Numerical Algorithms (2021) 87:1201–1228
https://doi.org/10.1007/s11075-020-01005-5

ORIGINAL PAPER

Stability and convergence analysis of stabilized finite element method for the Kelvin-Voigt viscoelastic fluid flow model

Tong Zhang[1] **· Mengmeng Duan**[2]

Received: 20 February 2019 / Accepted: 17 August 2020 / Published online: 7 September 2020

Abstract

In this paper, we consider the Galerkin finite element method (FEM) for the Kelvin-Voigt viscoelastic fluid flow model with the lowest equal-order pairs. In order to overcome the restriction of the so-called inf-sup conditions, a pressure projection method based on the differences of two local Gauss integrations is introduced. Under some suitable assumptions on the initial data and forcing function, we firstly present some stability and convergence results of numerical solutions in spatial discrete scheme. By constructing the dual linearized Kelvin-Voigt model, stability and optimal error estimates of numerical solutions in various norms are established. Secondly, a fully discrete stabilized FEM is introduced, the backward Euler scheme is adopted to treat the time derivative terms, the implicit scheme is used to deal with the linear terms and semi-implicit scheme is applied to treat the nonlinear term, unconditional stability and convergence results are also presented. Finally, some numerical examples are presented to verify the developed theoretical analysis and show the performances of the considered numerical schemes.

Keywords Stabilized method · Kelvin-Voigt viscoelastic fluid flow model · The lowest equal-order mixed elements · The L'Hospital rule · Negative norm technique

Mathematics Subject Classification (2010) 65N15 · 65N30 · 76D05

✉ Tong Zhang
hpumathzhang@163.com

[1] School of Mathematics and Information Sciences, Yantai University, Yantai, 264005, China

[2] School of Mathematics & Information Science, Henan Polytechnic University, Jiaozuo, 454003, China

1 Introduction

As an important component of the non-Newton fluids, the viscoelastic fluid flow model has been widely used in food products, molten plastic, biologic fluid, etc. In recent years, some important viscoelastic models have been researched not only from the viewpoint of theoretical analysis but also from the numerical simulations point. For example, we can refer to [1–3] and the reference therein. In this paper, we consider the following Kelvin-Voigt viscoelastic flow model

$$\begin{cases} \dfrac{\partial u}{\partial t} - \kappa \Delta u_t - \nu \Delta u + u \cdot \nabla u + \nabla p = f, & (x, t) \in \Omega \times R^+, \\ \operatorname{div} u = 0, & (x, t) \in \Omega \times R^+, \\ u = 0, & (x, t) \in \partial\Omega \times R^+, \\ u(x, 0) = u_0, & x \in \Omega, \end{cases} \tag{1}$$

where $\Omega \in R^d (d = 2, 3)$ be a bounded convex domain or the boundary $\partial\Omega \in C^2$, u and p are the fluid velocity and pressure, the positive parameters ν and κ are the kinematic coefficient viscosity and the retardation time or the time of relaxation of deformations, respectively, f is the prescribed body force, and u_0 is the initial velocity.

The Kelvin-Voigt model was first introduced by Pavlovskii [4], which can be used to describe the motion of weakly concentrated water-polymer solution. It was named the Kelvin-Voigt viscoelastic fluid model by Oskolkov and his collaborators [5]. Later, the Kelvin-Voigt model as a smooth, inviscid regularization of the 2/3D Navier-Stokes equations has been proposed in [6]. For applications of the Kelvin-Voigt flow in organic polymer and food industry and in the mechanisms of diffuse axonal injury, etc., we can refer to [7] and the reference therein. Many numerical works have also been done for the Kelvin-Voigt model. For example, under the assumption of the exact solution is asymptotically stable, Oskolkov analyzed the convergence of the spectral Galerkin approximation in [8]. Pani and his coauthors in [9] shown that the optimal error estimation was consistent and effective in time under the assumption of uniqueness by applying a variant of the nonlinear semi-discrete spectral Galerkin method. Later, they considered the first- and second-order backward difference methods for the Kelvin-Voigt model and established the global discrete attractor and optimal error estimates by the Sobolev-Stokes projection and Stolz-Cesaro classical result on sequences in [10–12]. Bajpai and his coauthors considered the BDF schemes and two-grid Crank-Nicolson method for the Kelvin-Voigt viscoelastic fluid model in [13, 14]; stability and optimal error estimates were provided and numerical tests were also presented to verify the performances of the considered numerical methods.

When we solve the incompressible flow problem numerically, an important restriction is the compatibility between the discrete velocity and pressure spaces [15, 16]. However, many simple construction and computationally convenience mixed elements do not satisfy the inf-sup condition may also work well, especially the equal-order mixed elements, because these pairs have the same node distributions and basis functions on the same meshing. In order to overcome the restriction of the discrete inf-sup condition and make full use of the equal-order mixed elements, researchers developed several stabilized techniques, for example, the polynomial pressure projection method [17, 18], the macro-element method [19], and the local Gauss integrations stabilized method [20]. In this work, we consider the stabilized method for the Kelvin-Voigt model based on the lowest equal-order mixed elements. The difference between two local Gauss integrations is used to bypass the inf-sup condition due to this method has some attractive features, such as parameter-free, no higher-order derivatives, and non-edge-based data structures.

The main contributions of this paper can be listed as follows:

(I) Compared with [10–14], some different theoretical analysis tools are adopted to avoid using the Sobolev-Stokes projection and Stolz-Cesaro's classical result on sequences.
(II) Compared with [19], optimal error estimates for velocity in $L^\infty(L^2)$-norm are established by constructing the dual problem and using the negative norm estimate technique.
(III) Compared with [20, 21], the unconditional stability and optimal convergence results of velocity in various norms are provided for all $t \geq 0$.

The outline of this paper is organized as follows. Section 2 introduces the Sobolev spaces, the stabilized FEM, and some preliminary results. Stability and convergence results of stabilized FEM are presented in Sections 3 and 4 by the energy method and L'Hospital rule. Section 5 devotes to the optimal $L^\infty(L^2)$-norm error estimates of velocity by constructing dual problem and using the negative norm technique. Section 6 presents the fully discrete stabilized FEM and establishes the error estimates of the fully discrete numerical approximations in various norms. Finally, some numerical results are presented to verify the established theoretical analysis and show the performances of the considered stabilized method.

2 Preliminaries

For the mathematical setting of problem (1), standard Sobolev spaces are used. Denote $H^i(\Omega)$ the function with square integrable distribution derivatives up to order i over the domain Ω, $H_0^1(\Omega)$ is the closed subspace of $H^1(\Omega)$ consisting of the functions with zero trace on Ω. We equip the spaces $H^i(\Omega)(i = 1, 2)$ with the norm $\|\cdot\|_i$, $L^i(\Omega)$ with the norm $\|\cdot\|_0$ and inner product $(\cdot,\cdot)$, $H_0^1(\Omega)$ with the scalar product $((u, v)) = (\nabla u.\nabla v)$ and norm $\|u\|_1 = ((u, u))^{1/2}$. Set

$$X = H_0^1(\Omega)^d, \quad D(A) = H^2(\Omega)^d \cap X, \quad Y = L^2(\Omega)^d \ (d = 2, 3),$$
$$M = L_0^2(\Omega) = \{q \in L^2(\Omega); \int_\Omega q dx = 0\},$$
$$V = \{v \in X; \mathrm{div} v = 0\}, \quad H = \{v \in Y; \mathrm{div} v = 0, \ v \cdot n|_{\partial\Omega} = 0\}$$

and denote the Stokes operator by $\bar{A} = -P\Delta$, P is L^2-orthogonal projection of Y onto H.

Some assumptions about the prescribed data for problem (1) are needed (see [13, 14, 22]).

(A1). The initial velocity $u_0 \in H^2(\Omega)^d \cap V$ and the body force f satisfy

$$\|u_0\|_2^2 + \sup_{t\geq 0}(\|f(t)\|_0^2 + \|f_t(t)\|_0^2) \leq c.$$

The continuous bilinear forms $a(\cdot,\cdot)$ on $X\times X$ and $d(\cdot,\cdot)$ on $X\times M$ are defined by

$$a(u,v) = \nu((u,v)) = \nu(\nabla u, \nabla v), \quad d(v,q) = (q, \nabla\cdot v) \;\; \forall\, u, v \in X, \; q \in M.$$

Define the trilinear form $b(\cdot,\cdot,\cdot)$ on $X\times X\times X$ with $B(u,v) = (u\cdot\nabla)v + \frac{1}{2}(\nabla\cdot u)v$ by

$$b(u,v,w) = \langle B(u,v), w\rangle_{X'\times X} = ((u\cdot\nabla)v, w) + \frac{1}{2}((\mathrm{div} u)v, w).$$

The following properties of trilinear term can be found in [16, 19]

$$b(u,v,w) = -b(u,w,v), \quad |b(u,v,w)| \leq N\|u\|_1\|v\|_1\|w\|_1, \tag{2}$$

$$\begin{aligned}&|b(u,v,w)| + |b(w,u,v)|\\ &\leq \frac{c_0}{2}(\|u\|_0^{1/2}\|u\|_1^{1/2}\|v\|_1 + \|v\|_0^{1/2}\|v\|_1^{1/2}\|u\|_1)\|w\|_0^{1/2}\|w\|_1^{1/2},\end{aligned} \tag{3}$$

for all $u, v, w \in X$ and

$$\begin{aligned}&|b(u,v,w)| + |b(v,u,w)| + |b(w,u,v)|\\ &\leq \frac{c_0}{2}(\|u\|_1\|v\|_0^{1/2}\|v\|_2^{1/2} + \|u\|_0^{1/2}\|u\|_1^{1/2}\|v\|_1^{1/2}\|v\|_2^{1/2})\|w\|_0,\end{aligned} \tag{4}$$

for all $u \in X, \; v \in D(A), w \in Y$.

With above notations, the variational formulation of problem (1) reads as: find $(u,p) \in X\times M$, for all $(v,q) \in X\times M$ such that

$$(u_t, v) + \kappa((u_t, v)) + \mathcal{B}((u,p);(v,q)) + b(u,u,v) = (f,v), \tag{5}$$

where

$$\mathcal{B}((u,p);(v,q)) = a(u,v) - d(v,p) + d(u,q).$$

The following results are valid for small κ in 2D and for 3D with data small.

Theorem 2.1 *(see* [9, 22, 23]*) Under the assumption (A1), there exists a positive constant* $c = c(\nu, \delta_0, \Omega, \lambda_1)$ *such that for* $0 < \delta_0 < \min\{\frac{\nu\lambda_1}{2(1+\lambda_1\kappa)}, \frac{\nu}{2\kappa}\}$*, for all* $t \geq 0$*, problem (1) admits a unique solution* (u,p) *and satisfies*

$$\|u\|_0^2 + \|p\|_0^2 + e^{-\delta_0 t}\int_0^t e^{\delta_0 s}(\|u\|_2^2 + \|p\|_1^2)ds \leq c,$$

$$\|u_t\|_0^2 + \kappa\|u_t\|_1^2 + \|p\|_1^2 + e^{-\delta_0 t}\int_0^t e^{\delta_0 s}(\|u_t\|_1^2 + \kappa\|u_t\|_2^2)ds \leq c,$$

where $\lambda_1^{-1} > 0$ *satisfies* $\|v\|_0^2 \leq \lambda_1^{-1}\|\nabla v\|_0^2$*, it is the best possible constant depending on* Ω*.*

Lemma 2.2 *Under the assumptions of Theorem 2.1, for all* $t \geq 0$ *it holds*

$$\tau(t)(\|u_t\|_2 + \|u_{tt}\|_1 + \kappa\|u_{tt}\|_2) \leq c,$$

with $\tau(t) = \min\{t, 1\}$.

Proof Differentiating (5) with respect to t, one finds that

$$(u_{tt}, v)+\kappa((u_{tt}, v))+a(u_t, v)-d(v, p_t) + b(u_t, u, v) + b(u, u_t, v) = (f_t, v). \quad (6)$$

Taking $v = -\bar{A}u_{tt} \in V$ in (6), it yields

$$\|u_{tt}\|_1^2 + \kappa\|u_{tt}\|_2^2 + \frac{\nu}{2}\frac{d}{dt}\|u_t\|_2^2 + b(u_t, u, -\bar{A}u_{tt}) + b(u, u_t, -\bar{A}u_{tt}) = (f_t, -\bar{A}u_{tt}). \quad (7)$$

Thanks to (2), (3), and the Cauchy inequality, we have

$$|b(u_t, u, -\bar{A}u_{tt})|+|b(u, u_t, -\bar{A}u_{tt})| \leq c\|u_t\|_2\|u\|_1\|u_{tt}\|_2 \leq \frac{\kappa}{4}\|u_{tt}\|_2^2+c\|u_t\|_2^2\|u\|_1^2,$$

$$|(f_t, -\bar{A}u_{tt})| \leq \frac{\kappa}{4}\|u_{tt}\|_2^2 + c\|f_t\|_0^2.$$

Combining above inequalities with (7) and dropping some unnecessary terms, multiplying by $e^{\delta_0 t}\tau(t)$, noting the fact $\tau(t) \leq 1$, $\frac{d\tau(t)}{dt} \leq 1$, integrating from $t = 0$ to t, one finds

$$\frac{\nu}{2}e^{\delta_0 t}\tau(t)\|u_t\|_2^2 + \int_0^t e^{\delta_0 s}\tau(s)(\|u_{tt}\|_1^2 + \kappa\|u_{tt}\|_2^2)ds \leq c\int_0^t e^{\delta_0 s}(\|u_t\|_2^2(\|u\|_1^2 + 1 + \delta_0) + \|f_t\|_0^2)ds. \quad (8)$$

Multiplying (8) by $e^{-\delta_0 t}$, using Theorem 2.1 and the L'Hospital rule, we finish the proof.

