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ORIGINAL PAPER

Superconvergence analysis of two-grid methods for bacteria equations

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Abstract

In this paper, two-grid methods (TGMs) are developed for a system of reaction-diffusion equations of bacterial infection with initial and boundary conditions. The backward Euler (B-E) and Crank–Nicolson (C-N) fully discrete schemes are established, and the existence and uniqueness of the solutions of these schemes are proved. Moreover, based on the combination technique of the interpolation and Ritz projection and derivative transfer trick which are important ingredients in the TGMs, the superclose estimates of order $O(h^2 + H^4 + \tau)$ and $O(h^2 + H^4 + \tau^2)$ in H^1-norm are deduced for the above schemes, respectively, where h is fine mesh size, H is coarse mesh size, and τ is time step size. Then, by the interpolated postprocessing approach, the corresponding global superconvergence results are obtained. Finally, some other popular finite elements are discussed and numerical results are provided to verify the theoretical analysis, which show that the computing cost of TGMs are only half of Galerkin finite element methods (FEMs) for the test problem.

Keywords Bacteria equations · B-E and C-N schemes · Galerkin FEMs and TGMs · Supercloseness and superconvergence

1 Introduction

We consider the following bacteria equations [1]:

$$\begin{cases} u_t = d_1 \Delta u - a_{11} u + a_{12} v, & (\mathbf{x}, t) \in \Omega \times J, \\ v_t = d_2 \Delta v - a_{22} v + g(u), & (\mathbf{x}, t) \in \Omega \times J, \\ u(\mathbf{x}, t) = v(\mathbf{x}, t) = 0, & (\mathbf{x}, t) \in \Omega \times J, \\ u(\mathbf{x}, 0) = u_0(\mathbf{x}), \ v(\mathbf{x}, 0) = v_0(\mathbf{x}), & \mathbf{x} \in \Omega, \end{cases} \tag{1}$$

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where $\Omega \subset R^2$ is a rectangle with boundaries parallel to the axis, $J = (0, T], T > 0$. Here, u and v represent the average concentration of bacteria and the infective human, respectively. Hence, $-a_{11}u$ describes the natural mortality rate of u, $a_{12}v$ is the growth rate of u due to v, and $-a_{12}v$ is the natural damping of v due to the finite duration of infectiousness of humans. $g(u)$ represents the force of infection of u against v, which is twice continuously differentiable functional and satisfies the Lipschitz condition. We assume that the initial values $u_0(\mathbf{x})$ and $v_0(\mathbf{x})$ are given smooth functions. At the same time, d_1, d_2, a_{11}, a_{12}, and a_{22} are positive constants.

The problem (1) established the spatial transmission model of bacteria under the environment pollution of human, which has important research significance and there have been some works about it. For instance, the existence of periodic plane wave solutions was analyzed in [2], and the asymptotic stability of plane wave solutions in the bounded and unbounded domains were studied for 1D case in [3]. Moreover, the nonconforming FEM with EQ_1^{rot} element was applied to this problem in [4], and optimal error estimate with order of $O(h^2)$ in L^2-norm and the superclose estimate with order of $O(h^2)$ in H^1-norm were deduced for the semi-discrete scheme, respectively. Besides, the superclose estimate of order $O(h^2+\tau)$ for the C-N fully discrete scheme was also obtained. The backward and central difference finite element schemes were proposed and their priori error estimates in L^2-norm were derived with order of $O(h^2+\tau)$ and $O(h^2+\tau^2)$, respectively in [5]. Meanwhile, the existence, uniqueness and stability of traditional solutions under different boundary conditions were studied in [6–9]. However, there is no consideration on the estimations of the solutions u and v on the energy norm.

As we know, TGM is a very efficient algorithm for solving nonlinear PDEs [10–12], in which the nonlinear problem is first solved on the coarse mesh with size H and then a simple linearized scheme (one Newton like iteration) is dealt with on the fine mesh with size h ($h \ll H$). Later on, some developments on this aspect were also achieved, such as unconditional optimal error estimate of order $O(h + H^3 + \tau)$ in H^1-norm and order $O(h^2 + H^3 + \tau)$ in L^2-norm were deduced in [13] by introducing an auxiliary time discrete system, and the superclose estimate in H^1-norm was improved to order $O(h^2 + H^4 + \tau)$ in [14]. Furthermore, TGM also has been applied to the reaction–diffusion problems [15, 16], Sobolev problems [17], Navier–Stokes problems [18, 19], Maxwell's problems [20], and so on.

In this paper, we shall take the bilinear finite element, for example, to develop two-grid algorithms for the problem (1), and then to investigate their superconvergent behavior in H^1-norm through the combination skill of the interpolation and Ritz projection of [21] and derivative transfer trick.

The remainder of the paper is organized as follows. In Section 2, we give the B-E and C-N schemes and their briefly proofs of the existence and uniqueness of solutions, and then deduce the superclose estimates for the above two schemes in H^1-norm, respectively. In Sections 3–4, we present the approximate schemes with TGMs and study their superconvergence properties. In the last section, we provide some numerical results to verify the theoretical analysis.

