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ORIGINAL PAPER

Taylor type and Hermite type interpolants in $\mathbb{R}^n$

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Abstract

We construct new polynomial interpolation schemes of Taylor and Hermite types in $\mathbb{R}^n$. The interpolation conditions are real parts and imaginary parts of certain differential operators. We also give formulas for the interpolation polynomials which are of Newton form and can be computed by an algorithm.

Keywords Interpolation polynomial · Hermite interpolation · Taylor type polynomial · Newton formulas

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1 Introduction

Let $\mathscr{P}_d(\mathbb{R}^n)$ be the vector space of all polynomials of degree at most d in $\mathbb{R}^n$. It is well-known that the dimension of $\mathscr{P}_d(\mathbb{R}^n)$ equals $\binom{n+d}{n}$. The Lagrange interpolation problem associated to A asks whether there is a unique polynomial p

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in $\mathscr{P}_d(\mathbb{R}^n)$ which matches preassigned data on A, where A consists of $\binom{n+d}{n}$ distinct points in $\mathbb{R}^n$. Let $\mathscr{B} = \{p_1, \ldots, p_N\}$ be a basis for $\mathscr{P}_d(\mathbb{R}^n)$ with $N = \binom{n+d}{n}$. Then, the Vandermonde determinant defined by

$$\mathrm{VDM}(\mathscr{B}; A) = \det[p_j(\mathbf{a}_k)]_{1 \le j,k \le N}$$

is a polynomial of interpolation points. Hence, it is non-zero for almost all choices of interpolation points. In other words, a subset $A \subset \mathbb{R}^n$ of $\binom{n+d}{n}$ distinct points is regular for almost all choices of A. On the other hand, it is difficult to check whether a given set of $\binom{n+d}{n}$ points is regular as soon as $n \ge 2$. Explicit interpolation schemes are available in literatures. Chung and Yao [5] gave a quasi-constructive description of locations of nodes in $\mathbb{R}^n$ for which Lagrange interpolation is regular. Here, the interpolation points are suitably distributed on hyperplanes and form a so-called nature lattice. Another type of regular sets was discovered by Bos [2], where the interpolation points are taken from algebraic varieties in $\mathbb{R}^n$.

We consider the problem of Hermite interpolation by polynomials in $\mathbb{R}^n$. More precisely, the problem is to find a polynomial which matches, on a set of distinct points in $\mathbb{R}^n$, the values of a function and its partial derivatives. We deal with the case where the number of interpolation conditions is equal to the dimension of $\mathscr{P}_d(\mathbb{R}^n)$. If the interpolation problem has a unique solution, then we say that the interpolation scheme is regular.

Roughly speaking, Hermite interpolation of type total degree defined below is the most natural generalization of univariate Hermite interpolation.

Problem 1 (Lorentz [8, 9]) Let m be a positive integer. Let $d, d_1, \ldots, d_m$ be natural numbers such that

$$\binom{n+d}{n} = \binom{n+d_1}{n} + \cdots + \binom{n+d_m}{n}.$$

Let $\{\mathbf{b}_1, \ldots, \mathbf{b}_m\}$ be a set of distinct points in $\mathbb{R}^n$. Find conditions for which the interpolation problem

$$\frac{\partial^{|\alpha|} f}{\partial \mathbf{x}^\alpha}(\mathbf{b}_j) = c_{j,\alpha}, \quad 1 \le j \le m, \quad |\alpha| \le d_j$$

has a unique solution for given values $c_{j,\alpha}$.

Many regular (resp., almost regular, singular) interpolation schemes of type total degree can be found in literature. For instance, it was shown in [8] that the only multivariate interpolation of type total degree regular for all choice of nodes is Taylor interpolation. Also in [8], it was proved that the Hermite interpolation is singular if the number of nodes satisfies $2 \le m \le n+1$ with $n \ge 2$ except for the case of Lagrange interpolation. Some interesting results focusing on almost regular interpolation schemes were given in [16].

Another general type of Hermite interpolation was defined in [16], which replaces derivatives in the coordinate directions by directional derivatives. More precisely, interpolation conditions at the point $\mathbf{a} \in \mathbb{R}^n$ are given by chains of directional derivatives of consecutive order

$$f \mapsto f(\mathbf{a}), \quad f \mapsto D_{\mathbf{y}_1} f(\mathbf{a}), \ldots, f \mapsto D_{\mathbf{y}_1} \cdots D_{\mathbf{y}_k} f(\mathbf{a}), \quad \mathbf{y}_1, \ldots, \mathbf{y}_k \in \mathbb{R}^n \setminus \{0\}.$$

In [16], the authors introduced the notations of trees and the blockwise structure to defined general Hermite interpolation problems. Using the Bézier representation of polynomials in barycentric coordinates, they proved necessary and sufficient conditions for almost regular interpolation problems. Moreover, they established the Newton formula and the remainder formula for the Hermite interpolation polynomials. For details, we refer the reader to [16].

Some explicit multivariate Hermite interpolation schemes in $\mathbb{R}^n$ were constructed. In [1], the authors gave bivariate regular schemes, where the nodes are equidistant points on concentric circles and the derivatives are the normal derivatives at these points. In [3], Bos and Calvi constructed new regular interpolation schemes of Hermite type in $\mathbb{C}^n$ and $\mathbb{R}^n$. Here, the interpolation points are distributed on algebraic hypersurfaces and the discrete differential conditions come from certain least spaces of finite-dimensional spaces of analytic functions. For a recent account of the theory of Hermite interpolation, we refer the readers to [4, 9].

We now state another general problem. Associated with a polynomial $Q(\mathbf{x}) = \sum_\alpha c_\alpha \mathbf{x}^\alpha$, $c_\alpha \in \mathbb{R}$ in $\mathbb{R}^n$, we define a differential operator $P(\mathrm{D})$ by

$$Q(\mathrm{D})f = \sum_\alpha c_\alpha \frac{\partial^{|\alpha|} f}{\partial \mathbf{x}^\alpha}.$$

In the case when $Q(\mathbf{x}) = c$, we set $Q(\mathrm{D})f = cf$.

Problem 2 Let $A = \{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$ be m distinct points in $\mathbb{R}^n$. Let $n_1, \ldots, n_m$ and d be positive integers such that $n_1 + n_2 + \cdots + n_m = \binom{n+d}{n}$. Find differential operators $P_{jk}(\mathrm{D})$ for $j = 1, \ldots, m$ and $k = 0, \ldots, n_j - 1$ for which the interpolation problem

$$P_{jk}(\mathrm{D})P(\mathbf{a}_j) = f_{jk}, \quad 1 \le j \le m, \quad 0 \le k \le n_j - 1, \tag{1}$$

has a unique solution $P \in \mathscr{P}_d(\mathbb{R}^n)$ for any given preassigned data $\{f_{jk}\}$.

When $m = \binom{n+d}{n}$ and all the differential operators are point-evaluation functionals, Problem 2 becomes a Lagrange interpolation problem.

Some special cases of Problem 2 were recently studied by the first author of this paper. In [11], we gave a solution of the problem. The differential operators are the real parts and imaginary parts of the complex derivatives $\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right)^k$, $k \ge 0$. In [12], we considered an analogous problem on the space of bivariate symmetric polynomials. It was showed that the differential operators can be taken as $\left(-\beta\frac{\partial}{\partial x_1} + \alpha\frac{\partial}{\partial x_2}\right)^k$ with $\alpha, \beta \in \mathbb{R}$ and $k \ge 0$. We showed in [13] a way to mix two types of differential operators mentioned above to get solutions of Problem 2. Our method relies on factorization results for polynomials and the construction of certain Taylor type polynomials. Moreover, we give a Newton formula for the interpolation polynomial and use it to prove that the Hermite interpolation polynomial is the limit of the Lagrange interpolation polynomials.

The aim of this paper is to generalize Hermite interpolation in [11]. The regular Hermite schemes are constructed as follows. We first give a criterion such that a polynomial in $\mathscr{P}_d(\mathbb{R}^n)$ is divisible by the polynomial $q_{\mathbf{a}}(\mathbf{x}) := |(x_m - a_m) + c(x_j - a_j)|^2$ with $\mathbf{a} = (a_1, \ldots, a_n)$, $j \neq m$ and $c \in \mathbb{C} \setminus \mathbb{R}$. The criterion contains differential operators of the forms

$$f \longmapsto \left(\prod_{k=1, k\neq j,m}^{n} \frac{\partial^{\alpha_k}}{\partial x_k^{\alpha_k}} \right) \left(\frac{\partial}{\partial x_j} - c \frac{\partial}{\partial x_m} \right)^{\alpha_j} f(\mathbf{a}). \tag{2}$$

We use these differential operators to construct a Taylor type polynomial at $\mathbf{a}$. The formula and properties of the new Taylor polynomial are similar to that of ordinary Taylor polynomial. Next, we use the above-mentioned divisibility criterion to get a factorization result which leads to the regularity of Hermite interpolation. The formula for the Hermite interpolation polynomial is of a Newton form and is written in terms of Taylor type polynomials. This enables us to create an algorithm to compute the interpolation polynomial. We also give some examples to illustrate our results. It is worth pointing out that if we take $n = 2$ and $c = i$, we recover the theory of Hermite interpolation in [11].

Finally, we note that an analogous problem is studied on the unit sphere $\mathbb{S}$ in $\mathbb{R}^3$. In recent works, we constructed some regular Hermite schemes on $\mathbb{S}$. For more details, we refer the readers to [13–15].

Notations and conventions We use bold symbols $\mathbf{x}$, $\mathbf{a}$, etc. to denote points in $\mathbb{R}^n$. We always assume that $n \geq 2$, $1 \leq j, m \leq n$ with $m \neq j$. The constant c is a non-real number. The multi-index in $\mathbb{N}^{n-1}$ is denoted by $\alpha' = (\alpha_1, \ldots, \alpha_{m-1}, \alpha_{m+1}, \ldots, \alpha_n)$ in which the m-entry does not appear. Throughout this paper, f and g are real-valued functions. All polynomials are of real coefficients except for polynomials of two types $\Pi_{\alpha'}$ and $B_{\beta'}$. By a suitably defined function f, we mean that the function f is sufficiently differentiable.

2 Taylor type polynomials

2.1 A divisibility criterion

Let $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$ and $c \in \mathbb{C} \setminus \mathbb{R}$. Let $1 \leq j, m \leq n$ and $j \neq m$. We define the following polynomial

$$q_{\mathbf{a}}(\mathbf{x}) = |(x_m - a_m) + c(x_j - a_j)|^2.$$

Easy computations gives

$$q_{\mathbf{a}}(\mathbf{x}) = (x_m - a_m)^2 + 2\mathrm{Re}(c)(x_m - a_m)(x_j - a_j) + |c|^2(x_j - a_j)^2.$$

Note that $q_{\mathbf{a}}$ depends not only on $\mathbf{a}$ but also on j, m and c. We adopt this setting for simplicity of notations. Clearly $q_{\mathbf{a}}$ is an irreducible polynomial of degree 2 in $\mathbb{R}^n$ and

$$V(q_{\mathbf{a}}) := \{\mathbf{x} \in \mathbb{R}^n : q_{\mathbf{a}}(\mathbf{x}) = 0\} = \{\mathbf{x} \in \mathbb{R}^n : x_m = a_m,\ x_j = a_j\}, \tag{3}$$

which is a flat of dimension $n-2$ in $\mathbb{R}^n$.

