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Filtered interpolation for solving Prandtl's integro-differential equations

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Abstract

In order to solve Prandtl-type equations we propose a collocation-quadrature method based on de la Vallée Poussin (briefly VP) filtered interpolation at Chebyshev nodes. Uniform convergence and stability are proved in a couple of Hölder-Zygmund spaces of locally continuous functions. With respect to classical methods based on Lagrange interpolation at the same collocation nodes, we succeed in reproducing the optimal convergence rates of the L^2 case and cut off the typical log factor which seemed inevitable dealing with uniform norms. Such an improvement does not require a greater computational effort. In particular, we propose a fast algorithm based on the solution of a simple 2-bandwidth linear system and prove that, as its dimension tends to infinity, the sequence of the condition numbers (in any natural matrix norm) tends to a finite limit.

Keywords Prandtl equation · Hypersingular integral equations · Polynomial interpolation · Filtered approximation · De la Vallée Poussin mean · Hölder-Zygmund spaces · Chebyshev nodes

Mathematics Subject Classification (2010) 41A10 · 65D05 · 33C45

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1 Introduction

In this paper we propose a numerical method for the following Prandtl-type equation

$$\sigma f(y) - \frac{1}{\pi}\int_{-1}^{1} \frac{f(x)}{(x-y)^2}\varphi(x)dx - \frac{1}{\pi}\int_{-1}^{1} \log|x-y| f(x)\varphi(x)dx + \frac{1}{\pi}\int_{-1}^{1} h(x,y) f(x)\varphi(x)dx = g(y), \quad y \in (-1,1), \tag{1}$$

where $\varphi(x) := \sqrt{1-x^2}$, σ is a real constant, h is a smooth kernel function and the first integral has to be understood as the following derivative

$$\int_{-1}^{1} \frac{f(x)}{(x-y)^2}\varphi(x)dx = \frac{d}{dy}\int_{-1}^{1} \frac{f(x)}{x-y}\varphi(x)dx, \tag{2}$$

the integral at the right-hand side being in the Cauchy principal value sense.

Integro-differential equations of the type (1) are models for many physics and engineering problems (see [17] and the references therein). Indeed, besides the well known Prandtl's Equation which governs the circulation air flow along the contour of a plane wing profile (see, e.g., [11, 12]), Prandtl-type equations (1) model, for instance, the load transfer problem from thin-walled elements to massive bodies (see, e.g., [16, 22]), water scattering problems of vertical or inclined plates merged in infinitely deep water (see, e.g., [27]), crack problems in composite material or in inhomogeneous bodies [34], etc.

We focus on the case that in (1) the right-hand side known-term g can be well approximated by polynomials. More precisely, for a given Jacobi weight u, we consider the error of best, Jacobi-weighted, uniform approximation of g by means of polynomials of degree at most n, namely

$$E_n(g)_u := \inf_{\deg(P)\le n} \|(g-P)u\|, \qquad \|gu\| := \sup_{|x|\le 1} |g(x)u(x)|,$$

and we suppose that

$$E_n(g)_u = O\left(\frac{1}{n^s}\right), \qquad \text{for a certain } s > 0. \tag{3}$$

The space of all functions satisfying (3) is here denoted by $Z_s(u)$ and it is a Banach space equipped with the norm

$$\|g\|_{Z_s(u)} := \|gu\| + \sup_{n>0}(1+n)^s E_n(g)_u, \qquad s > 0.$$

Such kind of B-spaces have been introduced in [10, 14]. They are equivalent to some Hölder-Zygmund-type spaces defined by means of Ditzian-Totik weighted φ-moduli of smoothness [9, 10] and several singular integral equations have been already studied in a couple of these, also known as Besov-type spaces (see, e.g., [4, 6–8, 14, 18–20]).

We prove that for any $g \in Z_s(\varphi)$, the equation (1) has a unique solution $f \in Z_{s+1}(\varphi)$ (cf. Theorem 2.4). Such a result can be proved by standard arguments, combining the Fredholm's alternative theorem with the analysis of the mapping properties of the operators involved in the equation [4, 8].

The main result of this paper is the construction of a numerical method in order to approximate the solution f by a polynomial f_n such that

$$\|(f - f_n)\varphi\| \leq \frac{\mathcal{C}}{n^s}\|g\|_{Z_s(\varphi)}$$

and

$$\|f - f_n\|_{Z_r(\varphi)} \leq \frac{\mathcal{C}}{n^{s-r}}\|g\|_{Z_s(\varphi)}, \qquad 0 < r \leq s,$$

where, in both the estimates, $\mathcal{C}$ denotes a positive constant independent of n and f.

The proposed numerical method falls into the class of polynomial projection methods (see, e.g., [1, 28]), but it is based on non-standard projections obtained by applying de la Vallée Poussin (VP)-type filters. They project the function into non-standard polynomial spaces, spanned by the so-called fundamental VP polynomials that share the interpolation property of fundamental Lagrange polynomials at Chebyshev nodes of second type [5, 29–31, 33]. The resulting filtered VP interpolation has been already applied to the field of Cauchy singular integral equation in order to solve the airfoil equation [20] and in other situations [7, 18, 32]. It results to be an useful device also in our case. The related approximation error has been recently characterized in [25] and error estimates in Hölder-Zygmund spaces can be found in [26].

Here, we are going to use the already known results on the near-best uniform approximation provided by filtered VP interpolation, in order to construct a convergent, stable and efficient method for the numerical solution of (1). In the particular case $h \equiv 0$, the proposed algorithm fast computes the numerical solution f_n by solving a linear system with a band matrix whose not null (i, j)-entries are those such that $|i - j| \in \{0, 2\}$ (cf. (79)). Moreover, we prove that the sequence of the condition numbers of such systems converges to a finite limit, whatever natural matrix norm we consider (cf. Theorem 5.4).

Several authors studied Prandtl's type equations by proposing different numerical approaches. Among them we recall [2–4, 8]. In [2] a Nyström method based on the local approximation by quasi-interpolatory splines is proposed for the particular case $h \equiv 0$. Nevertheless, the efficiency of this method seems to suffer from the low degree of approximation provided by splines functions. A global polynomial approximation based on the Lagrange interpolation at Jacobi zeros has been considered in [3, 4, 8]. In [3] a collocation and a collocation-quadrature method is studied in Sobolev subspaces of weighted-L^2 spaces, getting optimal convergence rates. For the same methods, convergence results in uniform norm have been obtained in [4] by investigating a regularized version of the equation in certain weighted Besov spaces of locally continuous functions. A direct collocation-quadrature method has been proposed in [8], considering the equation in the Hölder-Zygmund spaces $Z_s(u)$. In such spaces the authors prove the convergence and stability of their method and study also the conditioning of the final linear system. Nevertheless, dealing with the uniform norm, the previous methods based on Lagrange interpolation suffer from the unboundedness of the Lebesgue constants, which entails the presence of a log factor in the corresponding error estimates. This extra-factor disappears in the method here proposed, thanks to the uniform boundedness of the VP interpolating operators w.r.t. weighted uniform norms.

The paper is organized as follows. Section 2 is a survey of the theoretical results on the Prandtl-type equation (1): we analyze the spaces where the equation is studied and the related mapping properties of the continuous operators that compose the equation, arriving to state existence and uniqueness theorems for the solution of (1). In Section 3 we focus on the VP interpolating polynomials we use for the numerical method, recalling the related properties of interest. The method is described in the next two sections. Section 4 is devoted to the approximate equation and its unique solvability, providing convergence and stability theorems. Section 5 deals with the computation of the numerical solution by analyzing the computational aspects of the linear system obtained by our method. Finally, Section 6 provides the numerical experiments, by comparing the results of our method with those achieved by the methods in [2–4, 8]. For a better legibility, the proofs of the main results are given in the Appendix.

2 The equation in Hölder-Zygmund spaces

Defining the operators

$$D : f \to Df, \quad Df(y) := -\frac{1}{\pi}\int_{-1}^{1} \frac{f(x)}{(x-y)^2}\varphi(x)dx, \tag{4}$$

$$K : f \to Kf, \quad Kf(y) := -\frac{1}{\pi}\int_{-1}^{1} \log|x-y| f(x)\varphi(x)dx, \tag{5}$$

$$H : f \to Hf, \quad Hf(y) := \frac{1}{\pi}\int_{-1}^{1} h(x,y) f(x)\varphi(x)dx, \tag{6}$$

and setting $If = f$, the equation (1) can be shortly written as follows

$$(\sigma I + D + K + H)f = g. \tag{7}$$

Here we are going to study this equation and the related operators in some subspaces of the space C^0_φ of all locally continuous functions f (i.e., continuous in any $[a,b] \subset]-1,1[$) such that

$$\lim_{x\to\pm 1} f(x)\varphi(x) = 0.$$

It is well-known that C^0_φ is a Banach space equipped with the norm $\|f\|_{C^0_\varphi} := \|f\varphi\|$. Moreover, Weierstrass approximation theorem holds in C^0_φ, i.e., any $f \in C^0_\varphi$ can be approximated by polynomials with the desired accuracy and we have

$$f \in C^0_\varphi \Longleftrightarrow \lim_{n\to\infty} E_n(f)_\varphi = 0.$$

The rate of convergence to zero of $E_n(f)_\varphi$ can be characterized by means of certain weighted moduli of smoothness that Ditzian and Totik introduced in [9] in order to get Jackson and Stechkin-type inequalities.

In particular, for any $r \in \mathbb{N}$, if we consider the following Sobolev-type spaces

$$W_r(\varphi) := \left\{ f \in C^0_\varphi : f^{(r-1)} \text{ is locally absolutely continuous and } \|f^{(r)}\varphi^{r+1}\| < \infty \right\},$$

equipped with the norm

$$\|f\|_{W_r(\varphi)} := \|f\varphi\| + \|f^{(r)}\varphi^{r+1}\|,$$

then, taking into account the equivalence of Ditzian-Totik moduli with some K-functional [9, Th. 2.1.1], we get

$$E_n(f)_\varphi \le \frac{\mathcal{C}}{n^r}\|f\|_{W_r(\varphi)}, \quad \forall f \in W_r(\varphi), \quad r \in \mathbb{N}, \qquad \mathcal{C} \neq \mathcal{C}(n, f), \qquad (8)$$

where here and throughout the paper $\mathcal{C}$ denotes a positive constant having different meaning in different formulas and we write $\mathcal{C} \neq \mathcal{C}(n, f, \ldots)$ to say that $\mathcal{C}$ is independent of $n, f, \ldots$.

In order to generalize (8) to the case of any, not necessarily integer, exponent $r > 0$, we introduce the following spaces

$$Z_r(\varphi) := \{f \in C^0_\varphi \ : \ \sup_{n>0}(n+1)^r E_n(f)_\varphi < \infty\}, \qquad r > 0,$$

equipped with the following norm

$$\|f\|_{Z_r(\varphi)} := \|f\varphi\| + \sup_{n>0}(n+1)^r E_n(f)_\varphi, \qquad r > 0.$$

Obviously, by this definition, we have

$$E_n(f)_\varphi \le \frac{\mathcal{C}}{n^r}\|f\|_{Z_r(\varphi)}, \qquad \forall f \in Z_r(\varphi), \quad r > 0, \qquad \mathcal{C} \neq \mathcal{C}(n, f). \qquad (9)$$

Regarding the definition of the spaces $Z_r(\varphi)$, we recall they are also known as Hölder-Zygmund spaces and constitute a particular case of the Besov-type spaces studied in [10], where they have been equivalently defined in terms of Ditzian-Totik moduli of smoothness [10, Th. 2.1].

Moreover, we recall that the spaces $Z_r(\varphi)$ belong to the larger class of spaces $C^{\mathcal{B}}_\varphi$ introduced in [14, p. 204], where a more general behavior of the error $E_n(f)_\varphi$ is allowed by an arbitrary sequence $\mathcal{B} := \{b_n\}_n \subset \mathbb{R}^+$ such that

$$\lim_{n\to\infty} b_n = 0 \qquad \text{and} \qquad \sup_{n>0} \frac{E_n(f)_\varphi}{b_n} < \infty.$$

We will also consider the non-weighted space $C^0 = C[-1, 1]$ equipped with the uniform norm, and its Zygmund-type subspaces Z_r, $r > 0$, which are defined as $Z_r(\varphi)$ but replacing the function φ with the constant 1.