Through out this paper, without ambiguity, we use the standard Banach space $W^{m,p}(\Omega)$ with norm $\|\cdot\|_{m,p}$ and if $p = 2$, we can simply denote $W^{k,2}(\Omega)$ by $H^k(\Omega)$

with the inner product $(\cdot,\cdot)$. Then, we equip a function space $L^p(J;X)$ with the norm $\|g\|_{L^p(J;X)} := \left(\int_J \|g(\cdot,t)\|_X^p \,\mathrm{d}t\right)^{\frac{1}{p}}$ and if $p=\infty$, the integral is replaced by the essential supremum. The generic constant $C>0$ is independent of n (time level), H, h, and τ, and may have different values in different places.

2 Superclose estimates for Galerkin FEMs

2.1 B-E scheme case

Let $\mathscr{T}_h$ be a regular rectangular partition of Ω with h, and $\tilde{V}_h^0$ be the bilinear finite element space which vanishes on $\partial\Omega$. We define the Ritz projection operator R_h : $H_0^1(\Omega)\to\tilde{V}_h^0$ as follows:

$$(\nabla(R_h w - w), \nabla v_h) = 0, \quad \forall v_h \in \tilde{V}_h^0. \tag{2}$$

At the same time, let I_h be the associated interpolation operator over $\tilde{V}_h^0$, then, for $w \in H_0^1(\Omega)\bigcap H^3(\Omega)$, we can obtain the following estimates according lemma 2 in [21]:

$$\|w - R_h w\|_0 + h\|\nabla(w - R_h w)\|_0 \le Ch^2\|w\|_2, \tag{3}$$

$$\|I_h w - R_h w\|_1 \le Ch^2\|w\|_3. \tag{4}$$

The weak formulation of the problem (1) is to find $(u,v): J\to H_0^1(\Omega)\times H_0^1(\Omega)$, such that

$$\begin{cases} (u_t,\phi) + d_1(\nabla u,\nabla\phi) + a_{11}(u,\phi) - a_{12}(v,\phi) = 0, & \forall\phi\in H_0^1(\Omega),\\ (v_t,\psi) + d_2(\nabla v,\nabla\psi) + a_{22}(v,\psi) = (g(u),\psi), & \forall\psi\in H_0^1(\Omega),\\ u(\mathbf{x},0) = u_0(\mathbf{x}), \quad v(\mathbf{x},0) = v_0(\mathbf{x}), \quad \mathbf{x}\in\Omega. \end{cases} \tag{5}$$

Let $0=t_0<t_1<\ldots<t_N=T$ be a uniform partition on $[0,T]$ with τ and $\varphi^n = \varphi(\mathbf{x},t_n)$, for φ is a generalized function. Then, we may pose the following B-E fully discrete scheme of the problem (1) to find $\left(U_h^n, V_h^n\right)\in\tilde{V}_h^0\times\tilde{V}_h^0$ for $n=1,2\ldots,N$, such that

$$\begin{cases} \left(D_\tau U_h^n,\phi_h\right) + d_1\left(\nabla U_h^n,\nabla\phi_h\right) + a_{11}\left(U_h^n,\phi_h\right) - a_{12}\left(V_h^n,\phi_h\right) = 0, & \forall\phi_h\in\tilde{V}_h^0,\\ \left(D_\tau V_h^n,\psi_h\right) + d_2\left(\nabla V_h^n,\nabla\psi_h\right) + a_{22}\left(V_h^n,\psi_h\right) = \left(g\left(U_h^n\right),\psi_h\right), & \forall\psi_h\in\tilde{V}_h^0,\\ U_h^0 = R_h u_0(\mathbf{x}), \quad V_h^0 = R_h v_0(\mathbf{x}), \quad \mathbf{x}\in\Omega, \end{cases} \tag{6}$$

where $D_\tau\varphi^n = \left(\varphi^n - \varphi^{n-1}\right)/\tau$.

Now, we define a new finite element space $\mathbf{W}_h = \left\{\mathbf{x}_h = (U_h, V_h)^T : U_h, V_h\in\tilde{V}_h^0\right\}$ endowed with the scalar product $(\mathbf{x}1_h,\mathbf{x}2_h)_* := (U1_h, U2_h) + (V1_h, V2_h)$ where $\mathbf{x}i_h := (Ui_h, Vi_h)$ $(i=1,2)$ and the norm $\|\mathbf{x}_h\|_*^2 := \|U_h\|_0^2 + \|V_h\|_0^2$. Then, let

$$F(\mathbf{x}_h) = \begin{pmatrix} F_1(U_h,V_h)\\ F_2(U_h,V_h)\end{pmatrix} := \begin{pmatrix} -a_{11}U_h + a_{12}V_h\\ g(U_h) - a_{22}V_h\end{pmatrix}, \quad \mathbf{x}_h = \begin{pmatrix} U_h\\ V_h\end{pmatrix}.$$

Obviously, the function $F(\cdot)$ satisfies the Lipschitz condition, and the problem (6) can be rewritten as to find $\mathbf{x}_h^n \in \mathbf{W}_h$ for $n = 1, 2 \ldots, N$, such that

$$\begin{cases} \left(D_\tau \mathbf{x}_h^n, \Phi_h\right)_* + \left(D^* \nabla \mathbf{x}_h^n, \nabla \Phi_h\right)_* = \left(F\left(\mathbf{x}_h^n\right), \Phi_h\right)_*, \quad \forall \Phi_h \in \mathbf{W}_h, \\ \mathbf{x}_h^0 = \left(U_h^0, V_h^0\right), \end{cases} \tag{7}$$

where $D^* = \mathrm{diag}(d_1, d_2)$.

