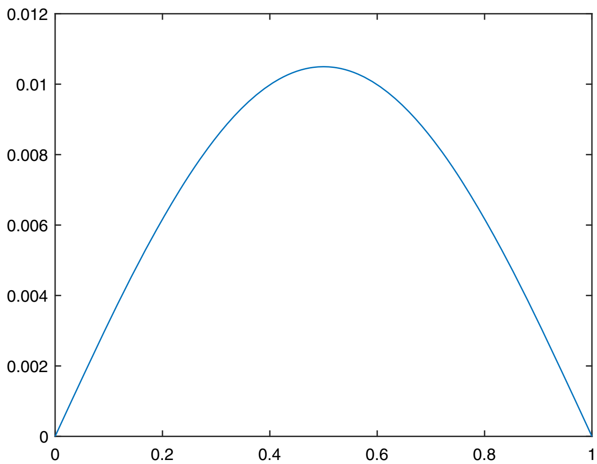

Fig. 1 The graph of the approximate solution in Example 2

It is worth emphasizing that in [17] *by the monotone method the author could only prove the convergence of the iterative sequences to extremal solutions of the problem but not the existence and uniqueness of solution.*

Using the discrete iterative method (20)–(22) on the grid with grid step $h = 0.01$ and the stopping criterion $\|\Phi_m - \Phi_{m-1}\| \leq 10^{-10}$ we found an approximate solution after 7 iterations. The graph of this approximate solution is depicted in Fig. 1.

5 Conclusion

In this paper we have established the existence and uniqueness of the solution for a fourth-order nonlinear integro-differential equation with the Navier boundary conditions and proposed an iterative method at both continuous and discrete levels for finding the solution. The second order of accuracy of the discrete method has been proved. Some examples, where the exact solution is known and is not known, demonstrate the validity of the obtained theoretical results and the efficiency of the iterative method. It should be emphasized that for the example of Wang in [17] we have established the existence and uniqueness of solution and found it numerically but Wang could prove only the convergence of the iterative sequences constructed by the monotone method to extremal solutions.

The method used in this paper with appropriate modifications can be applied to nonlinear integro-differential equations of any order with other boundary conditions and more complicated nonlinear terms. This is the direction of our research in the future.

Funding This work is supported by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under the grant number 102.01-2021.03.

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