



Minimum residual modified HSS iteration method for a class of complex symmetric linear systems

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Abstract

For solving a class of complex symmetric system of linear equations, we apply the minimum residual technique to the modified Hermitian and skew-Hermitian splitting (MHSS) iteration scheme and propose an iteration method referred to as minimum residual MHSS (MRMHSS) iteration method. Compared with the classical MHSS method, the MRMHSS method has two more iteration parameters, which can be automatically and easily computed. Then, some properties of the MRMHSS iteration method are carefully studied. Finally, we use four examples to test the performance of the MRMHSS iteration method by comparing its numerical results with three other iteration methods.

Keywords Complex symmetric matrix · Modified Hermitian and skew-Hermitian splitting iteration method · Minimum residual technique · Convergence property · Iteration parameter

Mathematics Subject Classification (2010) 65F10 · 65F50 · 65N22

1 Introduction

We consider the following systems of linear equations

$$Ax = \mathbf{b}, \quad A \in \mathbb{C}^{n \times n}, \quad \mathbf{x}, \mathbf{b} \in \mathbb{C}^n, \quad (1)$$

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where A is a complex symmetric matrix of the form

$$A = W + \iota T, \quad (2)$$

and $\iota = \sqrt{-1}$ denotes the imaginary unit, $W, T \in \mathbb{R}^{n \times n}$ are both real symmetric matrices with at least one of them being positive definite. Hereafter, without loss of generality, we suppose that W is positive definite and $T \neq 0$ is positive semi-definite. This kind of complex symmetric linear systems can be found in many scientific problems, such as fast Fourier transform-based (FFT) solution of certain time-dependent partial differential equations [9], diffuse optical tomography [1], quantum mechanics [20], molecular scattering [17], and structural dynamics [13]. For more applications of this kind of complex symmetric systems, we refer to [8] and the references therein.

Let $A = H(A) + S(A)$ be the Hermitian and skew-Hermitian splitting (HSS) of coefficient matrix A , where

$$H(A) = \frac{1}{2}(A + A^*) = W, \quad \text{and} \quad S(A) = \frac{1}{2}(A - A^*) = \iota T$$

are the Hermitian and the skew-Hermitian parts of matrix A , respectively. Here A^* denotes the conjugate transpose of matrix A . Since A is a non-Hermitian and positive definite matrix, one can straightforwardly employ the HSS iteration method [6] to solve the above complex symmetric system of linear equation (1). Nevertheless, as is known that there is a potential difficulty with the HSS iteration method since a shifted skew-Hermitian system of linear equations should be solved at each iteration step, which is time-consuming. To avoid this shortcoming, a modified HSS (MHSS) iteration method was presented lately [3] for solving this kind of complex symmetric systems of linear equations, in which the solution of the shifted skew-Hermitian linear system was replaced by solving a real symmetric positive definite systems of linear equations. The iterative scheme of MHSS iteration method can be written as

$$\begin{cases} (\alpha I + W)\mathbf{x}^{(k+\frac{1}{2})} = (\alpha I - \iota T)\mathbf{x}^{(k)} + \mathbf{b}, \\ (\alpha I + T)\mathbf{x}^{(k+1)} = (\alpha I + \iota W)\mathbf{x}^{(k+\frac{1}{2})} - \iota \mathbf{b}, \end{cases} \quad (3)$$

where α is a given positive constant and I is the identity matrix, $k = 0, 1, 2, \dots$ and $\mathbf{x}^{(0)}$ is an initial guess.

The MHSS method is very effective and reliable, thus has been extended and developed to many new variants, such as preconditioned MHSS (PMHSS) iteration method [4, 5], generalized PMHSS [12, 22], lopsided PMHSS [16] and so on. In addition, the MHSS iteration method and its variants have been used to solve complex symmetric nonlinear equations [24], complex singular linear systems [10, 25], and also been used as the smoother of the multigrid methods [15].

Recently, in another point of view, Yang, Cao and Wu [23] proposed a minimum residual HSS (MRHSS) iteration method to improve the efficiency of the HSS iteration method by making use of the minimum residual technique on HSS iteration scheme. Numerical results show that the MRHSS method is much more effective than the HSS iteration method. However, when we use it to solve complex symmetric system of linear (1), a shifted skew-Hermitian system of linear equations still needs to be solved in each step of the MRHSS method. In order to avoid this problem, we further use the minimum residual technique to the MHSS iteration scheme. Then, a

new method named as minimum residual MHSS (MRMHSS) iteration method is proposed in this work. In the implementation of this method, the shifted skew-Hermitian system of linear equations need to be solved is replaced by a symmetric positive definite systems of linear equations. Thus, we may expect that the MRMHSS iteration method outperforms the MRHSS iteration method.

Throughout this paper, we denote by $(\mathbf{x}, \mathbf{y}) = \mathbf{y}^* \mathbf{x}$ the Euclidean inner product for any complex vectors $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$, and denote by $\|\mathbf{x}\| = \sqrt{(\mathbf{x}, \mathbf{x})}$ the Euclidean norm for any complex vector $\mathbf{x} \in \mathbb{C}^n$. For an arbitrary matrix $X \in \mathbb{C}^{n \times n}$, $\|X\|$ denotes the spectral norm of X , which is actually induced by the Euclidean vector norms. The remainder of this work is organized as follows. In Section 2, based on the residual form of the MHSS method, we first construct the MRMHSS iteration method. Some essential properties of the involved parameters and the convergence property of the MRMHSS method are discussed in Section 3. Numerical experiments are presented in Section 4 to test the efficiency and robustness of the MRMHSS iteration method. Finally, in Section 5, we draw a brief conclusion for this work.

2 The MRMHSS iteration method

In this section, we first rewrite the MHSS method into an equivalent form, and thereafter introduce the MRMHSS iteration method by using the minimum residual technique onto the new form of the MHSS method.

Note that $(\alpha I + W)^{-1}(\alpha I - \iota T) = I - (\alpha I + W)^{-1}A$ and $(\alpha I + T)^{-1}(\alpha I + \iota W) = I + \iota(\alpha I + T)^{-1}A$, and denote $\mathbf{r}^{(k)} = \mathbf{b} - A\mathbf{x}^{(k)}$ and $\mathbf{r}^{(k+\frac{1}{2})} = \mathbf{b} - A\mathbf{x}^{(k+\frac{1}{2})}$, the MHSS iteration scheme (3) can be rewritten as

$$\begin{cases} \mathbf{x}^{(k+\frac{1}{2})} = \mathbf{x}^{(k)} + (\alpha I + W)^{-1}\mathbf{r}^{(k)}, \\ \mathbf{x}^{(k+1)} = \mathbf{x}^{(k+\frac{1}{2})} - \iota(\alpha I + T)^{-1}\mathbf{r}^{(k+\frac{1}{2})}, \end{cases} \tag{4}$$

which is mathematically equivalent to the direct-splitting scheme (3). However, the numerical behaviors of (3) and (4) are different and the latter generally performs more efficient than the former; See details in [7].