In [14] the following embedding properties have been proved

(i) $\forall r > 0$, $Z_r(\varphi)$ is compactly embedded in C^0_φ (cf. [14, Lemma 3.2]);
(ii) $\forall s > r > 0$, $Z_s(\varphi)$ is compactly embedded in $Z_r(\varphi)$ (cf. [14, Lemma 3.3]);
(iii) $\forall s > 0$, Z_s is continuously embedded in $Z_s(\varphi)$ (cf. [14, Remark 3.5]).

Moreover, recalling that the non-weighted error $E_n(f) := \inf_{\deg(P)\le n}\|f - P\|$ satisfies the Favard inequality (see, e.g., [21])

$$E_n(f) \le \frac{\mathcal{C}}{n}E_{n-1}(f')_\varphi, \qquad \mathcal{C} \neq \mathcal{C}(n, f),$$

we easily deduce that

$$\sup_{n>0}(n+1)^{r+1}E_n(f) \le \mathcal{C}\|f'\|_{Z_r(\varphi)} \le \mathcal{C}\|f'\|_{Z_r}, \qquad r>0, \qquad \mathcal{C} \ne \mathcal{C}(f). \quad (10)$$

In the following subsections we are going to recall the main properties of the operarors D, K and H in the previous spaces in order to get, finally, the unique solvability of the Prandtl equation (7) in Hölder-Zygmund spaces.

2.1 On the operator *D*

Recalling (2), the hypersingular integral operator D is strictly related to the Cauchy singular integral operator

$$Af(y) := -\frac{1}{\pi}\int_{-1}^{1}\frac{f(x)}{x-y}\,\varphi(x)dx, \quad (11)$$

being

$$Df(y) = \frac{d}{dy}Af(y), \qquad -1<y<1. \quad (12)$$

Consequently, some nice properties of the operator A (see, e.g., [28]) have been transferred on D.

In particular, let us consider the Chebyshev polynomials of the second kind of degree $n \in \mathbb{N}$

$$p_n(x) = \sqrt{\frac{2}{\pi}}\,\frac{\sin[(n+1)t]}{\sin t}, \qquad t = \arccos x, \quad |x| \le 1, \quad (13)$$

being understood that $p_n(\pm 1) = \sqrt{\frac{2}{\pi}}(n+1)$. In [3] the following mapping property has been proved

$$Dp_n(y) = (n+1)p_n(y), \qquad \forall n \in \mathbb{N}. \quad (14)$$

Moreover, in the recent paper [8], it has been proved the following

Theorem 2.1 *For any $r>0$, $D : Z_{r+1}(\varphi) \to Z_r(\varphi)$ is a bounded map having bounded inverse.*

Finally, the explicit form of $D^{-1} : Z_r(\varphi) \to Z_{r+1}(\varphi)$ has been found in [4, Prop. 2.3]

$$D^{-1}f(y) = -\hat{A}WAf(y), \qquad \forall f \in Z_r(\varphi), \quad (15)$$

where A is given by (11) and

$$\hat{A}f(y) := \frac{1}{\pi}\int_{-1}^{1}\frac{f(x)}{x-y}\frac{dx}{\varphi(x)}, \quad (16)$$

$$Wf(y) := \frac{1}{\pi}\int_{-1}^{1}\log|x-y|f(x)\frac{dx}{\varphi(x)}. \quad (17)$$

2.2 On the operator *K*

It is well known (see, e.g., [1, p. 8]) that

$$\sup_{|x|\le 1}\int_{-1}^{1}\log|x-y|dx<\infty.$$

This yields

$$\|Kf\|\le\mathcal{C}\|f\varphi\|,\quad \forall f\in C^0_\varphi,\qquad \mathcal{C}\ne\mathcal{C}(f),\tag{18}$$

namely, $K: C^0_\varphi\to C^0$ is a bounded map.

On the other hand, K is related to the Cauchy operator A given in (11) as follows

$$Af(y)=\frac{d}{dy}Kf(y),\qquad -1<y<1.\tag{19}$$

In [7, 18] the mapping properties of Cauchy singular integral operators have been studied. In particular, we recall that [7, Th. 3.4] for all $r>0$ the map $A: Z_r(\varphi)\to Z_r$ is bounded, i.e.,

$$\|Af\|_{Z_r}\le\mathcal{C}\|f\|_{Z_r(\varphi)},\qquad \forall f\in Z_r(\varphi),\quad r>0,\qquad \mathcal{C}\ne\mathcal{C}(f).\tag{20}$$

By means of (19) and (20), taking into account (18) and applying (10) with Kf instead of f, we easily get

$$\|Kf\|_{Z_{r+1}}\le\mathcal{C}\|f\|_{Z_r(\varphi)},\qquad \forall f\in Z_r(\varphi),\quad r>0,\qquad \mathcal{C}\ne\mathcal{C}(f),\tag{21}$$

i.e., the map $K: Z_r(\varphi)\to Z_{r+1}$ is bounded for any $r>0$.

Combining this result with the continuous embedding $Z_{r+1}\subset Z_{r+1}(\varphi)$ and the compact embedding $Z_{r+1}(\varphi)\subset Z_r(\varphi)$ (cf. (iii) and (ii) respectively), we obtain the following

Theorem 2.2 *For any $r>0$, the map $K: Z_r(\varphi)\to Z_{r+1}(\varphi)$ is bounded while the map $K: Z_s(\varphi)\to Z_r(\varphi)$ is a compact operator for all $s\ge r$.*

Finally, we recall that K maps the orthonormal Chebyshev-2th kind polynomials $\{p_n\}_n$ (cf. (13)) according with the following recurrence relation proved in [3, Corollary 4.3]

$$Kp_0(y)=\frac{1}{4}\left[\left(2\log 2+\frac{1}{2}\right)p_0-\frac{1}{2}p_2(y)\right],\tag{22}$$

$$Kp_\ell(y)=\frac{1}{4}\left[-\frac{1}{\ell}p_{\ell-2}(y)+\left(\frac{1}{\ell}+\frac{1}{\ell+2}\right)p_\ell(y)-\frac{1}{\ell+2}p_{\ell+2}(y)\right],\quad \ell>0.\tag{23}$$

2.3 On the operator *H*

The mapping properties of the operator H depend on the smoothness of its bivariate kernel $h(x,y)$. For instance, if h is continuous w.r.t. both the variables, i.e., if $h\in C([-1,1]^2)$ then, similarly to (18), we get

$$\|Hf\|\le\mathcal{C}\|f\varphi\|,\quad \forall f\in C^0_\varphi,\qquad \mathcal{C}\ne\mathcal{C}(f).\tag{24}$$

In view of Th. 2.1 and Th. 2.2, we aim to have that, for some $s > 0$, the map $H : Z_{s+1}(\varphi) \to Z_s(\varphi)$ is compact or, at least, such that

$$\|Hf\|_{Z_{s+1}(\varphi)} \le \mathcal{C}\|f\|_{Z_s(\varphi)}, \qquad \forall f \in Z_s(\varphi), \qquad \mathcal{C} \ne \mathcal{C}(f). \tag{25}$$

Using the notation $\overline{h}_x$ (resp. $\overline{\overline{h}}_y$) to look at the bivariate function $h(x, y)$ as an univariate function of the second (resp. first) variable, i.e., setting

$$\overline{h}_x(t) := h(x,t) \qquad (\text{resp. } \overline{\overline{h}}_y(t) := h(t,y)), \qquad -1 \le t \le 1, \tag{26}$$

a sufficient condition to get (25) is given by the following proposition.

Proposition 2.3 [14, Prop. 4.12] *Let $s > 0$ and $v(x) = (1-x)^\nu(1+x)^\zeta$ with $\nu, \zeta \in [0, 1[$. If the kernel h is s.t. $h(x, y)v(x)\varphi(y) \in C^0([-1, 1]^2)$ and $\overline{h}_x v(x) \in Z_s(\varphi)$ uniformly w.r.t. $x \in [-1, 1]$, then the map $H : C^0_\varphi \to Z_s(\varphi)$ is bounded, i.e.,*

$$\|Hf\|_{Z_s(\varphi)} \le \mathcal{C}\|f\varphi\|, \quad \forall f \in C^0_\varphi, \qquad \mathcal{C} \ne \mathcal{C}(f).$$

Combining this result with the embedding property (ii), we get, under the hypotheses of Proposition 2.3, that

$$H : C^0_\varphi \to Z_r(\varphi) \text{ is a compact operator for all } r \in]0, s[. \tag{27}$$

Similarly, recalling (i), under the assumptions of Proposition 2.3, we get

$$H : Z_r(\varphi) \to Z_s(\varphi) \text{ is a compact operator for all } r > 0. \tag{28}$$

2.4 On the unique solvability of the Prandtl equation

Let us write the Prandtl-type equation (7) in the shorter form

$$(D + U)f = g, \qquad U := \sigma I + K + H, \tag{29}$$

and suppose that, for a given $s > 0$ we have $g \in Z_s(\varphi)$.

In order to prove the existence and uniqueness of the solution f of (29), we use the result in [15, Cor. 3.8]) that follows from the classical Fredholm's alternative theorem.

More precisely, by taking into account the mapping properties of the operators in (29), we get

Theorem 2.4 *Let us assume that for some $s > 0$, we have that $g \in Z_s(\varphi)$ and that the map $H : Z_{s+1}(\varphi) \to Z_s(\varphi)$ is a compact operator. If the map $D + U : Z_{s+1}(\varphi) \to Z_s(\varphi)$ is injective (i.e., if $Ker(D + U) = \{0\}$ in $Z_{s+1}(\varphi)$) then equation $(D + U)f = g$ admits a unique, stable solution $f^* \in Z_{s+1}(\varphi)$.*

Recalling Proposition 2.3, we also have

Theorem 2.5 *Let us assume that $h(x, y)\varphi(y) \in C^0([-1, 1]^2)$ and that $\overline{h}_x \in Z_s(\varphi)$ uniformly w.r.t. $x \in [-1, 1]$, for some $s > 0$. Moreover, let $0 < r \le s$ and suppose that $D + U : Z_{r+1}(\varphi) \to Z_r(\varphi)$ is an injective map. Then for all $g \in Z_r(\varphi)$, the equation $(D + U)f = g$ admits a unique, stable solution $f^* \in Z_{r+1}(\varphi)$.*

We remark that the previous theorems ensure, besides the unisolvence, also the stability of problem (29), that means the continuity of $(D+U)^{-1}: Z_r(\varphi) \to Z_{r+1}(\varphi)$, $r \le s$.

3 Filtered VP interpolation

For any $n \in \mathbb{N}$, let us consider the zeros of the nth Chebyshev-2th kind polynomial p_n (cf. (13)) and the related Christoffel numbers, respectively given by the formulas

$$x_k{:=}x_{n,k}{=}\cos\left(\frac{k\pi}{n+1}\right) \quad \text{and} \quad \lambda_k{:=}\lambda_{n,k}{=}\frac{\pi}{n+1}\sin^2\left(\frac{k\pi}{n+1}\right), \qquad k{=}1,\ldots,n.$$

In alternative to the classical Lagrange interpolation at the nodes $\{x_k\}_k$, we consider the following VP interpolation polynomials [30]

$$V_n^m f(x) := \sum_{k=1}^{n} f(x_k)\Phi_{n,k}^m(x), \qquad |x| \le 1, \qquad m < n, \tag{30}$$

defined by means of the so-called fundamental VP polynomials

$$\Phi_{n,k}^m(x) = \lambda_k \sum_{j=0}^{n+m-1} \mu_{n,j}^m p_j(x_k) p_j(x), \qquad k = 1, \ldots, n, \tag{31}$$

where $\mu_{n,j}^m$ are the following VP filtering coefficients

$$\mu_{n,j}^m := \begin{cases} 1 & \text{if} \quad j = 0, \ldots, n-m, \\ \dfrac{n+m-j}{2m} & \text{if} \quad n-m < j < n+m. \end{cases} \tag{32}$$

Equivalently, we can define the fundamental VP polynomials as the following delayed arithmetic means of the Darboux kernels $K_r(x,y) := \sum_{j=0}^{r} p_j(x)p_j(y)$

$$\Phi_{n,k}^m(x) = \frac{\lambda_k}{2m} \sum_{r=n-m}^{n+m-1} K_r(x_k, x), \qquad k = 1, \ldots, n. \tag{33}$$

Moreover, for $k = 1, \ldots, n$ the following trigonometric form can be found in [25]

$$\Phi_{n,k}^m(\cos t) = \frac{(-1)^k \sin t_k}{4m(n+1)} \frac{\sin[(n+1)t]}{\sin t} \left[\frac{\sin[m(t-t_k)]}{\sin^2[(t-t_k)/2]} - \frac{\sin[m(t+t_k)]}{\sin^2[(t+t_k)/2]}\right], \tag{34}$$

where $t_k := \frac{k\pi}{n+1}$, $k = 1, \ldots, n$.