From now on, h is a real positive parameter tending to 0, we let $\mathcal{T}_h$ be a uniformly regular mesh of $\bar{\Omega}$ made of n-simplices K with mesh size h (see [15, 24]). Based on $\mathcal{T}_h$, we introduce the finite-dimensional subspaces $(X_h, M_h) \subset X \times M$. This paper focuses on the analysis for the unstable velocity-pressure pair of the lowest equal-order elements:

$$X_h = \{v_h \in C^0(\Omega)^2 \cap X : v_h|_K \in R_1(K)^2, \ \forall K \in \mathcal{T}_h\},$$
$$M_h = \{q_h \in C^0(\Omega) \cap M : q_h|_K \in R_1(K), \ \forall K \in \mathcal{T}_h\},$$

where $R_1(K) = P_1(K)$ if K is triangular and $R_1(K) = Q_1(K)$ if K is quadrilateral.

It is well known that the lowest equal-order elements do not satisfy the discrete inf-sup condition, so we use the stabilized method to overcome this restriction and set $\Pi_h : M \to R_0 = \{q_h \in M : q_h|_K$ is a constant,$\forall K \in \mathcal{T}_h\}$ be a L^2-projection and satisfy (see [20, 25]):

$$(p, q_h) = (\Pi_h p, q_h), \quad \|\Pi_h p\|_0 \leq c\|p\|_0, \quad \forall p \in M, \quad q_h \in R_0, \quad (9)$$

$$\|p - \Pi_h p\|_0 \leq ch\|p\|_1, \quad \forall p \in H^1(\Omega) \cap M. \quad (10)$$

Define the stabilized term based on the differences of two local Gauss integrations (see[17, 20])

$$G(p,q)=(p-\Pi_h p, q-\Pi_h q). \tag{11}$$

With above notations, the stabilized finite element variational formulation of problem (5) reads as: Find $(u_h, p_h)\in X_h\times M_h$, for all $(v_h, q_h)\in X_h\times M_h$ and $t>0$, it holds

$$(u_{ht}, v_h)+\kappa((u_{ht}, v_h))+\mathcal{B}_h((u_h, p_h);(v_h, q_h))+b(u_h, u_h, v_h)=(f, v_h), \tag{12}$$

where

$$\mathcal{B}_h((u_h, p_h);(v_h, q_h))=a(u_h, v_h)-d(v_h, p_h)+d(u_h, q_h)+G(p_h, q_h),$$

is the discrete generalized bilinear form.

Now, we present some assumptions about the spaces X_h and M_h: There exist operators π_h and ρ_h such that (see [17, 20, 25])

$$\|v-\pi_h v\|_0+h\|v-\pi_h v\|_1\le ch^2\|v\|_2,\quad \forall v\in D(A), \tag{13}$$

$$\|q-\rho_h q\|_0\le ch\|q\|_1,\quad \forall q\in H^1(\Omega)\cap M. \tag{14}$$

Define the discrete Stokes operator $A_h=-P_h\Delta_h$ by

$$(A_h u_h, v_h)=(\nabla u_h, \nabla v_h),\quad \forall u_h, v_h\in X_h,$$

where the L^2-orthogonal projection operators $P_h: Y\rightarrow X_h$ and $\rho_h: M\rightarrow M_h$ are defined by

$$(P_h v, v_h)=(v, v_h)\ \ \forall v\in Y,\ \ v_h\in X_h;\quad (\rho_h q, q_h)=(q, q_h)\ \ \forall q\in M,\ \ q_h\in M_h.$$

The discrete norm $\|v_h\|_\alpha=\|A_h^{\alpha/2}v_h\|_0$ with α-order can be defined, where

$$\|v_h\|_1=\|\nabla v_h\|_0,\quad \|v_h\|_2=\|A_h v_h\|_0,\quad \|v_h\|_{-1}=\|A_h^{-1/2}v_h\|_0.$$

Denote the subspaces V_h of X_h by

$$V_h=\{v_h\in X_h; d(v_h, q_h)=0,\ \ \forall q_h\in M_h\}.$$

The following theorem establishes the continuity and coercivity for $\mathcal{B}_h((\cdot,\cdot);(\cdot,\cdot))$. □

Theorem 2.3 *(see[20, 25]) There exists a constant $\beta>0$, independent of h, such that*

$$|\mathcal{B}_h((u,p);(v,q))|\le c(\|u\|_1+\|p\|_0)(\|v\|_1+\|q\|_0),\quad \forall (u,p),(v,q)\in X\times M,$$

$$\beta(\|u_h\|_1+\|p_h\|_0)\le \sup_{(v_h,q_h)\in X_h\times M_h}\frac{|\mathcal{B}_h((u_h, p_h);(v_h, q_h))|}{\|v_h\|_1+\|q_h\|_0},\quad \forall (u_h, p_h)\in X_h\times M_h,$$

$$|G(p,q)|\le c\|p-\Pi_h p\|_0\|q-\Pi_h q\|_0,\quad \forall p,q\in M.$$

3 Unconditional stability of spatial discrete numerical solutions

In this section, we establish some stability results of numerical solutions u_h and p_h. Firstly, for all $(u, p) \in X \times M$ and $(v_h, q_h) \in X_h \times M_h$, we define the Galerkin projection $(R_h, Q_h) : X \times M \to X_h \times M_h$ by

$$\mathcal{B}_h((R_h(u, p), Q_h(u, p)); (v_h, q_h)) = \mathcal{B}((u, p); (v_h, q_h)), \tag{15}$$

which is well defined and for all $(u, p) \in D(A) \times H^1(\Omega) \cap M$, it holds (see [19, 20])

$$\|u-R_h(u, p)\|_0+h(\|u-R_h(u, p)\|_1+\|p-Q_h(u, p)\|_0) \le ch^2(\|u\|_2+\|p\|_1). \tag{16}$$

Due to $u_0 \in D(A)$, one gets $p_0 \in H^1(\Omega) \cap M$. Defining $(u_{0h}, p_{0h}) = (R_h(u_0, p_0), Q_h(u_0, p_0))$, setting $w_h = u - R_h(u, p)$ and $r_h = p - Q_h(u, p)$, with Theorem 2.1 and (15)–(16), we have

Lemma 3.1 *(see [19, 20]) Under the assumptions of Theorem 2.1 and the following uniqueness condition*

$$\nu^{-2}N\|f_\infty\|_{-1} < 1, \quad N = \sup_{u,v,w\in X} \frac{|((u\cdot\nabla)v, w)|}{\|u\|_1\|v\|_1\|w\|_1}, \tag{17}$$

where $\|f_\infty\|_{-1} = \sup\limits_{v\in X} \frac{|(f_\infty, v)|}{\|v\|_1}$, *for all* $t \ge 0$, *it holds*

$$\|w_h(t)\|_0 + h(\|w_h(t)\|_1 + \|r_h(t)\|_0) \le ch^2,$$
$$e^{-\delta_0 t}\int_0^t e^{\delta_0 s}(\|w_{ht}\|_0^2 + h^2\|w_{ht}\|_1^2 + h^2\|r_{ht}\|_0^2)ds \le ch^4.$$

Theorem 3.2 *Under the assumptions of Theorem 2.1, for all* $t \ge 0$, *it holds*

$$\|u_h(t)\|_0^2 + \kappa\|u_h(t)\|_1^2 + e^{-\delta_0 t}\int_0^t e^{\delta_0 s}(\nu\|u_h\|_1^2 + G(p_h, p_h))ds \le c, \tag{18}$$

$$\|u_h(t)\|_1^2 + \kappa\|u_h(t)\|_2^2 + e^{-\delta_0 t}\int_0^t e^{\delta_0 s}\nu\|u_h\|_2^2 ds \le c, \tag{19}$$

$$\lim_{t\to\infty} sup(\nu\|u_h(t)\|_1^2 + G(p_h, p_h)) \le \nu^{-1}\|f_\infty\|_{-1}^2. \tag{20}$$

Proof Taking $(v_h, q_h) = (u_h, p_h)$ in (12), using (2) and the Cauchy inequality, we get

$$\frac{d}{dt}(\|u_h\|_0^2 + \kappa\|u_h\|_1^2) + \nu\|u_h\|_1^2 + 2G(p_h, p_h) \le \nu^{-1}\|f\|_0^2. \tag{21}$$

Multiplying (21) by $e^{\delta_0 t}$, noting the fact $\nu\|u_h\|_1^2 \ge \nu\lambda_1\|u_h\|_0^2 \ge 2\delta_0\|u_h\|_0^2$ and the assumption (A1), integrating with respect to time from 0 to t, we obtain (18) after a multiplication by $e^{-\delta_0 t}$.

Then, taking $v_h = A_h u_h \in V_h$, $q_h = 0$ and using the similar proof of (18), we get (19).

Furthermore, choosing $(v_h, q_h) = (u_h, p_h)$ in (12) and multiplying $e^{\delta_0 t}$, one finds

$$\begin{aligned} &\frac{d}{dt}(e^{\delta_0 t}\|u_h\|_0^2 + \kappa e^{\delta_0 t}\|u_h\|_1^2) + 2e^{\delta_0 t}(\nu\|u_h\|_1^2 + G(p_h, p_h)) \\ \le\ & \delta_0 e^{\delta_0 t}(\|u_h\|_0^2 + \kappa\|u_h\|_1^2) + 2e^{\delta_0 t}(f, u_h). \end{aligned} \tag{22}$$

Integrating above inequality from 0 to t and multiplying by $e^{-\delta_0 t}$, we have

$$\begin{aligned} &\|u_h(t)\|_0^2 + \kappa\|u_h(t)\|_1^2 + 2e^{-\delta_0 t}\int_0^t e^{\delta_0 s}(\nu\|u_h(s)\|_1^2 + G(p_h(s), p_h(s)))ds \\ \le\ & e^{-\delta_0 t}\int_0^t e^{\delta_0 s}(\delta_0(\|u_h\|_0^2 + \kappa\|u_h\|_1^2) + 2(f, u_h))ds + e^{-\delta_0 t}(\|R_h(u_0, p_0)\|_0^2 \\ +\ & \kappa\|R_h(u_0, p_0)\|_1^2). \end{aligned}$$

Letting $t \to \infty$ in above inequality, using the L'Hospital rule and noting the fact that

$$\lim_{t\to\infty} 2(\nu\|u_h(t)\|_1^2 + G(p_h, p_h)) \le \lim_{t\to\infty} \nu\|u_h(t)\|_1^2 + \nu^{-1}\|f_\infty\|_{-1}^2.$$

Then, we deduce the desired result (20). □

Theorem 3.3 *Under the assumptions of Theorem 2.1, for all $t \ge 0$, u_{ht} satisfies*

$$\|u_{ht}\|_1^2 + \kappa\|u_{ht}\|_2^2 + e^{-\delta_0 t}\int_0^t e^{\delta_0 s}(\|u_{ht}\|_1^2 + \kappa\|u_{ht}\|_2^2)ds \le c.$$

Proof Taking $v_h = A_h u_{ht} \in V_h, q_h = 0$ in (12), using (3), the Cauchy inequality, multiplying by $e^{\delta_0 t}$, integrating with respect to time from 0 to t, we finish the proof by multiplying $e^{-\delta_0 t}$ and using Theorem 3.2. □

Theorem 3.4 *Under the assumptions of Theorem 2.1, for all $t \ge 0$, it holds*

$$\|u - u_h\|_0^2 + \kappa\|u - u_h\|_1^2 + e^{-\delta_0 t}\int_0^t e^{\delta_0 s}\nu\|u - u_h\|_1^2 ds \le ch^2, \tag{23}$$

$$\nu\|u_h\|_1^2 + G(p_h, p_h) + e^{-\delta_0 t}\int_0^t e^{\delta_0 s}(\|u_{ht}\|_0^2 + \kappa\|u_{ht}\|_1^2)ds \le c. \tag{24}$$

Proof Differentiating the terms $d(u_h, q_h) + G(p_h, q_h)$ in (12) with respect to the time, taking $(v_h, q_h) = (u_{ht}, p_h)$, we get

$$\begin{aligned} \|u_{ht}\|_0^2 + \kappa\|u_{ht}\|_1^2 + \frac{1}{2}\frac{d}{dt}(\nu\|u_h\|_1^2 + G(p_h, p_h)) - b(u - u_h, u, u_{ht}) \\ -b(u_h, u - u_h, u_{ht}) + b(u, u, u_{ht}) = (f, u_{ht}). \end{aligned} \tag{25}$$