Now, denote

$$\mathbf{d}^{(k)} = (\alpha I + W)^{-1}\mathbf{r}^{(k)} \quad \text{and} \quad \mathbf{d}^{(k+\frac{1}{2})} = (\alpha I + T)^{-1}\mathbf{r}^{(k+\frac{1}{2})}, \tag{5}$$

(4) can be rewritten as

$$\mathbf{x}^{(k+\frac{1}{2})} = \mathbf{x}^{(k)} + \mathbf{d}^{(k)}, \quad \mathbf{x}^{(k+1)} = \mathbf{x}^{(k+\frac{1}{2})} - \iota\mathbf{d}^{(k+\frac{1}{2})}.$$

It is obvious that the search distances along the two updating directions $\mathbf{d}^{(k)}$ and $\mathbf{d}^{(k+\frac{1}{2})}$ are both unitary. Naturally, we can multiply two generalized parameters $\lambda_k, \theta_k \in \mathbb{C}$ in front of the updating directions $\mathbf{d}^{(k)}$ and $\mathbf{d}^{(k+\frac{1}{2})}$ to expect them work well to improve the convergence rates, which is similar with some other accelerating techniques too, such as the minimum residual smoothing [14, 21, 26]. Then, an improved iteration scheme is derived as

$$\mathbf{x}^{(k+\frac{1}{2})} = \mathbf{x}^{(k)} + \lambda_k \mathbf{d}^{(k)}, \quad \mathbf{x}^{(k+1)} = \mathbf{x}^{(k+\frac{1}{2})} - \iota \theta_k \mathbf{d}^{(k+\frac{1}{2})}. \tag{6}$$

For the reason of clearness in the narrative, we define two notations

$$G_1 := A(\alpha I + W)^{-1} \text{ and } G_2 := A(\alpha I + T)^{-1}, \tag{7}$$

then the residual form of the iteration scheme (6) can be written as

$$\mathbf{r}^{(k+\frac{1}{2})} = \mathbf{r}^{(k)} - \lambda_k G_1 \mathbf{r}^{(k)} \text{ and } \mathbf{r}^{(k+1)} = \mathbf{r}^{(k+\frac{1}{2})} + \iota \theta_k G_2 \mathbf{r}^{(k+\frac{1}{2})}. \tag{8}$$

Now our task is to determine proper values of the parameters λ_k and θ_k on minimizing the residual norms $\|\mathbf{r}^{(k+\frac{1}{2})}\|$ and $\|\mathbf{r}^{(k+1)}\|$ at each iteration step, respectively. By straightforwardly calculating, they can be formulated as

$$\begin{aligned} \|\mathbf{r}^{(k+\frac{1}{2})}\|^2 &= \|\mathbf{r}^{(k)}\|^2 - 2Re(\lambda_k)(\mathcal{H}(G_1)\mathbf{r}^{(k)}, \mathbf{r}^{(k)}) \\ &\quad - 2Im(\lambda_k)(\iota\mathcal{S}(G_1)\mathbf{r}^{(k)}, \mathbf{r}^{(k)}) + (Re(\lambda_k)^2 + Im(\lambda_k)^2)\|G_1\mathbf{r}^{(k)}\|^2 \end{aligned} \tag{9}$$

and

$$\begin{aligned} \|\mathbf{r}^{(k+1)}\|^2 &= \|\mathbf{r}^{(k+\frac{1}{2})}\|^2 - 2Im(\theta_k)(\mathcal{H}(G_2)\mathbf{r}^{(k+\frac{1}{2})}, \mathbf{r}^{(k+\frac{1}{2})}) \\ &\quad + 2Re(\theta_k)(\iota\mathcal{S}(G_2)\mathbf{r}^{(k+\frac{1}{2})}, \mathbf{r}^{(k+\frac{1}{2})}) + (Re(\theta_k)^2 + Im(\theta_k)^2)\|G_2\mathbf{r}^{(k+\frac{1}{2})}\|^2, \end{aligned} \tag{10}$$

where $Re(\cdot)$ and $Im(\cdot)$ represent the real and the imaginary parts of a complex number, $\mathcal{H}(\cdot)$ and $\mathcal{S}(\cdot)$ denote the Hermitian and the skew-Hermitian parts of a matrix, respectively. It is easily observed that these two norms can be viewed as two real valued convex functions of two variables $Re(\lambda_k)$ and $Im(\lambda_k)$, $Re(\theta_k)$ and $Im(\theta_k)$, respectively. Then, the minimum point of each functions can be directly derived as

$$Re(\lambda_k) = \frac{(\mathcal{H}(G_1)\mathbf{r}^{(k)}, \mathbf{r}^{(k)})}{\|G_1\mathbf{r}^{(k)}\|^2}, \quad Im(\lambda_k) = \frac{(\iota\mathcal{S}(G_1)\mathbf{r}^{(k)}, \mathbf{r}^{(k)})}{\|G_1\mathbf{r}^{(k)}\|^2} \tag{11}$$

and

$$Re(\theta_k) = -\frac{(\iota\mathcal{S}(G_2)\mathbf{r}^{(k+\frac{1}{2})}, \mathbf{r}^{(k+\frac{1}{2})})}{\|G_2\mathbf{r}^{(k+\frac{1}{2})}\|^2}, \quad Im(\theta_k) = \frac{(\mathcal{H}(G_2)\mathbf{r}^{(k+\frac{1}{2})}, \mathbf{r}^{(k+\frac{1}{2})})}{\|G_2\mathbf{r}^{(k+\frac{1}{2})}\|^2}. \tag{12}$$

It is certainly true that we get the formulas of two introduced parameters, however, we can hardly compute them by (11) and (12) in actual implementations because of the difficulty involved in calculating the matrix vector products $\mathcal{H}(G_1)\mathbf{r}^{(k)}$, $\mathcal{S}(G_1)\mathbf{r}^{(k)}$, $\mathcal{H}(G_2)\mathbf{r}^{(k+\frac{1}{2})}$ and $\mathcal{S}(G_2)\mathbf{r}^{(k+\frac{1}{2})}$, for given vectors $\mathbf{r}^{(k)}$ and $\mathbf{r}^{(k+\frac{1}{2})}$. Thus, we ought to find a practical manner for computing these parameters. Fortunately, they can be reformulated to easier forms in the following

$$\lambda_k = Re(\lambda_k) + \iota Im(\lambda_k) = \frac{(\mathcal{H}(G_1)\mathbf{r}^{(k)}, \mathbf{r}^{(k)})}{\|G_1\mathbf{r}^{(k)}\|^2} + \iota \frac{(\iota\mathcal{S}(G_1)\mathbf{r}^{(k)}, \mathbf{r}^{(k)})}{\|G_1\mathbf{r}^{(k)}\|^2} = \frac{(\mathbf{r}^{(k)}, G_1\mathbf{r}^{(k)})}{\|G_1\mathbf{r}^{(k)}\|^2} \tag{13}$$

and

$$\begin{aligned} \theta_k &= Re(\theta_k) + \iota Im(\theta_k) = -\frac{(\iota\mathcal{S}(G_2)\mathbf{r}^{(k+\frac{1}{2})}, \mathbf{r}^{(k+\frac{1}{2})})}{\|G_2\mathbf{r}^{(k+\frac{1}{2})}\|^2} + \iota \frac{(\mathcal{H}(G_2)\mathbf{r}^{(k+\frac{1}{2})}, \mathbf{r}^{(k+\frac{1}{2})})}{\|G_2\mathbf{r}^{(k+\frac{1}{2})}\|^2} \\ &= \frac{\iota(\mathbf{r}^{(k+\frac{1}{2})}, G_2\mathbf{r}^{(k+\frac{1}{2})})}{\|G_2\mathbf{r}^{(k+\frac{1}{2})}\|^2}. \end{aligned} \tag{14}$$

Using (7) and (5), the computing formulas (13) and (14) can be further rewritten as

$$\lambda_k = \frac{(\mathbf{r}^{(k)}, A\mathbf{d}^{(k)})}{\|A\mathbf{d}^{(k)}\|^2}, \quad \theta_k = \iota \frac{(\mathbf{r}^{(k+\frac{1}{2})}, A\mathbf{d}^{(k+\frac{1}{2})})}{\|A\mathbf{d}^{(k+\frac{1}{2})}\|^2}. \tag{15}$$

On the basis of previous analyses of iteration parameters λ_k and θ_k of the residual updating iteration scheme (6), the MRMHSS iteration method can be formulated as follows.