Theorem 1 *Let* $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$ *and* $q_{\mathbf{a}}(\mathbf{x}) = |(x_m - a_m) + c(x_j - a_j)|^2$ *with* $m \neq j$ *and* $c \in \mathbb{C} \setminus \mathbb{R}$. *Let* P *be a polynomial of degree at most* d *in* $\mathbb{R}^n$ *of real coefficients. Then* P *is a multiple of* $q_{\mathbf{a}}$ *if and only if*

$$\left(\prod_{k=1, k\neq j,m}^{n} \frac{\partial^{\alpha_k}}{\partial x_k^{\alpha_k}}\right)\left(\frac{\partial}{\partial x_j} - c\frac{\partial}{\partial x_m}\right)^{\alpha_j} P(\mathbf{a}) = 0, \quad |\alpha'| = \sum_{k=1, k\neq m}^{n} \alpha_k \leq d, \tag{4}$$

Proof Without loss of generality we assume that $j = n-1$ and $m = n$. We write $P(\mathbf{x}) = \sum_{k=0}^{d} C_k(\mathbf{x}')x_n^k$ with $C_k \in \mathscr{P}_{d-k}(\mathbb{R}^{n-1})$ and $\mathbf{x}' = (x_1, \ldots, x_{n-1})$. We can regard $q_{\mathbf{a}}$ and P as polynomials in $\mathbb{C}^n$, that is $\mathbf{x} \in \mathbb{C}^n$. Since

$$q_{\mathbf{a}}(\mathbf{x}) = ((x_n - a_n) + c(x_{n-1} - a_{n-1}))\,((x_n - a_n) + \overline{c}(x_{n-1} - a_{n-1}))\,,$$

we see that $q_{\mathbf{a}}$ divides P if and only if both $(x_n - a_n) + c(x_{n-1} - a_{n-1})$ and $(x_n - a_n) + \overline{c}(x_{n-1} - a_{n-1})$ divide P. By [10, Lemma 2.5], the condition reduces to

$$P\left(\mathbf{x}', a_n - c(x_{n-1} - a_{n-1})\right) \equiv 0 \tag{5}$$

and

$$P\left(\mathbf{x}', a_n - \overline{c}(x_{n-1} - a_{n-1})\right) \equiv 0. \tag{6}$$

Note that (5) and (6) hold for every $\mathbf{x} \in \mathbb{C}^n$ if and only if they are true for every $\mathbf{x} \in \mathbb{R}^n$. Hence we can return to work with polynomials in $\mathbb{R}^n$. We have

$$\begin{aligned}
P\left(\mathbf{x}', a_n - c(x_{n-1} - a_{n-1})\right) &= \sum_{k=0}^{d} C_k(\mathbf{x}')\,(a_n - c(x_{n-1} - a_{n-1}))^k \\
&= \sum_{k=0}^{d} C_k(\mathbf{x}')\,(a_n - \mathrm{Re}(c)(x_{n-1} - a_{n-1}) - i\mathrm{Im}(c)(x_{n-1} - a_{n-1}))^k \\
&= \sum_{k=0}^{d} C_k(\mathbf{x}')Q_k(\mathbf{x}') + iC_k(\mathbf{x}')R_k(\mathbf{x}'),
\end{aligned}$$

where

$$Q_k(\mathbf{x}') = \sum_{l=0,\, l \text{ even}}^{k} (-1)^{\frac{j}{2}} \binom{k}{l} (a_n - \mathrm{Re}(c)(x_{n-1} - a_{n-1}))^{k-l}\,(\mathrm{Im}(c)(x_{n-1} - a_{n-1}))^l$$

and

$$R_k(\mathbf{x}') = -\sum_{l=0,\, l \text{ odd}}^{k} (-1)^{\frac{l-1}{2}} \binom{k}{l} (a_n - \mathrm{Re}(c)(x_{n-1} - a_{n-1}))^{k-l}\,(\mathrm{Im}(c)(x_{n-1} - a_{n-1}))^l\,.$$

Note that both $C_k Q_k$ and $C_k R_k$ belong to $\mathscr{P}_d(\mathbb{R}^{n-1})$. In other words, the real part and the imaginary part of $P\left(\mathbf{x}', a_n - c(x_{n-1} - a_{n-1})\right)$ are polynomials of degree at most d in $\mathbb{R}^{n-1}$.

We consider the canonical basis for $\mathscr{P}_d(\mathbb{R}^{n-1})$

$$\mathscr{B} = \left\{ \prod_{k=1}^{n-1} x_k^{\alpha_k} : \alpha' = (\alpha_1, \ldots, \alpha_{n-1}), |\alpha'| \le d \right\} \tag{7}$$

Observe that relation (5) holds if and only if

$$Q(\mathrm{D}) P\left(\mathbf{x}', a_n - c(x_{n-1} - a_{n-1})\right)\Big|_{\mathbf{x}'=\mathbf{a}'} = 0, \quad Q \in \mathscr{B}. \tag{8}$$

By the chain rule, it is easily seen that

$$\frac{\partial}{\partial x_{n-1}} P\left(\mathbf{x}', a_n - c(x_{n-1} - a_{n-1})\right)\Big|_{\mathbf{x}'=\mathbf{a}'} = \left(\frac{\partial}{\partial x_{n-1}} - c\frac{\partial}{\partial x_n}\right) P(\mathbf{a})$$

and

$$\frac{\partial}{\partial x_k} P\left(\mathbf{x}', a_n - c(x_{n-1} - a_{n-1})\right)\Big|_{\mathbf{x}'=\mathbf{a}'} = \frac{\partial P}{\partial x_k}(\mathbf{a}), \quad 1 \le k \le n-2.$$

More generally, relation (8) can be rewritten as

$$\left(\prod_{k=1}^{n-2} \frac{\partial^{\alpha_k}}{\partial x_k^{\alpha_k}}\right)\left(\frac{\partial}{\partial x_{n-1}} - c\frac{\partial}{\partial x_n}\right)^{\alpha_{n-1}} P(\mathbf{a}) = 0, \quad |\alpha'| = \alpha_1 + \cdots + \alpha_{n-1} \le d, \tag{9}$$

Using similar arguments applying to relation (6), we get

$$\left(\prod_{k=1}^{n-2} \frac{\partial^{\alpha_k}}{\partial x_k^{\alpha_k}}\right)\left(\frac{\partial}{\partial x_{n-1}} - \bar{c}\frac{\partial}{\partial x_n}\right)^{\alpha_{n-1}} P(\mathbf{a}) = 0, \quad |\alpha'| = \alpha_1 + \cdots + \alpha_{n-1} \le d, \tag{10}$$

Note that P is a polynomial of real coefficients, two relations (9) and (10) are equivalent, and the proof is complete. □

We define

$$\Pi_{\alpha'}(\mathbf{x}) = (x_j - c x_m)^{\alpha_j} \prod_{k=1, k\neq j,m}^{n} x_k^{\alpha_k}, \quad \alpha' = (\alpha_1, \ldots, \alpha_{m-1}, \alpha_{m+1}, \ldots, \alpha_n).$$

Then,

$$\Pi_{\alpha'}(\mathrm{D}) = \left(\prod_{k=1, k\neq j,m}^{n} \frac{\partial^{\alpha_k}}{\partial x_k^{\alpha_k}}\right)\left(\frac{\partial}{\partial x_j} - c\frac{\partial}{\partial x_m}\right)^{\alpha_j}.$$

Theorem 1 can be restated as: the polynomial $P \in \mathscr{P}_d(\mathbb{R}^n)$ is divisible by $q_{\mathbf{a}}(\mathbf{x}) = |(x_m - a_m) + c(x_j - a_j)|^2$ if and only if it satisfies the following relations

$$\Pi_{\alpha'}(\mathrm{D}) P(\mathbf{a}) = 0, \quad |\alpha'| = \sum_{k=1, k\neq m}^{n} \alpha_k \le d. \tag{11}$$

In the case where $\alpha_j \neq 0$, the differential operator $\Pi_{\alpha'}(\mathrm{D})$ can be decomposed into non-zero real and imaginary parts. More precisely $\Pi_{\alpha'}(\mathrm{D}) = \mathrm{Re}\,(\Pi_{\alpha'}(\mathrm{D})) + i\mathrm{Im}\,(\Pi_{\alpha'}(\mathrm{D}))$ where

$$\mathrm{Re}\,(\Pi_{\alpha'}(\mathrm{D})) = \left(\prod_{k=1,k\neq j,m}^{n} \frac{\partial^{\alpha_k}}{\partial x_k^{\alpha_k}}\right) \sum_{l=0,l \text{ even}}^{\alpha_j} (-1)^{\frac{l}{2}} \binom{\alpha_j}{l} \left(\frac{\partial}{\partial x_j} - \mathrm{Re}(c)\frac{\partial}{\partial x_m}\right)^{\alpha_j - l} \left(\mathrm{Im}(c)\frac{\partial}{\partial x_m}\right)^{l}$$

and

$$\mathrm{Im}\,(\Pi_{\alpha'}(\mathrm{D})) = -\left(\prod_{k=1,k\neq j,m}^{n} \frac{\partial^{\alpha_k}}{\partial x_k^{\alpha_k}}\right) \sum_{l=0,l \text{ odd}}^{\alpha_j} (-1)^{\frac{l-1}{2}} \binom{\alpha_j}{l} \left(\frac{\partial}{\partial x_j} - \mathrm{Re}(c)\frac{\partial}{\partial x_m}\right)^{\alpha_j - l} \left(\mathrm{Im}(c)\frac{\partial}{\partial x_m}\right)^{l}.$$

The above remark enables us to compute the number of interpolation conditions in the real setting in (11).

Lemma 1 *Let* $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$ *and* $c \in \mathbb{C} \setminus \mathbb{R}$. *Let* $1 \leq j, m \leq n$ *and* $j \neq m$. *Then the set of functionals*

$$f \longmapsto \Pi_{\alpha'}(\mathrm{D}) f(\mathbf{a}), \quad |\alpha'| \leq d, \quad \alpha_j = 0$$

and

$$f \longmapsto \mathrm{Re}\,(\Pi_{\alpha'}(\mathrm{D}))\, f(\mathbf{a}), \quad f \longmapsto \mathrm{Im}\,(\Pi_{\alpha'}(\mathrm{D}))\, f(\mathbf{a}), \quad |\alpha'| \leq d, \quad \alpha_j \neq 0$$

consist of $\binom{n+d-1}{n-1} + \binom{n+d-2}{n-1}$ *elements.*

To prove Lemma 1, we need the following simple result. The proof are left to the reader.