Thanks to (2)–(3) and the inverse inequality, we obtain

$$|b(u-u_h,u,u_{ht})| \le ch^{-1}\|u-u_h\|_1\|u\|_1\|u_{ht}\|_0 \le \frac{1}{8}\|u_{ht}\|_0^2 + ch^{-2}\|u-u_h\|_1^2,$$
$$|b(u_h,u-u_h,u_{ht})| \le ch^{-1}\|u_h\|_1\|u-u_h\|_1\|u_{ht}\|_0 \le \frac{1}{8}\|u_{ht}\|_0^2 + ch^{-2}\|u-u_h\|_1^2,$$
$$|b(u,u,u_{ht})| \le c\|u\|_2\|u\|_1\|u_{ht}\|_0 \le \frac{1}{8}\|u_{ht}\|_0^2 + c\|u\|_2^2.$$

Combining above estimates with (25) and multiplying by $e^{\delta_0 t}$, we arrive at

$$\begin{aligned} &e^{\delta_0 t}(\|u_{ht}\|_0^2+\kappa\|u_{ht}\|_1^2)+\frac{d}{dt}(e^{\delta_0 t}(\nu\|u_h\|_1^2+G(p_h,p_h)))\\ &\le \delta_0 e^{\delta_0 t}(\nu\|u_h\|_1^2+G(p_h,p_h))+ce^{\delta_0 t}(h^{-2}\|u-u_h\|_1^2+\|u\|_2^2+\|f\|_0^2). \end{aligned} \quad (26)$$

Integrating (26) with respect to time from 0 to t, by Theorems 2.1 and 3.2, one finds

$$Y(t) \le ce^{\delta_0 t}+\nu\|u_{0h}\|_1^2+G(p_{0h},p_{0h})+ch^{-2}\int_0^t e^{\delta_0 s}\|u-u_h\|_1^2 ds,$$

where

$$Y(t)=e^{\delta_0 t}(\nu\|u_h(t)\|_1^2+G(p_h(t),p_h(t)))+\int_0^t e^{\delta_0 s}(\|u_{ht}\|_0^2+\kappa\|u_{ht}\|_1^2)ds. \quad (27)$$

From the definition of (u_{0h},p_{0h}), we have $\nu\|u_{0h}\|_1^2+G(p_{0h},p_{0h}) \le c\|u_0\|_1^2 + c\|p_0\|_0^2$, then it holds

$$Y(t) \le ce^{\delta_0 t}+ch^{-2}\int_0^t e^{\delta_0 s}\|u-u_h\|_1^2 ds.$$

Subtracting (12) from (5) with $(v,q)=(v_h,q_h)$, using the projection (R_h,Q_h), for all $(v_h,q_h)\in X_h\times M_h$, we get

$$\begin{aligned} &(u_t-u_{ht},v_h)+\kappa((u_t-u_{ht},v_h))+\mathcal{B}_h((e_h,\mu_h);(v_h,q_h))\\ &+b(u,u-u_h,v_h)+b(u-u_h,u,v_h)-b(u-u_h,u-u_h,v_h)=0 \end{aligned} \quad (28)$$

with $e_h=R_h(u,p)-u_h,\ \mu_h=Q_h(u,p)-p_h$ and $u-u_h=w_h+e_h$.

With $(v_h,q_h)=(e_h,\mu_h)$, we can rewrite (28) as

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|u-u_h\|_0^2+\kappa\|u-u_h\|_1^2)-(u_t-u_{ht},w_h)-\kappa((u_t-u_{ht},w_h))\\ &+\nu\|u-u_h\|_1^2+G(\mu_h,\mu_h)\\ =\ &2a(w_h,u-u_h)-a(w_h,w_h)-b(u,u-u_h,w_h)-b(u-u_h,u_h,w_h)\\ &+b(u-u_h,u_h,u-u_h). \end{aligned} \quad (29)$$

Thanks to (2), it is valid that

$$|b(u-u_h,u_h,w_h)|+|b(u,u-u_h,w_h)| \le N\|w_h\|_1(\|u\|_1+\|u_h\|_1)\|u-u_h\|_1,$$
$$|b(u-u_h,u_h,u-u_h)| \le N\|u_h\|_1\|u-u_h\|_1^2.$$

Combining above estimates with (28), multiplying by $e^{\delta_0 t}$ and using Theorem 2.1, we obtain

$$
\begin{aligned}
&\frac{d}{dt}(e^{\delta_0 t}(\|u-u_h\|_0^2+\kappa\|u-u_h\|_1^2))+2e^{\delta_0 t}G(\mu_h,\mu_h)\\
&+2e^{\delta_0 t}(\nu-N\|u_h\|_1)\|u-u_h\|_1^2\\
\le\ & \delta_0 e^{\delta_0 t}(\|u-u_h\|_0^2+\kappa\|u-u_h\|_1^2)+ce^{\delta_0 t}(\|u_t\|_0+\|u_{ht}\|_0)\|w_h\|_0 \qquad (30)\\
&+ce^{\delta_0 t}\|w_h\|_1(1+\|u\|_1+\|u_h\|_1)\|u-u_h\|_1+ce^{\delta_0 t}\|w_h\|_1^2\\
&+ce^{\delta_0 t}\kappa(\|u_t\|_2+\|u_{ht}\|_2)\|w_h\|_0.
\end{aligned}
$$

Integrating (30) with respect to time from 0 to t, using Theorem 2.1 and Lemma 3.1, after a final multiplication by $e^{-\delta_0 t}$, we have

$$
\begin{aligned}
&\|u-u_h\|_0^2+\kappa\|u-u_h\|_1^2+2e^{-\delta_0 t}\int_0^t e^{\delta_0 s}((\nu-N\|u_h\|_1)\|u-u_h\|_1^2\\
&+G(\mu_h,\mu_h))ds\\
\le\ & ch^2+\delta_0 e^{-\delta_0 t}\int_0^t e^{\delta_0 s}(\|u-u_h\|_0^2+\kappa\|u-u_h\|_1^2)ds\\
&+ch^2(e^{-\delta_0 t}\int_0^t e^{\delta_0 s}\kappa(\|u_t\|_2^2+\|u_{ht}\|_2^2)ds)^{1/2}\\
&+ch(e^{-\delta_0 t}\int_0^t e^{\delta_0 s}(1+\|u_h\|_1^2+\|u\|_1^2)\|u-u_h\|_1^2 ds)^{1/2}. \qquad (31)
\end{aligned}
$$

Letting $t\to\infty$ in (31), using the L'Hopital rule and Lemma 3.2, one gets

$$
\begin{aligned}
&(\nu-\nu^{-1}N\|f_\infty\|_{-1})\lim_{t\to\infty}\sup\|u-u_h\|_1^2\\
\le\ & ch\lim_{t\to\infty}\sup\|u-u_h\|_1+ch^2\lim_{t\to\infty}(e^{-\delta_0 t}\int_0^t e^{\delta_0 s}\kappa\|u_{ht}\|_2^2 ds)^{1/2}+ch^2. \qquad (32)
\end{aligned}
$$

Considering the uniqueness condition (17), we obtain

$$
\lim_{t\to\infty}\sup\|u-u_h\|_1^2\le ch^2+ch^2\lim_{t\to\infty}\sup(e^{-\delta_0 t}\int_0^t e^{\delta_0 s}\kappa\|u_{ht}\|_2^2 ds)^{1/2}. \qquad (33)
$$

Combining (27) with (33) and Theorem 3.3, using the L'Hopital rule, we find

$$
\lim_{t\to\infty}\sup\|u-u_h\|_1^2\le ch^2,\quad \lim_{t\to\infty}\sup Y(t)\le c. \qquad (34)
$$

As a consequence, using the fact $\|u\|_0\le c\|u\|_1$, we have

$$
\|u-u_h\|_0^2\le ch^2\quad \forall t\ge 0. \qquad (35)
$$

Combining (31) with (35), using Theorem 3.2, we get (23) after a multiplication by $e^{-\delta_0 t}$. □

Theorem 3.5 *Under the assumptions of Theorem 3.4, for all $t\ge 0$, it holds*

$$
\|u_{ht}\|_0^2+\kappa\|u_{ht}\|_1^2+e^{-\delta_0 t}\int_0^t e^{\delta_0 s}(\nu\|u_{ht}\|_1^2+G(p_{ht},p_{ht}))ds\le c. \qquad (36)
$$

Proof Differentiating (12) with respect to t, taking $(v_h, q_h) = (u_{ht}, p_{ht})$, using (3), the Cauchy inequality, integrating with respect to the time from 0 to t, multiplying by $e^{-\delta_0 t}$ and using Theorems 3.2 and 3.3, we complete the proof. □

Theorem 3.6 . *Under the assumptions of Theorem 3.4, for all* $t \geq 0$*, it holds*

$$\nu\|u_{ht}\|_1^2 + G(p_{ht}, p_{ht}) + e^{-\delta_0 t}\int_0^t e^{\delta_0 s}(\|u_{htt}\|_0^2 + \kappa\|u_{htt}\|_1^2)ds \leq c. \tag{37}$$

Proof Differentiating (12) with respect to t, then differentiating $d(u_{ht}, q_h) + G(p_{ht}, p_{ht})$ with respect to time again, taking $(v_h, q_h) = (u_{htt}, p_{ht})$, thanks to (2)–(4), the Cauchy inequality, integrating with respect to time from 0 to t, multiplying by $e^{-\delta_0 t}$, applying Theorems 3.3, 3.4, and 3.5, we finish the proof. □

Theorem 3.7 *Under the assumptions of Theorem 3.4, for all* $t \geq 0$*, it holds*

$$\tau(t)(\|u_{ht}\|_2 + \|u_{htt}\|_1 + \kappa\|u_{htt}\|_2) \leq c.$$

Proof Following the proof of Lemma 2.2, we derive the desired results. □

4 Error estimates of spatial discrete numerical approximations

This section is devoted to present the error estimates of the numerical solutions u_h and p_h in various norms. The main results of this section are the following theorem.

Theorem 4.1 *Under the assumptions of Theorem 3.4, for all* $t \geq 0$*, it holds*

$$\|u - u_h\|_1 + \tau^{1/2}(t)(\|p - p_h\|_0 + \|u_t - u_{ht}\|_0) \leq ch.$$

Proof The proof of the Theorem 4.1 consists of Lemmas 4.2, 4.3, and 4.4. □

Lemma 4.2 . *Under the assumptions of Theorem 3.4, for* $t \geq 0$*, it holds*

$$\|u - u_h\|_1^2 + e^{-\delta_0 t}\int_0^t e^{\delta_0 s}(\|u_t - u_{ht}\|_0^2 + \kappa\|u_t - u_{ht}\|_1^2)ds \leq ch^2.$$

Proof Differentiating the terms $d(e_h, q) + G(\mu_h, q_h)$ with respect to time t in (28) and taking $(v_h, q_h) = (e_{ht}, \mu_h)$, one deduces that

$$\begin{aligned}
&\|u_t - u_{ht}\|_0^2 - (u_t - u_{ht}, w_{ht}) + \kappa\|u_t - u_{ht}\|_1^2 - \kappa((u_t - u_{ht}, w_{ht}))\\
&+b(u - u_h, u, e_{ht})\\
&+\frac{1}{2}\frac{d}{dt}(\nu\|e_h\|_1^2 + G(\mu_h, \mu_h)) + b(u, u - u_h, e_{ht})\\
&-b(u - u_h, u - u_h, e_{ht}) = 0.
\end{aligned} \tag{38}$$

Thanks to (2)–(4) and the inverse inequality, we have

$$\begin{aligned}
&|b(u-u_h,u,e_{ht})|+|b(u,u-u_h,e_{ht})|\le c\|u\|_2\|u-u_h\|_1(\|u_t-u_{ht}\|_0+\|w_{ht}\|_0)\\
\le\ &\frac{1}{8}\|u_t-u_{ht}\|_0^2+c\|u\|_2^2\|u-u_h\|_1^2+c\|u\|_2(\|u\|_1+\|u_h\|_1)\|w_{ht}\|_0,\\
&|b(u-u_h,u-u_h,e_{ht})|\le ch^{-1/2}\|u-u_h\|_0^{1/2}\|u-u_h\|_1^{3/2}(\|u_t-u_{ht}\|_0+\|w_{ht}\|_0)\\
\le\ &\frac{1}{8}\|u_t-u_{ht}\|_0^2+ch^{-1}\|u-u_h\|_0\|u-u_h\|_1^3+ch^{-1/2}\|u-u_h\|_0^{1/2}\|u-u_h\|_1^{3/2}\|w_{ht}\|_0.
\end{aligned}$$

Combining above estimates with (38), we obtain

$$\begin{aligned}
&\|u_t-u_{ht}\|_0^2+\kappa\|u_t-u_{ht}\|_1^2+\frac{d}{dt}(\nu\|e_h\|_1^2+G(\mu_h,\mu_h))\\
\le\ &2(\|u_t\|_0+\|u_{ht}\|_0)\|w_{ht}\|_0+2\kappa(\|u_t\|_2+\|u_{ht}\|_2)\|w_{ht}\|_0\\
&+c\|u\|_2(\|u\|_1+\|u_h\|_1)\|w_{ht}\|_0\\
&+c\|u\|_2^2\|u-u_h\|_1^2+ch^{-1}\|u-u_h\|_0\|u-u_h\|_1^3\\
&+ch^{-1/2}\|u-u_h\|_0^{1/2}\|u-u_h\|_1^{3/2}\|w_{ht}\|_0.
\end{aligned}$$