Method 1 (The MRMHSS iteration method) Given an initial guess $\mathbf{x}^{(0)} \in \mathbb{C}^n$, compute $\mathbf{r}^{(0)} = \mathbf{b} - A\mathbf{x}^{(0)}$. For $k = 0, 1, 2, \dots$, until $\{\mathbf{x}^{(k)}\}$ converges

1. calculate $\mathbf{x}^{(k+\frac{1}{2})}$ from the iteration formula $\mathbf{x}^{(k+\frac{1}{2})} = \mathbf{x}^{(k)} + \lambda_k \mathbf{d}^{(k)}$, where

$$\mathbf{d}^{(k)} = (\alpha I + W)^{-1} \mathbf{r}^{(k)}, \quad \lambda_k = \frac{(\mathbf{r}^{(k)}, A\mathbf{d}^{(k)})}{\|A\mathbf{d}^{(k)}\|^2};$$

2. calculate $\mathbf{x}^{(k+1)}$ from the iteration formula $\mathbf{x}^{(k+1)} = \mathbf{x}^{(k+\frac{1}{2})} - \iota \theta_k \mathbf{d}^{(k+\frac{1}{2})}$, where

$$\mathbf{d}^{(k+\frac{1}{2})} = (\alpha I + T)^{-1} \mathbf{r}^{(k+\frac{1}{2})}, \quad \theta_k = \iota \frac{(\mathbf{r}^{(k+\frac{1}{2})}, A\mathbf{d}^{(k+\frac{1}{2})})}{\|A\mathbf{d}^{(k+\frac{1}{2})}\|^2},$$

where α is a given positive constant and I is the identity matrix.

Remark 1 When we choose $\lambda_k = \theta_k \equiv 1$ at each iteration step of Method 1, then the MRMHSS iteration method immediately reduces to the classical MHSS iteration method.

It is evident that there are only two real and symmetric linear subsystems need to be solved at each step of the MRMHSS iteration method, whose coefficient matrices are completely same as those in the MHSS. Moreover, since $W \in \mathbb{R}^{n \times n}$ is symmetric positive definite, $T \in \mathbb{R}^{n \times n}$ is symmetric positive semi-definite, and α is positive real scalar, we know that both matrices $\alpha I + W$ and $\alpha I + T$ are symmetric positive definite. Thus, the two linear subsystems involved in each step of the MRMHSS iteration can be solved effectively using Cholesky factorization or (preconditioned) conjugate gradient (CG) method.

Remark 2 The MRMHSS iteration method is also applicable to the case where matrix W is symmetric positive semi-definite and T is symmetric positive definite. More generally, if there exist real numbers β and δ such that both matrices $\tilde{W} := \beta W + \delta T$ and $\tilde{T} := \delta T - \beta W$ are symmetric positive semi-definite with at least one of them being positive definite, we can first multiply the complex symmetric system (1) by the complex number $\beta - \iota \delta$ to obtain the equivalent system

$$(\tilde{W} + \iota \tilde{T})\mathbf{x} = \tilde{\mathbf{b}}, \quad \text{with } \tilde{\mathbf{b}} := (\beta - \iota \delta)\mathbf{b}, \tag{16}$$

and then employ the MRMHSS iteration method to solve (16) approximately.

3 The convergence property of the MRMHSS iteration method

As we know, the iteration parameters λ_k and θ_k involved in the MRMHSS iteration method are chosen to minimize the corresponding residual norms $\|\mathbf{r}^{(k+\frac{1}{2})}\|$ and $\|\mathbf{r}^{(k+1)}\|$, respectively. In this section, we will show a closer relationship between λ_k, θ_k and $\|\mathbf{r}^{(k+1)}\|$. Furthermore, the convergence property of the MRMHSS iteration method will also be discussed.

To do so, we firstly present an important lemma.

Lemma 1 (See [23]) *Let $C = (c_{ij}) \in \mathbb{C}^{n \times n}$ be a Hermitian matrix. For any vector $\mathbf{z} \in \mathbb{C}^n$, we have*

$$\frac{\partial(C\mathbf{z}, \mathbf{z})}{\partial(\|\mathbf{z}\|^2)} = \frac{1}{\|\mathbf{z}\|^2}(C\mathbf{z}, \mathbf{z}),$$

where ∂ is the symbol of partial derivatives.

Now, by making use of the above lemma, we can obtain the following property of the iteration parameters λ_k and θ_k .

Theorem 1 *The quadruple $(Re(\lambda_k), Im(\lambda_k), Re(\theta_k), Im(\theta_k))$ determined by (11) and (12) is the minimum point of the function $\|\mathbf{r}^{(k+1)}\|$, which implies that the values of λ_k and θ_k defined by (13) and (14) are optimal in the complex field \mathbb{C} .*

Proof From (8) and (10), the residual norm $\|\mathbf{r}^{(k+1)}\|^2$ can be viewed as a real valued function of the complex variables λ_k and θ_k , or equivalently, of the real variables $Re(\lambda_k), Im(\lambda_k), Re(\theta_k)$ and $Im(\theta_k)$. Let $\tilde{\mathbf{r}}(\xi_1, \xi_2) := \mathbf{r}^{(k)} - (\xi_1 + i\xi_2)G_1\mathbf{r}^{(k)}$, and define a function ϕ as

$$\begin{aligned} \phi(\tilde{\mathbf{r}}(\xi_1, \xi_2), \eta_1, \eta_2) &:= \|\tilde{\mathbf{r}}\|^2 - 2\eta_2(\mathcal{H}(G_2)\tilde{\mathbf{r}}, \tilde{\mathbf{r}}) \\ &\quad + 2\eta_1(i\mathcal{S}(G_2)\tilde{\mathbf{r}}, \tilde{\mathbf{r}}) + (\eta_1^2 + \eta_2^2)\|G_2\tilde{\mathbf{r}}\|^2. \end{aligned} \tag{17}$$

Because of $\|G_2\tilde{\mathbf{r}}\|^2 = (G_2^*G_2\tilde{\mathbf{r}}, \tilde{\mathbf{r}})$, thus using (17) and Lemma 1, we have

$$\begin{aligned} \frac{\partial\phi}{\partial(\|\tilde{\mathbf{r}}\|^2)} &= \frac{\|\tilde{\mathbf{r}}\|^2}{\|\tilde{\mathbf{r}}\|^2} - \frac{2\eta_2}{\|\tilde{\mathbf{r}}\|^2}(\mathcal{H}(G_2)\tilde{\mathbf{r}}, \tilde{\mathbf{r}}) \\ &\quad + \frac{2\eta_1}{\|\tilde{\mathbf{r}}\|^2}(i\mathcal{S}(G_2)\tilde{\mathbf{r}}, \tilde{\mathbf{r}}) + \frac{\eta_1^2 + \eta_2^2}{\|\tilde{\mathbf{r}}\|^2}\|G_2\tilde{\mathbf{r}}\|^2 \\ &= \frac{1}{\|\tilde{\mathbf{r}}\|^2}\phi. \end{aligned} \tag{18}$$