Lemma 2 *For any* $1 \leq k \leq n$, *we have*

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}, \qquad \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-2}{k-1} + \cdots + \binom{k-1}{k-1}.$$

Proof (Proof of Lemma 1) For simplicity, we assume that $j = n-1$ and $m = n$. If $\alpha_{n-1} = 0$, then $\alpha_1 + \cdots + \alpha_{n-2} \leq d$, and hence, we get $\binom{n+d-2}{n-2}$ differential operators of the forms

$$f \longmapsto \prod_{k=1}^{n-2} \frac{\partial^{\alpha_k}}{\partial x_k^{\alpha_k}} f(\mathbf{a}), \quad \alpha_1 + \cdots + \alpha_{n-2} \leq d.$$

Otherwise, if $0 < k = \alpha_{n-1} \leq d$, then $\alpha_1 + \cdots + \alpha_{n-2} \leq d-k$. In this case, we have $\binom{d-k+n-2}{n-2}$ choices of $(\alpha_1, \ldots, \alpha_{n-2})$ and two choices corresponding to the real part and the imaginary part of $\left(\frac{\partial}{\partial x_{n-1}} - c\frac{\partial}{\partial x_n}\right)^{\alpha_{n-1}}$. Hence, we get

$2\binom{d-k+n-2}{n-2}$ differential operators. The number of functionals coincide with the number of differential operators which are equal to

$$\begin{aligned}\binom{n+d-2}{n-2}+2\sum_{k=1}^{d}\binom{d-k+n-2}{n-2} &= \sum_{k=0}^{d}\binom{d-k+n-2}{n-2}+\sum_{k=1}^{d}\binom{d-k+n-2}{n-2}\\ &= \binom{n+d-1}{n-1}+\binom{n+d-2}{n-1},\end{aligned}$$

where we use Lemma 2 in the second relation. The proof is complete. □

2.2 Construction of Taylor type operators

We give a dual basis for the differential operators and use it to construct a Taylor type operator.

Lemma 3 *Let* $\mathbf{a}=(a_1,\ldots,a_n)\in\mathbb{R}^n$ *and* $c\in\mathbb{C}\setminus\mathbb{R}$. *For* $1\le j,m\le n$ *and* $j\neq m$, $\beta'=(\beta_1,\ldots,\beta_{m-1},\beta_{m+1},\ldots,\beta_n)\in\mathbb{N}^{n-1}$ *we set*

$$B_{\beta'}(\mathbf{x})=\frac{|c|^{2\beta_j}}{(|c|^2-c^2)^{\beta_j}\prod_{k=1,k\neq m}^{n}\beta_k!}\left(\prod_{k=1,k\neq j,m}^{n}(x_k-a_k)^{\beta_k}\right)\left((x_j-a_j)+\frac{c}{|c|^2}(x_m-a_m)\right)^{\beta_j}. \tag{12}$$

Then,

$$\Pi_{\alpha'}(\mathrm{D})B_{\beta'}(\mathbf{a})=\begin{cases}0 & if\quad \alpha'\neq\beta'\\ 1 & if\quad \alpha'=\beta'.\end{cases}$$

Proof By definition, we have

$$\begin{aligned}\Pi_{\alpha'}(\mathrm{D})B_{\beta'}(\mathbf{a}) = {}& \frac{|c|^{2\beta_j}}{(|c|^2-c^2)^{\beta_j}\beta_j!}\left(\frac{\partial}{\partial x_j}-c\frac{\partial}{\partial x_m}\right)^{\alpha_j}\left((x_j-a_j)+\frac{c}{|c|^2}(x_m-a_m)\right)^{\beta_j}\Bigg|_{\mathbf{x}=\mathbf{a}}\\ &\times\prod_{k=1,k\neq j,m}^{n}\frac{1}{\beta_k!}\frac{\partial^{\alpha_k}}{\partial x_k^{\alpha_k}}(x_k-a_k)^{\beta_k}\Bigg|_{\mathbf{x}=\mathbf{a}}.\end{aligned}$$

It is easily to check that

$$\left(\frac{\partial}{\partial x_j}-c\frac{\partial}{\partial x_m}\right)^{\alpha_j}\left((x_j-a_j)+\frac{c}{|c|^2}(x_m-a_m)\right)^{\beta_j}\Bigg|_{\mathbf{x}=\mathbf{a}}=\begin{cases}\alpha_j!\left(1-\frac{c^2}{|c|^2}\right)^{\alpha_j} & \text{if}\quad \alpha_j=\beta_j\\ 0 & \text{if}\quad \alpha_j\neq\beta_j\end{cases}$$

and, for $1\le k\le n$, $k\neq j,m$,

$$\frac{\partial^{\alpha_k}}{\partial x_k^{\alpha_k}}(x_k-a_k)^{\beta_k}\Bigg|_{\mathbf{x}=\mathbf{a}}=\begin{cases}\alpha_k! & \text{if}\quad \alpha_k=\beta_k\\ 0 & \text{if}\quad \alpha_k\neq\beta_k.\end{cases}$$

The result follows directly from the above computations. □

Proposition 1 *Let* $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$ *and* $c \in \mathbb{C} \setminus \mathbb{R}$. *Let* $1 \le j, m \le n$ *and* $j \neq m$. *For a suitably defined function* f, *we set*

$$\mathrm{T}^d_{\mathbf{a},q_{\mathbf{a}}}(f)(\mathbf{x}) = f(\mathbf{a}) + \sum_{|\alpha'|=1,\alpha_j=0}^{d} \Pi_{\alpha'}(\mathrm{D})(f)(\mathbf{a})B_{\alpha'}(\mathbf{x}) + \sum_{|\alpha'|=1,\alpha_j>0}^{d} \left(\Pi_{\alpha'}(\mathrm{D})(f)(\mathbf{a})B_{\alpha'}(\mathbf{x}) + \overline{\Pi}_{\alpha'}(\mathrm{D})(f)(\mathbf{a})\overline{B}_{\alpha'}(\mathbf{x})\right), \tag{13}$$

where

$$\overline{\Pi}_{\alpha'}(\mathbf{x}) := (x_j - \overline{c}x_m)^{\alpha_j} \prod_{k=1,k\neq j,m}^{n} x_k^{\alpha_k}$$

and

$$\overline{B}_{\alpha'}(\mathbf{x}) = \frac{|c|^{2\alpha_j}}{(|c|^2 - \overline{c}^2)^{\alpha_j} \prod_{k=1,k\neq m}^{n} \alpha_k!} \left(\prod_{k=1,k\neq j,m}^{n} (x_k - a_k)^{\alpha_k} \right) \left((x_j - a_j) + \frac{\overline{c}}{|c|^2}(x_m - a_m) \right)^{\alpha_j}.$$

Then, $\mathrm{T}^d_{\mathbf{a},q_{\mathbf{a}}}(f)$ *belongs to* $\mathscr{P}_d(\mathbb{R}^n)$ *and*

$$\Pi_{\alpha'}(\mathrm{D})\left(\mathrm{T}^d_{\mathbf{a},q_{\mathbf{a}}}(f)\right)(\mathbf{a}) = \Pi_{\alpha'}(\mathrm{D})(f)(\mathbf{a}), \quad |\alpha'| \le d. \tag{14}$$

The polynomial $\mathrm{T}^d_{\mathbf{a},q_{\mathbf{a}}}(f)$ is called a Taylor type polynomial of f at $\mathbf{a}$ corresponding to $q_{\mathbf{a}}$.

Proof Firstly, observe that $\Pi_{\alpha'}(\mathrm{D})(f)(\mathbf{a}) \in \mathbb{R}$ and $B_{\alpha'} \in \mathscr{P}_d(\mathbb{R}^n)$ when $\alpha_j = 0$. By definition, $\Pi_{\alpha'}(\mathrm{D})(f)(\mathbf{a})$ and $B_{\alpha'}(\mathbf{x})$ are the complex conjugates of $\overline{\Pi}_{\alpha'}(\mathrm{D})(f)(\mathbf{a})$ and $\overline{B}_{\alpha'}(\mathbf{x})$ respectively. In addition, $B_{\alpha'}$ is a polynomial of degree $|\alpha'| \le d$. It follows that $\mathrm{T}^d_{\mathbf{a},q_{\mathbf{a}}}(f)$ is a polynomial of degree at most d with real coefficients. It remains to check the relation (14). From the formula, we consider three cases.

If $\alpha' = 0$, then $\mathrm{T}^d_{\mathbf{a},q_{\mathbf{a}}}(f)(\mathbf{a}) = f(\mathbf{a})$, because $B_{\beta'}(\mathbf{a}) = 0$ for every $|\beta'| > 0$.

Next, we assume that $|\alpha'| > 0$ and $\alpha_j = 0$. Then, for any β' with $|\beta'| > 0$ and $\beta_j > 0$, we have $\alpha' \neq \beta'$. It follows from Lemma 3 that $\Pi_{\alpha'}(\mathrm{D})\left(B_{\beta'}\right)(\mathbf{a}) = 0$ for such β'. Moreover

$$\Pi_{\alpha'}(\mathrm{D})\left(\overline{B}_{\beta'}\right)(\mathbf{a}) = \overline{\Pi_{\alpha'}(\mathrm{D})\left(B_{\beta'}\right)(\mathbf{a})} = 0.$$

On the other hand, if $|\beta'| > 0$ and $\beta_j = 0$ then, using Lemma 3 again, we obtain

$$\Pi_{\alpha'}(\mathrm{D})\left(B_{\beta'}\right)(\mathbf{a}) = \begin{cases} 0 \text{ if } & \alpha' \neq \beta' \\ 1 \text{ if } & \alpha' = \beta'. \end{cases}$$

Hence, relation (14) holds in this case.

Finally, we treat the case where $|\alpha'| > 0$ and $\alpha_j > 0$. It is easily seen that

$$\left(\frac{\partial}{\partial x_j} - c\frac{\partial}{\partial x_m}\right)^{\alpha_j}\left((x_j - a_j) + \frac{\overline{c}}{|c|^2}(x_m - a_m)\right)^{\beta_j}\Bigg|_{\mathbf{x}=\mathbf{a}} = 0.$$

It follows that

$$\Pi_{\alpha'}(\mathrm{D})\left(\overline{B}_{\beta'}\right)(\mathbf{a}) = 0.$$

Furthermore, we can use Lemma 3 to get

$$\Pi_{\alpha'}(\mathrm{D})\left(B_{\beta'}\right)(\mathbf{a}) = \begin{cases} 0 \text{ if } & \alpha' \neq \beta' \\ 1 \text{ if } & \alpha' = \beta'. \end{cases}$$

Combining the last two relations, we obtain the desired equation. The proof is complete. □

Corollary 1 *For a suitably defined function f, $\mathrm{T}^d_{\mathbf{a},q_{\mathbf{a}}}(f) = 0$ if and only if*

$$\Pi_{\alpha'}(\mathrm{D})(f)(\mathbf{a}) = 0, \quad \forall |\alpha'| \leq d.$$

Proof One direction is trivial. We assume that $\Pi_{\alpha'}(\mathrm{D})(f)(\mathbf{a}) = 0$ for every $|\alpha'| \leq d$. Then, its conjugate $\overline{\Pi}_{\alpha'}(\mathrm{D})(f)(\mathbf{a})$ also vanishes. Hence, the conclusion follows directly from (13). □

Remark 1 The definition of the Taylor type polynomials gives a recurrent relation which is useful in computations,

$$\begin{aligned}\mathrm{T}^d_{\mathbf{a},q_{\mathbf{a}}}(f)(\mathbf{x}) &= \mathrm{T}^{d-1}_{\mathbf{a},q_{\mathbf{a}}}(f)(\mathbf{x}) + \sum_{|\alpha'|=d,\alpha_j=0} \Pi_{\alpha'}(\mathrm{D})(f)(\mathbf{a})B_{\alpha'}(\mathbf{x}) \\ &\quad + \sum_{|\alpha'|=d,\alpha_j>0}\left(\Pi_{\alpha'}(\mathrm{D})(f)(\mathbf{a})B_{\alpha'}(\mathbf{x}) + \overline{\Pi}_{\alpha'}(\mathrm{D})(f)(\mathbf{a})\overline{B}_{\alpha'}(\mathbf{x})\right).\end{aligned} \tag{15}$$

2.3 Some properties of Taylor type operators

In this subsection, we show that the Taylor type polynomial has some expected properties. In particular, the Taylor type polynomial of any multiple of $q_{\mathbf{a}}$ is identically zero.