Throughout this paper, we assume that the solution (u, v) to problem (1) exists and satisfies the following:

$$\begin{aligned} &\|u\|_{L^\infty(J;H^3(\Omega))} + \|v\|_{L^\infty(J;H^3(\Omega))} + \|u_t\|_{L^2(J;H^2(\Omega))} + \|v_t\|_{L^2(J;H^2(\Omega))} \\ &+\|u_{\mathrm{tt}}\|_{L^\infty(J;H^1(\Omega))} + \|v_{\mathrm{tt}}\|_{L^\infty(J;H^1(\Omega))} + \|u_{\mathrm{ttt}}\|_{L^\infty(J;H^1(\Omega))} \\ &+\|v_{\mathrm{ttt}}\|_{L^\infty(J;H^1(\Omega))} \le C \end{aligned} \tag{8}$$

Theorem 1 *The problem (7) has a unique solution (U_h^n, V_h^n).*

Proof It is similar to the proof of [22] (see pages 236–237).

Now, we consider the following theorem which gives the superclose estimate of the problem (6). □

Theorem 2 *Let (u, v) and $\left(U_h^n, V_h^n\right)$ be the solutions of (5) and (6), respectively, then we have the following:*

$$\left\|U_h^n - I_h u^n\right\|_1 + \left\|V_h^n - I_h v^n\right\|_1 \le \quad C(h^2 + \tau).$$

Proof Let $t = t_n$ in (5), we get the following:

$$\begin{cases} (D_\tau u^n, \phi_h) + d_1 (\nabla u^n, \nabla \phi_h) + a_{11}(u^n, \phi_h) - a_{12}(v^n, \phi_h) = \left(r_1^n, \phi_h\right), \quad \forall \phi_h \in \tilde{V}_h^0, \\ (D_\tau v^n, \psi_h) + d_2(\nabla v^n, \nabla \psi_h) + a_{22}(v^n, \psi_h) = (g(u^n), \psi_h) + \left(r_2^n, \psi_h\right), \ \forall \psi_h \in \tilde{V}_h^0, \end{cases} \tag{9}$$

where

$$r_1^n = D_\tau u^n - u_t^n, \quad \text{and} \quad \left\|r_1^n\right\|_0^2 \le \tau \|u_{\mathrm{tt}}\|^2_{L^2([t_{n-1}, t_n]; L^2(\Omega))}, \tag{10}$$

$$r_2^n = D_\tau v^n - v_t^n, \quad \text{and} \quad \left\|r_2^n\right\|_0^2 \le \tau \|v_{\mathrm{tt}}\|^2_{L^2([t_{n-1}, t_n]; L^2(\Omega))}. \tag{11}$$

Denote $u^n - U_h^n = u^n - R_h u^n + R_h u^n - U_h^n := \eta^n + \xi^n$, $\;v^n - V_h^n = v^n - R_h v^n + R_h v^n - V_h^n := \chi^n + \gamma^n$.

Then, subtracting (6) from (9), we obtain the following:

$$\begin{aligned} &(D_\tau \xi^n, \phi_h) + d_1(\nabla \xi^n, \nabla \phi_h) + a_{11}(\xi^n, \phi_h) - a_{12}(\gamma^n, \phi_h) \\ =\; & -(D_\tau \eta^n, \phi_h) - d_1(\nabla \eta^n, \nabla \phi_h) - a_{11}(\eta^n, \phi_h) + a_{12}(\chi^n, \phi_h) + \left(r_1^n, \phi_h\right), \end{aligned} \tag{12}$$

$$\begin{aligned} &(D_\tau \gamma^n, \psi_h) + d_2(\nabla \gamma^n, \nabla \psi_h) + a_{22}(\gamma^n, \psi_h) \\ =\; & -(D_\tau \chi^n, \psi_h) - d_2(\nabla \chi^n, \nabla \psi_h) - a_{22}(\chi^n, \psi_h) + \left(g(u^n) - g\left(U_h^n\right), \psi_h\right) \\ & + \left(r_2^n, \psi_h\right). \end{aligned} \tag{13}$$

Noticing that,

$$(\nabla \xi^n, D_\tau \nabla \xi^n) = \left(\|\nabla \xi^n\|_0^2 - \|\nabla \xi^{n-1}\|_0^2 + \|\nabla \xi^n - \nabla \xi^{n-1}\|_0^2 \right) / (2\tau), \quad (14)$$

$$(\nabla \gamma^n, D_\tau \nabla \gamma^n) = \left(\|\nabla \gamma^n\|_0^2 - \|\nabla \gamma^{n-1}\|_0^2 + \|\nabla \gamma^n - \nabla \gamma^{n-1}\|_0^2 \right) / (2\tau). \quad (15)$$