Denote by

$$\varphi(\xi_1, \xi_2, \eta_1, \eta_2) := \phi(\tilde{\mathbf{r}}(\xi_1, \xi_2), \eta_1, \eta_2). \tag{19}$$

Then, the four first-order partial derivatives of $\varphi(\xi_1, \xi_2, \eta_1, \eta_2)$ can be calculated, which are

$$\begin{aligned} \frac{\partial \varphi}{\partial \xi_1} &= \frac{\partial \phi}{\partial (\|\tilde{\mathbf{r}}\|^2)} \frac{\partial (\|\tilde{\mathbf{r}}\|^2)}{\partial \xi_1} = \frac{2\phi}{\|\tilde{\mathbf{r}}\|^2} \left(\xi_1 \|G_1 \mathbf{r}^{(k)}\|^2 - (\mathcal{H}(G_1) \mathbf{r}^{(k)}, \mathbf{r}^{(k)}) \right), \\ \frac{\partial \varphi}{\partial \xi_2} &= \frac{\partial \phi}{\partial (\|\tilde{\mathbf{r}}\|^2)} \frac{\partial (\|\tilde{\mathbf{r}}\|^2)}{\partial \xi_2} = \frac{2\phi}{\|\tilde{\mathbf{r}}\|^2} \left(\xi_2 \|G_1 \mathbf{r}^{(k)}\|^2 - (\iota \mathcal{S}(G_1) \mathbf{r}^{(k)}, \mathbf{r}^{(k)}) \right), \\ \frac{\partial \varphi}{\partial \eta_1} &= 2 \left(\eta_1 \|G_2 \tilde{\mathbf{r}}\|^2 + (\iota \mathcal{S}(G_2) \tilde{\mathbf{r}}, \tilde{\mathbf{r}}) \right), \\ \frac{\partial \varphi}{\partial \eta_2} &= 2 \left(\eta_2 \|G_2 \tilde{\mathbf{r}}\|^2 - (\mathcal{H}(G_2) \tilde{\mathbf{r}}, \tilde{\mathbf{r}}) \right). \end{aligned}$$

It is easy to see, from (11) and (12), that the quadruple $(Re(\lambda_k), Im(\lambda_k), Re(\theta_k), Im(\theta_k))$ is the unique stationary point of the function φ defined by (19). Furthermore, in order to verify the quadruple is the minimum point of the function φ , its second-order partial derivatives are required. For the convenience of representation, we firstly define four notations as follows:

$$\begin{aligned} \Psi_1(\xi_1) &:= \xi_1 \|G_1 \mathbf{r}^{(k)}\|^2 - (\mathcal{H}(G_1) \mathbf{r}^{(k)}, \mathbf{r}^{(k)}), \\ \Psi_2(\xi_2) &:= \xi_2 \|G_1 \mathbf{r}^{(k)}\|^2 - (\iota \mathcal{S}(G_1) \mathbf{r}^{(k)}, \mathbf{r}^{(k)}), \\ \Psi_3(\xi_1, \xi_2, \eta_1) &:= \eta_1 \|G_2 \tilde{\mathbf{r}}\|^2 + (\iota \mathcal{S}(G_2) \tilde{\mathbf{r}}, \tilde{\mathbf{r}}), \\ \Psi_4(\xi_1, \xi_2, \eta_2) &:= \eta_2 \|G_2 \tilde{\mathbf{r}}\|^2 - (\mathcal{H}(G_2) \tilde{\mathbf{r}}, \tilde{\mathbf{r}}). \end{aligned}$$

Then, the second-order partial derivatives of $\varphi(\xi_1, \xi_2, \eta_1, \eta_2)$ read

$$\begin{aligned} \frac{\partial^2 \varphi}{\partial \xi_1^2} &= \Psi_1(\xi_1) \frac{\partial}{\partial \xi_1} \left(\frac{2\phi}{\|\tilde{\mathbf{r}}\|^2} \right) + \frac{2\phi}{\|\tilde{\mathbf{r}}\|^2} \|G_1 \mathbf{r}^{(k)}\|^2, & \frac{\partial^2 \varphi}{\partial \xi_1 \partial \xi_2} &= \Psi_1(\xi_1) \frac{\partial}{\partial \xi_2} \left(\frac{2\phi}{\|\tilde{\mathbf{r}}\|^2} \right), \\ \frac{\partial^2 \varphi}{\partial \xi_2^2} &= \Psi_2(\xi_2) \frac{\partial}{\partial \xi_2} \left(\frac{2\phi}{\|\tilde{\mathbf{r}}\|^2} \right) + \frac{2\phi}{\|\tilde{\mathbf{r}}\|^2} \|G_1 \mathbf{r}^{(k)}\|^2, & \frac{\partial^2 \varphi}{\partial \xi_2 \partial \xi_1} &= \Psi_2(\xi_2) \frac{\partial}{\partial \xi_1} \left(\frac{2\phi}{\|\tilde{\mathbf{r}}\|^2} \right), \\ \frac{\partial^2 \varphi}{\partial \eta_1 \partial \xi_1} &= 4\Psi_3(\xi_1, \xi_2, \eta_1) \frac{\Psi_1(\xi_1)}{\|\tilde{\mathbf{r}}\|^2}, & \frac{\partial^2 \varphi}{\partial \eta_1 \partial \xi_2} &= 4\Psi_3(\xi_1, \xi_2, \eta_1) \frac{\Psi_2(\xi_2)}{\|\tilde{\mathbf{r}}\|^2}, \\ \frac{\partial^2 \varphi}{\partial \eta_2 \partial \xi_1} &= 4\Psi_4(\xi_1, \xi_2, \eta_2) \frac{\Psi_1(\xi_1)}{\|\tilde{\mathbf{r}}\|^2}, & \frac{\partial^2 \varphi}{\partial \eta_2 \partial \xi_2} &= 4\Psi_4(\xi_1, \xi_2, \eta_2) \frac{\Psi_2(\xi_2)}{\|\tilde{\mathbf{r}}\|^2}, \\ \frac{\partial^2 \varphi}{\partial \xi_1 \partial \eta_1} &= \Psi_1(\xi_1) \frac{\partial}{\partial \eta_1} \left(\frac{2\phi}{\|\tilde{\mathbf{r}}\|^2} \right), & \frac{\partial^2 \varphi}{\partial \xi_1 \partial \eta_2} &= \Psi_1(\xi_1) \frac{\partial}{\partial \eta_2} \left(\frac{2\phi}{\|\tilde{\mathbf{r}}\|^2} \right), \\ \frac{\partial^2 \varphi}{\partial \xi_2 \partial \eta_1} &= \Psi_2(\xi_2) \frac{\partial}{\partial \eta_1} \left(\frac{2\phi}{\|\tilde{\mathbf{r}}\|^2} \right), & \frac{\partial^2 \varphi}{\partial \xi_2 \partial \eta_2} &= \Psi_2(\xi_2) \frac{\partial}{\partial \eta_2} \left(\frac{2\phi}{\|\tilde{\mathbf{r}}\|^2} \right), \\ \frac{\partial^2 \varphi}{\partial \eta_1^2} &= \frac{\partial^2 \varphi}{\partial \eta_2^2} = 2\|G_2 \tilde{\mathbf{r}}\|^2, & \frac{\partial^2 \varphi}{\partial \eta_1 \partial \eta_2} &= \frac{\partial^2 \varphi}{\partial \eta_2 \partial \eta_1} = 0. \end{aligned}$$