Similarly to Lagrange polynomials, the interpolation property (see [30, Section 4])

$$\Phi_{n,k}^m(x_h) = \delta_{h,k} = \begin{cases} 1 & h = k, \\ 0 & h \ne k, \end{cases} \qquad k = 1, \ldots, n, \tag{35}$$

holds for any choice of the positive integer $m < n$. Hence, like the Lagrange interpolation polynomial, we have

$$V_n^m f(x_k) = f(x_k), \qquad k = 1, \ldots, n, \qquad \forall m < n. \tag{36}$$

Nevertheless, denoted by $\mathbb{P}_n$ the space of all algebraic polynomials of degree at most n, the map $V_n^m : f \to V_n^m f \in \mathbb{P}_{n+m-1}$ is not a polynomial projection in the classical sense. It is usually defined as a *polynomial quasi-projector*, since its codomain is $\mathbb{P}_{n+m-1}$ but it preserves all the polynomials of (lower) degree at most $n-m$. Indeed, it has been proved in [30, Th.4.2] that V_n^m is a projection onto a non-canonical polynomial space, the so-called VP space:

$$\mathrm{S}_n^m := \mathrm{span}\left\{\Phi_{n,k}^m :\ k = 1, \dots, n\right\},$$

which is nested between the classical polynomial spaces as follows

$$\mathbb{P}_{n-m} \subset S_n^m \subset \mathbb{P}_{n+m-1}.$$

We remark that $\dim \mathrm{S}_n^m = \dim \mathbb{P}_{n-1} = n$. Moreover, we have [30, Th.4.2]

$$f \in \mathrm{S}_n^m \iff f = V_n^m f.$$

We recall that, w.r.t. the scalar product

$$< f, g >:= \int_{-1}^{1} f(x)g(x)\varphi(x)dx,$$

an orthogonal basis of S_n^m is given by [30, Th.4.3]

$$q_{n,j}^m(x) := \begin{cases} p_j(x) & \text{if} \quad j = 0, \dots, n-m, \\ \dfrac{m+n-j}{2m} p_j(x) - \dfrac{m-n+j}{2m} p_{2n-j}(x) & \text{if} \quad n-m < j < n, \end{cases} \tag{37}$$

i.e., we have

$$\mathrm{S}_n^m = \mathrm{span}\left\{q_{n,j}^m, \quad j = 0, \dots, n-1\right\}, \tag{38}$$

and

$$< q_{n,i}^m, q_{n,j}^m >= \delta_{i,j} \cdot \begin{cases} 1 & 0 \le j \le n-m, \\ \frac{m^2+(n-j)^2}{2m^2} & n-m < j < n. \end{cases} \tag{39}$$

In [30, Th. 4.3] the following change of basis formula was proved

$$\Phi_{n,k}^m(x) = \lambda_k \sum_{j=0}^{n-1} p_j(x_k) q_{n,j}^m(x). \tag{40}$$

This formula yields the following orthogonal expansion

$$V_n^m f(x) = \sum_{j=0}^{n-1} c_{n,j}(f) q_{n,j}^m(x), \qquad 0 < m < n, \tag{41}$$

where

$$c_{n,j}(f) = \sum_{k=1}^{n} \lambda_k p_j(x_k) f(x_k). \tag{42}$$

We point out that the coefficients $c_{n,j}(f)$ do not depend on the choice of m. Indeed, they coincide with the discretization of the Chebyshev-Fourier coefficients

$$c_j(f) = \int_{-1}^{1} p_j(y) f(y)\varphi(y)dy, \qquad j = 0, \dots, n-1,$$

by means of the Gauss-Chebyshev quadrature rule based on the zeros of p_n.

Taking into account (14), in the sequel we will also consider the following modified polynomials

$$\tilde{q}^m_{n,j}(x) := \begin{cases} \frac{p_j(x)}{j+1} & \text{if} \quad j = 0, \dots, n-m, \\ \dfrac{m+n-j}{2m}\,\dfrac{p_j(x)}{j+1} - \dfrac{m-n+j}{2m}\,\dfrac{p_{2n-j}(x)}{2n-j+1} & \text{if} \quad n-m < j < n, \end{cases} \tag{43}$$

which yield an orthogonal basis for the following modified VP space

$$\tilde{S^m_n} := \text{span}\{\tilde{q}^m_{n,j}(x) : \; j = 0, \dots, n-1\}.$$

Moreover, we note that

$$< q^m_{n,j}, \tilde{q}^m_{n,i} >= 0, \qquad \forall i \neq j, \tag{44}$$

and, for any $j = 1, \dots, n$, we have

$$< q^m_{n,j}, \tilde{q}^m_{n,j} >= \begin{cases} \frac{1}{j+1} & \text{if} \quad j = 0, \dots, n-m, \\ \dfrac{(m+n-j)^2}{4m^2(j+1)} + \dfrac{(m-n+j)^2}{4m^2(2n-j+1)} & \text{if} \quad n-m < j < n. \end{cases} \tag{45}$$

The motivation of considering such modified basis $\tilde{q}^m_{n,j}$ and space $\tilde{S^m_n}$ comes from (14) which yields

$$D\tilde{q}^m_{n,j} = q^m_{n,j}, \qquad j = 0, \dots, n-1, \tag{46}$$

and this identity implies the following

Proposition 3.1 *The map $D : f \in \tilde{S^m_n} \to Df \in S^m_n$ is a bijective map and we have*

$$V^m_n Df = Df, \qquad \forall f \in \tilde{S^m_n}.$$

About the approximation provided by VP interpolation operators $V^m_n : f \to V^m_n f$, under the assumption

$$m = \lfloor \theta n \rfloor, \qquad \text{for a fixed } 0 < \theta < 1 \qquad (\text{ briefly } n \sim m), \tag{47}$$

the following theorem holds [25, 26].

Theorem 3.2 *For any $n, m \in \mathbb{N}$ satisfying (47), the map $V^m_n : C^0_\varphi \to C^0_\varphi$ is bounded and we have*

$$\|(f - V^m_n f)\varphi\| \leq \mathcal{C} E_{n-m}(f)_\varphi, \quad \forall f \in C^0_\varphi, \qquad \mathcal{C} \neq \mathcal{C}(n, f). \tag{48}$$

Moreover, $\forall r > 0$*, also the map* $V_n^m : Z_r(\varphi) \to Z_r(\varphi)$ *is bounded and the following error estimate holds in Hölder-Zygmund spaces*

$$\|f - V_n^m f\|_{Z_r(\varphi)} \le \frac{\mathcal{C}}{n^{s-r}} \|f\|_{Z_s(\varphi)}, \qquad \forall f \in Z_s(\varphi),\ s \ge r > 0, \qquad \mathcal{C} \neq \mathcal{C}(n, f). \tag{49}$$

Note that for any $n \sim m$, (48) and (9) yield

$$\|(f - V_n^m f)\varphi\| \le \frac{\mathcal{C}}{n^r} \|f\|_{Z_r(\varphi)}, \qquad \forall f \in Z_r(\varphi),\ r > 0, \qquad \mathcal{C} \neq \mathcal{C}(n, f). \tag{50}$$

In the sequel, we are going to apply the projection map $V_n^m : f \to V_n^m f \in S_n^m$ in order to get a discrete version of the Prandtl equation (7).

Concerning the choice to do on the degree parameters $n > m$, we remark that n will represent the dimension of the discrete problem while m yields the action ray of VP filter, which we choose according with (47). The fixed parameter $0 < \theta < 1$ acts as a *localization parameter* [30].

In Section 6 we take, for simplicity, VP filters acting from an even integer N to $2N - 1$. In this case $\theta = \frac{1}{3}$, the action ray is $m = N/2$, and $n = 3N/2$ is the number of interpolation nodes that are the Chebyshev zeros of $p_{3N/2}(x)$.

4 The approximate equation

Let us assume that the conditions ensuring existence and uniqueness of the solution of the equation (1) in Hölder-Zygmund spaces are satisfied. In this section we are going to construct a finite dimensional equation to approximate (1). Moreover, we deduce the unique solvability of such approximate equation from that one of original problem (cf. Theorem 4.3). Finally we estimate the error we make by approximating the solution of the original problem with that one of the approximated problem.

Let us start by introducing the following discrete approximation of the operator K defined in (5)

$$K_n^m f(y) := V_n^m (Kf)(y) = \sum_{j=0}^{n-1} \left[\sum_{k=1}^{n} \lambda_k p_j(x_k) Kf(x_k) \right] q_{n,j}^m(y). \tag{51}$$

By applying Th. 3.2 and Th. 2.2, we easily deduce the following result concerning the previous approximation.

Theorem 4.1 *For all* $r > 0$ *and any pair of positive integers* $n \sim m$*, the map* $K_n^m : Z_r(\varphi) \to Z_{r+1}(\varphi)$ *is bounded and the map* $K_n^m : Z_r(\varphi) \to Z_r(\varphi)$ *is compact. Moreover, for each* $f \in Z_s(\varphi)$*,* $s > 0$*, and any* $0 < r \le s + 1$*, we have*

$$\|(K - K_n^m) f\|_{Z_r(\varphi)} \le \frac{\mathcal{C}}{n^{s+1-r}} \|f\|_{Z_s(\varphi)}, \qquad \mathcal{C} \neq \mathcal{C}(n, m, f), \tag{52}$$

$$\|(K - K_n^m) f\|_\infty \le \frac{\mathcal{C}}{n^{s+1}} \|f\|_{Z_s(\varphi)}, \qquad \mathcal{C} \neq \mathcal{C}(n, m, f). \tag{53}$$

In order to approximate the operator H, in the definition (6), we regard the bivariate kernel function h as the univariate function $\overline{\overline{h}}_y$ (cf. (26)), which we are going to approximate by $V_n^m \overline{\overline{h}}_y$. Recalling (41)–(42), we have

$$V_n^m \overline{\overline{h}}_y(x) = \sum_{j=0}^{n-1} \left[\sum_{k=1}^{n} \lambda_k p_j(x_k) h(x_k, y) \right] q_{n,j}^m(x), \qquad |x| \le 1, \tag{54}$$

and using in (6) this polynomial instead of $\overline{\overline{h}}_y(x)$, we get the operator

$$\tilde{H}_n^m f(y) := \frac{1}{\pi} \int_{-1}^{1} V_n^m \overline{\overline{h}}_y(x) f(x) \varphi(x) dx, \tag{55}$$

which we approximate as follows

$$H_n^m f(y) := V_n^m (\tilde{H}_n^m f)(y) = \sum_{j=0}^{n-1} \left[\sum_{k=1}^{n} \lambda_k p_j(x_k) \tilde{H}_n^m f(x_k) \right] q_{n,j}^m(y). \tag{56}$$

Concerning the previous approximate operators, we prove (see Appendix) the following

Theorem 4.2 *Let $s > 0$ and $n \sim m$. If $h(x, y)\varphi(y) \in C^0([-1, 1]^2)$ and $\overline{h}_x \in Z_s(\varphi)$ uniformly w.r.t. $x \in [-1, 1]$, then the maps $\tilde{H}_n^m : C_\varphi^0 \to Z_s(\varphi)$ and $H_n^m : C_\varphi^0 \to Z_s(\varphi)$ are bounded. Moreover, for each $f \in C_\varphi^0$ and any $0 < r \le s$, we have*

$$\|(H - H_n^m) f\|_{Z_r(\varphi)} \le \frac{\mathcal{C}}{n^{s-r}} \|f\varphi\|, \qquad \mathcal{C} \ne \mathcal{C}(n, m, f), \tag{57}$$

$$\|(H - H_n^m) f\varphi\| \le \frac{\mathcal{C}}{n^s} \|f\varphi\|, \qquad \mathcal{C} \ne \mathcal{C}(n, m, f). \tag{58}$$

By using the previous approximations, we define the following approximate equation

$$\left(\sigma V_n^m + D + K_n^m + H_n^m \right) \tilde{f} = V_n^m g. \tag{59}$$

For the time being, let us suppose that (59) has a solution $\tilde{f}$. In this case, we have

$$D\tilde{f} = V_n^m g - \sigma V_n^m \tilde{f} - K_n^m \tilde{f} - H_n^m \tilde{f} \in S_n^m,$$

and recalling Proposition 3.1, the map $D : \tilde{S_n^m} \to S_n^m$ is bijective. Hence, we conclude that if a solution of (59) exists, then it belongs to $\tilde{S_n^m}$.