Lemma 4 *The set of polynomials*

$$\mathscr{F} = \mathrm{span}_{\mathbb{C}}\left\{\Pi_{\alpha'} : |\alpha'| \leq d\right\}$$

is D*-invariance. In other words,*

$$\mathrm{D}^\beta \Pi_{\alpha'} \in \mathscr{F}, \quad \forall \beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n.$$

Proof It suffices to check that

$$\frac{\partial}{\partial x_k}\Pi_{\alpha'} \in \mathrm{span}_{\mathbb{C}}\left\{\Pi_{\alpha'} : |\alpha'| \leq d\right\}, \quad 1 \leq k \leq n. \tag{16}$$

Relation (16) is trivial when $\alpha' = 0$. Hence, we can assume that $|\alpha'| > 0$. Direct computations gives

$$\frac{\partial}{\partial x_m}\Pi_{\alpha'}(\mathbf{x}) = -c\alpha_j(x_j - cx_m)^{\alpha_j - 1}\prod_{k=1,k\neq j,m}^{n} x_k^{\alpha_k},$$

$$\frac{\partial}{\partial x_j}\Pi_{\alpha'}(\mathbf{x}) = \alpha_j(x_j - cx_m)^{\alpha_j - 1}\prod_{k=1,k\neq j,m}^{n} x_k^{\alpha_k}.$$

and

$$\frac{\partial}{\partial x_k}\Pi_{\alpha'}(\mathbf{x}) = \alpha_k(x_j - cx_m)^{\alpha_j}\prod_{l=1,l\neq j,m}^{n} x_l^{\alpha_l - \delta_{kl}}, \quad k \neq j, m,$$

where δ_{kl} is the Kronecker symbol. It follows from the above formulas for $\frac{\partial}{\partial x_k}\Pi_{\alpha'}$ that relation (16) holds in any cases. This finishes the proof of the lemma. □

Corollary 2 *For any $\beta \in \mathbb{N}^n$ and $|\alpha'| \leq d$, we have*

$$\left(\mathrm{D}^\beta \Pi_{\alpha'}\right)(\mathrm{D})\left(\mathrm{T}^d_{\mathbf{a},q_\mathbf{a}}(f)\right)(\mathbf{a}) = \left(\mathrm{D}^\beta \Pi_{\alpha'}\right)(\mathrm{D})(f)(\mathbf{a}). \tag{17}$$

Proof By Lemma 4, $\mathrm{D}^\beta \Pi_{\alpha'}$ belongs to $\mathscr{F}$. Hence, $\left(\mathrm{D}^\beta \Pi_{\alpha'}\right)(\mathrm{D})$ is a linear combination of the operators $\Pi_{\gamma'}(\mathrm{D})$, $|\gamma'| \leq d$. The desired relation now follows directly from (14). □

The result below asserts that the Taylor type operators obey the weak Leibniz rule.

Lemma 5 *For suitably defined functions f and g, we have*

$$\mathrm{T}^d_{\mathbf{a},q_\mathbf{a}}(fg) = \mathrm{T}^d_{\mathbf{a},q_\mathbf{a}}\left(f\mathrm{T}^d_{\mathbf{a},q_\mathbf{a}}(g)\right).$$

In particular, $\mathrm{T}^d_{\mathbf{a},q_\mathbf{a}}(g) = \mathrm{T}^d_{\mathbf{a},q_\mathbf{a}}\left(\mathrm{T}^d_{\mathbf{a},q_\mathbf{a}}(g)\right)$.

Proof It is sufficient to show that

$$(fg)(\mathbf{a}) = (f\mathrm{T}^d_{\mathbf{a},q_\mathbf{a}}(g))(\mathbf{a}), \tag{18}$$

$$\Pi_{\alpha'}(\mathrm{D})(fg)(\mathbf{a}) = \Pi_{\alpha'}(\mathrm{D})(f\mathrm{T}^d_{\mathbf{a},q_\mathbf{a}}(g))(\mathbf{a}), \quad 0 < |\alpha'| \leq d, \quad \alpha_j = 0 \tag{19}$$

and, for $\alpha_j > 0$,

$$\Pi_{\alpha'}(\mathrm{D})(fg)(\mathbf{a}) = \Pi_{\alpha'}(\mathrm{D})(f\mathrm{T}^d_{\mathbf{a},q_\mathbf{a}}(g))(\mathbf{a}), \quad \overline{\Pi}_{\alpha'}(\mathrm{D})(fg)(\mathbf{a}) = \overline{\Pi}_{\alpha'}(\mathrm{D})(f\mathrm{T}^d_{\mathbf{a},q_\mathbf{a}}(g))(\mathbf{a}). \tag{20}$$

Since $\mathrm{T}^d_{\mathbf{a},q_\mathbf{a}}(g)(\mathbf{a}) = g(\mathbf{a})$, relation (18) is trivial. For $0 < |\alpha'| \leq d$, we can use the Leibniz-Hörmander formula (see, e.g., [7, p. 177] or [6, p. 243]) to get

$$\Pi_{\alpha'}(\mathrm{D})(f\mathrm{T}^d_{\mathbf{a},q_\mathbf{a}}(g))(\mathbf{a}) = \sum_{|\beta|\leq \deg \Pi_{\alpha'}} \left(\mathrm{D}^\beta \Pi_{\alpha'}(\mathrm{D})\right)(\mathrm{T}^d_{\mathbf{a},q_\mathbf{a}}(g))(\mathbf{a})\frac{1}{\beta!}\mathrm{D}^\beta f(\mathbf{a}). \tag{21}$$

Now, it follows from Corollary 2 that

$$\left(\mathrm{D}^{\beta}\Pi_{\alpha'}(\mathrm{D})\right)(\mathrm{T}^{d}_{\mathbf{a},q_{\mathbf{a}}}(g))(\mathbf{a}) = \left(\mathrm{D}^{\beta}\Pi_{\alpha'}(\mathrm{D})\right)(g)(\mathbf{a}), \quad |\beta| \le \deg \Pi_{\alpha'}. \tag{22}$$

Consequently,

$$\Pi_{\alpha'}(\mathrm{D})(f\mathrm{T}^{d}_{\mathbf{a},q_{\mathbf{a}}}(g))(\mathbf{a}) = \sum_{|\beta|\le \deg \Pi_{\alpha'}} \left(\mathrm{D}^{\beta}\Pi_{\alpha'}(\mathrm{D})\right)(g)(\mathbf{a})\frac{1}{\beta!}\mathrm{D}^{\beta}f(\mathbf{a}) = \Pi_{\alpha'}(\mathrm{D})(fg)(\mathbf{a}), \tag{23}$$

where, in the second relation, we use the Leibniz-Hörmander formula again. This proves (19) and the first relation in (20). The second relation in (20) follows directly from the first one since

$$\overline{\Pi}_{\alpha'}(\mathrm{D})(fg)(\mathbf{a}) = \overline{\Pi_{\alpha'}(\mathrm{D})(fg)(\mathbf{a})}, \quad \overline{\Pi}_{\alpha'}(\mathrm{D})(f\mathrm{T}^{d}_{\mathbf{a},q_{\mathbf{a}}}(g))(\mathbf{a}) = \overline{\Pi_{\alpha'}(\mathrm{D})(f\mathrm{T}^{d}_{\mathbf{a},q_{\mathbf{a}}}(g))(\mathbf{a})}.$$

The proof is complete. □

Corollary 3 *If* $\Pi_{\alpha'}(\mathrm{D})(f)(\mathbf{a}) = 0$ *for any* $|\alpha'| \le d$*, then* $\Pi_{\alpha'}(\mathrm{D})(fg)(\mathbf{a}) = 0$ *for any* $|\alpha'| \le d$*.*

Proof By Corollary 1, we have $\mathrm{T}^{d}_{\mathbf{a},q_{\mathbf{a}}}(f) = 0$. Hence, using Lemma 5, we can write

$$\mathrm{T}^{d}_{\mathbf{a},q_{\mathbf{a}}}(fg) = \mathrm{T}^{d}_{\mathbf{a},q_{\mathbf{a}}}\left(g\mathrm{T}^{d}_{\mathbf{a},q_{\mathbf{a}}}(f)\right) = \mathrm{T}^{d}_{\mathbf{a},q_{\mathbf{a}}}(g\cdot 0) = 0.$$

Corollary 1 now gives the desired relations. □

Lemma 6 *If* Q *is a multiple of* $q_{\mathbf{a}}$*, then* $\mathrm{T}^{d}_{\mathbf{a},q_{\mathbf{a}}}(Q) = 0$*.*

Proof Without loss of generality we assume that $j = n-1$ and $m = n$. We first prove that $\mathrm{T}^{d}_{\mathbf{a},q_{\mathbf{a}}}(q_{\mathbf{a}}) = 0$. We see that

$$q_{\mathbf{a}}(\mathbf{x}) = |(x_n - a_n) + c(x_{n-1} - a_{n-1})|^2 = h_{\mathbf{a}}(\mathbf{x})\overline{h}_{\mathbf{a}}(\mathbf{x}).$$

Here, $h_{\mathbf{a}}(\mathbf{x}) = (x_n - a_n) + c(x_{n-1} - a_{n-1})$ and $\overline{h}_{\mathbf{a}}(\mathbf{x}) = (x_n - a_n) + \overline{c}(x_{n-1} - a_{n-1})$. Evidently, $\Pi_{\alpha'}(\mathrm{D})(q_{\mathbf{a}})(\mathbf{a}) = 0$ when $\alpha' = 0$ or $|\alpha'| \ge 3$. Direct computations show that

$$\mathrm{D}_1(q_{\mathbf{a}})(\mathbf{a}) = 0 \tag{24}$$

and

$$\mathrm{D}_1\mathrm{D}_2(q_{\mathbf{a}})(\mathbf{a}) = 0 \tag{25}$$

for every

$$\mathrm{D}_1, \mathrm{D}_2 \in \left\{\frac{\partial}{\partial x_{n-1}} - c\frac{\partial}{\partial x_n}, \frac{\partial}{\partial x_k} : 1 \le k \le n-2\right\}. \tag{26}$$

Indeed, relation (24) follows directly from the fact that $\frac{\partial}{\partial x_k}q_{\mathbf{a}}(\mathbf{a}) = 0$ for every $1 \le k \le n$. Moreover, we see that

$$\left(\frac{\partial}{\partial x_{n-1}} - c\frac{\partial}{\partial x_n}\right)(h_{\mathbf{a}}) = 0, \qquad \left(\frac{\partial}{\partial x_{n-1}} - c\frac{\partial}{\partial x_n}\right)(\overline{h}_{\mathbf{a}}) = \overline{c} - c$$

and

$$\frac{\partial}{\partial x_k}(h_{\mathbf{a}}) = \frac{\partial}{\partial x_k}(\overline{h}_{\mathbf{a}}) = 0, \quad 1 \le k \le n-2.$$

It follows that

$$\begin{aligned} \mathrm{D}_1\mathrm{D}_2(q_{\mathbf{a}})(\mathbf{a}) &= \mathrm{D}_1\mathrm{D}_2(h_{\mathbf{a}}\overline{h}_{\mathbf{a}})(\mathbf{a}) \\ &= \mathrm{D}_1\left(\overline{h}_{\mathbf{a}}\mathrm{D}_2 h_{\mathbf{a}} + h_{\mathbf{a}}\mathrm{D}_2\overline{h}_{\mathbf{a}}\right)(\mathbf{a}) \\ &= \left(\mathrm{D}_1\overline{h}_{\mathbf{a}}\mathrm{D}_2 h_{\mathbf{a}} + \mathrm{D}_1 h_{\mathbf{a}}\mathrm{D}_2\overline{h}_{\mathbf{a}}\right)(\mathbf{a}) \\ &= 0. \end{aligned}$$

Consequently, $\Pi_{\alpha'}(\mathrm{D})(q_{\mathbf{a}})(\mathbf{a}) = 0$ for every α'. Now, by the definition in Proposition 1 we get $\mathrm{T}^d_{\mathbf{a},q_{\mathbf{a}}}(q_{\mathbf{a}}) = 0$.