Meanwhile, it is easy to see that $\Psi_1(Re(\lambda_k)) = \Psi_2(Im(\lambda_k)) = 0$ and

$$\Psi_3(Re(\lambda_k), Im(\lambda_k), Re(\theta_k)) = \Psi_4(Re(\lambda_k), Im(\lambda_k), Im(\theta_k)) = 0.$$

Hence, the Hessian matrix of the function φ at the unique stationary point $(Re(\lambda_k), Im(\lambda_k), Re(\theta_k), Im(\theta_k))$ reads

$$\mathbf{H} = \begin{bmatrix} \frac{2\|\mathbf{r}^{(k+1)}\|^2\|G_1\mathbf{r}^{(k)}\|^2}{\|\mathbf{r}^{(k+\frac{1}{2})}\|^2} & 0 & 0 & 0 \\ 0 & \frac{2\|\mathbf{r}^{(k+1)}\|^2\|G_1\mathbf{r}^{(k)}\|^2}{\|\mathbf{r}^{(k+\frac{1}{2})}\|^2} & 0 & 0 \\ 0 & 0 & 2\|G_2\mathbf{r}^{(k+\frac{1}{2})}\|^2 & 0 \\ 0 & 0 & 0 & 2\|G_2\mathbf{r}^{(k+\frac{1}{2})}\|^2 \end{bmatrix},$$

which is a Hermitian and positive definite matrix.

Therefore, the quadruple $(Re(\lambda_k), Im(\lambda_k), Re(\theta_k), Im(\theta_k))$ determined by (11) and (12) is the minimum point of the residual norm $\|\mathbf{r}^{(k+1)}\|$. \square

Now, we study the convergence property of the MRMHSS iteration method. Denote by $\mathcal{F}(M)$ the field values of the complex matrix $M \in \mathbb{C}^{n \times n}$, i.e.,

$$\mathcal{F}(M) = \{(M\mathbf{y}, \mathbf{y}) / (\mathbf{y}, \mathbf{y}) : \mathbf{0} \neq \mathbf{y} \in \mathbb{C}^n\},$$

where (\cdot, \cdot) represents the Euclidean inner product of two vectors in \mathbb{C}^n .

Theorem 2 *The MRMHSS iteration method used for solving the complex symmetric system of linear (1) is convergent for any initial guess $\mathbf{x}^{(0)} \in \mathbb{C}^n$ if and only if*

$$0 \notin \mathcal{F}(A(\alpha I + W)^{-1}) \cap \mathcal{F}(A(\alpha I + T)^{-1}). \tag{20}$$

Furthermore, when (20) holds, the residuals satisfy

$$\|\mathbf{r}^{(k+1)}\| \leq \frac{\sqrt{\|A(\alpha I + W)^{-1}\|^2 - \delta_1^2}}{\|A(\alpha I + W)^{-1}\|} \frac{\sqrt{\|A(\alpha I + T)^{-1}\|^2 - \delta_2^2}}{\|A(\alpha I + T)^{-1}\|} \cdot \|\mathbf{r}^{(k)}\| \tag{21}$$

for any nonnegative integer k , where δ_1 and δ_2 represent the distances from the origin to $\mathcal{F}(A(\alpha I + W)^{-1})$ and $\mathcal{F}(A(\alpha I + T)^{-1})$, respectively.

Proof Denote by $\mathcal{R}(\cdot)$ and superscript \perp the range space of a given matrix and the orthogonal complement of a vector space contained in \mathbb{C}^n , respectively. From the construction of MRMHSS iteration method in Section 2, we can find that each step of MRMHSS method contains two residual projection procedures, i.e., $\mathbf{r}^{(k+\frac{1}{2})}$ and $\mathbf{r}^{(k+1)}$ are the projections of $\mathbf{r}^{(k)}$ and $\mathbf{r}^{(k+\frac{1}{2})}$ onto the subspaces $\mathcal{R}(A(\alpha I + W)^{-1})^\perp$ and $\mathcal{R}(A(\alpha I + T)^{-1})^\perp$, respectively. Hence, $\|\mathbf{r}^{(k+\frac{1}{2})}\| \leq \|\mathbf{r}^{(k)}\|$ with equality if and only if $\mathbf{r}^{(k)}$ is already orthogonal to $A(\alpha I + W)^{-1}\mathbf{r}^{(k)}$, i.e.,

$$(\mathbf{r}^{(k)}, A(\alpha I + W)^{-1}\mathbf{r}^{(k)}) = 0.$$

This implies $\|\mathbf{r}^{(k+\frac{1}{2})}\| < \|\mathbf{r}^{(k)}\|$ if and only if $0 \notin \mathcal{F}((\alpha I + W)^{-*}A^*)$. Because, for any matrix $M \in \mathbb{C}^{n \times n}$, the field of values of M^* is no other than the complex conjugate of the field of values of M , so

$$\|\mathbf{r}^{(k+\frac{1}{2})}\| < \|\mathbf{r}^{(k)}\| \text{ if and only if } 0 \notin \mathcal{F}(A(\alpha I + W)^{-1}).$$

Similarly, we can obtain that

$$\|\mathbf{r}^{(k+1)}\| < \|\mathbf{r}^{(k+\frac{1}{2})}\| \text{ if and only if } 0 \notin \mathcal{F}(A(\alpha I + T)^{-1}).$$

Therefore, we can finally obtain that $\|\mathbf{r}^{(k+1)}\| < \|\mathbf{r}^{(k)}\|$ if and only if

$$0 \notin \mathcal{F}(A(\alpha I + W)^{-1}) \cap \mathcal{F}(A(\alpha I + T)^{-1}). \tag{22}$$

Recalling that the field of values is a closed set, the distances of $\mathcal{F}(A(\alpha I + W)^{-1})$ and $\mathcal{F}(A(\alpha I + T)^{-1})$ from the origin can be, respectively, written as

$$\delta_1 = \min_{0 \neq y \in \mathbb{C}^n} \left| \frac{(A(\alpha I + W)^{-1}y, y)}{(y, y)} - 0 \right| \text{ and } \delta_2 = \min_{0 \neq y \in \mathbb{C}^n} \left| \frac{(A(\alpha I + T)^{-1}y, y)}{(y, y)} - 0 \right|.$$

Substituting (11) and (12), respectively, into (9) and (10), we obtain

$$\begin{aligned} \|\mathbf{r}^{(k+\frac{1}{2})}\|^2 &= \|\mathbf{r}^{(k)}\|^2 - \frac{|(A(\alpha I + W)^{-1}\mathbf{r}^{(k)}, \mathbf{r}^{(k)})|^2}{\|A(\alpha I + W)^{-1}\mathbf{r}^{(k)}\|^2} \\ &= \|\mathbf{r}^{(k)}\|^2 \left(1 - \left| \frac{(A(\alpha I + W)^{-1}\mathbf{r}^{(k)}, \mathbf{r}^{(k)})}{(\mathbf{r}^{(k)}, \mathbf{r}^{(k)})} \right|^2 \frac{\|\mathbf{r}^{(k)}\|^2}{\|A(\alpha I + W)^{-1}\mathbf{r}^{(k)}\|^2} \right) \\ &\leq \|\mathbf{r}^{(k)}\|^2 \left(1 - \frac{\delta_1^2}{\|A(\alpha I + W)^{-1}\|^2} \right) \end{aligned}$$

and

$$\|\mathbf{r}^{(k+1)}\|^2 \leq \|\mathbf{r}^{(k+\frac{1}{2})}\|^2 \left(1 - \frac{\delta_2^2}{\|A(\alpha I + T)^{-1}\|^2} \right).$$

Combining the above two inequalities, the result (21) is obtained, which completes the proof of the theorem. \square

Based on the above theorem, we in the following show a specific criterion for guaranteeing the convergence of Method 1. To this end, we firstly give an useful lemma.