On the other hand, the fact that $\tilde{f} \in \tilde{S_n^m}$ allows us to look at the approximate equation (59) as the equation resulting by a standard projection method, based on the VP projection $V_n^m : f \to V_n^m f \in S_m^n$ and applied to the following approximate equation

$$\left(\sigma I + D + K + \tilde{H}_n^m \right) \tilde{f} = g, \tag{60}$$

which is deduced from (7) by approximating its last integral. Of course in the case $h(x, y) \equiv 0$ (see Subsection 5.1) this equation coincides with (7) and the same holds if $\overline{\overline{h}}_y \in \mathbb{P}_{n-m}$ being $\tilde{H}_n^m = H$ in this case.

Hence, the equation (59), whose solution $\tilde{f} \in \tilde{S}_n^m$, can be obtained by applying the VP projection V_n^m at both the sides of the equation (60). Consequently, we can use standard arguments of projection methods in order to deduce the unique solvability of the approximate equation

$$T_n^m \tilde{f} = V_n^m g, \qquad T_n^m := D + U_n^m, \qquad U_n^m := \sigma V_n^m + K_n^m + H_n^m,$$

from the unique solvability of the original Prandtl equation

$$Tf = g, \qquad T := D + U, \qquad U := \sigma I + K + H.$$

Indeed, by Th. 3.2, Th. 4.1 and Th. 4.2 we get

$$\lim_{n\to+\infty} \|U - U_n^m\|_{Z_r(\varphi)\to Z_r(\varphi)} = 0, \qquad n \sim m, \tag{61}$$

where $\|U\|_{A\to B}$ denotes the norm of the operator map $U : A \to B$.

Hence, by standard arguments (see, e.g., [20, Th. 4.2]) we can prove the following

Theorem 4.3 *Let the assumptions of Theorem 2.5 be satisfied. For every* $0 < r \le s$ *and for all* $n \sim m$ *sufficiently large (say* $n > n_0$*), the operator* $T_n^m : Z_{r+1}(\varphi) \to Z_r(\varphi)$ *has a bounded inverse and*

$$\sup_{n\sim m} \|(T_n^m)^{-1}\|_{Z_r(\varphi)\to Z_{r+1}(\varphi)} < +\infty.$$

Moreover, the condition number of T_n^m *tends to the condition number of* T*, i.e.,*

$$\lim_{\substack{n\to\infty \\ n\sim m}} \frac{\|T_n^m\|_{Z_{r+1}(\varphi)\to Z_r(\varphi)} \|(T_n^m)^{-1}\|_{Z_r(\varphi)\to Z_{r+1}(\varphi)}}{\|T\|_{Z_{r+1}(\varphi)\to Z_r(\varphi)} \|T^{-1}\|_{Z_r(\varphi)\to Z_{r+1}(\varphi)}} = 1.$$

Hence, under the hypotheses of the previous theorem, for any $n \sim m$ sufficiently large and for any $g \in Z_r(\varphi)$, $r > 0$, there exists a unique solution $\tilde{f}_n^m$ of (59), which is a stable polynomial approximation of the solution $f^* \in Z_{r+1}(\varphi)$ of the Prandtl-type equation (7). The following theorem ensures us that as $n \sim m \to \infty$, we have $\tilde{f}_n^m \to f^*$ and the error estimates give the same convergence rate of the error of best approximation of g.

Theorem 4.4 *Let the assumption of Theorem 2.5 be satisfied. For every* $g \in Z_s(\varphi)$, $0 < r \le s$ *and for all* $n \sim m$ *sufficiently large (say* $n > n_0$*), the unique solutions* f^* *and* $\tilde{f}_n^m$ *of (7) and (59), respectively, satisfy the following estimates*

$$\|f^* - \tilde{f}_n^m\|_{Z_r(\varphi)} \le \frac{\mathcal{C}}{n^{s-r}} \|g\|_{Z_s(\varphi)}, \qquad \mathcal{C} \ne \mathcal{C}(n, m, f^*), \tag{62}$$

$$\|(f^* - \tilde{f}_n^m)\varphi\| \le \frac{\mathcal{C}}{n^s} \|g\|_{Z_s(\varphi)}, \qquad \mathcal{C} \ne \mathcal{C}(n, m, f^*). \tag{63}$$

Comparing this result with the convergence estimates of the methods in [4, 8], we succeed in cutting off the typical $\log n$ factors occurring with uniform norms, reproducing the estimates of the L^2 case (see [3]).

5 On the computation of the approximate solution

In this section, we suppose that the finite dimensional equation (59) has the unique solution $\tilde{f}_n^m$ and we face the problem of computing it.

We know that $\tilde{f}_n^m \in \tilde{S}_n^m$, hence choosing a basis in $\tilde{S}_n^m$, the problem is reduced to the computation of the $n = \dim \tilde{S}_n^m$ coefficients of $\tilde{f}_n^m$ in such basis. This can be done by using the approximate equation (59). Indeed, this is an equality in S_n^m and hence, by choosing a basis in S_n^m, (59) can be reduced to a system of $n = \dim S_n^m$ equations. In this way, once we have chosen a couple of bases in $\tilde{S}_n^m$ and S_n^m, we get a linear system of n equations in n unknowns that are the coefficients of $\tilde{f}_n^m$ in the chosen basis of $\tilde{S}_n^m$.

Obviously, different choices of the bases give rise to different linear systems to solve. Here we propose to choose the orthogonal bases (37) and (43) in S_n^m and $\tilde{S}_n^m$, respectively. We will see that this choice gives us some advantages.

In the sequel, we adopt simplified notations and set

$$q_j := q_{n,j}^m, \qquad \tilde{q}_j = \tilde{q}_{n,j}^m.$$

Hence, we represent the solution $\tilde{f} := \tilde{f}_n^m \in \widetilde{\mathrm{S}}_n^m$ of the approximate equation (59) as follows

$$\tilde{f}(y) = \sum_{j=0}^{n-1} \tilde{f}_j \tilde{q}_j(y). \tag{64}$$

The coefficients vector $\tilde{\mathbf{f}} = [\tilde{f}_0, \dots, \tilde{f}_{n-1}]^T$ constitutes the unknown of the linear system we are building.

Lemma 5.1 *We have*

$$\tilde{f}(y) = \sum_{j=0}^{n-1} \tilde{f}_j \tilde{q}_j(y) \quad \Longrightarrow \quad V_n^m \tilde{f}(y) = \sum_{j=0}^{n-1} \tilde{f}_j w_j q_j(y), \tag{65}$$

where

$$w_j := \begin{cases} \frac{1}{j+1} & 0 \le j \le n-m, \\ \frac{1}{2m}\left(\frac{m+n-j}{j+1} + \frac{j-n+m}{2n-j+1}\right) & n-m < j < n. \end{cases} \tag{66}$$

By virtue of this lemma, setting

$$\tilde{\mathbf{q}}(y) := [\tilde{q}_0(y), \dots, \tilde{q}_{n-1}(y)], \qquad \mathbf{q}(y) := [q_0(y), \dots, q_{n-1}(y)],$$
$$\mathcal{V}_n := \mathrm{diag}(w_j)_{j=0,..,n-1},$$

we have that

$$\tilde{f}(y) = \tilde{\mathbf{q}}(y) \cdot \tilde{\mathbf{f}} \quad \Longrightarrow \quad V_n^m \tilde{f}(y) = \mathbf{q}(y) \cdot \mathcal{V}_n \tilde{\mathbf{f}}. \tag{67}$$

Lemma 5.2 *If* $\tilde{f}(y) = \sum_{j=0}^{n-1} \tilde{f}_j \widetilde{q}_j(y)$, *then for any* $m < n$ *we have*

$$K_n^m \tilde{f}(y) = \sum_{j=0}^{n-1} \left[\alpha_{j+2}\tilde{f}_{j+2} + \beta_j \tilde{f}_j + \gamma_{j-2}\tilde{f}_{j-2}\right] q_j(y), \tag{68}$$

where

$$\alpha_\ell := \begin{cases} -\frac{1}{4\ell(\ell+1)} & 2 \le \ell \le n-m, \\ -\frac{1}{8m}\left[\frac{n+m-\ell}{\ell(\ell+1)} + \frac{\ell-n+m}{(2n-\ell+1)(2n-\ell+2)}\right] & n-m < \ell < n, \\ 0 & otherwise, \end{cases} \tag{69}$$

$$\beta_\ell := \begin{cases} \frac{1}{4}\left(\frac{1}{2} + 2\log 2\right) & \ell = 0, \\ \frac{1}{2\ell(\ell+2)} & \ell = 1, 2, \ldots, n-m, \\ \frac{1}{4m}\left[\frac{n+m-\ell}{\ell(\ell+2)} + \frac{\ell-n+m}{(2n-\ell)(2n-\ell+2)}\right] & n-m < \ell \le n-2, \\ \frac{1}{8m}\left[\frac{(m+1)(3n-1)}{n(n^2-1)} + \frac{(m-1)(3n+7)}{(n+1)(n+2)(n+3)}\right] & \ell = n-1, \end{cases} \tag{70}$$

$$\gamma_\ell := \begin{cases} -\frac{1}{8} & \ell = 0, \\ -\frac{1}{4(\ell+1)(\ell+2)} & \ell = 1, 2, \ldots, n-m, \\ -\frac{1}{8m}\left[\frac{n+m-\ell}{(\ell+1)(\ell+2)} + \frac{\ell-n+m}{(2n-\ell+1)(2n-\ell)}\right] & \ell = n-m+1, \ldots, n-3, \\ 0 & otherwise. \end{cases} \tag{71}$$

In vector form, Lemma 5.2 states that

$$\tilde{f}(y) = \tilde{\mathbf{q}}(y) \cdot \tilde{\mathbf{f}} \quad \Longrightarrow \quad K_n^m \tilde{f}(y) = \mathbf{q}(y) \cdot \mathcal{A}_n \tilde{\mathbf{f}}, \tag{72}$$

where

$$\mathcal{A}_n := \begin{pmatrix} \beta_0 & 0 & \alpha_2 & & & & \\ 0 & \beta_1 & 0 & \alpha_3 & & 0 & \\ \gamma_0 & 0 & \beta_2 & 0 & \alpha_4 & & \\ & \gamma_1 & 0 & \beta_3 & 0 & \alpha_5 & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots \\ & & & \ddots & \ddots & \ddots & \ddots \; \alpha_{n-1} \\ & & 0 & & \ddots & \ddots & \ddots \; 0 \\ & & & & \gamma_{n-3} & 0 & \beta_{n-1} \end{pmatrix}. \tag{73}$$

Lemma 5.3 *Let* $Q_i := < q_i, \widetilde{q}_i >_\varphi$, $i = 0, \ldots, n-1$ (cf. (45)). *If* $\tilde{f}$ *is given by* (64), *then*

$$H_n^m \tilde{f}(y) = \sum_{j=0}^{n-1} \left[\frac{1}{\pi} \sum_{i=0}^{n-1} \sum_{s=1}^{n} \sum_{k=1}^{n} \lambda_k \lambda_s p_i(x_s) p_j(x_k) h(x_s, x_k) \tilde{f}_i Q_i\right] q_j(y). \tag{74}$$