Next, we write $Q = q_{\mathbf{a}}Q_1$. Then, Lemma 5 enables us to write

$$\mathrm{T}^d_{\mathbf{a},q_{\mathbf{a}}}(Q) = \mathrm{T}^d_{\mathbf{a},q_{\mathbf{a}}}(q_{\mathbf{a}}Q_1) = \mathrm{T}^d_{\mathbf{a},q_{\mathbf{a}}}\left(Q_1\mathrm{T}^d_{\mathbf{a},q_{\mathbf{a}}}(q_{\mathbf{a}})\right) = \mathrm{T}^d_{\mathbf{a},q_{\mathbf{a}}}(Q_1\cdot 0) = 0.$$

The proof is complete. □

3 Hermite interpolation in $\mathbb{R}^n$

3.1 Hermite interpolation schemes

In the main theorem below, we show that interpolation conditions corresponding to Taylor type polynomials can be collected to obtain regular Hermite interpolation schemes in $\mathbb{R}^n$.

Theorem 2 *Let $d \ge 2$ be a positive integer and $m = [d/2]+1$. Let $s_k = d-2k+2$ for $k = 1,\ldots,m$. Let $1 \le j_k, m_k \le n$ with $j_k \ne m_k$ and $c^{[k]} \in \mathbb{C}\setminus\mathbb{R}$ for $k = 1,\ldots,m$. Let $A = \{\mathbf{a}_1,\ldots,\mathbf{a}_m\}$ be m distinct points in $\mathbb{R}^n$ such that $q_{\mathbf{a}_k}(\mathbf{a}_j) \ne 0$ for $j > k$, where*

$$q_{\mathbf{a}_k}(\mathbf{x}) = |\left(x_{m_k} - a^{[k]}_{m_k}\right) + c^{[k]}\left(x_{j_k} - a^{[k]}_{j_k}\right)|^2, \quad \mathbf{a}_k = \left(a^{[k]}_1,\ldots,a^{[k]}_n\right).$$

For each $1 \le k \le m$, let

$$\Pi^{[k]}_{(\alpha^{[k]})'}(\mathbf{x}) = (x_{j_k} - c^{[k]}x_{m_k})^{\alpha^{[k]}_{j_k}} \prod_{l=1,l\ne j_k,m_k}^{n} x_l^{\alpha^{[k]}_l}$$

where $\left(\alpha^{[k]}\right)' = \left(\alpha^{[k]}_1,\ldots,\alpha^{[k]}_{m_k-1},\alpha^{[k]}_{m_k+1},\ldots,\alpha^{[k]}_n\right)$. Then, for any function f is of class C^{s_k} in neighborhoods of the $\mathbf{a}_k$'s, there exists a unique polynomial $P \in \mathscr{P}_d(\mathbb{R}^n)$ such that

$$\Pi^{[k]}_{(\alpha^{[k]})'}(\mathrm{D})(P)(\mathbf{a}_k) = \Pi^{[k]}_{(\alpha^{[k]})'}(\mathrm{D})(f)(\mathbf{a}_k), \quad 1 \le k \le m, \quad |(\alpha^{[k]})'| \le s_k, \qquad (27)$$

Moreover, $P=\sum_{k=1}^m P_k$*, where* $P_1(\mathbf{x})=\mathrm{T}^{s_1}_{\mathbf{a}_1,q_{\mathbf{a}_1}}(f)(\mathbf{x})$*,*

$$P_k(\mathbf{x})=\prod_{j=1}^{k-1} q_{\mathbf{a}_j}(\mathbf{x})\mathrm{T}^{s_k}_{\mathbf{a}_k,q_{\mathbf{a}_k}}\left(\frac{f-P_1-\cdots-P_{k-1}}{\prod_{j=1}^{k-1} q_{\mathbf{a}_j}}\right)(\mathbf{x}),\quad 2\le k\le m.$$

Proof For each $1\le k\le m$, Lemma 1 shows that the number of interpolation conditions in (27) is

$$\binom{n+s_k-1}{n-1}+\binom{n+s_k-2}{n-1}=\binom{n+s_k}{n}-\binom{n+s_k-2}{n},$$

where we use Lemma 2 in the above binomial relation. It follows that the number of interpolation conditions in the Hermite scheme is equal to

$$\sum_{k=1}^m\left[\binom{n+s_k}{n}-\binom{n+s_k-2}{n}\right]=\sum_{k=1}^m\left[\binom{n+d-2k+2}{n}-\binom{n+d-2k}{n}\right]=\binom{n+d}{n},$$

which matches the dimension of $\mathscr{P}_d(\mathbb{R}^n)$. Hence, to prove the regularity of the Hermite scheme, it is sufficient to check that if $H\in\mathscr{P}_d(\mathbb{R}^n)$ and

$$\Pi^{[k]}_{(\alpha^{[k]})'}(\mathrm{D})H(\mathbf{a}_k)=0,\quad 1\le k\le m,\quad |(\alpha^{[k]})'|\le s_k, \tag{28}$$

then $H=0$. Since $s_1=d$, relation (28) along with Theorem 1 asserts that H divides $q_{\mathbf{a}_1}$. Hence, we can write $H=q_{\mathbf{a}_1}H_1$ with $\deg H_1\le d-2=s_2$. Using Corollary 3 for $f=H$ and $g=\frac{1}{q_{\mathbf{a}_1}}$, we get from (28) the following relations

$$\Pi^{[k]}_{(\alpha^{[k]})'}(\mathrm{D})(H_1)(\mathbf{a}_k)=0,\quad 2\le k\le m,\quad |(\alpha^{[k]})'|\le s_k. \tag{29}$$

By similar arguments, we have $H_1=q_{\mathbf{a}_2}H_2$ with $\deg H_2\le d-4$. We continue in this fashion to obtain

$$H(\mathbf{x})=\prod_{k=1}^m q_{\mathbf{a}_k}(\mathbf{x})H_{m+1}(\mathbf{x}),\quad H_{m+1}\in\mathscr{P}(\mathbb{R}^n).$$

It follows from the last relation that $H=0$. Conversely, suppose that $H\ne 0$. Then, the degree of the polynomial on the right hand side is at least $2m>d$. This contradicts to the fact that $\deg H\le d$, and the proof the first part of the theorem is complete.

It remains to prove the formula for the interpolation polynomial. We first check that the polynomial $P=\sum_{k=1}^m P_k$ belongs to $\mathscr{P}_d(\mathbb{R}^n)$. By definition we have $P_1\in\mathscr{P}_{s_1}(\mathbb{R}^n)=\mathscr{P}_d(\mathbb{R}^n)$. For $2\le k\le m$, since $\mathrm{T}^{s_k}_{\mathbf{a}_k,q_{\mathbf{a}_k}}(g)\in\mathscr{P}_{s_k}(\mathbb{R}^n)$, we get $\deg P_k\le 2(k-1)+s_k=d$. It follows that $\deg P\le d$. By Corollary 1, it is sufficient to show that

$$\mathrm{T}^{s_k}_{\mathbf{a}_k,q_{\mathbf{a}_k}}(P)(\mathbf{x})=\mathrm{T}^{s_k}_{\mathbf{a}_k,q_{\mathbf{a}_k}}(f)(\mathbf{x}),\quad 1\le k\le m. \tag{30}$$

To prove relation (30) we first treat the case $k=1$. In this case, we see that $q_{\mathbf{a}_1}$ divides P_k for any $k\ge 2$. Hence, Lemma 6 gives $\mathrm{T}^{s_1}_{\mathbf{a}_1,q_{\mathbf{a}_1}}(P_k)=0$ for $2\le k\le m$. This enables us to write

$$\mathrm{T}^{s_1}_{\mathbf{a}_1,q_{\mathbf{a}_1}}(P)=\mathrm{T}^{s_1}_{\mathbf{a}_1,q_{\mathbf{a}_1}}(P_1)=\mathrm{T}^{s_1}_{\mathbf{a}_1,q_{\mathbf{a}_1}}(f),$$

where we use Lemma 5 in the second relation. Next, we assume that $2 \leq k \leq m$. Using the fact that P_j contains the factor $q_{\mathbf{a}_k}$ for $k+1 \leq j \leq m$ and Lemma 6 we get $\mathrm{T}^{s_k}_{\mathbf{a}_k, q_{\mathbf{a}_k}}(P_j) = 0$ for $k+1 \leq j \leq m$. Therefore

$$\mathrm{T}^{s_k}_{\mathbf{a}_k, q_{\mathbf{a}_k}}(P) = \sum_{j=1}^{k} \mathrm{T}^{s_k}_{\mathbf{a}_k, q_{\mathbf{a}_k}}(P_j) = \mathrm{T}^{s_k}_{\mathbf{a}_k, q_{\mathbf{a}_k}}(P_k) + \sum_{j=1}^{k-1} \mathrm{T}^{s_k}_{\mathbf{a}_k, q_{\mathbf{a}_k}}(P_j). \tag{31}$$

From Lemma 5, we have

$$\begin{aligned}
\mathrm{T}^{s_k}_{\mathbf{a}_k, q_{\mathbf{a}_k}}(P_k) &= \mathrm{T}^{s_k}_{\mathbf{a}_k, q_{\mathbf{a}_k}}\left(\prod_{j=1}^{k-1} q_{\mathbf{a}_k} \mathrm{T}^{s_k}_{\mathbf{a}_k, q_{\mathbf{a}_k}}\left(\frac{f - P_1 - \cdots - P_{k-1}}{\prod_{j=1}^{k-1} q_{\mathbf{a}_j}}\right)\right) \\
&= \mathrm{T}^{s_k}_{\mathbf{a}_k, q_{\mathbf{a}_k}}\left(\prod_{j=1}^{k-1} q_{\mathbf{a}_k} \frac{f - P_1 - \cdots - P_{k-1}}{\prod_{j=1}^{k-1} q_{\mathbf{a}_j}}\right) \\
&= \mathrm{T}^{s_k}_{\mathbf{a}_k, q_{\mathbf{a}_k}}(f - P_1 - \cdots - P_{k-1}) \\
&= \mathrm{T}^{s_k}_{\mathbf{a}_k, q_{\mathbf{a}_k}}(f) - \sum_{j=1}^{k-1} \mathrm{T}^{s_k}_{\mathbf{a}_k, q_{\mathbf{a}_k}}\left(P_j\right).
\end{aligned}$$

Combining the last relation with (31) we finally obtain $\mathrm{T}^{s_k}_{\mathbf{a}_k, q_{\mathbf{a}_k}}(P) = \mathrm{T}^{s_k}_{\mathbf{a}_k, q_{\mathbf{a}_k}}(f)$, which proves the claim. The proof is complete. □

Corollary 4 *The interpolation polynomial $P \in \mathscr{P}_d(\mathbb{R}^n)$ in Theorem 2 is determined by the following relation*

$$\mathrm{T}^{s_k}_{\mathbf{a}_k, q_{\mathbf{a}_k}}(P)(\mathbf{x}) = \mathrm{T}^{s_k}_{\mathbf{a}_k, q_{\mathbf{a}_k}}(f)(\mathbf{x}), \quad 1 \leq k \leq m. \tag{32}$$

Remark 2 The condition $q_{\mathbf{a}_k}(\mathbf{a}_j) \neq 0$ for $j > k$ is used in the proof of Theorem 2. From (3), we see that it is equivalent to

$$\left(a^{[k]}_{m_k}, a^{[k]}_{j_k}\right) \neq \left(a^{[j]}_{m_k}, a^{[j]}_{j_k}\right), \quad j > k.$$

In other words, the m_k-coordinate and the j_k-coordinate of $\mathbf{a}_k$ and $\mathbf{a}_j$ are not simultaneously identical for any $j > k$.