Lemma 2 [18, Proposition 1.18] *Let $M \in \mathbb{C}^{n \times n}$. Then the field of values of M is a convex set which contains the convex hull of its spectrum. It is equal to the convex hull of its spectrum when M is normal, i.e., $\mathcal{F}(M) = Co(\sigma(M))$, where $Co(\cdot)$ denotes the convex hull of a given set and $\sigma(\cdot)$ indicates the spectrum of a matrix.*

Corollary 1 *Assume that the matrix $A = W + \iota T$ is normal, then the MRMHSS iteration method used for solving the complex symmetric system of linear equations (1) is convergent for any initial guess $\mathbf{x}^{(0)} \in \mathbb{C}^n$.*

Proof Because $A = W + \iota T \in \mathbb{C}^{n \times n}$ is normal, by straightforward computations we find that the real symmetric matrices W and T are communicative, i.e., it holds that $WT = TW$. Thus, we can easily verify that both the matrices $A(\alpha I + W)^{-1}$ and $A(\alpha I + T)^{-1}$ are normal. From Lemma 2, it follows

$$\mathcal{F}(A(\alpha I + W)^{-1}) = Co(\sigma(A(\alpha I + W)^{-1}))$$

and

$$\mathcal{F}(A(\alpha I + T)^{-1}) = Co(\sigma(A(\alpha I + T)^{-1})).$$

Hence, (20) reduces to

$$0 \notin Co(\sigma(A(\alpha I + W)^{-1}) \cap Co(\sigma(A(\alpha I + T)^{-1})),$$

while this is automatically valid since in this situation the spectrum of both the matrices $A(\alpha I + W)^{-1}$ and $A(\alpha I + T)^{-1}$ are located in the right half plane of complex field. Therefore, the MRMHSS iteration method used for solving the complex symmetric system of linear equations (1) is convergent for any initial guess $\mathbf{x}^{(0)} \in \mathbb{C}^n$ when the system matrix A is normal. □

Corollary 2 *Assume that the matrix $A = W + \iota T$ is normal, then we have an upper bound of the convergence rate presented in (21), which is followed by*

$$\|\mathbf{r}^{(k+1)}\| \leq \frac{\sqrt{\kappa^2 - 1}}{\kappa} \|\mathbf{r}^{(k)}\|, \tag{23}$$

where κ_1 and κ_2 are the condition numbers of matrices $A(\alpha I + W)^{-1}$ and $A(\alpha I + T)^{-1}$, respectively, $\kappa = \min(\kappa_1, \kappa_2)$.

Proof Since $A = W + \iota T$ is normal, it holds that $WT = TW$. Hence, there exists an orthogonal matrix $Q \in \mathbb{R}^{n \times n}$ can be viewed as the eigenvector matrices of matrices W and T ,

$$W = QDQ^T \quad \text{and} \quad T = Q\Gamma Q^T,$$

where

$$D = \text{diag}(\xi_1, \xi_2, \dots, \xi_n) \quad \text{and} \quad \Gamma = \text{diag}(\gamma_1, \gamma_2, \dots, \gamma_n)$$

are diagonal matrices and $\lambda_j > 0, \gamma_j \geq 0, j = 1, 2, \dots, n$. It follows that

$$A(\alpha I + W)^{-1} = Q(D + \iota\Gamma)(\alpha I + D)^{-1}Q^T$$

let $\tilde{\mathbf{y}} = Q^T \mathbf{y} = (\tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_n)$, it yields that

$$\frac{(A(\alpha I + W)^{-1} \mathbf{y}, \mathbf{y})}{(\mathbf{y}, \mathbf{y})} = \frac{((D + \iota\Gamma)(\alpha I + D)^{-1} \tilde{\mathbf{y}}, \tilde{\mathbf{y}})}{(\tilde{\mathbf{y}}, \tilde{\mathbf{y}})} = \sum_{j=1}^n \mu_j \beta_j$$

where $\mu_j = (\xi_j + \iota\gamma_j)(\alpha + \xi_j)^{-1}$ and $\beta_j = \tilde{y}_j^2 / \sum_{j=1}^n \tilde{y}_j^2, j = 1, 2, \dots, n$. By straightforward computations, we have

$$|\sum_{j=1}^n \mu_j \beta_j| \geq \min_j \{|\mu_j|\} =: \mu_{min}, \quad \text{for any } \beta_j \in [0, 1].$$

Thus, we can obtain $\delta_1 \geq \mu_{min}$, and then it follows that

$$\frac{\sqrt{\|A(\alpha I + W)^{-1}\|^2 - \delta_1^2}}{\|A(\alpha I + W)^{-1}\|} \leq \frac{\sqrt{\mu_{max}^2 - \mu_{min}^2}}{\mu_{max}} = \frac{\sqrt{\kappa_1^2 - 1}}{\kappa_1},$$

where $\mu_{max} := \max_j \{|\mu_j|\}$ is equal to the 2-norm of matrix $A(\alpha I + W)^{-1}$ and

$\kappa_1 = \frac{\mu_{max}}{\mu_{min}}$ denotes the condition number of matrix $A(\alpha I + W)^{-1}$.

Similarly we can obtain an estimate of the second term in the right-hand side of (21), denoting by κ_2 the condition number of matrix $A(\alpha I + T)^{-1}$, we have

$$\|\mathbf{x}^{(k+1)}\| \leq \frac{\sqrt{\kappa_1^2 - 1}}{\kappa_1} \frac{\sqrt{\kappa_2^2 - 1}}{\kappa_2} \cdot \|\mathbf{x}^{(k)}\|.$$

Furthermore, both $(\kappa_1^2 - 1)^{1/2}/\kappa_1$ and $(\kappa_2^2 - 1)^{1/2}/\kappa_2$ are less than unity; thus, the inequality of (23) can be obtained. The proof is completed. \square

4 Numerical experiments

In this section, we use four examples to illustrate the feasibility and effectiveness of the MRMHSS iteration method.

We will compare the numerical results including numbers of iteration steps (denoted as IT) and elapsed CPU times (in seconds, denoted as CPU) of the MRMHSS iteration method with those of the MHSS, the generalized successive over-relaxation (GSOR) [19], the preconditioned MHSS (PMHSS) [4] and the MRHSS methods. In the tests, the MRMHSS, the MRHSS, the PMHSS and the MHSS iteration methods are used to solve the original complex system (1), while the GSOR method being used to solve the equivalent real system of linear equations. For more details about the equivalent real system, one can refer to [2, 11, 19] and the references therein.

The subsystems of linear equations with the coefficient matrices $\alpha I + W$ and $\alpha I + \iota T$ in each step of the MRHSS iteration method are solved directly by using the Cholesky factorization and the LU decomposition, respectively. The subsystems of linear equations of the MHSS, GSOR, PMHSS and MRMHSS iteration methods are solved directly by using the Cholesky factorization since their coefficient matrices are symmetric positive definite. It is noteworthy to mention that the symmetric approximate minimum degree (SYMAMD) reordering [18] is used before the Cholesky or the LU decomposition.