In vector form, Lemma 5.3 states that

$$\tilde{f}(y) = \tilde{\mathbf{q}}(y) \cdot \tilde{\mathbf{f}} \quad \Longrightarrow \quad H_n^m \tilde{f}(y) = \mathbf{q}(y) \cdot \mathcal{B}_n \tilde{\mathbf{f}}, \tag{75}$$

with

$$\mathcal{B}_n := \frac{1}{\pi} \left(\mathcal{P}_n \Lambda_n\right) \mathcal{H}_n \left(\mathcal{P}_n \Lambda_n\right)^T \mathcal{Q}_n,$$

where Λ_n and $\mathcal{Q}_n$ are the following diagonal matrices

$$\Lambda_n := \operatorname{diag}(\lambda_j)_{j=0,..,n-1}, \qquad \mathcal{Q}_n := \operatorname{diag}(< q_j, \tilde{q}_j >)_{j=0,..,n-1},$$

and $\mathcal{P}_n, \mathcal{H}_n$ are the following square matrices of order n

$$\mathcal{P}_n(i, j) = p_{i-1}(x_j), \qquad \mathcal{H}_n(i, j) = h(x_j, x_i), \qquad 1 \le i, j \le n.$$

Now let us write the approximate equation (59) as follows

$$(D + U_n^m)\tilde{f}(y) = V_n^m g(y), \qquad U_n^m := \sigma V_n^m + K_n^m + H_n^m. \tag{76}$$

The previous lemmas and (46) ensure that

$$\tilde{f}(y) = \tilde{\mathbf{q}}(y) \cdot \tilde{\mathbf{f}} \quad \Longrightarrow \quad \begin{cases} U_n^m \tilde{f}(y) = \mathbf{q}(y) \cdot (\sigma \mathcal{V}_n + \mathcal{A}_n + \mathcal{B}_n)\tilde{\mathbf{f}}, \\ D\tilde{f}(y) = \mathbf{q}(y) \cdot \tilde{\mathbf{f}}. \end{cases}$$

Hence, taking into account that

$$V_n^m g(y) = \mathbf{q}(y) \cdot \mathbf{g}, \qquad \mathbf{g} := \left[g_0, \ldots, g_{n-1}\right]^T, \qquad g_j := c_{n,j}(g) = \sum_{k=1}^{n} \lambda_k p_j(x_k) g(x_k), \tag{77}$$

we rewrite (76) as follows

$$\mathbf{q}(y) \cdot (\mathcal{I}_n + \sigma \mathcal{V}_n + \mathcal{A}_n + \mathcal{B}_n)\tilde{\mathbf{f}} = \mathbf{q}(y) \cdot \mathbf{g}.$$

Consequently, the unknown vector $\tilde{\mathbf{f}}$ results to be the unique solution of the following linear system

$$(\mathcal{I}_n + \sigma \mathcal{V}_n + \mathcal{A}_n + \mathcal{B}_n)\tilde{\mathbf{f}} = \mathbf{g} \tag{78}$$

which yields the unique solution $\tilde{f}(y) = \tilde{\mathbf{q}}(y) \cdot \tilde{\mathbf{f}}$ of the approximate equation (76).

5.1 The special case $H \equiv 0$

The case $H \equiv 0$ is a particularly favorite case since, in (78), $\mathcal{B}_n$ is the null matrix and the linear system (78) becomes

$$\mathcal{A}_n' \tilde{\mathbf{f}} = \mathbf{g}, \qquad \mathcal{A}_n' := \mathcal{I}_n + \sigma \mathcal{V}_n + \mathcal{A}_n. \tag{79}$$

The matrix $\mathcal{A}_n'$ of such system has not null only the (i, j)-entries s.t. $|i - j| \in \{0, 2\}$. It differs from the matrix $\mathcal{A}_n$ in (73) only for the diagonal entries that are given by

$$\delta_j := \mathcal{A}_n'(j, j) = 1 + \sigma w_j + \beta_j \qquad j = 0, \ldots, n-1. \tag{80}$$

The matrix $\mathcal{A}_n'$ is strictly diagonally dominant, since by (66) and (69)–(71) it easily follows that for any $\sigma \in \mathbb{R}$, we have

$$|1 + \sigma w_0 + \beta_0| > |\alpha_2|, \quad |1 + \sigma w_1 + \beta_1| > |\alpha_3|,$$

$$|1+\sigma w_j+\beta_j| > |\gamma_{j-2}|+|\alpha_{j+2}|, \quad 2 \le j \le n-3,$$
$$|1+\sigma w_{n-2}+\beta_{n-2}| > |\gamma_{n-3}|, \quad |1+\sigma w_{n-1}+\beta_{n-1}| > |\gamma_{n-2}|.$$

As a consequence, the Gaussian elimination method for solving the system (79) can be applied avoiding pivoting strategy to ensure the stability of the algorithm (see, e.g.,[13, §3.4.10]). Therefore, the 2-bandwidth of the matrix is preserved and the Gaussian elimination requires $5n$ long operations for solving (79).

The condition number of the matrix $\mathcal{A}'_n$ is defined in the usual way

$$\kappa(\mathcal{A}'_n) := \|\mathcal{A}'_n\| \| \left(\mathcal{A}'_n\right)^{-1} \|,$$

where $\|\cdot\|$ denotes any matrix norm satisfying the submultiplicative property (i.e., $\|AB\| \le \|A\|\|B\|$). The following theorem proves that $\kappa(\mathcal{A}'_n)$ tends to a finite limit as the dimension of the system $n \to +\infty$.

Theorem 5.4 *Under the previous setting, there exists the limit*

$$\lim_{n\to+\infty} \kappa(\mathcal{A}'_n) < \infty.$$

We recall that the linear system conditioning has been studied also for the method in [8] where the authors, in a more general context, proved that the condition numbers sequence diverges as $n \log^3 n$.

6 Numerical tests

For simplicity, in order to test the behavior of the numerical method for increasing values of $n \sim m$, we take increasing values of an even $N \in \mathbb{N}$ and focus on the particular case

$$m = \frac{N}{2}, \qquad \text{and} \qquad n = \frac{3}{2}N, \qquad \text{with } N \in \mathbb{N} \text{ even.} \tag{81}$$

This corresponds to take $\theta = 1/3$ in $m = \theta n$, and VP means from $N = (n-m)$ to $2N-1 = (n+m)-1$.

We are going to show the performance of the proposed method by some numerical experiments, making comparisons with the method introduced in [8] and the collocation-quadrature method studied in [3, 4], all of them based on the Lagrange interpolation at the same nodes. Moreover, we also make a comparison with the method proposed in [2] (cf. Example 6.4).

In all the cases the better performance of our method is displayed.

Although the well-conditioning of our linear systems has been proved only in the special case $H \equiv 0$, the numerical evidence shows the uniform boundedness of the condition numbers also in the general case.

For each example, denoting by X a sufficiently large uniform mesh of $[-1, 1]$, we compute the absolute weighted errors

$$\mathcal{E}_n := \max_{x\in X} \left(|f(x) - f_n(x)|\varphi(x) \right), \tag{82}$$

where n indicates the number of the collocation nodes and f_n denotes the numerical solution of the applied method. Moreover, we compute the condition numbers cond_n, w.r.t. the infinity matrix norm, of the involved linear systems of dimension n.

In order to distinguish the implemented numerical methods, we use the superscripts VP, [4, 8] and [2] in order to indicate our method and the ones in [2–4, 8], respectively.

We point out that all the computations were performed in 16-digit arithmetic and the solutions of the linear systems have been computed by Gaussian elimination method.

Example 6.1 Consider the integral equation

$$(I + D + H)f = g,$$

where

$$h(x, y) = x(y^2|y| + x|x|),$$
$$g(y) = y\left[\left(1 + \frac{4y}{15\pi}\right)|y| + \frac{6}{\pi} + \frac{3y^2 - 2}{\pi\sqrt{1-y^2}}\log\left(\frac{1+\sqrt{1-y^2}}{1-\sqrt{1-y^2}}\right)\right].$$

This example can be found in [3] and the exact solution is $f(y) = y|y|$. In Table 1 we display the results obtained implementing our method and those in [3, 4, 8].

Example 6.2 Consider the integral equation (1) with

$$\sigma = 1, \quad h(x, y) = \left|\cos\left(y - \frac{\pi}{4}\right)\right|^{\frac{9}{2}} + |\sin(x)|^{\frac{7}{2}}, \quad g(y) = |y|^{\frac{11}{2}}.$$

Since in this case the exact solution is unknown, the errors showed in Table 2 have been computed according to (82) but replacing f by f_n^{VP} with $n = 1024$.

Example 6.3 Consider the integral equation (1) with

$$\sigma = 1, \quad h(x, y) = |y| + |x|,$$
$$g(y) = 2 + \frac{|y|}{2} + \frac{2}{3\pi} + \frac{1}{4}(1 - 2y^2 + \log 4),$$

Table 1 Example 6.1

n	$\text{cond}_n^{\text{VP}}$	$\mathcal{E}_n^{\text{VP}}$	cond_n [8]	$\mathcal{E}_n$ [8]	cond_n [4]	$\mathcal{E}_n$ [4]
8	1.95	2.91e−3	4.89	8.50e−3	4.36	3.13
16	2.05	8.08e−4	9.12	2.27e−3	8.24	8.26e−3
32	2.10	2.13e−4	1.75e+1	6.03e−4	1.59e+1	2.15e−3
64	2.12	5.48e−4	3.44e+1	1.56e−4	3.14e+1	5.52e−4
128	2.13	1.34e−5	6.81e+1	3.94e−5	6.23e+1	1.40e−4
256	2.14	2.83e−6	1.35e+2	9.24e−6	1.24e+2	3.53e−5
512	2.15	3.81e−7	2.70e+2	2.54e−6	2.47e+2	8.86e−6

Table 2 Example 6.2

n	cond_n^{VP}	$\mathcal{E}_n^{VP}$	cond_n [8]	$\mathcal{E}_n$ [8]	cond_n [4]	$\mathcal{E}_n$ [4]
8	2.47	6.91e−6	6.13	9.19e−5	5.48	3.78e−04
16	2.60	8.50e−8	1.16e+1	9.22e−7	1.02e+1	1.44e−06
32	2.66	1.08e−9	2.24e+1	2.18e−8	1.98e+1	2.18e−08
64	2.69	1.39e−11	4.39e+1	7.97e−10	3.91e+1	7.97e−10
128	2.71	2.03e−13	8.67e+1	3.48e−11	7.74e+1	3.48e−11
256	2.72	4.53e−15	1.72e+2	1.48e−12	1.54e+2	1.63e−11

whose exact solution is $f(y) = 1$. We compare our method with those in [8] and [3, 4], reporting the results in Table 3.

Example 6.4 Consider the integral equation

$$(2I + D + K)f = g,$$

with $g(x)$ such that the exact solution is $f(x) = \sqrt{(1-x^2)^3}$. This example can be found in [2] where the proposed method provides an approximation of f with at most 2 exact decimal digits. In Table 4, besides the results of our method also the results of the methods in [3, 4, 8] are presented.

In this case $H \equiv 0$ and, as pointed up in Section 5.1, our method leads to solve the system (79) in about $5n$ long operations, whereas the linear systems associated with the methods in [3, 4, 8] have a full matrix and can be solved by Gaussian elimination method with pivoting, requiring at least $n^3/3$ long operation.