Definition 1 The interpolation polynomial $P \in \mathscr{P}_d(\mathbb{R}^n)$ in Theorem 2 is called a Hermite type interpolation polynomial of f at A. We write

$$P = \mathbf{H}[\{(\mathbf{a}_1, q_{\mathbf{a}_1}, s_1), \ldots, (\mathbf{a}_m, q_{\mathbf{a}_m}, s_m)\}; f].$$

From the Newton type formula in Theorem 2, we obtain an algorithm to compute the polynomial $\mathbf{H}[\{(\mathbf{a}_1, q_{\mathbf{a}_1}, s_1), \ldots, (\mathbf{a}_m, q_{\mathbf{a}_m}, s_m)\}; f]$.

Step 1. Compute $P_1 = \mathrm{T}^{s_1}_{\mathbf{a}_1, q_{\mathbf{a}_1}}(f)$ by using (13);

Step 2. Compute $P_k = \prod_{j=1}^{k-1} q_{\mathbf{a}_j} \mathrm{T}^{s_k}_{\mathbf{a}_k, q_{\mathbf{a}_k}}\left(\frac{f - P_1 - \cdots - P_{k-1}}{\prod_{j=1}^{k-1} q_{\mathbf{a}_j}}\right)$ for $k = 2, \ldots, m$ respectively by using (13);

Step 3. Compute the sum $\mathbf{H}[\{(\mathbf{a}_1, q_{\mathbf{a}_1}, s_1), \ldots, (\mathbf{a}_m, q_{\mathbf{a}_m}, s_m)\}; f] = P_1 + \cdots + P_m$.

Remark 3 We have known that some kinds of Hermite interpolants are the limits of Lagrange interpolants when interpolation points coalesce (see, e.g., [11–14]). One may ask whether there are regular Lagrange interpolation schemes such that the corresponding Lagrange interpolation polynomials of sufficiently smooth functions converge to the Hermite interpolation polynomial constructed in Theorem 2.

3.2 Some examples

In this subsection, we compute the set differential operators of degree 2. We also give explicit formulas for Hermite interpolation polynomials of degree 2 and degree 3 in $\mathbb{R}^3$.

Example 1 This example gives interpolation conditions for Hermite interpolation of degree 2 at two points in $\mathbb{R}^n$. Let $d = 2$. Then $m = 2$, $s_1 = 2$ and $s_2 = 0$. Let $\mathbf{a}_k = \left(a_1^{[k]}, \ldots, a_n^{[k]}\right)$ for $k = 1, 2$. Set $c^{[k]} = u_k + iv_k$ with $v_k \neq 0$, $k = 1, 2$. Take $1 \leq j_k, m_k \leq n$ such that $j_k \neq m_k$ for $k = 1, 2$. Set

$$q_{\mathbf{a}_k}(\mathbf{x}) = |\left(x_{m_k} - a_{m_k}^{[k]}\right) + c^{[k]}(x_{j_k} - a_{j_k}^{[k]})|^2, \quad k = 1, 2.$$

We have

$$\begin{aligned}\left\{\Pi^{[1]}_{(\alpha^{[1]})'} : |(\alpha^{[1]})'| \leq 2\right\} = \{1\} \bigcup \{x_l, x_{j_1} - u_1 x_{m_1} - iv_1 x_{m_1} : l \neq j_1, m_1\} \\ \bigcup \left\{(x_{j_1} - u_1 x_{m_1})^2 - v_1^2 x_{m_1}^2 - 2iv_1(x_{j_1} - u_1 x_{m_1})x_{m_1}\right\} \\ \bigcup \left\{\left(x_{j_1} - u_1 x_{m_1} - iv_1 x_{m_1}\right) x_l : l \neq j_1, m_1\right\} \bigcup \{x_\ell x_l : 1 \leq \ell \leq l \leq n, \ell, l \neq j_1, m_1\}.\end{aligned}$$

Since $s_2 = 0$, we get

$$\left\{\Pi^{[2]}_{(\alpha^{[2]})'} : |(\alpha^{[2]})'| \leq 0\right\} = \{1\}.$$

The interpolation conditions in (27) corresponding the following differential operators

$$f \mapsto f(\mathbf{a}_1), \quad f \mapsto f(\mathbf{a}_2), \quad f \mapsto \frac{\partial f}{\partial x_l}(\mathbf{a}_1), \quad 1 \leq l \leq n, \ l \neq j_1, m_1,$$

$$f \mapsto \left(\frac{\partial}{\partial x_{j_1}} - u_1 \frac{\partial}{\partial x_{m_1}} - iv_1 \frac{\partial}{\partial x_{m_1}}\right) f(\mathbf{a}_1),$$

$$f \mapsto \left(\left(\frac{\partial}{\partial x_{j_1}} - u_1 \frac{\partial}{\partial x_{m_1}}\right)^2 - v_1^2 \frac{\partial^2}{\partial x_{m_1}^2} - 2iv_1 \left(\frac{\partial}{\partial x_{j_1}} - u_1 \frac{\partial}{\partial x_{m_1}}\right) \frac{\partial}{\partial x_{m_1}}\right) f(\mathbf{a}_1),$$

$$f \mapsto \left(\frac{\partial}{\partial x_{j_1}} - u_1 \frac{\partial}{\partial x_{m_1}} - iv_1 \frac{\partial}{\partial x_{m_1}}\right) \frac{\partial}{\partial x_l} f(\mathbf{a}_1), \quad 1 \leq l \leq n, \ l \neq j_1, m_1,$$

$$f \mapsto \frac{\partial^2}{\partial x_\ell \partial x_l} f(\mathbf{a}_1), \quad 1 \leq \ell \leq l \leq n, \ \ell, l \neq j_1, m_1.$$

Easy computations show that the above relations are equivalent to $\binom{n+2}{2}$ interpolation conditions in the real settings:

$$f \mapsto f(\mathbf{a}_1), \quad f \mapsto f(\mathbf{a}_2), \quad f \mapsto \frac{\partial f}{\partial x_l}(\mathbf{a}_1), \quad 1 \le l \le n,$$

$$f \mapsto \left(\left(\frac{\partial}{\partial x_{j_1}} - u_1 \frac{\partial}{\partial x_{m_1}} \right)^2 - v_1^2 \frac{\partial^2}{\partial x_{m_1}^2} \right) f(\mathbf{a}_1),$$

$$f \mapsto \left(\frac{\partial}{\partial x_{j_1}} - u_1 \frac{\partial}{\partial x_{m_1}} \right) \frac{\partial}{\partial x_{m_1}} f(\mathbf{a}_1),$$

$$f \mapsto \frac{\partial^2}{\partial x_{j_1} \partial x_l} f(\mathbf{a}_1), \quad f \mapsto \frac{\partial^2}{\partial x_{m_1} \partial x_l} f(\mathbf{a}_1), \quad 1 \le l \le n, \ l \ne j_1, m_1,$$

$$f \mapsto \frac{\partial^2}{\partial x_\ell \partial x_l} f(\mathbf{a}_1), \quad 1 \le \ell \le l \le n, \ell, l \ne j_1, m_1.$$

Example 2 We construct a formula for the Hermite interpolation polynomial of degree 2 at two points in $\mathbb{R}^3$. Let $n = 3$ and $d = 2$. Then, $m = 2$, $s_1 = 2$ and $s_2 = 0$. Let $\mathbf{a}_1 = (0, 0, 0)$ and $\mathbf{a}_2 = (0, 1, 0)$. We choose $c^{[1]} = 2i$ and $c^{[2]} = 3i$. Take $j_1 = 1, m_1 = j_2 = 2, m_2 = 3$. Then,

$$q_{\mathbf{a}_1}(\mathbf{x}) = 4x_1^2 + x_2^2, \quad q_{\mathbf{a}_2}(\mathbf{x}) = 9(x_2 - 1)^2 + x_3^2.$$

The polynomial $P = \mathbf{H}[\{(\mathbf{a}_1, q_{\mathbf{a}_1}, 2), (\mathbf{a}_2, q_{\mathbf{a}_2}, 0)\}](f)$ belonging to $\mathscr{P}_2(\mathbb{R}^3)$ interpolates f at the following 10 conditions

$$g \mapsto g(\mathbf{a}_1), \quad g \mapsto g(\mathbf{a}_2), \quad g \mapsto \frac{\partial g}{\partial x_l}(\mathbf{a}_1), \quad 1 \le l \le 3, \tag{33}$$

$$g \mapsto \left(\frac{\partial^2}{\partial x_1^2} - 4 \frac{\partial^2}{\partial x_2^2} \right) g(\mathbf{a}_1), \quad g \mapsto \frac{\partial^2}{\partial x_3^2} g(\mathbf{a}_1), \tag{34}$$

$$g \mapsto \frac{\partial^2}{\partial x_\ell \partial x_l} g(\mathbf{a}_1), \quad 1 \le \ell < l \le 3. \tag{35}$$

By definition of Taylor type polynomial in (13), we have

$$\begin{aligned} \mathrm{T}^2_{\mathbf{a}_1, q_{\mathbf{a}_1}}(f)(\mathbf{x}) = \; & f(\mathbf{a}_1) + \sum_{1 \le \alpha_3 \le 2} \Pi_{(0,\alpha_3)}(\mathrm{D})(f)(\mathbf{a}_1) B_{(0,\alpha_3)}(\mathbf{x}) \\ & + \sum_{0 \le \alpha_3 \le 1} \left(\Pi_{(1,\alpha_3)}(\mathrm{D})(f)(\mathbf{a}_1) B_{(1,\alpha_3)}(\mathbf{x}) + \overline{\Pi}_{(1,\alpha_3)}(\mathrm{D})(f)(\mathbf{a}_1) \overline{B}_{(1,\alpha_3)}(\mathbf{x}) \right) \\ & + \left(\Pi_{(2,0)}(\mathrm{D})(f)(\mathbf{a}_1) B_{(2,0)}(\mathbf{x}) + \overline{\Pi}_{(2,0)}(\mathrm{D})(f)(\mathbf{a}_1) \overline{B}_{(2,0)}(\mathbf{x}) \right), \end{aligned}$$

where

$$\Pi_{(\alpha_1,\alpha_3)}(\mathbf{x}) = (x_1 - 2i x_2)^{\alpha_1} x_3^{\alpha_3},$$

and

$$B_{(\alpha_1,\alpha_3)}(\mathbf{x}) = \frac{1}{2^{\alpha_1} \alpha_1! \alpha_3!} \left(x_1 + \frac{i}{2} x_2 \right)^{\alpha_1} x_3^{\alpha_3}.$$