Then, let $\phi_h = D_\tau \xi^n$ in (12) and $\psi_h = D_\tau \gamma^n$ in (13) to have the following:

$$\begin{aligned} &\|D_\tau \xi^n\|_0^2 + d_1 \left(\|\nabla \xi^n\|_0^2 - \|\nabla \xi^{n-1}\|_0^2 \right) / (2\tau) + a_{11} \left(\|\xi^n\|_0^2 - \|\xi^{n-1}\|_0^2 \right) / \tau \\ \le\ & | - d_1(\nabla \eta^n, D_\tau \nabla \xi^n) - (D_\tau \eta^n, D_\tau \xi^n) - a_{11}(\eta^n, D_\tau \xi^n) + a_{12}(\gamma^n + \chi^n, D_\tau \xi^n) \\ &+ (r_1^n, D_\tau \xi^n)| := \sum_{i=1}^{5} D_i, \end{aligned} \quad (16)$$

$$\begin{aligned} &\|D_\tau \gamma^n\|_0^2 + d_2 \left(\|\nabla \gamma^n\|_0^2 - \|\nabla \gamma^{n-1}\|_0^2 \right) / (2\tau) + a_{22} \left(\|\gamma^n\|_0^2 - \|\gamma^{n-1}\|_0^2 \right) / \tau \\ \le\ & \left| -d_2(\nabla \chi^n, D_\tau \nabla \gamma^n) - (D_\tau \chi^n, D_\tau \gamma^n) - a_{22}(\chi^n, D_\tau \gamma^n) \right. \\ &\left. + \left(g(u^n) - g\left(U_h^n\right), D_\tau \gamma^n \right) + \left(r_2^n, D_\tau \gamma^n \right) \right| := \sum_{i=1}^{5} E_i. \end{aligned} \quad (17)$$

Here, by use of (2), we have $D_1 = 0$, $E_1 = 0$. And observing that $\|D_\tau \phi^n\|_0^2 \le \tau^{-1} \int_{t_{n-1}}^{t_n} \|\phi_t^n\|_0^2 \,\mathrm{ds}$ (see [13]) and (3), we can get by Cauchy inequality and Young inequality that,

$$\begin{aligned} & |\sum_{i=2}^{4} D_i + \sum_{i=2}^{3} E_i| \\ \le\ & Ch^4 \left(a_{11}^2 \|u^n\|_2^2 + \left(2a_{12}^2 + a_{22}^2 \right) \|v^n\|_2^2 + \tau^{-1} \int_{t_{n-1}}^{t_n} \left(\|u_t\|_2^2 + \|v_t\|_2^2 \right) \mathrm{ds} \right) \\ & + 2a_{12}^2 \|\gamma^n\|_0^2 + \frac{3}{4} \|D_\tau \xi^n\|_0^2 + \frac{1}{4} \|D_\tau \gamma^n\|_0^2 \\ \le\ & Ch^4 \left(\|u^n\|_2^2 + \|v^n\|_2^2 + \tau^{-1} \int_{t_{n-1}}^{t_n} \left(\|u_t\|_2^2 + \|v_t\|_2^2 \right) \mathrm{ds} \right) + 2a_{12}^2 \|\gamma^n\|_0^2 \\ & + \frac{3}{4} \|D_\tau \xi^n\|_0^2 + \frac{1}{4} \|D_\tau \gamma^n\|_0^2. \end{aligned}$$

Considering (10)–(11), we find that

$$\begin{aligned} |D_5 + E_5| \le\ & \tau \left(\|u_{\mathrm{tt}}\|_{L^2([t_{n-1}, t_n]; L^2(\Omega))}^2 + \|v_{\mathrm{tt}}\|_{L^2([t_{n-1}, t_n]; L^2(\Omega))}^2 \right) \\ & + \frac{1}{4} \|D_\tau \xi^n\|_0^2 + \frac{1}{4} \|D_\tau \gamma^n\|_0^2. \end{aligned}$$

Under the assumption of $g(\cdot)$, we have at once the following:

$$\begin{aligned} |E_4| \le\ & c\|u^n - U_h^n\|_0 \, \|D_\tau \gamma^n\|_0 \le 2c^2 \left(\|\xi^n\|_0^2 + \|\eta^n\|_0^2 \right) + \frac{1}{4} \|D_\tau \gamma^n\|_0^2 \\ \le\ & Ch^4 \|u^n\|_2^2 + 2c^2 \|\xi^n\|_0^2 + \frac{1}{4} \|D_\tau \gamma^n\|_0^2. \end{aligned}$$

Then, substituting above results into (16)–(17), we can get the following:

$$
\begin{aligned}
& c_0\left(\|\xi^n\|_1^2-\|\xi^{n-1}\|_1^2\right)/(2\tau)+c_0\left(\|\gamma^n\|_1^2-\|\gamma^{n-1}\|_1^2\right)/(2\tau)\\
\le\ & Ch^4\left(\tau^{-1}\int_{t_{n-1}}^{t_n}\left(\|u_t\|_2^2+\|v_t\|_2^2\right)\mathrm{ds}+\|u^n\|_2^2+\|v^n\|_2^2\right)\\
& +C\tau\left(\|u_{\mathrm{tt}}\|^2_{L^2([t_{n-1},t_n];L^2(\Omega))}+\|u_{\mathrm{tt}}\|^2_{L^2([t_{n-1},t_n];L^2(\Omega))}\right)\\
& +2a_{12}^2\|\gamma^n\|_1^2+2c^2\|\xi^n\|_1^2, \qquad (18)
\end{aligned}
$$

where $c_0=\min\{d_1,d_2,2a_{11},2a_{12}\}$.