In actual implementations, we choose the initial guess as zero vector and all the computing processes are terminated once the current iterate $\mathbf{x}^{(k)}$ satisfies the stopping criterion

$$\frac{\|\mathbf{b} - A\mathbf{x}^{(k)}\|}{\|\mathbf{b}\|} \leq 10^{-6}, \tag{24}$$

or the number of iteration steps exceeds $k_{max} = 1000$. In addition, all the numerical experiments are computed in MATLAB [version 9.1.0.441655 (R2016b)] in double precision on a personal laptop, with 2.53GHz central processing unit (Intel® Core™ i3 M380), 4.87 G memory and Windows 10 operating system.

For the choice of the parameter values of the MHSS, the MRHSS and the MRMHSS methods, we use the experimentally found optimal ones, which lead to the least number of iteration steps. If the optimal parameter values form an interval, then we use the one that belongs to this interval and leads to the least CPU time. While, for the GSOR method, the optimal iteration parameter values are chosen as presented in [19], and for the PMHSS method, those are chosen as presented in [4]

except for Example 4 and for the cases of $m = 512$ in Examples 1, 2 and 3, which are determined as above-mentioned manner used for the MHSS, the MRHSS and the MRMHSS methods.

Example 1 Consider the complex symmetric linear systems of the form (1) as below (cf. [3])

$$\left[\left(K + \frac{3 - \sqrt{3}}{\tau} I \right) + \iota \left(K + \frac{3 + \sqrt{3}}{\tau} I \right) \right] \mathbf{x} = \mathbf{b}, \tag{25}$$

where τ is the time step-size and K is the matrix of a standard five point centered difference formula, approximating the negative Laplacian operator with homogeneous Dirichlet boundary conditions on an uniform mesh in the two dimensional unit square $[0, 1] \times [0, 1]$, and we set $h = 1/(m + 1)$, $\tau = h$, where m is a number of inner grid-points in one direction. Then, we can know that the matrix $K \in \mathbb{R}^{n \times n}$ is with the tensor product form $K = V_m \otimes I + I \otimes V_m$, with $V_m = h^{-2} \text{tridiag}(-1, 2, -1) \in \mathbb{R}^{m \times m}$. What’s more, it is obvious that K is an $n \times n$ block tridiagonal matrix and $n = m^2$. In our tests, we take

$$W = K + \frac{3 - \sqrt{3}}{\tau} I \text{ and } T = K + \frac{3 + \sqrt{3}}{\tau} I,$$

and the right-hand side vector \mathbf{b} with the j -th entry b_j being the form as follows

$$b_j = \frac{(1 - \iota)j}{\tau(j + 1)^2}, \quad j = 1, 2, \dots, n.$$

In addition, we normalize the system by multiplying two sides by h^2 . And for other details, we can refer to [3].

Example 2 Consider the complex symmetric linear systems of the form (1) as below (cf. [3])

$$[(-\omega^2 M + K) + \iota(\omega C_V + C_H)] \mathbf{x} = \mathbf{b}, \tag{26}$$

where M is the inertia matrix, which is typically real symmetric, possibly singular and we take $M = I$, K is the (real symmetric) stiffness matrix, C_V is the viscous damping matrix (real, diagonal, “highly singular” and possibly zero) and we choose $C_V = 10I$, C_H is the hysteretic damping matrix (also real symmetric) and we here take $C_H = \epsilon K$, with ϵ is a damping coefficient, ω is the circular frequency. Besides, we assume that K is the five-point centered difference matrix approximating the negative Laplacian operator with homogeneous Dirichlet boundary conditions on a uniform mesh in the unit square $[0, 1] \times [0, 1]$ whose mesh-size is $h = 1/(m + 1)$. Then, we can know that the matrix $K \in \mathbb{R}^{n \times n}$ possesses the tensor product form $K = V_m \otimes I + I \otimes V_m$, where $V_m = h^{-2} \text{tridiag}(-1, 2, -1) \in \mathbb{R}^{m \times m}$ and I is the identical matrix in $\mathbb{R}^{m \times m}$. Thus, K is an $n \times n$ block tridiagonal matrix, $n = m^2$. Moreover, we take $\omega = \pi$, $\epsilon = 0.02$, and the right-hand side vector b as $b = (1 + \iota)A\mathbf{1}$, where $\mathbf{1}$ being the vector whose all entries equal to one. Furthermore, we normalize the (26) by multiplying both sides by h^2 . For more details, we refer to [3].

Example 3 The system of linear equations (1) is of the form $(W + \iota T)\mathbf{x} = \mathbf{b}$, where (cf. [3])

$T = I \otimes V + V \otimes I$ and $W = 10(I \otimes V_c + V_c \otimes I) + 9(\mathbf{e}_1 \mathbf{e}_m^T + \mathbf{e}_m \mathbf{e}_1^T) \otimes I$ with $V = \text{tridiag}(-1, 2, -1) \in \mathbb{R}^{m \times m}$, $V_c = V - \mathbf{e}_1 \mathbf{e}_m^T - \mathbf{e}_m \mathbf{e}_1^T \in \mathbb{R}^{m \times m}$, and \mathbf{e}_1 and \mathbf{e}_m are the first and the last unit vectors in \mathbb{R}^m , respectively. We take the right-hand side vector b to be $\mathbf{b} = (1 + \iota)\mathbf{A}\mathbf{1}$, with $\mathbf{1}$ being the vector of all entries equal to one.

Here T and W correspond to the five-point centered difference matrices approximating the negative Laplacian operator with homogeneous Dirichlet boundary conditions and periodic boundary conditions, respectively, on a uniform mesh in the unit squares $[0, 1] \times [0, 1]$ with the mesh-size $h = 1/(m + 1)$.

Example 4 We consider the complex Helmholtz equation (cf. [9])

$$-\Delta u + \sigma_1 u + \iota \sigma_2 u = f,$$

where σ_1 and σ_2 are real coefficient functions, u satisfies Dirichlet boundary conditions in $D = [0, 1] \times [0, 1]$. We discretize the problem with finite differences on a $m \times m$ grid with mesh-size $h = 1/(m + 1)$. This leads to a system of linear equations

$$((K + \sigma_1 I) + \iota \sigma_2 I)\mathbf{x} = \mathbf{b},$$

where $K = I \otimes V_m + V_m \otimes I$ is the discretization of $-\Delta$ by means of centered differences, wherein $V_m = h^{-2} \text{tridiag}(-1, 2, -1) \in \mathbb{R}^{m \times m}$. The right-hand side vector b is taken to be $\mathbf{b} = (1 + \iota)\mathbf{A}\mathbf{1}$, with $\mathbf{1}$ being the the vector of all entries equal to one. Furthermore, before solving the system we normalize the coefficient matrix and the right-hand side vector by multiplying both by h^2 . For the numerical tests we set $\sigma_1 = \sigma_2 = 100$.

The experimentally found optimal iteration parameters α_{exp} of the MHSS, GSOR, PMHSS, MRHSS and MRMHSS iteration methods for all the numerical examples with different mesh numbers are listed in Table 1. We observe that, for all the iteration methods and all the numerical examples except for the PMHSS iteration method, the optimal parameters α_{exp} decrease as the mesh number increasing. In addition, almost for all the numerical examples with different mesh numbers, the optimal parameter values of the MRHSS and the MRMHSS iteration methods are much smaller than those of the MHSS, the PMHSS and the GSOR iteration methods.