Table 3 Example 6.3

n	cond^{VP}	$\mathcal{E}_n^{VP}$	cond [8]	$\mathcal{E}_n$ [8]	cond [4]	$\mathcal{E}_n$ [4]
8	2.64	5.44e−4	5.48	1.82e−3	5.01	1.82e−3
16	2.77	1.46e−4	1.05e+1	5.05e−4	9.50	5.05e−4
32	2.84	3.80e−5	2.04e+1	1.33e−4	1.84e+1	1.33e−4
64	2.87	9.70e−6	4.00e+1	3.44e−5	3.62e+1	3.44e−5
128	2.89	2.45e−6	7.91e+1	8.74e−6	7.19e+1	8.74e−6
256	2.89	6.15e−7	1.57e+2	2.20e−6	1.43e+2	2.20e−6
512	2.90	1.54e−7	3.13e+2	5.53e−7	2.85e+2	5.53e−7

Table 4 Example 4

n	cond$^{\text{VP}}$	$\mathcal{E}_n^{\text{VP}}$	cond [8]	$\mathcal{E}_n$ [8]	cond [4]	$\mathcal{E}_n$ [4]
8	2.93	3.24e−4	4.99	1.53e−3	5.24	1.57e−2
16	3.22	1.97e−5	9.01	1.01e−4	1.00e+1	1.97e−3
32	3.36	5.87e−7	1.68e+1	2.56e−6	1.93e+1	2.59e−4
64	3.43	1.84e−8	3.24e+1	9.67e−8	3.77e+1	3.35e−5
128	3.47	6.05e−10	6.35e+1	1.57e−9	7.42e+1	4.28e−6
256	3.49	4.69e−12	1.25e+2	4.94e−11	1.47e+2	5.41e−7
512	3.50	7.41e−14	2.50e+2	1.99e−12	2.93e+2	6.80e−8

Appendix

Proof of Theorem 4.2

Firstly note that, by (55) and (30), we have

$$\tilde{H_n^m} f(y) = \int_{-1}^{1} \mathcal{H}(x,y) f(x)\varphi(x)dx, \qquad \mathcal{H}(x,y) := \sum_{k=1}^{n} h(x_k,y)\Phi_{n,k}^m(x).$$

Let us prove that the boundedness of $\tilde{H_n^m} : C_\varphi^0 \to Z_s(\varphi)$ follows from Proposition 2.3. Indeed, we recall that (cf. Theorem 3.2)

$$\|V_n^m\|_{C_\varphi^0 \to C_\varphi^0} := \sup_{\|f\varphi\|\le 1} \|(V_n^m f)\varphi\| = \sup_{|x|\le 1} \sum_{k=1}^{n} \frac{|\Phi_{n,k}^m(x)|\varphi(x)}{\varphi(x_k)} \le \mathcal{C} \ne \mathcal{C}(n,m). \tag{83}$$

Hence, from the assumptions on h, we deduce that

(a) $\mathcal{H}(x,y)\varphi(x)\varphi(y)$ is a continuous function w.r.t. both the variables $x, y \in [-1,1]$;

(b) for all $n \in \mathbb{N}$ and $|x| \le 1$, taking into account that

$$E_n(\overline{h}_{x_k})_\varphi = \|(\overline{h}_{x_k} - P_{x_k}^*)\varphi\| \le \frac{\mathcal{C}}{n^s}, \quad k = 1,\dots,n, \qquad \mathcal{C} \ne \mathcal{C}(n,k),$$

we set

$$P^*(y) := \sum_{k=1}^{n} P_{x_k}^*(y)\Phi_{n,k}^m(x)\varphi(x), \qquad |y| \le 1.$$

Hence, we have

$$\begin{aligned}
E_n(\overline{\mathcal{H}}_x\varphi(x))_\varphi &= \inf_{P\in\mathbb{P}_n}\sup_{|y|\le 1}|\mathcal{H}(x,y)\varphi(x)-P(y)|\,\varphi(y)\\
&\le \sup_{|y|\le 1}\left|\mathcal{H}(x,y)\varphi(x)-P^*(y)\right|\varphi(y)\\
&= \sup_{|y|\le 1}\left|\sum_{k=1}^n\left[h(x_k,y)-P^*_{x_k}(y)\right]\Phi^m_{n,k}(x)\varphi(x)\right|\varphi(y)\\
&\le \sum_{k=1}^n\frac{|\Phi^m_{n,k}(x)|\varphi(x)}{\varphi(x_k)}\sup_{|y|\le 1}\left|\overline{h}_{x_k}(y)-P^*_{x_k}(y)\right|\varphi(y)\\
&\le \frac{C}{n^s}\sum_{k=1}^n\frac{|\Phi^m_{n,k}(x)|\varphi(x)}{\varphi(x_k)}\le\frac{\mathcal{C}}{n^s},\qquad \mathcal{C}\ne\mathcal{C}(n,x).
\end{aligned}$$

In conclusion, by virtue of (a) and (b), the kernel $\mathcal{H}$ satisfies the assumptions of Proposition 2.3 with $v=\varphi$, so that the operator defined by this kernel, namely $\tilde{H}^m_n : C^0_\varphi \to Z_s(\varphi)$, is bounded.

Consequently, the map $H^m_n : C^0_\varphi \to Z_s(\varphi)$ is bounded too, since it is the following composition of bounded maps (cf. Theorem 3.2)

$$H^m_n : C^0_\varphi \xrightarrow{\tilde{H}^m_n} Z_s(\varphi) \xrightarrow{V^m_n} Z_s(\varphi).$$

In order to prove (58), let us arbitrarily fix $f\in C^0_\varphi$ and $|y|\le 1$.

By using (30), we note that

$$H^m_n f(y)=\sum_{j=1}^n \tilde{H}^m_n f(x_j)\Phi^m_{n,j}(y)=\int_{-1}^1\left(\sum_{j=1}^n\sum_{k=1}^n h(x_k,x_j)\Phi^m_{n,k}(x)\Phi^m_{n,j}(y)\right)f(x)\varphi(x)dx,$$

that means $H^m_n f$ can be obtained by replacing h with its bivariate VP interpolation polynomial based on tensor-product Chebyshev nodes of the second kind (see [23, 24]).

Hence, we write

$$\begin{aligned}
Hf(y)-H^m_n f(y)&=\int_{-1}^1\left(h(x,y)-\sum_{j=1}^n h(x,x_j)\Phi^m_{n,j}(y)\right)f(x)\varphi(x)dx\\
&\quad+\int_{-1}^1\sum_{j=1}^n\Phi^m_{n,j}(y)\left(h(x,x_j)-\sum_{k=1}^n h(x_k,x_j)\Phi^m_{n,k}(x)\right)f(x)\varphi(x)dx.
\end{aligned}$$

Then, using (30), (26) and (cf. (36))

$$\sum_{k=1}^n\Phi^m_{n,k}(x)=1,\qquad \forall |x|\le 1, \tag{84}$$

we get

$$Hf(y) - H_n^m f(y) = \int_{-1}^{1} \Big(\overline{h}_x(y) - V_n^m(\overline{h}_x)(y)\Big) f(x)\varphi(x)dx$$
$$+ \int_{-1}^{1} \sum_{j=1}^{n} \Phi_{n,j}^m(y) \sum_{k=1}^{n} \Phi_{n,k}^m(x) \Big(\overline{h}_x(x_j) - \overline{h}_{x_k}(x_j)\Big) f(x)\varphi(x)dx.$$

Consequently

$$|Hf(y) - H_n^m f(y)|\varphi(y) \le \int_{-1}^{1} \Big|\overline{h}_x(y) - V_n^m(\overline{h}_x)(y)\Big|\varphi(y)|f(x)|\varphi(x)dx$$
$$+ \sum_{j=1}^{n} |\Phi_{n,j}^m(y)|\varphi(y) \int_{-1}^{1} \sum_{k=1}^{n} |\Phi_{n,k}^m(x)| \Big|\overline{h}_x(x_j) - \overline{h}_{x_k}(x_j)\Big| |f(x)|\varphi(x)dx$$
$$=: A + B.$$

On the other hand, the hypothesis $\overline{h}_x \in Z_s(\varphi)$ uniformly w.r.t. $x \in [-1, 1]$ implies that $\forall n \in \mathbb{N}$ and $\forall |x| \le 1$ we have

$$E_n(\overline{h}_x)_\varphi \le \frac{\mathcal{C}}{n^s}, \qquad \mathcal{C} \ne \mathcal{C}(n, x), \tag{85}$$

and in particular, there exists $P^* \in \mathbb{P}_n$ such that

$$\sup_{|y|\le 1} \Big|\overline{h}_x(y) - P^*(y)\Big|\varphi(y) \le \frac{\mathcal{C}}{n^s}, \qquad \mathcal{C} \ne \mathcal{C}(n, x). \tag{86}$$

Hence, concerning A, by (48) and (85), recalling that $m = \lfloor \theta n \rfloor$ with a fixed $0 < \theta < 1$, we get

$$A \le \int_{-1}^{1} E_{n-m}(\overline{h}_x)_\varphi |f(x)|\varphi(x)dx \le \frac{\mathcal{C}}{(n-m)^s} \int_{-1}^{1} |f(x)|\varphi(x)dx \le \frac{\mathcal{C}}{n^s}\|f\varphi\|.$$

Moreover, regarding B, by (86) and (83), we have

$$B \le \|f\varphi\| \sum_{j=1}^{n} \frac{|\Phi_{n,j}^m(y)|\varphi(y)}{\varphi(x_j)} \int_{-1}^{1} \sum_{k=1}^{n} \frac{|\Phi_{n,k}^m(x)|\varphi(x)}{\varphi(x_k)} \Big|(\overline{h}_x - P^*)(x_j) - (\overline{h}_{x_k} - P^*)(x_j)\Big| \frac{\varphi(x_j)}{\varphi(x)} dx$$
$$\le \frac{\mathcal{C}}{n^s}\|f\varphi\| \sum_{j=1}^{n} \frac{|\Phi_{n,j}^m(y)|\varphi(y)}{\varphi(x_j)} \int_{-1}^{1} \sum_{k=1}^{n} \frac{|\Phi_{n,k}^m(x)|\varphi(x)}{\varphi(x_k)} \frac{dx}{\varphi(x)}$$
$$\le \frac{\mathcal{C}}{n^s}\|f\varphi\| \int_{-1}^{1} \frac{dx}{\varphi(x)} \le \frac{\mathcal{C}}{n^s}\|f\varphi\|, \qquad \text{with } \mathcal{C} \ne \mathcal{C}(n, f, y).$$

Finally, let us prove (57). Taking into account that $H_n^m f \in \mathbb{P}_{n+m-1}$ and recalling that $H : C_\varphi^0 \to Z_s(\varphi)$ is bounded (cf. Proposition 2.3), for any $f \in C_\varphi^0$ we note that

$$\sup_{k \ge n+m-1} (k+1)^r E_k(Hf - H_n^m f)_\varphi = \sup_{k \ge n+m-1} (k+1)^r E_k(Hf)_\varphi$$
$$\le \mathcal{C} \sup_{k \ge n+m-1} \frac{\|Hf\|_{Z_s(\varphi)}}{k^{s-r}} \le \mathcal{C}\frac{\|f\varphi\|}{n^{s-r}}.$$

Moreover, by (58) we get

$$\sup_{k<n+m-1}(k+1)^r E_k(Hf-H_n^m f)_\varphi \le \|(Hf-H_n^m f)\varphi\| \sup_{k<n+m-1}(k+1)^r$$
$$\le \mathcal{C}\frac{\|f\varphi\|}{n^s}(n+m)^r \le \mathcal{C}\frac{\|f\varphi\|}{n^{s-r}}.$$

Consequently, by (58) we have

$$\|(H-H_n^m)f\|_{Z_r(\varphi)} = \|(Hf-H_n^m f)\varphi\| + \sup_{k\in\mathbb{N}}(k+1)^r E_k(Hf-H_n^m f)_\varphi \le \mathcal{C}\frac{\|f\varphi\|}{n^{s-r}}$$

and (57) follows.

Proof of Theorem 4.4

Note that we can write

$$\begin{aligned} f^*-\tilde{f}_n^m &= (D+U_n^m)^{-1}\left[(D+U_n^m)f^*-V_n^m g\right] \\ &= (D+U_n^m)^{-1}\left[(g-V_n^m g)+(U_n^m-U)f^*\right]. \end{aligned}$$

Hence, by the uniform boundedness of $(D+U_n^m)^{-1}: Z_r(\varphi)\to Z_{r+1}(\varphi)$ (cf. Th. 4.3), for any $0<r\le s$ we deduce

$$\|f^*-\tilde{f}_n^m\|_{Z_{r+1}(\varphi)} \le \mathcal{C}\left[\|g-V_n^m g\|_{Z_r(\varphi)} + \|(U_n^m-U)f^*\|_{Z_r(\varphi)}\right]$$

and (62) follows from Th. 3.2, Th. 4.1 and Th. 4.2, taking into account that, by hypothesis, $g\in Z_s(\varphi)$, $f^*\in Z_{s+1}(\varphi)$ and $\|g\|_{Z_s(\varphi)}\sim\|f^*\|_{Z_{s+1}(\varphi)}$.