Multiplying with $e^{\delta_0 t}$, integrating with respect to the time from 0 to t and noting the fact

$$\|u-u_h\|_1^2\le\|w_h\|_1^2+\|e_h\|_1^2,$$

then one finds by applying Lemma 3.1, Theorems 2.1, 3.4 and multiplying by $e^{-\delta_0 t}$

$$\begin{aligned}
&\nu\|u-u_h\|_1^2+G(\mu_h,\mu_h)+e^{-\delta_0 t}\int_0^t e^{\delta_0 s}(\|u_t-u_{ht}\|_0^2+\kappa\|u_t-u_{ht}\|_1^2)ds\\
\le\ &ch^2(e^{-\delta_0 t}\int_0^t e^{\delta_0 s}(\|u_t\|_0^2+\|u_{ht}\|_0^2+\|u\|_2^2(\|u\|_0+\|u_h\|_0)^2)ds)^{1/2}\\
&+ch^2(e^{-\delta_0 t}\int_0^t e^{\delta_0 s}\kappa(\|u_t\|_2^2+\|u_{ht}\|_2^2)ds)^{1/2}\\
&+ch^{-1}e^{-\delta_0 t}\int_0^t e^{\delta_0 s}\|u-u_h\|_0\|u-u_h\|_1^3 ds\\
&+ch^2(e^{-\delta_0 t}\int_0^t e^{\delta_0 s}h^{-1}\|u-u_h\|_0\|u-u_h\|_1^3 ds)^{1/2}+ch^2.
\end{aligned}\tag{39}$$

Thanks to Theorems 2.1, 3.3, and 3.4, we finish the proof of Lemma 4.2. □

Lemma 4.3 *Under the assumptions of Theorem 3.4, for $t\ge 0$, it holds*

$$\begin{aligned}
\tau(t)(\|u_t-u_{ht}\|_0^2+\kappa\|u_t-u_{ht}\|_1^2)+e^{-\delta_0 t}\int_0^t e^{\delta_0 s}\tau(t)(\nu\|u_t-u_{ht}\|_1^2\\
+G(\mu_{ht},\mu_{ht}))ds\le ch^2.
\end{aligned}$$

Proof Differentiating (28) with respect to t, taking $(v_h, q_h) = (e_{ht}, \mu_{ht})$, using (3)–(4), the Cauchy inequality, multiplying by $e^{\delta_0 t}\tau(t)$, integrating with respect to t, applying Theorems 2.1, 3.4 and Lemmas 3.1, 4.2, we have by multiplying $e^{-\delta_0 t}$

$$\begin{aligned}
&\tau(t)(\|u_t - u_{ht}\|_0^2 + \kappa\|u_t - u_{ht}\|_1^2) + e^{-\delta_0 t}\int_0^t e^{\delta_0 s}\tau(t)(\nu\|e_{ht}\|_1^2 + G(\mu_{ht}, \mu_{ht}))ds \\
\leq\ & ce^{-\delta_0 t}\int_0^t e^{\delta_0 s}\tau(t)(1 + \|u_h\|_1^2)\|u_t - u_{ht}\|_0^2 ds \\
&+c\sup_{0\leq s\leq t}(\|u(s) - u_h(s)\|_0^2 + h^2\|u - u_h\|_1^2) \\
&+ce^{-\delta_0 t}\int_0^t e^{\delta_0 s}\tau(t)\kappa\|u_t - u_{ht}\|_1^2 ds + ch^2(e^{-\delta_0 t}\int_0^t e^{\delta_0 s}\tau(t)\|u_{htt}\|_0^2 ds)^{1/2} \\
&+ch^2(e^{-\delta_0 t}\int_0^t e^{\delta_0 s}\tau(t)\kappa(\|u_{tt}\|_2^2 + \|u_{htt}\|_2^2)ds)^{1/2} + ch^2.
\end{aligned} \tag{40}$$

Thanks to Lemma 2.2, the triangle inequality, Theorem 3.6, and the fact $e^{-\delta_0 t}\int_0^t e^{\delta_0 s}\|w_{ht}\|_1^2 ds \leq ch^2$, we deduce that

$$\begin{aligned}
&\tau(t)(\|u_t - u_{ht}\|_0^2 + \kappa\|u_t - u_{ht}\|_1^2) \\
&+e^{-\delta_0 t}\int_0^t e^{\delta_0 s}\tau(t)(\nu\|u_t - u_{ht}\|_1^2 + G(\mu_{ht}, \mu_{ht}))ds \\
\leq\ & ce^{-\delta_0 t}\int_0^t e^{\delta_0 s}\tau(t)(1 + \|u_h\|_1^2)\|u_t - u_{ht}\|_0^2 ds + ce^{-\delta_0 t}\int_0^t e^{\delta_0 s}\tau(t)\kappa\|u_t - u_{ht}\|_1^2 ds \\
&+ch^2 + c\sup_{0\leq s\leq t}(\|u(s) - u_h(s)\|_0^2 + h^2\|u - u_h\|_1^2).
\end{aligned}$$

Combining above inequality with Theorem 3.2 and Lemma 4.2, we complete the proof. □

Lemma 4.4 *Under the assumptions of Theorem 3.4, for all* $t \geq 0$, *it holds*

$$\tau^{1/2}(t)\|p - p_h\|_0 \leq ch.$$

Proof The inf-sup condition (15) and (28) guarantee that

$$\begin{aligned}
\beta\|\mu_h\|_0 &\leq \sup_{(v_h,q_h)\in(X_h,M_h)} \frac{\mathcal{B}_h((e_h, \mu_h); (v_h, q_h))}{\|v_h\|_1 + \|q_h\|_0} \\
&\leq c\|u_t - u_{ht}\|_0 + c\kappa\|u_t - u_{ht}\|_1 + c(\|u\|_1 + \|u_h\|_1)\|u - u_h\|_1.
\end{aligned}$$

Combining above estimate with Lemma 3.1 and the triangle inequality, one finds

$$\begin{aligned}
\tau^{1/2}(t)\|p - p_h\|_0 \leq ch + c\tau^{1/2}(t)(\|u_t - u_{ht}\|_0 + \kappa\|u_t - u_{ht}\|_1) \\
+c(1 + \|u_h\|_1)\|u - u_h\|_1.
\end{aligned} \tag{41}$$

We finish the proof by combining (41), Theorem 3.2, Lemmas 4.2 and 4.3. □

5 Optimal L^2-norm error estimates of velocity in spatial discrete scheme

This section is devoted to establish the optimal L^2-norm error estimates of numerical solution u_h in stabilized finite element method (12).

Theorem 5.1 *Under the assumptions of Theorem 3.4, for all $t \geq 0$, it holds*

$$\|u-u_h\|_0 + h(\|u-u_h\|_1 + \tau^{1/2}(t)\|p-p_h\|_0) \leq ch^2.$$

Proof The proof of Theorem 5.1 consists of Theorems 2.1, 3.2, 4.1 and Lemma 5.5.

In order to get the convergence of $e(t) = u - u_h$ in L^2-norm, we begin with a technical result concerning a linear Kelvin-Voigt problem, we can refer to this technique from Heywood & Rannacher [26] and Hill & Süli [27] for the linear Navier-Stokes equations.

For any given $s > 0$ and $g = e^{\delta_0 t} e(t) \in L^2(0, s; L^2(\Omega)^2)$, consider the following problem: Find $(\Phi, \Psi) \in X \times M$ such that for $0 < t < s$

$$\begin{aligned}&(v, \Phi_t) + \kappa((v, \Phi_t)) - a(v, \Phi) - d(v, \Psi)\\&+ d(\Phi, q) - b(u, v, \Phi) - b(v, u, \Phi) = (v, g),\end{aligned} \tag{42}$$

for all $(v, q) \in X \times M$ with $\Phi(s) = 0$.

Since $u_0 \in V$ and $f, f_t \in L^2(R^+, Y)$, Theorem 2.1 ensures that u is sufficiently smooth so that (Φ, Ψ) is correctly defined for all $0 < t \leq s$. Thus for every $\sigma \in (0, s)$, (42) is a well-posed problem and has a unique solution (Φ, Ψ) with

$$\Phi \in C(0, s; V) \cap L^2(\sigma, s; D(A)) \cap H^1(\sigma, s; Y), \Psi \in L^2(\sigma, s; H^1(\Omega) \cap M).$$

This following result can be obtained by the similar methods used by [27, 28]. □

Lemma 5.2 *Let (Φ, Ψ) be the solution of problem (42) with $g = e^{\delta_0 t} e(t)$, it holds*

$$\sup_{0 \leq t \leq s} e^{-\delta_0 s}\|\Phi(s)\|_1^2 + \int_0^s e^{-\delta_0 t}(\|\Phi\|_2^2 + \|\Phi_t\|_0^2 + \|\Psi\|_1^2)dt \leq ce^{cs}\int_0^s e^{\delta_0 s}\|e(t)\|_0^2 dt.$$

Lemma 5.3 *Under the assumptions of Theorem 3.4, the error $u - u_h$ satisfies*

$$e^{-\delta_0 s}\int_0^s e^{\delta_0 t}\|u(t) - u_h(t)\|_0^2 dt \leq ch^4, \quad \text{for all } s > 0.$$

Proof Given $g = e^{\delta_0 t} e(t) \in L^2(0, s; Y)$, let $(\Phi, \Psi) \in X \times M$ be the solution of problem (42). Taking $(v, q) = (e(t), p - p_h)$ in (42), we have

$$\begin{aligned}e^{\delta_0 t}\|e(t)\|_0^2 = {}& \frac{d}{dt}((e, \Phi) + \kappa((e, \Phi))) - (e_t, \Phi) - \kappa((e_t, \Phi))\\&- \mathcal{B}_h((e, p-p_h); (\Phi, \Psi)) - b(u, e, \Phi) - b(e, u, \Phi) + G(p-p_h, \Psi).\end{aligned} \tag{43}$$

For all $0 < t < s$, we consider the dual Galerkin projection $(\Phi_h, \Psi_h) \in X_h \times M_h$

$$\mathcal{B}_h((v_h, q_h); (\Phi - \Phi_h, \Psi - \Psi_h)) = G(q_h, \Psi), \quad \forall (v_h, q_h) \in X_h \times M_h. \tag{44}$$

By a similar approach to the proof of Lemma 3.1, we get

$$\|\Phi(t) - \Phi_h(t)\|_0 + h\|\Phi(t) - \Phi_h(t)\|_1 + h\|\Psi(t) - \Psi_h(t)\|_0 \le ch^2(\|\Phi(t)\|_2 + \|\Psi(t)\|_1). \tag{45}$$

Let us recall the error identity (28) with $(v_h, q_h) = (\Phi_h, \Psi_h)$, we have

$$(e_t, \Phi_h) + \kappa((e_t, \Phi_h)) + \mathcal{B}_h((e, p - p_h); (\Phi_h, \Psi_h)) + b(u, e, \Phi_h) + b(e, u, \Phi_h) - b(e, e, \Phi_h) = 0. \tag{46}$$

Adding (46) to (43) and using the relationship (44) with $(v_h, q_h) = (e_h, \mu_h)$, where $e_h = R_h(u, p) - u_h$, $\mu_h = Q_h(u, p) - p_h$ and $u - u_h = w_h + e_h$, one finds

$$\begin{aligned} e^{\delta_0 t}\|e\|_0^2 = & \frac{d}{dt}((e, \Phi) + \kappa((e, \Phi))) - (e_t, \Phi - \Phi_h) - \kappa((e_t, \Phi - \Phi_h)) \\ & - \mathcal{B}_h((w_h, r_h); (\Phi - \Phi_h, \Psi - \Psi_h)) + b(u, e, \Phi - \Phi_h) \\ & + b(e, u, \Phi - \Phi_h) + b(e, e, \Phi - \Phi_h) - b(e, e, \Phi) + G(p - Q_h, \Psi). \end{aligned} \tag{47}$$

By (2), (11), (13), (14), (16), and (45), we deduce that

$$\begin{aligned} e^{\delta_0 t}\|e\|_0^2 \le & \frac{d}{dt}(((e, \Phi) + \kappa(\nabla e, \nabla\Phi))) + c\|e_t\|_0\|\Phi - \Phi_h\|_0 \\ & + c\kappa\|e_t\|_2\|\Phi - \Phi_h\|_0 + ch^2\|\Psi\|_1 \\ & + c(\|u\|_1\|e\|_1 + \|w_h\|_1 + \|r_h\|_0)(\|\Phi - \Phi_h\|_1 + \|\Psi - \Psi_h\|_0) + c\|e\|_1^2\|\Phi\|_1 \\ \le & \frac{d}{dt}((e, \Phi) + \kappa(\nabla e, \nabla\Phi)) + ch^2\|e_t\|_0\|\Phi\|_2 + c\|e\|_1^2\|\Phi\|_1 + ch^2\|\Psi\|_1 \\ & + ch(\|u\|_1\|e\|_1 + \|w_h\|_1 + \|r_h\|_0)(\|\Phi\|_2 + \|\Psi\|_1) + c\kappa h^2\|e_t\|_2\|\Phi\|_2. \end{aligned} \tag{48}$$