Hence,

$$\Pi_{(0,\alpha_3)}(\mathrm{D})(f)(\mathbf{a}_1)B_{(0,\alpha_3)}(\mathbf{x}) = \frac{1}{\alpha_3!}\frac{\partial^{\alpha_3} f(\mathbf{a}_1)}{\partial x_3^{\alpha_3}} x_3^{\alpha_3}, \quad \alpha_3 = 1, 2,$$

$$\Pi_{(1,\alpha_3)}(\mathrm{D})(f)(\mathbf{a}_1)B_{(1,\alpha_3)}(\mathbf{x}) + \overline{\Pi}_{(1,\alpha_3)}(\mathrm{D})(f)(\mathbf{a}_1)\overline{B}_{(1,\alpha_3)}(\mathbf{x}) = \begin{cases} \frac{\partial f(\mathbf{a}_1)}{\partial x_1}x_1 + \frac{\partial f(\mathbf{a}_1)}{\partial x_2}x_2 & \text{if} \quad \alpha_3 = 0 \\ \frac{\partial^2 f(\mathbf{a}_1)}{\partial x_1 \partial x_3}x_1x_3 + \frac{\partial^2 f(\mathbf{a}_1)}{\partial x_2 \partial x_3}x_2x_3 & \text{if} \quad \alpha_3 = 1 \end{cases}$$

and

$$\Pi_{(2,0)}(\mathrm{D})(f)(\mathbf{a}_1)B_{(2,0)}(\mathbf{x}) + \overline{\Pi}_{(2,0)}(\mathrm{D})(f)(\mathbf{a}_1)\overline{B}_{(2,0)}(\mathbf{x}) = \frac{1}{16}\left(\frac{\partial^2 f(\mathbf{a}_1)}{\partial x_1^2} - 4\frac{\partial^2 f(\mathbf{a}_1)}{\partial x_2^2}\right)\left(4x_1^2 - x_2^2\right) + \frac{\partial^2 f(\mathbf{a}_1)}{\partial x_1 \partial x_2}x_1x_2.$$

Combining the above computations we obtain

$$P_1(\mathbf{x}) = \mathrm{T}^2_{\mathbf{a}_1,q_{\mathbf{a}_1}}(f)(\mathbf{x}) = f(\mathbf{a}_1) + \sum_{l=1}^{3}\frac{\partial f(\mathbf{a}_1)}{\partial x_l}x_l + \sum_{1\le \ell < l \le 3}\frac{\partial^2 f(\mathbf{a}_1)}{\partial x_\ell \partial x_l}x_\ell x_l + \frac{1}{2}\frac{\partial^2 f(\mathbf{a}_1)}{\partial x_3^2}x_3^2 + \frac{1}{16}\left(\frac{\partial^2 f(\mathbf{a}_1)}{\partial x_1^2} - 4\frac{\partial^2 f(\mathbf{a}_1)}{\partial x_2^2}\right)(4x_1^2 - x_2^2).$$

On the other hand,

$$\mathrm{T}^0_{\mathbf{a}_2,q_{\mathbf{a}_2}}\left(\frac{f - P_1}{q_{\mathbf{a}_1}}\right)(\mathbf{x}) = \frac{f(\mathbf{a}_2) - P_1(\mathbf{a}_2)}{q_{\mathbf{a}_1}(\mathbf{a}_2)} = f(\mathbf{a}_2) - f(\mathbf{a}_1) - \frac{\partial f(\mathbf{a}_1)}{\partial x_2} + \frac{1}{16}\left(\frac{\partial^2 f(\mathbf{a}_1)}{\partial x_1^2} - 4\frac{\partial^2 f(\mathbf{a}_1)}{\partial x_2^2}\right).$$

It follows that

$$\begin{aligned} P(\mathbf{x}) &= \mathrm{T}^2_{\mathbf{a}_1,q_{\mathbf{a}_1}}(f)(\mathbf{x}) + q_{\mathbf{a}_1}(\mathbf{x})\mathrm{T}^0_{\mathbf{a}_2,q_{\mathbf{a}_2}}\left(\frac{f - P_1}{q_{\mathbf{a}_1}}\right)(\mathbf{x}) \\ &= f(\mathbf{a}_1) + \sum_{l=1}^{3}\frac{\partial f(\mathbf{a}_1)}{\partial x_l}x_l + \sum_{1\le \ell < l \le 3}\frac{\partial^2 f(\mathbf{a}_1)}{\partial x_\ell \partial x_l}x_\ell x_l \\ &\quad + \frac{1}{2}\frac{\partial^2 f(0)}{\partial x_3^2}x_3^2 + \frac{1}{16}\left(\frac{\partial^2 f(\mathbf{a}_1)}{\partial x_1^2} - 4\frac{\partial^2 f(\mathbf{a}_1)}{\partial x_2^2}\right)\left(4x_1^2 - x_2^2\right) \\ &\quad + \left(4x_1^2 + x_2^2\right)\left(f(\mathbf{a}_2) - f(\mathbf{a}_1) - \frac{\partial f(\mathbf{a}_1)}{\partial x_2} + \frac{1}{16}\left(\frac{\partial^2 f(\mathbf{a}_1)}{\partial x_1^2} - 4\frac{\partial^2 f(\mathbf{a}_1)}{\partial x_2^2}\right)\right) \end{aligned}$$

$$= f(\mathbf{a}_1) + \sum_{l=1}^{3} \frac{\partial f(\mathbf{a}_1)}{\partial x_l} x_l + \sum_{1\le \ell < l \le 3} \frac{\partial^2 f(\mathbf{a}_1)}{\partial x_\ell \partial x_l} x_\ell x_l + \frac{1}{2}\frac{\partial^2 f(\mathbf{a}_1)}{\partial x_3^2} x_3^2$$
$$+ \left(4f(\mathbf{a}_2) - 4f(\mathbf{a}_1) - 4\frac{\partial f(\mathbf{a}_1)}{\partial x_2} + \frac{1}{2}\frac{\partial^2 f(\mathbf{a}_1)}{\partial x_1^2} - 2\frac{\partial^2 f(\mathbf{a}_1)}{\partial x_2^2}\right) x_1^2$$
$$+ \left(f(\mathbf{a}_2) - f(\mathbf{a}_1) - \frac{\partial f(\mathbf{a}_1)}{\partial x_2}\right) x_2^2.$$

Example 3 We give a formula for the Hermite interpolation polynomial of degree 3 at two points in $\mathbb{R}^3$. Let $n = 3$ and $d = 3$. We have $m = 2, s_1 = 3$ and $s_2 = 1$. Let $\mathbf{a}_1 = (0, 0, 0)$ and $\mathbf{a}_2 = (0, 1, 0)$. We choose $c^{[1]} = 2i$ and $c^{[2]} = 3i$. Take $j_1 = 1, m_1 = j_2 = 2, m_2 = 3$. Then

$$q_{\mathbf{a}_1}(\mathbf{x}) = 4x_1^2 + x_2^2, \quad q_{\mathbf{a}_2}(\mathbf{x}) = 9(x_2 - 1)^2 + x_3^2.$$

The polynomial $P = \mathbf{H}[\{(\mathbf{a}_1, q_{\mathbf{a}_1}, 2), (\mathbf{a}_2, q_{\mathbf{a}_2}, 1)\}](f)$ belonging to $\mathscr{P}_3(\mathbb{R}^3)$ satisfies the following conditions in the complex setting

$$\left(\frac{\partial}{\partial x_1} - 2i\frac{\partial}{\partial x_2}\right)^{\alpha_1} \frac{\partial^{\alpha_3}}{\partial x_3^{\alpha_3}} P(\mathbf{a}_1) = \left(\frac{\partial}{\partial x_1} - 2i\frac{\partial}{\partial x_2}\right)^{\alpha_1} \frac{\partial^{\alpha_3}}{\partial x_3^{\alpha_3}} f(\mathbf{a}_1), \quad \alpha_1 + \alpha_3 \le 3,$$

and

$$\left(\frac{\partial}{\partial x_2} - 3i\frac{\partial}{\partial x_3}\right)^{\beta_2} \frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}} P(\mathbf{a}_2) = \left(\frac{\partial}{\partial x_2} - 3i\frac{\partial}{\partial x_3}\right)^{\beta_2} \frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}} f(\mathbf{a}_2), \quad \beta_1 + \beta_2 \le 1.$$

They are equivalent to 20 conditions which consist of 10 conditions in (33)–(35) along with the following functionals

$$g \mapsto \frac{\partial g}{\partial x_l}(\mathbf{a}_2), \quad 1 \le l \le 3, \quad g \mapsto \frac{\partial^3}{\partial x_3^3} g(\mathbf{a}_1), \quad g \mapsto \frac{\partial^3}{\partial x_l \partial x_3^2} g(\mathbf{a}_1), \quad 1 \le l \le 2$$
$$g \mapsto \left(\frac{\partial^2}{\partial x_1^2} - 4\frac{\partial^2}{\partial x_2^2}\right)\frac{\partial}{\partial x_3} g(\mathbf{a}_1), \quad g \mapsto \frac{\partial^3}{\partial x_1 \partial x_2 \partial x_3} g(\mathbf{a}_1),$$
$$g \mapsto \left(\frac{\partial^3}{\partial x_1^3} - 12\frac{\partial^3}{\partial x_1 \partial x_2^2}\right) g(\mathbf{a}_1), \quad g \mapsto \left(3\frac{\partial^3}{\partial x_1^2 \partial x_2} - 4\frac{\partial^3}{\partial x_2^3}\right) g(\mathbf{a}_1).$$

Theorem 2 gives the formula for the interpolation polynomial

$$\mathbf{H}[\{(\mathbf{a}_1, q_{\mathbf{a}_1}, 3), (\mathbf{a}_2, q_{\mathbf{a}_2}, 1)\}](\mathbf{x}) = \mathrm{T}^3_{\mathbf{a}_1, q_{\mathbf{a}_1}}(f)(\mathbf{x}) + q_{\mathbf{a}_1}(\mathbf{x}) \mathrm{T}^1_{\mathbf{a}_2, q_{\mathbf{a}_2}}(g)(\mathbf{x}). \tag{36}$$