Multiplying (18) by 2τ and summing from $n=1,\ldots,m$ $(1\le m\le N)$, we have the following:

$$
\begin{aligned}
c_0\left(\|\xi^m\|_1^2+\|\gamma^m\|_1^2\right)\le\ & Ch^4\left(\|u_t\|^2_{L^2(J;H^2(\Omega))}+\|v_t\|^2_{L^2(J;H^2(\Omega))}\right.\\
& \left.+\|u\|^2_{L^\infty(J;H^2(\Omega))}+\|v\|^2_{L^\infty(J;H^2(\Omega))}\right)\\
& +C\tau^2\left(\|u_{\mathrm{tt}}\|^2_{L^\infty(J;L^2(\Omega))}+\|v_{\mathrm{tt}}\|^2_{L^\infty(J;L^2(\Omega))}\right)\\
& +4\tau a_{12}^2\sum_{n=1}^m\|\gamma^n\|_1^2+4c^2\tau\sum_{n=1}^m\|\xi^n\|_1^2.
\end{aligned}
$$

Thanks to discrete Gronwall's lemma, when $c_0-4a_{12}^2\tau>0$ and $c_0-4c^2\tau>0$, we have the following:

$$
\begin{aligned}
\|\xi^m\|_1^2+\|\gamma^m\|_1^2\le\ & Ch^4\left(\|u_t\|^2_{L^2(J;H^2(\Omega))}+\|v_t\|^2_{L^2(J;H^2(\Omega))}+\|u\|^2_{L^\infty(J;H^2(\Omega))}\right.\\
& \left.+\|v\|^2_{L^\infty(J;H^2(\Omega))}\right)+C\tau^2\left(\|u_{\mathrm{tt}}\|^2_{L^\infty(J;L^2(\Omega))}+\|v_{\mathrm{tt}}\|^2_{L^\infty(J;L^2(\Omega))}\right)\\
\le\ & C(h^4+\tau^2),
\end{aligned}
$$

which together with (4) completes the proof. □

2.2 C-N scheme case

For the purpose of obtaining higher accuracy in time, we shall pose the following C-N fully discrete scheme of the problem (1) to find $\left(U_h^n,V_h^n\right)\in\tilde V_h^0\times\tilde V_h^0$ for $n=1,2\ldots,N$, such that

$$
\begin{cases}
\left(D_\tau U_h^n,\phi_h\right)+d_1\left(\nabla\bar U_h^n,\nabla\phi_h\right)+a_{11}\left(\bar U_h^n,\phi_h\right)-a_{12}\left(\bar V_h^n,\phi_h\right)=0, & \forall\phi_h\in\tilde V_h^0,\\
\left(D_\tau V_h^n,\psi_h\right)+d_2\left(\nabla\bar V_h^n,\nabla\psi_h\right)+a_{22}\left(\bar V_h^n,\psi_h\right)=\left(g\left(\bar U_h^n\right),\psi_h\right), & \forall\psi_h\in\tilde V_h^0,\\
U_h^0=R_hu_0(\mathbf{x}),\quad V_h^0=R_hv_0(\mathbf{x}),\quad \mathbf{x}\in\Omega,
\end{cases}
\qquad (19)
$$

where $\bar\varphi^n=(\varphi^n+\varphi^{n-1})/2$.

Theorem 3 *The problem* (19) *has a unique solution* $\left(U_h^n,V_h^n\right)$.

Proof Similar to (7), we shall rewrite the problem (19) as to find $\mathbf{x}_h^n \in \mathbf{W}_h$ for $n = 1, 2 \ldots, N$, such that

$$\begin{cases} \left(D_\tau \mathbf{x}_h^n, \Phi_h\right)_* + \left(D^* \nabla \bar{\mathbf{x}}_h^n, \nabla \Phi_h\right)_* = \left(F\left(\bar{\mathbf{x}}_h^n\right), \Phi_h\right)_*, \quad \forall \Phi_h \in \mathbf{W}_h, \\ \mathbf{x}_h^0 = \left(U_h^0, V_h^0\right). \end{cases} \tag{20}$$

Here, we shall first multiply (20) by 2τ and remark it as $\left(G_h\left(\mathbf{x}_h^n\right), \Phi_h\right)_* = 0$, where $G_h : \mathbf{W}_h \to \mathbf{W}_h$ is a continuous functional. Then, with the Lipschitz continuous property of $F(\cdot)$, we have for given $\mathbf{x}_h^{n-1}$ as follows:

$$\begin{aligned}
(G_h(\Phi_h), \Phi_h)_* &= 2\left(\Phi_h - \mathbf{x}_h^{n-1}, \Phi_h\right)_* + \tau\left(D\nabla\left(\Phi_h + \mathbf{x}_h^{n-1}\right), \nabla\Phi_h\right)_* \\
&\quad -2\tau\left(F\left(\frac{\Phi_h + \mathbf{x}_h^{n-1}}{2}\right), \Phi_h\right)_* \\
&\geq \|\Phi_h\|_*^2 - \left\|\mathbf{x}_h^{n-1}\right\|_*^2 + \tau\left(\min_{\lambda\in\sigma(D)} |\lambda| \|\nabla\Phi_h\|_*^2\right. \\
&\quad \left. - \max_{\lambda\in\sigma(D)} |\lambda| \|\nabla\Phi_h\|_* \left\|\nabla\mathbf{x}_h^{n-1}\right\|_*\right) \\
&\quad -C\tau(\|F(0)\|_* + \|\Phi_h\|_* + \|\mathbf{x}_h^{n-1}\|_*)\|\Phi_h\|_* \\
&\geq \left(1 - \frac{3}{2}C\tau\right)\|\Phi_h\|_*^2 - (1 + C\tau)\left\|\mathbf{x}_h^{n-1}\right\|_*^2 - C\tau\|F(0)\|_*^2 \\
&\quad +\tau\left(\min_{\lambda\in\sigma(D)} |\lambda| \|\nabla\Phi_h\|_*^2 - \max_{\lambda\in\sigma(D)} |\lambda| \|\nabla\Phi_h\|_* \|\nabla\mathbf{x}_h^{n-1}\|_*\right) \\
&\geq \left(1 - \frac{3}{2}C\tau\right)\|\Phi_h\|_*^2 - (1 + C\tau)\|\mathbf{x}_h^{n-1}\|_*^2 - C\tau\|F(0)\|_*^2 \\
&\quad -\tau\left(\frac{\max_{\lambda\in\sigma(D)} |\lambda|}{4 \min_{\lambda\in\sigma(D)} |\lambda|} \left\|\nabla\mathbf{x}_h^{n-1}\right\|_*\right).
\end{aligned}$$

$q^2 = \frac{1}{2}\left[(1 + C\tau)\left\|\mathbf{x}_h^{n-1}\right\|_*^2 + C\tau\|F(0)\|_*^2 + \tau\left(\frac{\max_{\lambda\in\sigma(D)} |\lambda|}{4 \min_{\lambda\in\sigma(D)} |\lambda|}\left\|\nabla\mathbf{x}_h^{n-1}\right\|_*\right)\right]$ to ensure $(G_h(\Phi_h), \Phi_h)_* > 0$ for $\|\Phi_h\|_* = q$, when $\tau \leq \tau_0 < \frac{2}{3C}$.

Further, by the Brouwer's fixed point theorem, we see that the equation $G_h(\mathbf{x}) = 0$ has a solution $\mathbf{x} \in B_q = \{\Phi_h \in \mathbf{W}_h; \|\Phi_h\|_* \leq q\}$. In fact, if we assume that $G_h(\Phi_h) \neq 0$ in B_q, then the mapping $\tilde{G}_h(\Phi_h) = -qG_h(\Phi_h)/\|G_h(\Phi_h)\|_*$ is continuous from B_q to itself, and therefore has a fixed point $\tilde{\Phi}_h \in B_q$, with $q^2 = \|\tilde{\Phi}_h\|_*^2 = -q(G_h(\tilde{\Phi}_h, \tilde{\Phi}_h))_*/\|G_h(\tilde{\Phi}_h)\|_*$, which contradicts $(G_h(\tilde{\Phi}_h), \tilde{\Phi}_h)_* > 0$. So, there exists a solution $\mathbf{x}_h^n$ of (20) in B_q, namely the problem (19) has a solution $\left(U_h^n, V_h^n\right)$.

Now, we give a briefly proof of the uniqueness of the solution $\left(U_h^n, V_h^n\right)$ of the problem (19), when the solution (u, v) of the problem (1) is smooth and τ is sufficiently small. In fact, let $\mathbf{x}1 = (U_1, V_1)$ and $\mathbf{x}2 = (U_2, V_2)$ be two solutions of the problem (19). Then by subtraction, we have as follows:

$$2(\mathbf{x}_1 - \mathbf{x}_2, \phi_h)_* + \tau(D\nabla(\mathbf{x}_1 - \mathbf{x}_2), \nabla\phi_h)_*$$
$$= 2\tau\left(F\left(\frac{\mathbf{x}_1 + \mathbf{x}_h^{n-1}}{2}\right) - F\left(\frac{\mathbf{x}_2 + \mathbf{x}_h^{n-1}}{2}\right), \phi_h\right)_* .$$

Choosing $\phi_h = \mathbf{x}_1 - \mathbf{x}_2$, we find the following:

$$2\|\mathbf{x}_1 - \mathbf{x}_2\|_*^2 + \tau \min_{\lambda\in\sigma(D)} |\lambda| \, \|\nabla(\mathbf{x}_1 - \mathbf{x}_2)\|_*^2 \le C\tau\|\mathbf{x}_1 - \mathbf{x}_2\|_*^2.$$

When $\tau \le \tau_0$, we can conclude that $\|\mathbf{x}_1 - \mathbf{x}_2\|_* = 0$, thus, $U_1 = U_2$, $V_1 = V_2$, and the problem (19) has a unique solution $\left(U_h^n, V_h^n\right)$. The proof is completed. □

Theorem 4 *Let (u, v) and $\left(U_h^n, V_h^n\right)$ be the solutions of (5) and (19), respectively, then we have the following:*

$$\left\|U_h^n - I_h u^n\right\|_1 + \left\|V_h^n - I_h v^n\right\|_1 \le C(h^2 + \tau^2).$$