Integrating (48) about t from 0 to s, using Theorems 2.1, 3.2, 3.3 and Lemma 3.1, we have

$$\begin{aligned} \int_0^s e^{\delta_0 t}\|e\|_0^2 dt \le & -(e(0), \Phi(0)) - \kappa(\nabla e(0), \nabla\Phi(0)) + ch^2 e^{cs} \sup_{0\le t\le s} e^{-\frac{1}{2}\delta_0 t}\|\Phi\|_1 \\ & + ch^2 e^{cs}\Big(\int_0^s e^{-\delta_0 t}(\|\Phi\|_2^2 + \|\Psi\|_1^2)dt\Big)^{1/2}. \end{aligned} \tag{49}$$

Furthermore, we have

$$\begin{aligned} |(e(0), \Phi(0))| &= |(u_0 - R_h(u_0, p_0), \Phi(0) - R_h(\Phi(0), \Psi(0)))| \\ &\le ch^2\|u_0\|_1\|\Phi(0)\|_1 \le ch^2\Big(\int_0^s e^{\delta_0 t}\|e(t)\|_0^2 dt\Big)^{1/2}, \end{aligned} \tag{50}$$

$$\begin{aligned} \kappa|(\nabla e(0), \nabla\Phi(0))| &= \kappa|(\nabla(u_0 - R_h(u_0, p_0)), \nabla(\Phi(0) - R_h(\Phi(0), \Psi(0))))| \\ &\le c\kappa h^2\|u_0\|_2\|\Phi(0)\|_2 \le c\kappa h^2\Big(\int_0^s e^{\delta_0 t}\|e(t)\|_0^2 dt\Big)^{1/2}. \end{aligned} \tag{51}$$

Combining Lemma 5.2 with (49)–(51), we finish the proof. □

Lemma 5.4 *Under the assumptions of Theorem 3.4, the error $e(t) = u - u_h$ satisfies*

$$(\|u-u_h\|_{-1}^2 + \kappa\|u-u_h\|_0^2) + e^{-\delta_0 t}\int_0^t e^{\delta_0 s}(\nu\|e_h\|_0^2 + G(\mu_h, A_h^{-1}\mu_h))ds$$
$$\leq ch^4 + ce^{-\delta_0 t}\int_0^t e^{\delta_0 s}\|u-u_h\|_0^2 ds + ce^{-\delta_0 t}\int_0^t e^{\delta_0 s}\|u-u_h\|_1^4 ds.$$

Proof Recalling the error equality (28) with $v_h = A_h^{-1}e_h \in V_h, q_h = 0$ and noting the fact that $(u_t - u_{ht}, A_h^{-1}e_h) = (e_{ht}, A_h^{-1}e_h) + (w_{ht}, A_h^{-1}(u-u_h-w_h))$, we obtain

$$\frac{1}{2}\frac{d}{dt}(\|e_h\|_{-1}^2 + \kappa\|e_h\|_0^2 - \|w_h\|_{-1}^2 - \kappa\|w_h\|_0^2) + \nu\|e_h\|_0^2$$
$$\leq |(w_{ht}, A_h^{-1}(u-u_h))| + |\kappa((w_{ht}, A_h^{-1}(u-u_h)))| + |b(u, u-u_h, A_h^{-1}e_h)$$
$$+b(u-u_h, u, A_h^{-1}e_h) - b(u-u_h, u-u_h, A_h^{-1}e_h)|. \quad (52)$$

Thanks to the inverse inequality, (2), and Theorem 2.1, we have

$$|b(u-u_h, u, A_h^{-1}e_h)| + |b(u, u-u_h, A_h^{-1}e_h)| \leq \frac{\nu}{8}\|e_h\|_{-1}^2 + c\|u\|_2^2\|u-u_h\|_0^2,$$
$$|b(u-u_h, u-u_h, A_h^{-1}e_h)| \leq c\|u-u_h\|_1^2\|e_h\|_{-1} \leq \frac{\nu}{8}\|e_h\|_{-1}^2 + c\|u-u_h\|_1^4.$$

Combining above estimates with (52), multiplying with $e^{\delta_0 t}$, integrating for t from 0 to t, using the triangle inequality and Lemma 3.1, we obtain

$$e^{\delta_0 t}(\|u-u_h\|_{-1}^2 + \kappa\|u-u_h\|_0^2) + \nu\int_0^t e^{\delta_0 s}\|e_h\|_0^2 ds$$
$$\leq ce^{\delta_0 t}h^4 + c\int_0^t e^{\delta_0 s}\|u-u_h\|_0^2 ds + c\int_0^t e^{\delta_0 s}\|u-u_h\|_1^4 ds. \quad (53)$$

Multiplying (53) by $e^{-\delta_0 t}$, we complete the proof of Lemma 5.4. □

Lemma 5.5 *Under the assumptions of Theorem 3.4, for all $t \geq 0$, it holds*

$$\|u(t) - u_h(t)\|_0^2 \leq ch^4. \quad (54)$$

Proof Choosing $v_h = A_h^{-1}e_h \in V_h, q_h = 0$ in (28) and using (2), one derives that

$$\frac{d}{dt}(\|u-u_h\|_{-1}^2 + \kappa\|u-u_h\|_0^2) - 2(u_t - u_{ht}, A_h^{-1}w_h)$$
$$-2\kappa((u_t - u_{ht}, A_h^{-1}w_h)) + 2\nu\|e_h\|_0^2$$
$$+2b(u-u_h, u, A_h^{-1}e_h) + 2b(u_h, e_h, A_h^{-1}e_h) + 2b(u_h, w_h, A_h^{-1}e_h) = 0. \quad (55)$$

For the trilinear terms of (55), using (2)–(4), we arrive at

$$|b(u_h, w_h, A_h^{-1}e_h)| \leq c\|u\|_1\|e_h\|_0\|w_h\|_0, \quad |b(u_h, e_h, A_h^{-1}e_h)| \leq N\|u_h\|_1\|e_h\|_0^2,$$
$$|b(u-u_h, u, A_h^{-1}e_h)| \leq c\|u\|_1\|u-u_h\|_0\|e_h\|_0.$$

Combining above estimates with (55), integrating for t from 0 to s, using Theorem 2.1 and Lemmas 3.1, 4.2, 5.3, and 5.4, after multiplying by $e^{-\delta_0 s}$, we have

$$\begin{aligned}
&(\|u-u_h\|_{-1}^2+\kappa\|u-u_h\|_0^2)+2e^{-\delta_0 s}\int_0^s e^{\delta_0 t}(\nu-N\|u_h\|_1)\|e_h\|_0^2dt\\
\le\ & ch^2(e^{-\delta_0 s}\int_0^s e^{\delta_0 t}\kappa\|u_t-u_{ht}\|_0^2dt)^{1/2}\\
&+\delta_0 e^{-\delta_0 s}\int_0^s e^{\delta_0 t}(\|u-u_h\|_{-1}^2+\kappa\|u-u_h\|_0^2)dt\\
&+ch^2(e^{-\delta_0 s}\int_0^s e^{\delta_0 t}\|e_h\|_0)^2dt)^{1/2}+ch^2(e^{-\delta_0 s}\int_0^s e^{\delta_0 t}\|u_t-u_{ht}\|_{-1}^2dt)^{1/2}. \quad (56)
\end{aligned}$$

Setting

$$Z=\lim_{s\to\infty}\sup\|e_h(s)\|_0^2,\quad Y=\lim_{s\to\infty}\sup e^{-\delta_0 s}\int_0^s e^{\delta_0 t}(\|u_t-u_{ht}\|_{-1}^2+\kappa\|u_t-u_{ht}\|_0^2)dt.$$

Letting $s\to\infty$ in (56), applying the L'Hospital Rule, Lemma 4.2, and (17), one finds

$$Z\le ch^4+ch^2Y^{1/2}. \quad (57)$$

Taking $v_h=A^{-1}e_{ht}\in V_h, q_h=0$ in (28), one deduces that

$$\begin{aligned}
&\|u_t-u_{ht}\|_{-1}^2+\|e_{ht}\|_{-1}^2-\|w_{ht}\|_{-1}^2+\kappa\|u_t-u_{ht}\|_0^2+\kappa\|e_{ht}\|_0^2-\kappa\|w_{ht}\|_0^2\\
&+\frac{d}{dt}\nu\|e_h\|_0^2+2\frac{d}{dt}(b(w_h,u,A_h^{-1}e_h)+b(u,w_h,A_h^{-1}e_h))\\
&+2b(e_h,u,A_h^{-1}e_{ht})+2b(u,e_h,A_h^{-1}e_{ht})-2b(w_{ht},u,A_h^{-1}e_h)\\
&-2b(u,w_{ht},A_h^{-1}e_h)\\
&-2b(u-u_h,u-u_h,A_h^{-1}e_{ht})-2b(w_h,u_t,A_h^{-1}e_h)\\
&-2b(u_t,w_h,A_h^{-1}e_h)=0. \quad (58)
\end{aligned}$$

Thanks to (3)–(4), the inverse inequality, and Theorems 2.1 and 4.1, we have

$$\begin{aligned}
|b(e_h,u,A_h^{-1}e_{ht})|+|b(u,e_h,A_h^{-1}e_{ht})| &\le \frac{1}{8}\|e_{ht}\|_{-1}^2+c\|u\|_2^2\|e_h\|_0^2,\\
|b(u-u_h,u-u_h,A_h^{-1}e_{ht})| &\le \frac{\kappa}{8}\|e_{ht}\|_0^2+ch^2(\|e_h\|_0+\|w_h\|_0),\\
|b(w_{ht},u,A_h^{-1}e_h)|+|b(u,w_{ht},A_h^{-1}e_h)| &\le \frac{\nu}{8}\|e_h\|_0^2+c\|u\|_1^2\|w_{ht}\|_0^2,\\
|b(w_h,u_t,A_h^{-1}e_h)|+|b(u_t,w_h,A_h^{-1}e_h)| &\le \frac{\nu}{8}\|e_h\|_0^2+c\|u_t\|_1^2\|w_h\|_0^2.
\end{aligned}$$

Combining above estimates with (58), multiplying by $e^{\delta_0 t}$, integrating from 0 to s, applying Theorem 2.1 and Lemma 3.1, we have by a final multiplication $e^{-\delta_0 s}$,

$$\begin{aligned} &\nu\|e_h\|_0^2 + 2(b(w_h, u, A_h^{-1}e_h) + b(u, w_h, A_h^{-1}e_h)) \\ &+e^{-\delta_0 s}\int_0^s e^{\delta_0 t}(\|u_t - u_{ht}\|_{-1}^2 + \kappa\|u_t - u_{ht}\|_0^2)dt \\ \le\ & 2e^{-\delta_0 s}(b(w_h(0), u_0, A_h^{-1}e_h(0)) + b(u_0, w_h(0), A_h^{-1}e_h(0))) + 2e^{-\delta_0 s}\nu\|e_h(0)\|_0^2 \\ &+ch^4 + ce^{-\delta_0 t}\int_0^s e^{\delta_0 t}\|e_h\|_0^2 dt. \end{aligned} \tag{59}$$

Due to (2), Theorem 2.1, and Lemma 3.1, we know that for all $s > 0$

$$|b(w_h(s), u(s), A_h^{-1}e_h(s))| + |b(u(s), w_h(s), A_h^{-1}e_h(s))| \le \frac{\nu}{4}\|e_h\|_0^2 + ch^4.$$

Combining above estimate with (59), we arrive at

$$e^{-\delta_0 t}\int_0^s e^{\delta_0 t}(\|u_t - u_{ht}\|_{-1}^2 + \kappa\|u_t - u_{ht}\|_0^2)ds \le ch^4 + ce^{-\delta_0 t}\int_0^s e^{\delta_0 t}\|e_h\|_0^2 dt. \tag{60}$$

Setting $s \to \infty$ in (60) and using the L'Hospital rule, it holds that

$$Y \le ch^4 + cZ. \tag{61}$$

Using (57) with (61) that $Z \le ch^4$. Furthermore, by Lemma 3.1, we have

$$\lim_{s\to\infty}\sup\|u(s) - u_h(s)\|_0^2 \le 2\lim_{s\to\infty}\sup\|w_h(s)\|_0^2 + cZ \le ch^4.$$

Then, we finish the proof. □

6 Fully discrete stabilized finite element method

In this section, we take the time step Δt and denote the discrete times $t_n = n\Delta t,\ n = 0, 1, \cdots$. Then, the fully discrete stabilized FEM for the Kelvin-Voigt problem (5) reads as: For all $n \ge 1$, find $(u_h^n, p_h^n) \in X_h \times M_h$ such that

$$(d_t u_h^n, v_h) + \kappa((d_t u_h^n, v_h)) + \mathcal{B}_h((u_h^n, p_h^n); (v_h, q_h)) + b(u_h^{n-1}, u_h^n, v_h) = (f^n, v_h), \tag{62}$$

for all $(v_h, q_h) \in X_h \times M_h$ with $u_h^0 = u_{0h}$, where $d_t u_h^n = \frac{1}{\Delta t}(u_h^n - u_h^{n-1}),\quad f^n = f(t_n)$.