Finally, (63) follows from (62) when $r\to 0^+$.

Proof of Lemma 5.1

Let $\tilde{f}(y)=\sum_{j=0}^{n-1}\tilde{f}_j\tilde{q}_j(y)$. Recalling (41)–(42), we have $V_n^m\tilde{f}(y)=\sum_{r=0}^{n-1}q_r(y)c_{n,r}(\tilde{f})$, where

$$\begin{aligned} c_{n,r}(\tilde{f}) &= \sum_{k=1}^{n}\lambda_k p_r(x_k)\tilde{f}(x_k) \\ &= \left(\sum_{j=0}^{n-m}+\sum_{j=n-m+1}^{n-1}\right)\tilde{f}_j\sum_{k=1}^{n}\lambda_k p_r(x_k)\tilde{q}_j(x_k) \\ &= \sum_{j=0}^{n-m}\tilde{f}_j\sum_{k=1}^{n}\lambda_k p_r(x_k)\frac{p_j(x_k)}{j+1} \\ &+ \sum_{j=n-m+1}^{n-1}\tilde{f}_j\sum_{k=1}^{n}\lambda_k p_r(x_k)\left(\frac{m+n-j}{2m(j+1)}p_j(x_k)-\frac{(j-n+m)}{2m(2n-j+1)}p_{2n-j}(x_k)\right). \end{aligned}$$

Consequently, by using [30]

$$p_{2n-j}(x_k) = -p_j(x_k), \qquad k=1,\ldots,n, \qquad n-m<j<n, \tag{87}$$

for any $r = 0, \ldots, n-1$, we get

$$c_{n,r}(\tilde{f}) = \sum_{j=0}^{n-m} \tilde{f}_j < p_r, p_j > \frac{1}{j+1} + \sum_{j=n-m+1}^{n-1} \tilde{f}_j < p_r, p_j > \frac{m+n-j}{2m(j+1)} + \sum_{j=n-m+1}^{n-1} \tilde{f}_j < p_r, p_j > \frac{(j-n+m)}{2m(2n-j+1)}.$$

Hence, the orthogonality relation $< p_r, p_j >= \delta_{r,j}$ implies that

$$V_n^m \tilde{f}(y) = \sum_{r=0}^{n-m} q_r(y) \tilde{f}_r \left[\frac{1}{r+1}\right] + \sum_{r=n-m+1}^{n-1} q_r(y) \tilde{f}_r \left[\frac{1}{2m}\left(\frac{m+n-r}{r+1} + \frac{r-n+m}{2n-r+1}\right)\right],$$

and the statement follows.

Proof of Lemma 5.2

Due to the assumption $\tilde{f}(y) = \sum_{\ell=0}^{n-1} \tilde{f}_\ell \, \tilde{q}_\ell(y)$, the core of the proof consists in stating that

$$K\widetilde{q}_\ell(x_k) = \alpha_\ell p_{\ell-2}(x_k) + \beta_\ell p_\ell(x_k) + \gamma_\ell p_{\ell+2}(x_k), \quad \ell = 0, 1, \ldots, n-1. \tag{88}$$

This is an immediate consequence of (43), (22) and (23) in the case that $\ell = 0, \ldots, (n-m)$. In order to prove (88) also in the case $(n-m) < \ell < n$, we observe that in this case the previous equations yield

$$\begin{aligned} K\widetilde{q}_\ell(x_k) &= \frac{n+m-\ell}{2m(\ell+1)} K p_\ell(x_k) - \frac{\ell-n+m}{2m(2n-\ell+1)} K p_{2n-\ell}(x_k) \\ &= \frac{n+m-\ell}{8m(\ell+1)} \left[-\frac{1}{\ell} p_{\ell-2}(x_k) + \left(\frac{1}{\ell} + \frac{1}{\ell+2}\right) p_\ell(x_k) - \frac{1}{\ell+2} p_{\ell+2}(x_k)\right] \\ &- \frac{\ell-n+m}{8m(2n-\ell+1)} \left[-\frac{1}{2n-\ell} p_{2n-\ell-2}(x_k) + \left(\frac{1}{2n-\ell} + \frac{1}{2n-\ell+2}\right) p_{2n-\ell}(x_k) \right. \\ &\left. - \frac{1}{2n-\ell+2} p_{2n-\ell+2}(x_k)\right], \end{aligned}$$

and using (87), we get

$$\begin{aligned} K\widetilde{q}_\ell(x_k) = &- \left[\frac{n+m-\ell}{8m\ell(\ell+1)} + \frac{\ell-n+m}{8m(2n-\ell+1)(2n-\ell+2)}\right] p_{\ell-2}(x_k) \\ &+ \left[\frac{n+m-\ell}{4m\ell(\ell+2)} + \frac{\ell-n+m}{4m(2n-\ell)(2n-\ell+2)}\right] p_\ell(x_k) \\ &- \left[\frac{n+m-\ell}{8m(\ell+1)(\ell+2)} + \frac{\ell-n+m}{8m(2n-\ell+1)(2n-\ell)}\right] p_{\ell+2}(x_k). \end{aligned} \tag{89}$$

By (89), the statement follows if $(n-m) < \ell \leq (n-3)$ and the same holds if $\ell = (n-2)$, being in this case $p_{\ell+2}(x_k) = p_n(x_k) = 0$.

If $\ell = (n-1)$ then (89) implies that

$$
\begin{aligned}
K\widetilde{q}_{n-1}(x_k) = &-\left[\frac{m+1}{8mn(n-1)} + \frac{m-1}{8m(n+2)(n+3)}\right] p_{n-3}(x_k)\\
&+\left[\frac{m+1}{4m(n-1)(n+1)} + \frac{m-1}{4m(n+1)(n+3)}\right] p_{n-1}(x_k)\\
&-\left[\frac{m+1}{8mn(n+1)} + \frac{m-1}{8m(n+2)(n+1)}\right] p_{n+1}(x_k),
\end{aligned}
$$

and taking into account that $p_{n-1}(x_k) = -p_{n+1}(x_k)$ (cf. (87)), we get

$$
\begin{aligned}
K\widetilde{q}_{n-1}(x_k) = &-\left[\frac{m+1}{8mn(n-1)} + \frac{m-1}{8m(n+2)(n+3)}\right] p_{n-3}(x_k)\\
&+\left[\frac{(m+1)(3n-1)}{8mn(n-1)(n+1)} + \frac{(m-1)(3n+7)}{8m(n+1)(n+2)(n+3)}\right] p_{n-1}(x_k),
\end{aligned}
$$

which concludes the proof of (88).

Finally, by (88), we get the statement as follows

$$
\begin{aligned}
K_n^m \tilde{f}(y) &= \sum_{j=0}^{n-1} q_j(y)\left[\sum_{k=1}^{n} \lambda_k p_j(x_k) \sum_{\ell=0}^{n-1} \tilde{f}_\ell K\widetilde{q}_\ell(x_k)\right]\\
&= \sum_{j=0}^{n-1} q_j(y)\left[\sum_{k=1}^{n} \lambda_k p_j(x_k) \sum_{\ell=0}^{n-1} \tilde{f}_\ell \Big(\alpha_\ell p_{\ell-2}(x_k) + \beta_\ell p_\ell(x_k) + \gamma_\ell p_{\ell+2}(x_k)\Big)\right]\\
&= \sum_{j=0}^{n-1} q_j(y) \sum_{\ell=0}^{n-1} \tilde{f}_\ell \Big[\alpha_\ell < p_{\ell-2}, p_j > + \beta_\ell < p_\ell, p_j > + \gamma_\ell < p_{\ell+2}, p_j > \Big]\\
&= \sum_{j=0}^{n-1} q_j(y)\Big[\alpha_{j+2}\tilde{f}_{j+2} + \beta_j \tilde{f}_j + \gamma_{j-2}\tilde{f}_{j-2}\Big].
\end{aligned}
$$

Proof of Lemma 5.3

The statement can be deduced from (54)–(56) and (44), as follows

$$
\begin{aligned}
H_n^m \tilde{f}(y) &= \sum_{j=0}^{n-1} q_j(y)\left[\sum_{k=1}^{n} \lambda_k p_j(x_k) \widetilde{H}_n \tilde{f}(x_k)\right]\\
&= \frac{1}{\pi}\sum_{j=0}^{n-1} q_j(y) \sum_{k=1}^{n} \lambda_k p_j(x_k)\left[\sum_{i=0}^{n-1}\sum_{s=1}^{n} \lambda_s p_i(x_s) h(x_s, x_k) \int_{-1}^{1} \tilde{f}(x) q_i(x)\varphi(x)dx\right]\\
&= \frac{1}{\pi}\sum_{j=0}^{n-1} q_j(y)\left[\sum_{i=0}^{n-1}\sum_{k=1}^{n}\sum_{s=1}^{n} \lambda_k\lambda_s p_j(x_k) p_i(x_s) h(x_s, x_k) \sum_{\ell=0}^{n-1} \tilde{f}_\ell < \widetilde{q}_\ell, q_i > \right]\\
&= \frac{1}{\pi}\sum_{j=0}^{n-1} q_j(y)\left[\sum_{i=0}^{n-1}\sum_{k=1}^{n}\sum_{s=1}^{n} \lambda_k\lambda_s p_j(x_k) p_i(x_s) h(x_s, x_k) \tilde{f}_i < \widetilde{q}_i, q_i > \right].
\end{aligned}
$$

Proof of Theorem 5.4

Firstly, let us focus on the elements $\alpha_k, \gamma_k, \beta_k$ and w_k that have been introduced by Lemmas 5.2 and 5.1, respectively. We note that they are decreasing in modulus and tend to 0 as $n \to \infty$, $|\alpha_i|, |\gamma_i|, \beta_i$ with order $1/n^2$ and w_i with order $1/n$. Consequently, for any matrix norm $\|\cdot\|$ and for any $\varepsilon > 0$, there exists ν_ε such that the matrix $\mathcal{A}'_n$ can be splitted into the following sum

$$\mathcal{A}'_n = \mathcal{R}_n + \mathcal{E}_n, \qquad \text{with } \|\mathcal{E}_n\| < \varepsilon, \quad \forall n > \nu_\epsilon. \tag{90}$$

More precisely, for any $n > \nu_\varepsilon$ we represent the matrix $\mathcal{A}'_n$ in the following block form

$$\mathcal{A}'_n =: \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix},$$

where

$$A_{1,1} := \begin{pmatrix} \delta_0 & 0 & \alpha_2 & & & \\ 0 & \delta_1 & 0 & \alpha_3 & & 0 \\ \gamma_0 & 0 & \delta_2 & 0 & \ddots & \\ & \ddots & \ddots & \ddots & \ddots & \alpha_{\nu_\varepsilon} \\ & 0 & \ddots & \ddots & \ddots & 0 \\ & & & \gamma_{\nu_\varepsilon-2} & 0 & \delta_{\nu_\varepsilon} \end{pmatrix} \in \mathbb{R}^{(\nu_\varepsilon+1)\times(\nu_\varepsilon+1)}$$

$$A_{1,2} := \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ \vdots & 0 & 0 & 0 & 0 \\ 0 & \vdots & 0 & \vdots & \vdots \\ \alpha_{\nu_\varepsilon+1} & 0 & \vdots & 0 & 0 \\ 0 & \alpha_{\nu_\varepsilon+2} & 0 & \dots & 0 \end{pmatrix} \in \mathbb{R}^{(\nu_\varepsilon+1)\times(n-\nu_\varepsilon-1)}$$

$$A_{2,1} := \begin{pmatrix} 0 & 0 & 0 & \gamma_{\nu_\varepsilon-1} & 0 \\ 0 & \vdots & \vdots & 0 & \gamma_{\nu_\varepsilon} \\ \vdots & \vdots & 0 & \dots & 0 \\ 0 & 0 & 0 & \vdots & 0 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix} \in \mathbb{R}^{(n-\nu_\varepsilon-1)\times(\nu_\varepsilon+1)}$$

$$A_{2,2} := \begin{pmatrix} \delta_{\nu_\varepsilon+1} & 0 & \alpha_{\nu_\varepsilon+3} & & & & \\ 0 & \delta_{\nu_\varepsilon+2} & 0 & \alpha_{\nu_\varepsilon+4} & & 0 & \\ \gamma_{\nu_\varepsilon+1} & 0 & \delta_{\nu_\varepsilon+3} & 0 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \alpha_n & \\ & 0 & & \ddots & \ddots & 0 & \\ & & & & \gamma_{n-2} & 0 & \delta_{n-1} \end{pmatrix} \in \mathbb{R}^{(n-\nu_\varepsilon-1)\times(n-\nu_\varepsilon-1)}$$

and we recall that $\delta_k = 1 + \sigma w_k + \beta_k$ has been defined in (80).