Proof Let $t = t_{n-\frac{1}{2}}$ in (5), then we can get the following:

$$\begin{cases} (D_\tau u^n, \phi_h) + d_1\left(\nabla u^{n-\frac{1}{2}}, \nabla\phi_h\right) + a_{11}\left(u^{n-\frac{1}{2}}, \phi_h\right) - a_{12}\left(v^{n-\frac{1}{2}}, \phi_h\right) = \left(r_1^{n-\frac{1}{2}}, \phi_h\right), & \forall\phi_h \in \tilde{V}_h^0, \\ (D_\tau v^n, \psi_h) + d_2\left(\nabla v^{n-\frac{1}{2}}, \nabla\psi_h\right) + a_{22}\left(v^{n-\frac{1}{2}}, \psi_h\right) = \left(g\left(u^{n-\frac{1}{2}}\right), \psi_h\right) + \left(r_2^{n-\frac{1}{2}}, \psi_h\right), & \forall\psi_h \in \tilde{V}_h^0, \end{cases} \tag{21}$$

where

$$r_1^{n-\frac{1}{2}} = D_\tau u^n - u_t^{n-\frac{1}{2}}, \quad \text{and} \quad \left\|r_1^{n-\frac{1}{2}}\right\|_0^2 \le \frac{\tau^3}{64}\|u_{ttt}\|_{L^2([t_{n-1},t_n];L^2(\Omega))}^2, \tag{22}$$

$$r_2^{n-\frac{1}{2}} = D_\tau v^n - v_t^{n-\frac{1}{2}}, \quad \text{and} \quad \left\|r_2^{n-\frac{1}{2}}\right\|_0^2 \le \frac{\tau^3}{64}\|v_{ttt}\|_{L^2([t_{n-1},t_n];L^2(\Omega))}^2. \tag{23}$$

We write as before $u^n - U_h^n = u^n - R_h u^n + R_h u^n - U_h^n := \eta^n + \xi^n$, $v^n - V_h^n = v^n - R_h v^n + R_h v^n - V_h^n := \chi^n + \gamma^n$.

Then subtracting (19) from (21) and choosing $\phi_h = D_\tau\xi^n$, $\psi_h = D_\tau\gamma^n$, we can get the following:

$$\begin{aligned} &(D_\tau\xi^n, D_\tau\xi^n) + d_1(\nabla\bar{\xi}^n, \nabla D_\tau\xi^n) + a_{11}(\bar{\xi}^n, D_\tau\xi^n) \\ &= -(D_\tau\eta^n, D_\tau\xi^n) - d_1(\nabla\bar{\eta}^n, \nabla D_\tau\xi^n) - a_{11}(\bar{\eta}^n, D_\tau\xi^n) + a_{12}(\bar{\chi}^n + \bar{\gamma}^n, D_\tau\xi^n) \\ &\quad + \left(r_1^{n-\frac{1}{2}}, D_\tau\xi^n\right) - d_1\left(\nabla\bar{u}^n - \nabla u^{n-\frac{1}{2}}, \nabla D_\tau\xi^n\right) - a_{11}\left(\bar{u}^n - u^{n-\frac{1}{2}}, D_\tau\xi^n\right) \\ &\quad + a_{12}\left(\bar{v}^n - v^{n-\frac{1}{2}}, D_\tau\xi^n\right) := \sum_{i=1}^{8} I_i^n, \end{aligned} \tag{24}$$

$$
\begin{aligned}
&(D_\tau \gamma^n, D_\tau \gamma^n) + d_2(\nabla \bar{\gamma}^n, \nabla D_\tau \gamma^n) + a_{22}(\bar{\gamma}^n, D_\tau \gamma^n) \\
= &-(D_\tau \chi^n, D_\tau \gamma^n) - d_2(\nabla \bar{\chi}^n, \nabla D_\tau \gamma^n) - a_{22}(\bar{\chi}^n, D_\tau \gamma^n) \\
&+ \left(g(\bar{u}^n) - g(\bar{U}_h^n), D_\tau \gamma^n\right) + \left(r_2^{n-\frac{1}{2}}, D_\tau \gamma^n\right) - d_2\left(\nabla \bar{v}^n - \nabla v^{n-\frac{1}{2}}, \nabla D_\tau \gamma^n\right) \\
&-a_{22}\left(\bar{v}^n - v^{n-\frac{1}{2}}, D_\tau \gamma^n\right) + \left(g\left(u^{n-\frac{1}{2}}\right) - g(\bar{u}^n), D_\tau \gamma^n\right) := \sum_{i=1}^{8} K_i^n. \qquad (25)
\end{aligned}
$$

Collecting (24)–(25), the left side of the equation (LHS) can be estimated as follows in the same way as (14)–(15):

$$
\begin{aligned}
\text{LHS} \geq\ & \|D_\tau \xi^n\|_0^2 + d_1\left(\|\nabla \xi^n\|_0^2 - \|\nabla \xi^{n-1}\|_0^2\right)/(2\tau) \\
&+a_{11}\left(\|\xi^n\|_0^2 - \|\xi^{n-1}\|_0^2\right)/(2\tau) + \|D_\tau \gamma^n\|_0^2 \\
&+d_2\left(\|\nabla \gamma^n\|_0^2 - \|\nabla \gamma^{n-1}\|_0^2\right)/(2\tau) \\
&+a_{22}\left(\|\gamma^n\|_0^2 - \|\gamma^{n-1}\|_0^2\right)/(2\tau). \qquad (26)
\end{aligned}
$$

So, our purpose now is thus to drive error estimates of I_i^n and K_i^n $(i = 1, \ldots, 8)$.