Based on Theorem 2.3 and the classical Lax-Milgram theorem, we know that problem (62) admits a unique solution. By choosing different test functions $(v_h, q_h) = (u_h^n, p_h^n)$ and $v_h = A_h u_h^n \in V_h, q_h = 0$ in (62) with energy method, we can establish the following stability results of numerical solutions. Here we omit these proof due to the standard process.

Theorem 6.1 *Under the assumptions of Theorem 3.4, there exists a positive constant $c = c(\nu, \Omega, \lambda_1)$ such that for all $m \geq 1$, it holds*

$$\|u_h^m\|_0^2 + \kappa\|\nabla u_h^n\|_0^2 + \nu\|\nabla u_h^n\|_0^2 + G(p_h^n, p_h^n) \leq \|u_h^0\|_0 + \kappa\|\nabla u_h^0\|_0^2 + c\Delta t\sum_{n=1}^{m}\|f^n\|_0^2,$$

$$\|\nabla u_h^m\|_0^2 + \kappa\|A_h u_h^n\|_0^2 + \nu\|A_h u_h^n\|_0^2 \leq \|\nabla u_h^0\|_0 + \kappa\|A_h u_h^0\|_0^2 + c\Delta t\sum_{n=1}^{m}\|f^n\|_0^2.$$

In order to analyze the discretization errors $e_h^n = u_h(t_n) - u_h^n$ and $\mu_h^n = p_h(t_n) - p_h^n$, for all $(v_h, q_h) \in X_h \times M_h$, we discrete (12) at nth time level, it holds

$$\begin{aligned}(d_t u_h(t_n), v_h) + \kappa((d_t u_h(t_n), v_h)) + \mathcal{B}_h((u_h(t_n), p_h(t_n)); (v_h, q_h))\\ +b(u_h(t_n), u_h(t_n), v_h) = (f^n, v_h) + (E_n, v_h),\end{aligned} \tag{63}$$

with

$$E_n = \frac{1}{\Delta t}\int_{t_{n-1}}^{t_n}(t - t_{n-1})u_{htt}dt - \frac{\kappa}{\Delta t}\int_{t_{n-1}}^{t_n}(t - t_{n-1})\Delta u_{htt}dt. \tag{64}$$

Subtracting (62) from (63), we obtain with $(e_h^0, \mu_h^0) = (0, 0)$

$$\begin{aligned}(d_t e_h^n, v_h) + \kappa((d_t e_h^n, v_h)) + \mathcal{B}_h((e_h^n, \mu_h^n); (v_h, q_h)) + \Delta t b(u_t, u(t_n), v_h)\\ +b(e_h^{n-1}, u_h(t_n), v_h) + b(u_h^{n-1}, e_h^n, v_h) = (E_n, v_h).\end{aligned} \tag{65}$$

Lemma 6.2 *Under the assumptions of Theorem 6.1, it holds*

$$\|e_h^n\|_0^2 + \kappa\|e_h^n\|_1^2 + \Delta t\sum_{n=1}^{m}(\|e_h^n\|_1^2 + G(\mu_h^n, \mu_h^n)) \leq c\Delta t^2, \quad \forall\, m \geq 1.$$

Proof Taking $(v_h, q_h) = (e_h^n, \mu_h^n)$ in (65) and using (2), we have

$$\begin{aligned}&\frac{1}{2\Delta t}(\|e_h^n\|_0^2 - \|e_h^{n-1}\|_0^2 + \|e_h^n - e_h^{n-1}\|_0^2) + \nu\|e_h^n\|_1^2 + G(\mu_h^n, \mu_h^n) + b(e_h^{n-1}, u_h(t_n), e_h^n)\\ &+\frac{\kappa}{2\Delta t}(\|e_h^n\|_1^2 - \|e_h^{n-1}\|_1^2 + \|e_h^n - e_h^{n-1}\|_1^2) + \Delta t b(u_t, u(t_n), e_h^n) = (E_n, e_h^n).\end{aligned} \tag{66}$$

For the trilinear terms and the right-hand side term, by (3) and Theorem 3.2, one finds

$$\begin{aligned}|b(e_h^{n-1}, u_h(t_n), e_h^n)| &\leq c\|e_h^{n-1}\|_0\|A_h u_h(t_n)\|_0\|e_h^n\|_1 \leq \frac{\nu}{4}\|e_h^n\|_1^2 + c\|A_h u_h(t_n)\|_0^2\|e_h^{n-1}\|_0^2,\\ |\Delta t b(u_t, u(t_n), e_h^n)| &\leq c\Delta t\|u_t\|_0\|Au(t_n)\|_0\|e_h^n\|_1 \leq \frac{\nu}{4}\|e_h^n\|_1^2 + c\Delta t^2\|Au(t_n)\|_0^2\|u_t\|_0^2,\\ (E_n, e_h^n) &= \frac{1}{\Delta t}\int_{t_{n-1}}^{t_n}(t - t_{n-1})(u_{htt}, e_h^n)dt - \frac{\kappa}{\Delta t}\int_{t_{n-1}}^{t_n}(t - t_{n-1})(\nabla u_{htt}, \nabla e_h^n)dt\\ &\leq \frac{\nu}{4}\|e_h^n\|_1^2 + \Delta t\int_{t_{n-1}}^{t_n}(\|u_{htt}\|_0^2 + \kappa\|\nabla u_{htt}\|_0^2)dt.\end{aligned}$$

Combining above estimate with (66), summing from $n = 1$ to m, applying the Gronwall Lemma and Theorem 3.7, we complete the proof. □

Lemma 6.3 *Under the assumptions of Theorem 6.1, for all $m \geq 1$, we have*

$$\|e_h^m\|_1^2 + \Delta t \sum_{n=1}^{m}(\|d_t e_h^n\|_0^2 + \kappa\|d_t e_h^n\|_1^2) \leq c\Delta t^2.$$

Proof Taking $v_h = d_t e_h^n \Delta t \in V_h$, $q_h = 0$ in (65), one finds

$$\begin{aligned}&\frac{\nu}{2}(\|e_h^n\|_1^2 - \|e_h^{n-1}\|_1^2 + \|e_h^n - e_h^{n-1}\|_1^2) + \Delta t\|d_t e_h^n\|_0^2 + \Delta t^2 b(u_t, u(t_n), d_t e_h^n)\\ &+\kappa\Delta t\|d_t e_h^n\|_1^2 + b(e_h^{n-1}, u_h(t_n), d_t e_h^n)\Delta t + b(u_h^{n-1}, e_h^n, d_t e_h^n)\Delta t = (E_n, d_t e_h^n)\Delta t.\end{aligned} \tag{67}$$

For the trilinear terms and the right-hand side term, by (3) and Cauchy inequality, it holds

$$\begin{aligned}|\Delta t^2 b(u_t, u(t_n), d_t e_h^n)| &\leq \Delta t^2\|u_t\|_1\|u(t_n)\|_2\|d_t e_h^n\|_0 \leq \frac{\Delta t}{8}\|d_t e_h^n\|_0^2 + c\Delta t^3\|Au(t_n)\|_0^2\|u_t\|_1^2,\\
|\Delta t b(e_h^{n-1}, u_h(t_n), d_t e_h^n)| &\leq c\Delta t\|e_h^{n-1}\|_1\|A_h u_h(t_n)\|_0\|d_t e_h^n\|_0 \leq \frac{\Delta t}{8}\|d_t e_h^n\|_0^2 + c\Delta t\|A_h u_h(t_n)\|_0^2\|e_h^{n-1}\|_1^2,\\
|\Delta t b(u_h^{n-1}, e_h^n, d_t e_h^n)| &\leq c\Delta t\|A_h u_h^{n-1}\|_0\|e_h^n\|_1\|d_t e_h^n\|_0 \leq \frac{\Delta t}{8}\|d_t e_h^n\|_0^2 + c\Delta t\|A_h u_h^{n-1}\|_0^2\|e_h^n\|_1^2,\\
|\Delta t(E_n, d_t e_h^n)| &= \int_{t_{n-1}}^{t_n}(t - t_{n-1})(u_{htt}, d_t e_h^n)dt - \kappa\int_{t_{n-1}}^{t_n}(t - t_{n-1})(A_h u_{htt}, d_t e_h^n)dt\\
&\leq \frac{\Delta t}{8}\|d_t e_h^n\|_0^2 + \Delta t^2\int_{t_{n-1}}^{t_n}(\|u_{htt}\|_0^2 + \kappa\|A_h u_{htt}\|_0^2)dt.\end{aligned}$$

Combining above estimate with (67), summing from $n = 1$ to m, applying Gronwall Lemma and Theorem 3.7, we finish the proof. □

Theorem 6.4 *Under the assumptions of Theorem 6.1, for all $m \geq 1$, the errors $u(t_n) - u_h^n$ and $p(t_n) - p_h^n$ satisfy*

$$\|u(t_m) - u_h^m\|_0 \leq c(h^2 + \Delta t), \quad \|u(t_m) - u_h^m\|_1 \leq c(h + \Delta t), \tag{68}$$

$$\Delta t \sum_{n=1}^{m} \tau(t_n)\|p(t_n) - p_h^n\|_0^2 \leq c(h^2 + \Delta t^2). \tag{69}$$

Proof Thanks to the triangle inequality, Theorem 5.1, and Lemmas 6.2 and 6.3, we finish the proof of (68).

For (69), by using (2)–(4) and Theorem 2.3, one finds

$$\begin{aligned}\beta(\|e_h^n\|_1 + \|\mu_h^n\|_0) &\leq \sup_{(v_h,q_h)\in X_h\times M_h}\frac{|\mathcal{B}_h((e_h^n, \mu_h^n); (v_h, q_h))|}{\|v_h\|_1 + \|q_h\|_0}\\
&\leq \sup_{(v_h,q_h)\in X_h\times M_h}\frac{1}{\|v_h\|_1 + \|q_h\|_0}|(d_t e_h^n, v_h) + \kappa((d_t e_h^n, v_h)) + \Delta t b(u_t, u(t_n), v_h)\\
&\qquad + b(e_h^{n-1}, u_h(t_n), v_h) + b(u_h^{n-1}, e_h^n, v_h) - (E_n, v_h)|\\
&\leq C(\|d_t e_h^n\|_0 + \kappa\|d_t e_h^n\|_1 + \|e_h^{n-1}\|_1\|u_h(t_n)\|_1 + \|e_h^n\|_1\|u_h^n\|_1 + \|E_n\|_0\\
&\quad + \Delta t\|u_t\|_0\|u(t_n)\|_2).\end{aligned}$$

Squaring above inequality, summing from $n = 0$ to m, multiplying Δt and combing Theorem 5.1 with Lemma 6.3, we complete the proof of (69). □

Remark 6.1 In Theorem 6.4, we obtain the optimal error estimates of velocity in L^2 and H^1-norms. The spatial convergence order for pressure in L^2 norm is $\mathcal{O}(h)$, which is optimal, but the time convergence order for pressure in L^2 norm is one order weakly. The reason is that we need a such estimate $\|d_t e_h^n\|_0 + \kappa \|d_t e_h^n\|_1 \leq C\Delta t$. We can obtain this result by taking the differences of (65) at n and $n-1$ time level and choosing $v_h = d_t e_h^n \in V_h, q_h = 0$. Here we omit this proof due to the limitation of pages and just verify from the point of numerical results.

Remark 6.2 In our results, we did not distinguish κ^{-1} from the constant c, how to establish the stability and convergence uniformly as $\kappa \mapsto 0$ is a meaningful topic, we can refer to [11, 12, 23]. In these literature, authors developed their novel stability and convergence analysis and avoided using the constant in a priori bound and a priori error estimates which depends on κ^{-1} with some suitable weight functions, as a consequence, the estimates are valid uniformly as κ goes to zero. How to combine our results with the techniques provided in [11, 12, 23] to obtain the desired results will be our next work.

7 Numerical experiments

In this section, we present some numerical results to verify the performances of the stabilized FEM for the Kelvin-Voigt viscoelastic fluid model with different parameters. The partition of domain Ω uses the triangle mesh with the different mixed elements for the velocity and pressure. The mesh is obtained by dividing Ω into squares and then drawing a diagonal in each square. The Euler backward scheme is adopted to treat the time derivative term.