In Table 2, we list numerical results of the four tested iteration methods for all the four examples. From the numerical results, we see that all the tested iteration methods are convergent within k_{max} iteration steps for all the four examples with different mesh numbers m . For Examples 1, 2 and 4, the GSOR and the MRHSS iteration methods perform much better than the MHSS iteration method no matter in iteration number or in computing time. However, among the PMHSS, the GSOR and the MRHSS iteration methods, the MRHSS becomes the worst one for Example 3 when mesh number is large. Meanwhile, the PMHSS costs more computing time than the MHSS for Example 4 since in this example the matrix W is more complicated to decompose than the matrix T . Finally, we should note that the MRMHSS

Table 1 The optimal parameters α_{exp} for the MHSS, GSOR, MRHSS and MRMHSS methods

Example	Method	Grid			
		64×64	128×128	256×256	512×512
No. 1	MHSS	0.33	0.25	0.18	0.16
	GSOR	0.457	0.432	0.428	0.412
	PMHSS	1.35	1.05	1.44	1.154
	MRHSS	0.05	0.02	0.01	0.002
	MRMHSS	0.05	0.02	0.01	0.005
No. 2	MHSS	0.024	0.006	0.0015	0.0004
	GSOR	0.455	0.455	0.455	0.457
	PMHSS	0.93	1.10	0.97	0.91
	MRHSS	0.0064	0.001	0.0005	0.0001
	MRMHSS	0.0048	0.0005	0.0005	0.0001
No. 3	MHSS	0.415	0.166	0.07	0.054
	GSOR	0.566	0.353	0.199	0.105
	PMHSS	0.57	0.78	0.73	0.73
	MRHSS	0.19	0.082	0.05	0.034
	MRMHSS	0.165	0.145	0.105	0.065
No. 4	MHSS	0.03	0.005	0.002	0.0004
	GSOR	0.862	0.862	0.862	0.862
	PMHSS	0.93	0.93	0.94	0.86
	MRHSS	0.03	0.005	0.002	0.0002
	MRMHSS	0.02	0.005	0.002	0.0002

iteration method discussed in this work is always the most efficient one for all the four tested examples, since it costs the least computing time and the number of iteration steps to achieve convergence. Moreover, for Examples 1, 2 and 4, the number of iteration steps of the MRMHSS, the MRHSS and the PMHSS methods are nearly h -independent.

In Fig. 1, we draw the curves of the computing times of the three iteration methods, i.e., MHSS, MRHSS and MRMHSS, with respect to the values of iteration parameter α when $m = 128$. The curves vividly illustrate that the MRMHSS iteration method always takes much less computing times than the MHSS and the MRHSS iteration methods in all tested cases. In addition, compared with the MHSS method, the MRMHSS and the MRHSS methods are not very sensitive to the value of parameter α . The MRMHSS iteration method always achieves its minimum computing time when α becomes very small, which is similar to the MRHSS iteration method.

Therefore, we can conclude that the MRMHSS iteration method discussed in this work is very efficient and robust when applied to solve the complex symmetric systems of linear equations (1).

Table 2 The numerical results of the tested iteration methods with respective optimal parameters

Example	Method		Grid			
			64 × 64	128 × 128	256 × 256	512 × 512
No. 1	MHSS	IT	134	172	222	332
		CPU	0.7134	5.6440	34.3782	356.7656
	GSOR	IT	23	25	46	26
		CPU	0.2162	1.2450	9.7464	30.2256
	PMHSS	IT	21	22	21	21
		CPU	0.1337	0.6093	3.1509	20.2668
	MRHSS	IT	7	6	6	5
		CPU	0.0544	0.3705	0.8704	4.5600
	MRMHSS	IT	5	5	5	4
		CPU	0.0499	0.2331	0.7859	4.4229
No. 2	MHSS	IT	110	99	88	77
		CPU	0.6464	3.0333	14.0404	85.0229
	GSOR	IT	22	22	22	22
		CPU	0.2025	1.0749	4.1669	32.0400
	PMHSS	IT	40	40	40	40
		CPU	0.2140	1.1011	5.7979	40.9831
	MRHSS	IT	20	21	21	19
		CPU	0.1196	0.5336	2.9660	19.9335
	MRMHSS	IT	15	14	16	14
		CPU	0.0949	0.4640	2.6106	14.8514
No. 3	MHSS	IT	58	84	106	254
		CPU	0.3372	3.2627	22.4705	448.6054
	GSOR	IT	19	34	70	130
		CPU	0.1511	1.5697	19.6162	260.1505
	PMHSS	IT	32	32	35	36
		CPU	0.2399	1.5121	9.0656	59.8396
	MRHSS	IT	25	60	176	526
		CPU	0.1354	1.8167	27.1334	696.2792
	MRMHSS	IT	26	45	91	251
		CPU	0.0988	1.04584	11.4727	257.3556
No. 4	MHSS	IT	37	40	41	40
		CPU	0.0929	0.9415	3.3191	29.3439
	GSOR	IT	7	7	7	6
		CPU	0.0490	0.2499	0.9761	7.5036
	PMHSS	IT	39	40	40	40
		CPU	0.1505	1.1340	6.0465	40.6596
	MRHSS	IT	12	12	13	11
		CPU	0.0375	0.2139	1.0963	8.0725
	MRMHSS	IT	10	10	11	8
		CPU	0.0313	0.1772	0.8975	5.6121

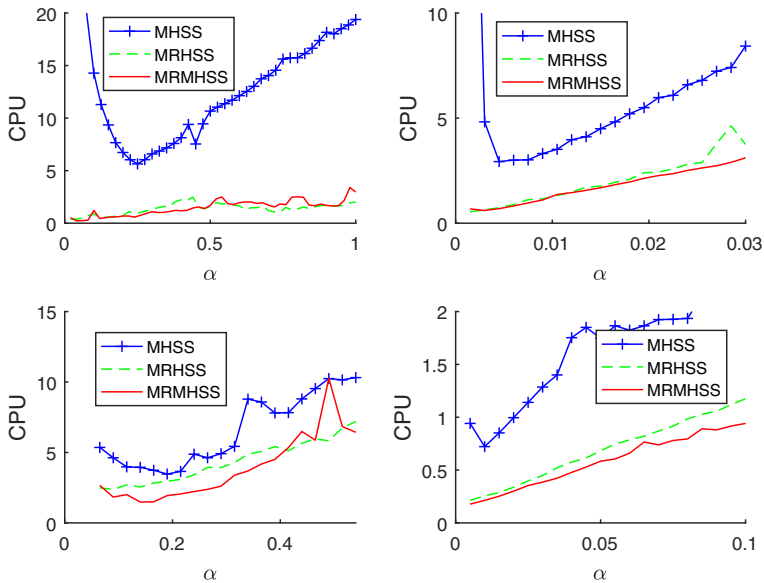


Fig. 1 Computing times vs iteration parameter α for the MHSS, MRHSS and MRMHSS methods of four tested examples; *top-left*: Example 1, *top-right*: Example 2, *down-left*: Example 3, *down-right*: Example 4

5 Conclusions

For the large sparse complex symmetric system of linear equations (1), we proposed an efficient iteration method named as minimum residual MHSS (MRMHSS) iteration method, by applying the minimum residual technique to the MHSS iteration scheme. Although the MRMHSS method involves two more iteration parameters than the classical MHSS iteration scheme, they can be computed easily and automatically. The convergence property of the MRMHSS iteration method was carefully studied, and the numerical results illustrated that the MRMHSS is very robust and powerful.

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