We need to compute $\mathrm{T}^3_{\mathbf{a}_1,q_{\mathbf{a}_1}}(f)$ and $\mathrm{T}^1_{\mathbf{a}_2,q_{\mathbf{a}_2}}(g)$. From (15), we can write

$$\begin{aligned}
\mathrm{T}^3_{\mathbf{a}_1,q_{\mathbf{a}_1}}(f)(\mathbf{x}) \;=\; & \mathrm{T}^2_{\mathbf{a}_1,q_{\mathbf{a}_1}}(f)(\mathbf{x}) + \Pi_{(0,3)}(\mathrm{D})(f)(\mathbf{a}_1)B_{(0,3)}(\mathbf{x}) \\
& + \big(\Pi_{(1,2)}(\mathrm{D})(f)(\mathbf{a}_1)B_{(1,2)}(\mathbf{x}) + \overline{\Pi}_{(1,2)}(\mathrm{D})(f)(\mathbf{a}_1)\overline{B}_{(1,2)}(\mathbf{x})\big) \\
& + \big(\Pi_{(2,1)}(\mathrm{D})(f)(\mathbf{a}_1)B_{(2,1)}(\mathbf{x}) + \overline{\Pi}_{(2,1)}(\mathrm{D})(f)(\mathbf{a}_1)\overline{B}_{(2,1)}(\mathbf{x})\big) \\
& + \big(\Pi_{(3,0)}(\mathrm{D})(f)(\mathbf{a}_1)B_{(3,0)}(\mathbf{x}) + \overline{\Pi}_{(3,0)}(\mathrm{D})(f)(\mathbf{a}_1)\overline{B}_{(3,0)}(\mathbf{x})\big) \\
=: \; & \mathrm{T}^2_{\mathbf{a}_1,q_{\mathbf{a}_1}}(f)(\mathbf{x}) + \Sigma_0 + \Sigma_1 + \Sigma_1 + \Sigma_3,
\end{aligned}$$

where Σ_l denotes the $(l+1)$-th term at the right hand side for $0 \le l \le 3$,

$$\Pi_{(\alpha_1,\alpha_3)}(\mathbf{x}) = (x_1 - 2ix_2)^{\alpha_1} x_3^{\alpha_3},$$

and

$$B_{\alpha_1,\alpha_3}(\mathbf{x}) = \frac{1}{2^{\alpha_1}\alpha_1!\alpha_3!}(x_1 + \frac{i}{2}x_2)^{\alpha_1} x_3^{\alpha_3}.$$

Direct computations give

$$\Sigma_0 = \frac{1}{6}\frac{\partial^3}{\partial x_3^3} f(\mathbf{a}_1)x_3^3, \quad \Sigma_1 = \frac{1}{2}\left(\frac{\partial^3}{\partial x_1 \partial x_3^2} f(\mathbf{a}_1)x_1x_3^2 + \frac{\partial^3}{\partial x_2 \partial x_3^2} f(\mathbf{a}_1)x_2x_3^2\right),$$

$$\Sigma_2 = \frac{1}{4}\left(\frac{\partial^2}{\partial x_1^2} - 4\frac{\partial^2}{\partial x_2^2}\right)\frac{\partial}{\partial x_3} f(\mathbf{a}_1)\left(x_1^2 - \frac{x_2^2}{4}\right)x_3 + \frac{\partial^3 P}{\partial x_1 \partial x_2 \partial x_3} f(\mathbf{a}_1)x_1x_2x_3,$$

$$\Sigma_3 = \frac{1}{24}\left[\left(\frac{\partial^3}{\partial x_1^3} - 12\frac{\partial^3}{\partial x_1 \partial x_2^2}\right) f(\mathbf{a}_1)\left(x_1^3 - \frac{3x_1x_2^2}{4}\right) + \left(3\frac{\partial^3}{\partial x_1^2 \partial x_2} - 4\frac{\partial^3}{\partial x_2^3}\right) f(\mathbf{a}_1)\left(3x_1^2x_2 - \frac{x_2^3}{4}\right)\right].$$

It follows that

$$\begin{aligned}
\mathrm{T}^3_{\mathbf{a}_1,q_{\mathbf{a}_1}}(f)(\mathbf{x}) \;=\; & f(\mathbf{a}_1) + \sum_{l=1}^{3}\frac{\partial f(\mathbf{a}_1)}{\partial x_l}x_l + \sum_{1\le \ell < l \le 3}\frac{\partial^2 f(\mathbf{a}_1)}{\partial x_\ell \partial x_l}x_\ell x_l + \frac{1}{2}\frac{\partial^2 f(\mathbf{a}_1)}{\partial x_3^2}x_3^2 \\
& + \frac{1}{16}\left(\frac{\partial^2 f(\mathbf{a}_1)}{\partial x_1^2} - 4\frac{\partial^2 f(\mathbf{a}_1)}{\partial x_2^2}\right)(4x_1^2 - x_2^2) + \frac{1}{6}\frac{\partial^3}{\partial x_3^3} f(\mathbf{a}_1)x_3^3 \\
& + \frac{1}{2}\frac{\partial^3}{\partial x_1 \partial x_3^2} f(\mathbf{a}_1)x_1x_3^2 + \frac{1}{2}\frac{\partial^3}{\partial x_2 \partial x_3^2} f(\mathbf{a}_1)x_2x_3^2 \\
& + \frac{1}{4}\left(\frac{\partial^2}{\partial x_1^2} - 4\frac{\partial^2}{\partial x_2^2}\right)\frac{\partial}{\partial x_3} f(\mathbf{a}_1)\left(x_1^2 - \frac{x_2^2}{4}\right)x_3 + \frac{\partial^3}{\partial x_1 \partial x_2 \partial x_3} f(\mathbf{a}_1)x_1x_2x_3 \\
& + \frac{1}{24}\left(\frac{\partial^3}{\partial x_1^3} - 12\frac{\partial^3}{\partial x_1 \partial x_2^2}\right) f(\mathbf{a}_1)\left(x_1^3 - \frac{3x_1x_2^2}{4}\right) \\
& + \frac{1}{24}\left(3\frac{\partial^3}{\partial x_1^2 \partial x_2} - 4\frac{\partial^3}{\partial x_2^3}\right) f(\mathbf{a}_1)\left(3x_1^2x_2 - \frac{x_2^3}{4}\right).
\end{aligned}$$

Next, we calculate

$$\mathrm{T}^1_{\mathbf{a}_2,q_{\mathbf{a}_2}}(g)(\mathbf{x}) = g(\mathbf{a}_2) + \sum_{l=1}^{3}\frac{\partial}{\partial x_l}g(\mathbf{a}_2)x_l, \quad g = \frac{f - \mathrm{T}^3_{\mathbf{a}_1,q_{\mathbf{a}_1}}(f)}{q_{\mathbf{a}_1}}.$$

From the formula for $\mathrm{T}^3_{\mathbf{a}_1,q_{\mathbf{a}_1}}(f)$, we see that

$$\mathrm{T}^3_{\mathbf{a}_1,q_{\mathbf{a}_1}}(f)(\mathbf{a}_2)=f(\mathbf{a}_1)+\frac{\partial f(\mathbf{a}_1)}{\partial x_2}-\frac{1}{16}\left(\frac{\partial^2 f(\mathbf{a}_1)}{\partial x_1^2}-4\frac{\partial^2 f(\mathbf{a}_1)}{\partial x_2^2}\right)-\frac{1}{96}\left(3\frac{\partial^3}{\partial x_1^2\partial x_2}-4\frac{\partial^3}{\partial x_2^3}\right)f(\mathbf{a}_1),$$

$$\frac{\partial}{\partial x_1}\mathrm{T}^3_{\mathbf{a}_1,q_{\mathbf{a}_1}}(f)(\mathbf{a}_2)=\frac{\partial f(\mathbf{a}_1)}{\partial x_1}+\frac{\partial^2 f(\mathbf{a}_1)}{\partial x_1\partial x_2}-\frac{1}{32}\left(\frac{\partial^3}{\partial x_1^3}-12\frac{\partial^3}{\partial x_1\partial x_2^2}\right)f(\mathbf{a}_1),$$

$$\frac{\partial}{\partial x_2}\mathrm{T}^3_{\mathbf{a}_1,q_{\mathbf{a}_1}}(f)(\mathbf{a}_2)=\frac{\partial f(\mathbf{a}_1)}{\partial x_2}-\frac{1}{8}\left(\frac{\partial^2 f(\mathbf{a}_1)}{\partial x_1^2}-4\frac{\partial^2 f(\mathbf{a}_1)}{\partial x_2^2}\right)-\frac{1}{32}\left(3\frac{\partial^3}{\partial x_1^2\partial x_2}-4\frac{\partial^3}{\partial x_2^3}\right)f(\mathbf{a}_1),$$

$$\frac{\partial}{\partial x_3}\mathrm{T}^3_{\mathbf{a}_1,q_{\mathbf{a}_1}}(f)(\mathbf{a}_2)=\frac{\partial f(\mathbf{a}_1)}{\partial x_3}+\frac{\partial^2 f(\mathbf{a}_1)}{\partial x_2\partial x_3}-\frac{1}{16}\left(\frac{\partial^2}{\partial x_1^2}-4\frac{\partial^2}{\partial x_2^2}\right)\frac{\partial}{\partial x_3}f(\mathbf{a}_1).$$

Direct computations give

$$q_{\mathbf{a}_1}(\mathbf{a}_2)=1,\quad \frac{\partial}{\partial x_2}q_{\mathbf{a}_1}(\mathbf{a}_2)=2,\quad \frac{\partial}{\partial x_l}q_{\mathbf{a}_1}(\mathbf{a}_2)=0,\quad l=1,3.$$

Hence,

$$\begin{aligned}\mathrm{T}^1_{\mathbf{a}_2,q_{\mathbf{a}_2}}(g)(\mathbf{x}) \;=\;& f(\mathbf{a}_2)-f(\mathbf{a}_1)-\frac{\partial f(\mathbf{a}_1)}{\partial x_2}+\frac{1}{16}\left(\frac{\partial^2 f(\mathbf{a}_1)}{\partial x_1^2}-4\frac{\partial^2 f(\mathbf{a}_1)}{\partial x_2^2}\right)\\ &+\frac{1}{96}\left(3\frac{\partial^3}{\partial x_1^2\partial x_2}-4\frac{\partial^3}{\partial x_2^3}\right)f(\mathbf{a}_1)\\ &+\left(\frac{\partial f(\mathbf{a}_2)}{\partial x_1}-\frac{\partial f(\mathbf{a}_1)}{\partial x_1}-\frac{\partial^2 f(\mathbf{a}_1)}{\partial x_1\partial x_2}+\frac{1}{32}\left(\frac{\partial^3}{\partial x_1^3}-12\frac{\partial^3}{\partial x_1\partial x_2^2}\right)f(\mathbf{a}_1)\right)x_1\\ &+\left(2f(\mathbf{a}_1)-2f(\mathbf{a}_2)+\frac{\partial f(\mathbf{a}_2)}{\partial x_2}+\frac{\partial f(\mathbf{a}_1)}{\partial x_2}+\frac{1}{32}\left(3\frac{\partial^3}{\partial x_1^2\partial x_2}-4\frac{\partial^3}{\partial x_2^3}\right)f(\mathbf{a}_1)\right.\\ &\left.-\frac{1}{48}\left(3\frac{\partial^3}{\partial x_1^2\partial x_2}-4\frac{\partial^3}{\partial x_2^3}\right)f(\mathbf{a}_1)\right)x_2\\ &+\left(\frac{\partial f(\mathbf{a}_2)}{\partial x_3}-\frac{\partial f(\mathbf{a}_1)}{\partial x_3}-\frac{\partial^2 f(\mathbf{a}_1)}{\partial x_2\partial x_3}+\frac{1}{16}\left(\frac{\partial^2}{\partial x_1^2}-4\frac{\partial^2}{\partial x_2^2}\right)\frac{\partial}{\partial x_3}f(\mathbf{a}_1)\right)x_3.\end{aligned}$$

The above computations along with (36) lead to a formula for $\mathbf{H}[\{(\mathbf{a}_1,q_{\mathbf{a}_1},3),(\mathbf{a}_2,q_{\mathbf{a}_2},1)\}]$. The precise formula is left to the readers.

Remark 4 The interpolation conditions in the previous two examples and the corresponding Hermite interpolation polynomials do not depend on the choices of j_2 and m_2.

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