7.1 An analytical solution: convergence validation

In this test, we consider the domain $\Omega = [0, 1]^2$, $T = 1.0$ and present some numerical results with the following analytical solutions for the velocity $u = (u_1, u_2)$ and pressure p

$$u_1 = 2\pi \sin^2(\pi x) \sin(\pi y) \cos(\pi y), \quad u_2 = -2\pi \sin(\pi x) \cos(\pi x) \sin^2(\pi y),$$
$$p = 10 \cos(\pi x) \cos(\pi y) \cos(t).$$

In order to show the performances of the developed stabilized FEM with the lowest equal-order elements, the standard Galerkin method is also introduced with stable mixed elements, such as the MINI element (refer to Arnold et al. [29] and Gerbeau [30]). Let $\hat{b} \in H_0^1(K)^2$ take the value 1 at the barycenter of K and satisfy that $0 \leq \hat{b} \leq 1$, which is called a "bubble function," we define

$$X_h = \{v_h \in C^0(\overline{\Omega})^2 \cap X; v_h|_K \in (P_1(K) \oplus \text{span}\{\hat{b}\})^2, \ \forall K \in \mathcal{T}_h\},$$
$$M_h = \{q_h \in C^0(\overline{\Omega}) \cap M; q_h|_K \in P_1(K), \ \forall K \in \mathcal{T}_h\}.$$

Table 1 Stabilized FEM for problem (62) with P_1-P_1 element for u and p with $\nu = 1,\ \Delta t = h^2,\ \kappa = 1$

$1/h$	$\frac{\|u-u_h\|_0}{\|u\|_0}$	Rate	$\frac{\|\nabla(u-u_h)\|_0}{\|\nabla u\|_0}$	Rate	$\frac{\|p-p_h\|_0}{\|p\|_0}$	Rate	CPU (s)
16	0.024353	-	0.159541	-	0.151513	-	4.4
20	0.0157088	1.9648	0.127907	0.9904	0.107177	1.5514	10.3
32	0.00618088	1.9846	0.0800996	0.9958	0.0458273	1.8076	64.8
40	0.00395808	1.9974	0.0641041	0.9983	0.0317864	1.6395	163.6
64	0.00154285	2.0045	0.0400787	0.9993	0.0153253	1.5522	1051.6

The P_2-P_1 element (refer to Boffi et al. [31]).

$$
\begin{aligned}
X_h &= \{v_h \in C^0(\overline{\Omega})^2 \cap X;\, v_h|_K \in (P_2(K))^2,\ \forall K \in \mathcal{T}_h\},\\
M_h &= \{q_h \in C^0(\overline{\Omega}) \cap M;\, q_h|_K \in P_1(K),\ \forall K \in \mathcal{T}_h\}.
\end{aligned}
$$

The P_2-P_0 element (refer to Boffi et al. [31] and Girault [15])

$$
\begin{aligned}
X_h &= \{v_h \in C^0(\overline{\Omega})^2 \cap X;\, v_h|_K \in (P_2(K))^2,\ \forall K \in \mathcal{T}_h\},\\
M_h &= \{q_h \in M;\, q_h|_K \in P_0(K),\ \forall K \in \mathcal{T}_h\}.
\end{aligned}
$$

Firstly, we set $\nu = 1$ and present the computational results of stabilized FEM with P_1-P_1 element for the Kelvin-Voigt model (62) with different κ. From Tables 1, 2, 3, and 4, we can see that the convergence orders of velocity in L^2- and H^1-norms are 2 and 1, which confirm the provided theoretical analysis results of Theorem 6.4 well. The convergence order of pressure in L^2-norm is about 1.5, which show some superconvergence, that maybe due to the smoothness of the analytical solutions. Compared with the standard Galerkin method with MINI element, we can see that as κ decreases, the differences of the relative errors between the stabilized FEM with P_1-P_1 element and the Galerkin method with MINI element become smaller and smaller. However, 23% the CPU time of stabilized method with different κ is saved than the standard Galerkin method's.

Next, we fix the parameter $\kappa = 1$ and present the relative errors of different numerical schemes with different mixed elements. From Tables 5 and 6, we see that the L^2-relative errors for the velocity and pressure in Galerkin method are $\frac{1}{2}$ and $\frac{1}{5}$ times

Table 2 Galerkin FEM for problem (62) with MINI element for u and p with $\nu = 1,\ \Delta t = h^2,\ \kappa = 1$

$1/h$	$\frac{\|u-u_h\|_0}{\|u\|_0}$	Rate	$\frac{\|\nabla(u-u_h)\|_0}{\|\nabla u\|_0}$	Rate	$\frac{\|p-p_h\|_0}{\|p\|_0}$	Rate	CPU (s)
16	0.0136219	-	0.224001	-	0.115033	-	5.5
20	0.00870514	2.0066	0.179631	0.9893	0.0801895	1.6170	13.3
32	0.00338758	2.0081	0.112534	0.9950	0.0382815	1.5732	83.4
40	0.00216191	2.0127	0.0900671	0.9980	0.0271891	1.5333	223.2
64	0.000837597	2.0175	0.0563117	0.9993	0.0135077	1.4884	1355.7

Table 3 Stabilized FEM for problem (62) with P_1-P_1 element for u and p with $\nu = 1,\ \Delta t = h^2,\ \kappa = 0.01$

$1/h$	$\frac{\|u-u_h\|_0}{\|u\|_0}$	Rate	$\frac{\|\nabla(u-u_h)\|_0}{\|\nabla u\|_0}$	Rate	$\frac{\|p-p_h\|_0}{\|p\|_0}$	Rate	CPU (s)
16	0.0475575	-	0.174421	-	0.098042	-	4.9
32	0.0117822	2.0131	0.0803021	1.1191	0.0283513	1.7900	65.3
40	0.00755908	1.9890	0.0642164	1.0018	0.0196654	1.6394	162.6
64	0.00295714	1.9969	0.0401124	1.0012	0.00962675	1.5198	1054.2

Table 4 Galerkin FEM for problem (62) with MINI element for u and p with $\nu = 1,\ \Delta t = h^2,\ \kappa = 0.01$

$1/h$	$\frac{\|u-u_h\|_0}{\|u\|_0}$	Rate	$\frac{\|\nabla(u-u_h)\|_0}{\|\nabla u\|_0}$	Rate	$\frac{\|p-p_h\|_0}{\|p\|_0}$	Rate	CPU (s)
16	0.0329686	-	0.151691	-	0.131481	-	5.5
32	0.00833616	1.9836	0.0757752	1.0013	0.0434872	1.5962	83.2
40	0.00533729	1.9982	0.0605911	1.0012	0.0308537	1.5381	224.6
64	0.00208164	2.0033	0.0378351	1.0019	0.0153027	1.4920	1362.2

Table 5 Stabilized FEM for problem (62) with P_1-P_1 element for u and p with $\kappa = 1,\ \Delta t = h^2,\ \nu = 0.001$

$1/h$	$\frac{\|u-u_h\|_0}{\|u\|_0}$	Rate	$\frac{\|\nabla(u-u_h)\|_0}{\|\nabla u\|_0}$	Rate	$\frac{\|p-p_h\|_0}{\|p\|_0}$	Rate	CPU (s)
16	0.022269	-	0.159701	-	0.131104	-	4.5
20	0.014338	1.9731	0.127989	0.9920	0.0888887	1.7415	10.4
32	0.0056261	1.9904	0.0801195	0.9966	0.0398781	1.7054	65.6
40	0.0035996	2.0014	0.0641143	0.9987	0.0281581	1.5595	164.5
64	0.0014012	2.0074	0.0400814	0.9995	0.0139651	1.4921	1059.2

Table 6 Galerkin FEM for problem (62) with MINI element for u and p with $\kappa = 1,\ \Delta t = h^2,\ \nu = 0.001$

$1/h$	$\frac{\|u-u_h\|_0}{\|u\|_0}$	Rate	$\frac{\|\nabla(u-u_h)\|_0}{\|\nabla u\|_0}$	Rate	$\frac{\|p-p_h\|_0}{\|p\|_0}$	Rate	CPU (s)
16	0.0120832	-	0.237078	-	0.0156107	-	5.6
20	0.00771355	2.0114	0.190148	0.9885	0.0111208	1.5198	13.5
32	0.00299844	2.0104	0.119147	0.9946	0.00601306	1.3083	83.4
40	0.00191307	2.0139	0.0953666	0.9977	0.00479782	1.0118	208.3
64	0.000761734	1.9593	0.0595964	1.0003	0.00324371	0.8329	1508.3

Table 7 Galerkin FEM for problem (62) with P_2-P_1 element for u and p with $\kappa = 1,\ \Delta t = h^3,\ \nu = 0.001$

$1/h$	$\frac{\|u-u_h\|_0}{\|u\|_0}$	Rate	$\frac{\|\nabla(u-u_h)\|_0}{\|\nabla u\|_0}$	Rate	$\frac{\|p-p_h\|_0}{\|p\|_0}$	Rate	CPU (s)
16	0.00300551	-	0.0118417	-	0.0237408	-	126.2
20	0.00193263	1.9788	0.0076137	1.9793	0.0145808	2.1846	382.6
32	0.000757116	1.9939	0.0029982	1.9828	0.00651497	1.7140	4238.5
40	0.000485252	1.9936	0.00192649	1.9822	0.00491699	1.2611	13073.4
50	0.000311052	1.9929	0.00123851	1.9798	0.00398591	0.9408	32832.7

Table 8 Galerkin FEM for problem (62) with P_2-P_0 element for u and p with $\kappa = 1,\ \Delta t = h^3,\ \nu = 0.001$

$1/h$	$\frac{\|u-u_h\|_0}{\|u\|_0}$	Rate	$\frac{\|\nabla(u-u_h)\|_0}{\|\nabla u\|_0}$	Rate	$\frac{\|p-p_h\|_0}{\|p\|_0}$	Rate	CPU (s)
16	0.000677924	-	0.0116438	-	0.0661771	-	110.9
20	0.000368875	2.7273	0.00754774	1.9428	0.0528222	1.0101	350.4
32	0.000111011	2.5549	0.00308967	1.9004	0.0329648	1.0032	3750.8
40	6.60253e−005	2.3285	0.00205559	1.8262	0.0264004	0.9951	11361.6
50	4.00221e−005	2.2434	0.00139002	1.7533	0.0211747	0.9885	27824.5

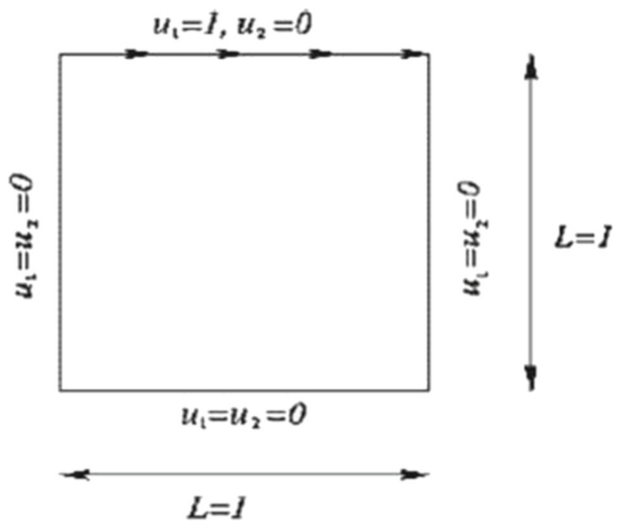

Fig. 1 Lid-driven cavity flow

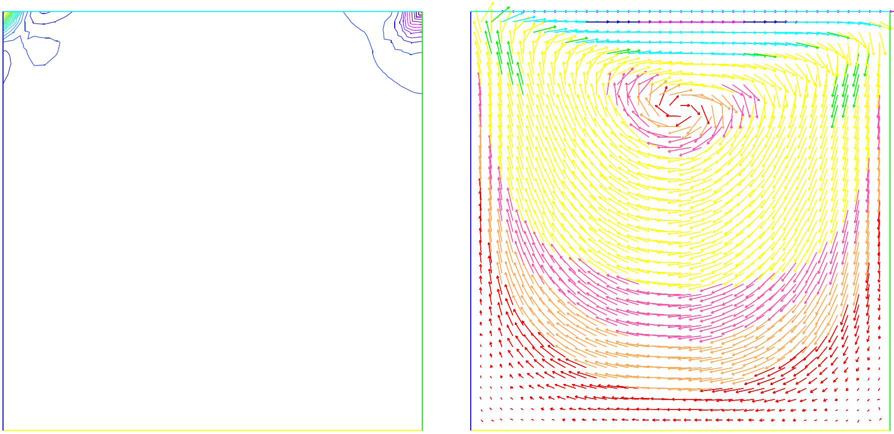

Fig. 2 The pressure lines and velocity vectors of driven cavity flow with $\nu = 1, \kappa = 10$

than that obtained by the stabilized FEM with $\nu = 0.001$, while the computational time of stabilized method can save $\frac{1}{3}$ than that obtained by the Galerkin method.

Finally, we present the computational results of Galerkin method with stable P_2-P_1 and P_2-P_0 elements in Tables 7 and 8. From these data, we can see that more accurate computational results are obtained with the stable higher mixed elements at the cost of high CPU overhead. For the data of Table 7, the desired convergence orders for velocity in L^2-norm and H^1-norm and pressure in L^2-norm should be 3,2,2, respectively. However, the computational results far from ideal, the reason may lie in that (1) the accumulation errors of computer destroys the accuracy of computational results, (2) the truncation errors of Euler scheme take the dominant position.