Hence, for a suitable choice of ν_ϵ (depending on the choice of the matrix norm and $\epsilon > 0$) we get (90) by taking

$$\mathcal{R}_n := \begin{pmatrix} A_{1,1} & \mathbf{O}_{(\nu_\varepsilon+1)\times(n-\nu_\varepsilon-1)} \\ \mathbf{O}_{(n-\nu_\varepsilon-1)\times(\nu_\varepsilon+1)} & \mathcal{I}_{n-\nu_\varepsilon-1} \end{pmatrix},$$

$$\mathcal{E}_n := \begin{pmatrix} \mathbf{O}_{(\nu_\varepsilon+1)\times(\nu_\varepsilon+1)} & A_{1,2} \\ A_{2,1} & A_{2,2} - \mathcal{I}_{n-\nu_\varepsilon-1} \end{pmatrix},$$

where $\mathbf{O}_{k\times h}$ denotes the null matrix in $\mathbb{R}^{k\times h}$ and $\mathcal{I}_k$ denotes the identity $k \times k$ matrix.

From (90), using $\|\mathcal{E}_n\| < \varepsilon$, we easily deduce

$$\|\mathcal{R}_n\| - \varepsilon \leq \|\mathcal{A}'_n\| \leq \|\mathcal{R}_n\| + \varepsilon, \quad \forall \epsilon > 0, \quad \forall n > \nu_\epsilon, \tag{91}$$

where we remark that $\|\mathcal{R}_n\|$ is independent of $n > \nu_\varepsilon$, since increasing the order n of $\mathcal{R}_n$, the order of the identity block increases, while $A_{1,1}$ remains unchanged. Consequently, by (91) we get

$$\forall \varepsilon > 0\ \exists \nu_\varepsilon :\ \forall n_1, n_2 > \nu_\varepsilon,\ \left| \|\mathcal{A}'_{n_1}\| - \|\mathcal{A}'_{n_2}\| \right| < 2\varepsilon,$$

i.e., the sequence $\{\|\mathcal{A}'_n\|\}_n$ converges since it is a Cauchy sequence.

In order to complete the proof, let us prove that also the sequence $\{\|(\mathcal{A}'{}_n)^{-1}\|\}_n$ is a Cauchy sequence.

Starting again from (90), we observe the matrix $\mathcal{R}_n$ is nonsingular and, under the previous settings, we have

$$\mathcal{R}_n^{-1} = \begin{pmatrix} A_{1,1}^{-1} & \mathbf{O} \\ \mathbf{O} & \mathcal{I} \end{pmatrix}, \tag{92}$$

where, for the sake of brevity, we omit to explicit the dimensions of the null and identity blocks.

Consequently, by (90) we get

$$\mathcal{A}'_n = \mathcal{R}_n(\mathcal{I}_n + \mathcal{R}_n^{-1}\mathcal{E}_n).$$

Let us prove that ν_ϵ can be chosen in such a way that also $\|\mathcal{R}_n^{-1}\mathcal{E}_n\| < \epsilon$ holds true for any $n > \nu_\epsilon$. Indeed, we note that

$$\mathcal{R}_n^{-1}\mathcal{E}_n = \begin{pmatrix} \mathbf{O} & A_{1,1}^{-1}A_{1,2} \\ A_{2,1} & A_{22} - \mathcal{I} \end{pmatrix} \tag{93}$$

differs from $\mathcal{E}_n$ only for the block matrix in the position (1,2), which is $A_{1,1}^{-1}A_{1,2}$ instead of $A_{1,2}$. Nevertheless, in terms of the column vectors, denoted by $\mathbf{e}_i \in \mathbb{R}^{\nu_\varepsilon+1}$

the $i-th$ vector of the canonical basis of $\mathbb{R}^{\nu_\varepsilon+1}$ and by $\mathbf{0}\in\mathbb{R}^{\nu_\epsilon+1}$ the null vector, we note that

$$\begin{aligned} A_{1,2} &= [\alpha_{\nu_\varepsilon+1}\mathbf{e}_{\nu_\varepsilon},\ \alpha_{\nu_\varepsilon+2}\mathbf{e}_{\nu_\varepsilon+1},\ \mathbf{0},\dots,\mathbf{0}],\\ A_{1,1}^{-1}A_{1,2} &= [\alpha_{\nu_\varepsilon+1}\mathbf{P}^0_{\nu_\varepsilon},\ \alpha_{\nu_\varepsilon+2}\mathbf{P}^1_{\nu_\varepsilon+1},\ \mathbf{0},\dots,\mathbf{0}],\end{aligned}$$

where we set

$$\mathbf{P}^0 := A_{1,1}^{-1}\cdot\mathbf{e}_{\nu_\varepsilon},\quad \mathbf{P}^1 := A_{1,1}^{-1}\cdot\mathbf{e}_{\nu_\varepsilon+1}.$$

In order to obtain an explicit expression of these vectors, we need only the last two columns of $A_{1,1}^{-1}$, which we deduce from the following LU factorization of $A_{1,1}$

$$A_{1,1}=LU=\begin{pmatrix}1&&&&&\\0&1&&&&0\\v_0&0&1&&&\\&v_1&0&1&&\\&&\ddots&\ddots&\ddots&\\&0&&v_{\nu_\varepsilon-2}&0&1\end{pmatrix}\begin{pmatrix}d_0&0&\alpha_2&&&\\&d_1&0&\alpha_3&&0\\&&d_2&0&\ddots&\\&&&\ddots&\ddots&\alpha_{\nu_\varepsilon}\\&&0&&d_{\nu_\epsilon-1}&0\\&&&&&d_{\nu_\varepsilon}\end{pmatrix},$$

where $d_0=\delta_0$, $d_1=\delta_1$, and for $k=2,3,\dots\nu_\varepsilon$, we have

$$\begin{aligned} v_{k-2} &:= \frac{\gamma_{k-2}}{d_{k-2}},\\ d_k &:= \delta_k - v_{k-2}\alpha_k = 1+\sigma w_k+\beta_k - v_{k-2}\alpha_k.\end{aligned}$$

More precisely, it can be checked that the elements $\mathbf{P}^0_i$ and $\mathbf{P}^1_i$, $i=0,\dots,\nu_\epsilon$, of the previous vectors $\mathbf{P}^0$ and $\mathbf{P}^1$ can be determined by induction as follows

$$\begin{aligned} \mathbf{P}^0_0 &= \frac{1-\alpha_2\mathbf{P}^0_2}{d_0},\\ \mathbf{P}^0_{2i+1} &= 0,\quad i=0,1,\dots,\frac{\nu_\varepsilon}{2}-1,\\ \mathbf{P}^0_{2i} &= \frac{(-1)^i\prod_{j=0}^{i-1}v_{2j}-\alpha_{2i+2}\mathbf{P}^0_{2i+2}}{d_{2i}}\quad i=\frac{\nu_\varepsilon}{2}-1,\frac{\nu_\varepsilon}{2}-2,\dots,1,\\ \mathbf{P}^0_{\nu_\varepsilon} &= \frac{(-1)^{\frac{\nu_\varepsilon}{2}}\prod_{j=0}^{\frac{\nu_\varepsilon}{2}-1}v_{2j}}{d_{\nu_\varepsilon}},\\ \mathbf{P}^1_1 &= \frac{1-\alpha_3\mathbf{P}^1_3}{d_1},\\ \mathbf{P}^1_{2i+1} &= \frac{(-1)^i\prod_{j=0}^{i-1}v_{2j+1}-\alpha_{2i+3}\mathbf{P}^1_{2i+3}}{d_{2i+1}}\quad i=\frac{\nu_\varepsilon}{2}-3,\dots,2,\\ \mathbf{P}^1_{2i} &= 0,\quad i=0,1,\dots,\frac{\nu_\varepsilon}{2},\\ \mathbf{P}^1_{\nu_\varepsilon-1} &= \frac{(-1)^{\frac{\nu_\varepsilon}{2}-1}\prod_{j=0}^{\frac{\nu_\varepsilon}{2}-2}v_{2j+1}}{d_{\nu_\varepsilon-1}}.\end{aligned}$$

Thus, recalling the behavior of the sequences $\alpha_k, \beta_k, \gamma_k, w_k$, we can conclude that, under a suitable choice of ν_ε, all the entries of $\mathcal{R}_n^{-1}\mathcal{E}_n$ are so small that we have

$$\mathcal{A}_n' = \mathcal{R}_n(\mathcal{I}_n + \mathcal{R}_n^{-1}\mathcal{E}_n), \qquad \text{with } \|\mathcal{R}_n^{-1}\mathcal{E}_n\| < \varepsilon, \qquad \forall n > \nu_\epsilon.$$

Consequently, taking $0 < \epsilon < 1$ and recalling that $\|(\mathcal{I}_n + \mathcal{R}_n^{-1}\mathcal{E}_n)^{-1}\| \leq \frac{1}{1-\|\mathcal{R}_n^{-1}\mathcal{E}_n\|}$ (see, e.g., [13, Lemma 2.3.3, p. 59]) we have

$$\|(\mathcal{A}'_n)^{-1}\| = \|(\mathcal{I}_n + \mathcal{R}_n^{-1}\mathcal{E}_n)^{-1}\mathcal{R}_n^{-1}\| \leq \frac{\|\mathcal{R}_n^{-1}\|}{1-\varepsilon}, \qquad \forall n > \nu_\varepsilon,$$

as well as, we get

$$\|\mathcal{R}_n^{-1}\| = \|(\mathcal{I}_n + \mathcal{R}_n^{-1}\mathcal{E}_n)(\mathcal{A}_n')^{-1}\| \leq (1+\epsilon)\|(\mathcal{A}_n')^{-1}\|, \qquad \forall n > \nu_\varepsilon.$$

Summing up, we have stated that

$$\frac{\|\mathcal{R}_n^{-1}\|}{1+\varepsilon} \leq \|(\mathcal{A}'_n)^{-1}\| \leq \frac{\|\mathcal{R}_n^{-1}\|}{1-\varepsilon}, \qquad \forall n > \nu_\varepsilon, \quad 0 < \epsilon < 1. \tag{94}$$

On the other hand, in view of (92), we can say that $\|\mathcal{R}_n^{-1}\|$ is independent of $n > \nu_\epsilon$, like $\|\mathcal{R}_n\|$, and in particular there exists a constant M independent of n s.t. $\|\mathcal{R}_n^{-1}\| \leq M$. Hence, by (94) we get for any $n_1, n_2 > \nu_\varepsilon$ and $0 < \varepsilon < 1$

$$\left|\|(\mathcal{A}'_{n_1})^{-1}\| - \|(\mathcal{A}'_{n_2})^{-1}\|\right| \leq \frac{2\varepsilon}{1-\varepsilon^2}\|\mathcal{R}_n^{-1}\| \leq \frac{2M\varepsilon}{1-\varepsilon^2},$$

by which

$$\lim_{n_1,n_2\to\infty}\left|\|(\mathcal{A}'_{n_1})^{-1}\| - \|(\mathcal{A}'_{n_2})^{-1}\|\right| = 0,$$

and the statement follows.

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