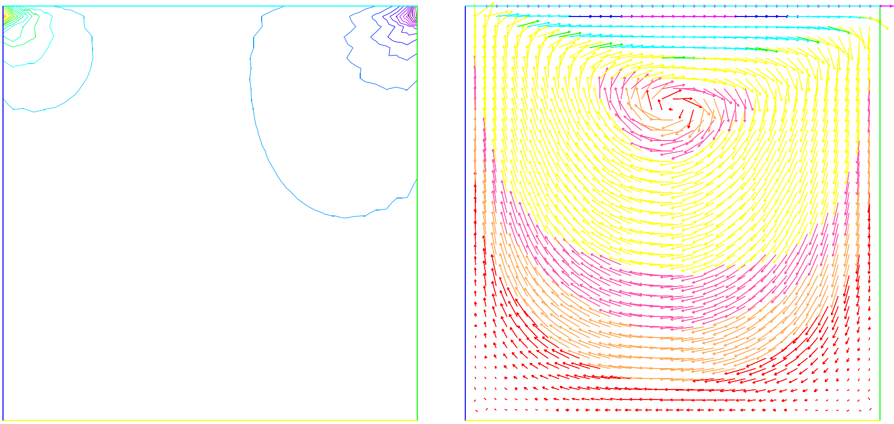

Fig. 3 The pressure lines and velocity vectors of driven cavity flow with $\nu = 1, \kappa = 0$

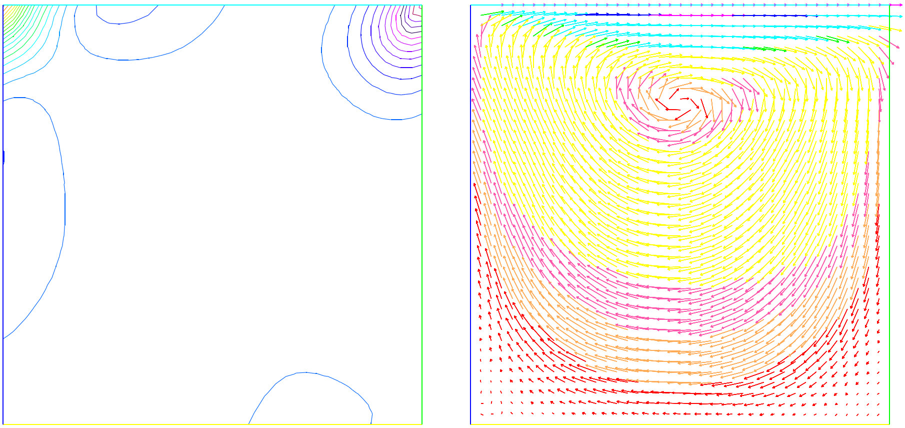

Fig. 4 The pressure lines and velocity vectors of driven cavity flow with $\nu = 0.0025$, $\kappa = 10$

7.2 Lid-driven cavity problem

In this test, we consider the incompressible lid-driven cavity flow problem defined on the unit square. Setting $f = 0$ and the boundary condition $u = 0$ on $[\{0\} \times (0, 1)] \cup [(0, 1) \times \{0\}] \cup [\{1\} \times (0, 1)]$ and $u = (1, 0)^T$ on $(0, 1) \times \{1\}$ (see Fig. 1). The mesh consists of triangular element and the mesh size $h = \frac{1}{40}$, the final time $T = 10$ and the time step $\Delta t = 0.01$.

Figures 2, 3, 4, and 5 show the velocity vectors and pressure contours of driven cavity flow with different ν and κ. From these figures, we can see that the velocity vectors and pressure contours are almost the same with $\nu = 1$ and show great differences with $\nu = \frac{1}{400}$ with different κ, which show that our stabilized method has an

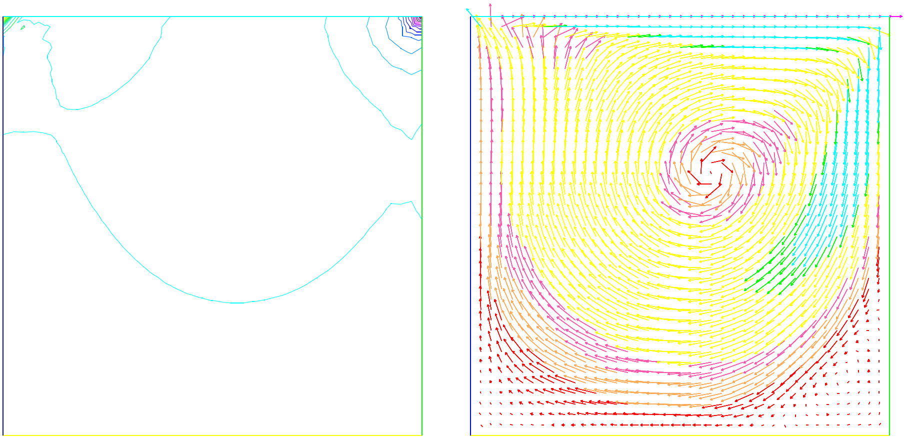

Fig. 5 The pressure lines and velocity vectors of driven cavity flow with $\nu = 0.0025$, $\kappa = 0$

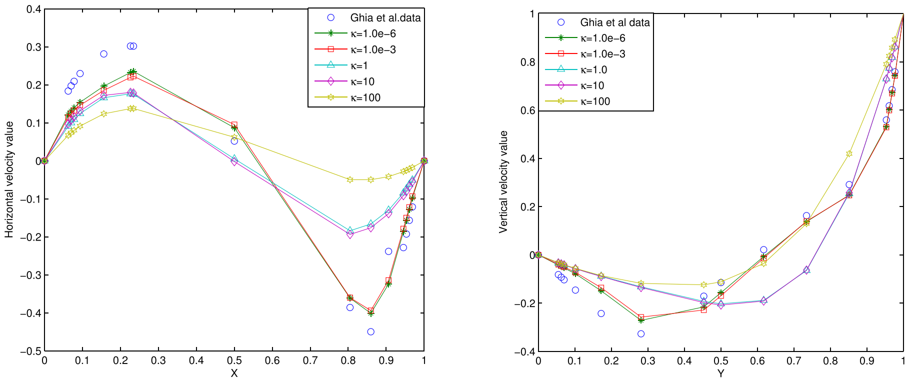

Fig. 6 The computed velocity profiles through the geometric center at $\nu = \frac{1}{400}$ with different κ

effect to stabilize flow field. This also means the time derivatives of diffusion term has little influence on the numerical solutions with large ν.

Furthermore, as the parameter κ decreases, the Kelvin-Voigt model tends to the Navier-Stokes equations. Figure 6 presents the data of numerical velocity obtained by stabilized FEM for the Kelvin-Voigt model at $y = 0.5$ and $x = 0.5$, respectively. Compared with the results given in [32], we can see that the velocity vectors and pressure contours are close to the lid-driven cavity problem of Navier-Stokes as κ decreases. For large κ, the time derivatives of diffusion term plays an important role in stabilizing the flow field with small viscosity parameter ν.

Funding This work was supported by NSF of China (No. 11971152).

References

1. Pani, A.K., Yuan, J.Y.: Semidiscrete finite element Galerkin approximations to the equations of motion arising in the oldroyd model. IMA J. Numer. Anal. **25**, 750–782 (2005)
2. Pani, A.K., Yuan, J.Y., Damazio, P.: On a linearized backward Euler method for the equations of motion of oldroyd fluids of order one. SIAM J. Numer. Anal. **44**, 804–825 (2006)
3. Zhang, T., Yuan, J.Y.: A stabilized characteristic finite element method for the viscoelastic oldroyd fluid motion problem. Int. J. Numer. Anal. Model. **12**, 617–635 (2015)
4. Pavlovskii, V.A.: To the question of theoretical description of weak aqueous polymer solutions. Soviet physics-Doklady **200**, 809–812 (1971)
5. Oskolkov, A.P.: The uniqueness and global solvability of boundary-value problems for the equations of motion for aqueous solutions of polymers. Journal of Soviet Mathematics **8**(4), 427–455 (1977)
6. Cao, Y., Lunasin, E., Titi, E.S.: Global wellposedness of the three dimensional viscous and inviscid simplified Bardina turbulence models. Commun. Math. Sci. **4**, 823–848 (2006)
7. Cotter, C.S., Smolarkiewicz, P.K., Szezyrba, I.N.: A viscoelastic model from brain injuries. Int. J. Numer. Methods Fluids **40**, 303–311 (2002)
8. Oskolkov, A.P.: On an estimate, uniform on the semiaxis $t \geq 0$, for the rate of convergence of Galerkin approximations for the equations of motion of Kelvin-Voight fluids. Journal of Soviet Mathematics **62**, 2802–2806 (1992)
9. Pani, A.K., Pany, A.K., Damazio, P., Yuan, J.Y.: A modified nonlinear spectral Galerkin method for the equations of motion arising in the Kelvin-Voigt fluids. Appl. Anal. **93**, 1587–1610 (2014)

10. Bajpai, S., Nataraj, N., Pani, A.K.: On fully discrete finite element schemes for equations of motion of Kelvin-Voigt fluids. Int. J. Numer. Anal. Model. **10**, 481–507 (2013)
11. Pany, A.K.: Fully discrete second-order backward difference method for Kelvin-Voigt fluid flow model. Numerical Algorithms **78**, 1061–1086 (2018)
12. Pany, A.K., Paikray, A.K., Padhy, S., Pani, A.K.: Backward Euler schemes for the Kelvin-Voigt viscoelastic fluid flow model. Int. J. Numer. Anal. Model. **14**, 126–151 (2017)
13. Bajpai, S., Nataraj, N.: On a two-grid finite element scheme combined with Crank-Nicolson method for the equations of motion arising in the Kelvin-Voigt model. Computer & Mathematics with Applications **68**, 2277–2291 (2014)
14. Bajpai, S., Nataraj, N., Pani, A.K.: On a two-grid finite element scheme for the equations of motion arising in Kelvin-Voigt model. Adv. Comput. Math. **40**, 1043–1071 (2014)
15. Giraut, V., Raviart, P.: Finite element approximation of the Navier-Stokes equations. Springer, Berlin (1979)
16. Temam, R.: Navier-Stokes equations, theory and numerical analysis. North-Holland Pub. Co, Amsterdam (1984)
17. Bochev, P., Dohrmann, C., Gunzburger, M.: Stabilization of low-order mixed finite elements for the Stokes equations. SIAM J. Numer. Anal. **44**, 82–101 (2006)
18. Dohrmann, C., Bochev, P.: A stabilized finite element method for the Stokes problem based on polynomial pressure projections. Int. J. Numer. Methods Fluids **46**, 183–201 (2004)
19. He, Y.N., Lin, Y.P., Sun, W.W.: Stabilized finite element method for the non-stationary Navier-Stokes problem. Discrete and Continuous Dynamical Systems-B **6**, 41–68 (2006)
20. Li, J., He, Y.N., Chen, Z.X.: A new stabilized finite element method for the transient Navier-Stokes equations. Comput. Methods Appl. Mech. Eng. **197**, 22–35 (2007)
21. Huang, P.Z., Feng, X.L., Liu, D.M.: A stabilized finite element method for the time-dependent Stokes equations based on Crank-Nicolson scheme. Appl. Math. Model. **37**, 1910–1919 (2013)
22. Bajpai, S., Nataraj, N., Pani, A.K., Damazio, P., Yuan, J.Y.: Semidiscrete Galerkin method for equations of motion arising in Kelvin-Voigt model of viscoelastic fluid flow. Numerical Methods for Partial Differential Equations **29**, 857–883 (2013)
23. Pany, A.K., Bajpai, S., Pani, A.K.: Optimal error estimates for semidiscrete Galerkin approximations to equations of motion described by Kelvin-Voigt viscoelastic fluid flow model. J. Comput. Appl. Math. **302**, 234–257 (2016)
24. Ciarlet, P.: The Finite Element Method for Elliptic Problems. SIAM Publications, North-Holland, Amsterdam (2002)
25. Feng, X.L., He, Y.N., Huang, P.Z.: A stabilized implicit fractional-step method for the time-dependent Navier-Stokes equations using equal-order pairs. J. Math. Anal. Appl. **392**, 209–224 (2012)
26. Heywood, J.G., Rannacher, R.: Finite element approximation of the nonstationary Navier-Stokes problem. I. Regularity of solutions and second-order error estimates for spatial discretization. SIAM J. Numer. Anal. **19**, 275–311 (1982)
27. Hill, A.T., Süli, E.: Approximation of the global attractor for the incompressible Navier-Stokes problem. IMA J. Numer. Anal. **20**, 633–667 (2000)
28. He, Y.N.: A fully discrete stabilized finite-element method for the time-dependent Navier-Stokes problem. IMA J. Numer. Anal. **23**, 665–691 (2003)
29. Arnold, D.N., Brezzi, F., FortinLi, M.: A stable finite element for the Stokes equations. Calcolo **21**, 337–344 (1984)
30. Gerbeau, J.F., Bris, C., Lelievre, T.: Mathematical Method for the Magnetohydrodynamics of Liquid Metals. Oxford University Press, Oxford (2006)
31. Boffi, D., Brezzi, F., Fortin, M.: Mixed Finite Element Methods and Applications. Springer, Heidelberg, New York, Dordrecht, London (2010)
32. Ghia, U., Ghia, K.N., Shin, C.T.: High-resolutions for incompressible flow using the Navier-Stokes equations and a multigrid method. J. Comput. Phys. **48**, 387–411 (1982)

Publisher's note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.