As in the estimation of D_i, E_j $(i = 1, \ldots, 4,\ j = 1, \ldots, 3)$, we have the following:

$$
\begin{aligned}
\left|\sum_{i=1}^{4} I_i^n + \sum_{i=1}^{3} K_i^n\right| \leq\ & Ch^4\left(\frac{3}{2\tau}\int_{t_{n-1}}^{t_n}\left(\|u_t\|_2^2 + \|v_t\|_2^2\right)\mathrm{ds} + \frac{3a_{11}^2}{2}\|\bar{u}^n\|_2^2\right. \\
&\left.+\left(3a_{12}^2 + 2a_{22}^2\right)\|\bar{v}^n\|_2^2\right) + \frac{3a_{12}^2}{2}\left(\|\gamma^n\|_0^2 + \|\gamma^{n-1}\|_0^2\right) \\
&+\frac{1}{2}\|D_\tau \xi^n\|_0^2 + \frac{1}{3}\|D_\tau \gamma^n\|_0^2 \\
\leq\ & Ch^4\left(\tau\int_{t_{n-1}}^{t_n}\left(\|u_t\|_2^2 + \|v_t\|_2^2\right)\mathrm{ds} + \|\bar{u}^n\|_2^2 + \|\bar{v}^n\|_2^2\right) \\
&+\frac{3a_{12}^2}{2}\left(\|\gamma^n\|_0^2 + \|\gamma^{n-1}\|_0^2\right) + \frac{1}{2}\|D_\tau \xi^n\|_0^2 + \frac{1}{3}\|D_\tau \gamma^n\|_0^2,
\end{aligned}
$$

and by use of (22)–(23), we can get that

$$
\begin{aligned}
\left|I_5^n + K_5^n\right| \leq\ & \frac{\tau^3}{128}\left(\|u_{ttt}\|_{L^2([t_{n-1},t_n];L^2(\Omega))}^2 + \|v_{ttt}\|_{L^2([t_{n-1},t_n];L^2(\Omega))}^2\right) \\
&+\frac{1}{6}\|D_\tau \xi^n\|_0^2 + \frac{1}{6}\|D_\tau \gamma^n\|_0^2.
\end{aligned}
$$

Then, we have the following estimation with $\|\bar{\phi}^n - \phi^{n-\frac{1}{2}}\|_0^2 \le \frac{\tau^3}{16}\int_{t_{n-1}}^{t_n} \|\phi_{\rm tt}\|_0 \,{\rm ds}$,

$$|I_7^n + I_8^n + K_7^n| \le \frac{3a_{11}^2\tau^3}{32}\left(\int_{t_{n-1}}^{t_n}\left(\|u_{\rm tt}\|_0^2 + \|v_{\rm tt}\|_0^2\right){\rm ds}\right) + \frac{1}{3}\|D_\tau\xi^n\|_0^2 + \frac{1}{6}\|D_\tau\gamma^n\|_0^2.$$

Furthermore, by the Lipschitz continuous property of $g(\cdot)$, K_4^n and K_8^n can be estimated as follow:

$$\begin{aligned}
\left|K_4^n + K_8^n\right| &\le c\left\|\bar{u}^n - \bar{U}_h^n\right\|_0\|D_\tau\gamma^n\|_0 + c\left\|\bar{u}^n - u^{n-\frac{1}{2}}\right\|_0\|D_\tau\gamma^n\|_0\\
&\le 3c^2\left(\|\bar{\xi}^n\|_0^2 + \|\bar{\eta}^n\|_0^2\right) + \frac{3c^2\tau^3}{32}\int_{t_{n-1}}^{t_n}\|u_{\rm tt}\|_0^2{\rm ds} + \frac{1}{3}\|D_\tau\gamma^n\|_0^2\\
&\le Ch^4\|\bar{u}^n\|_2^2 + C\tau^3\int_{t_{n-1}}^{t_n}\|u_{\rm tt}\|_0^2{\rm ds} + \frac{3c^2}{2}\left(\|\xi^n\|_0^2 + \|\xi^{n-1}\|_0^2\right)\\
&\quad + \frac{1}{3}\|D_\tau\gamma^n\|_0^2.
\end{aligned}$$

Hence, substituting above results into (24)–(25), we can get the following:

$$\begin{aligned}
&c_0\left(\|\xi^n\|_1^2 - \|\xi^{n-1}\|_1^2\right)/(2\tau) + c_0\left(\|\gamma^n\|_1^2 - \|\gamma^{n-1}\|_1^2\right)/(2\tau)\\
\le\ & 3a_{12}^2\left(\|\gamma^n\|_0^2 + \|\gamma^{n-1}\|_0^2\right) + \frac{3c^2}{2}\left(\|\xi^n\|_0^2 + \|\xi^{n-1}\|_0^2\right)\\
&+Ch^4\left(\tau^{-1}\int_{t_{n-1}}^{t_n}\left(\|u_t\|_2^2 + \|v_t\|_2^2\right){\rm ds} + \|\bar{u}^n\|_2^2 + \|\bar{v}^n\|_2^2\right)\\
&+C\tau^3\int_{t_{n-1}}^{t_n}\left(\|u_{\rm ttt}\|_0^2 + \|v_{\rm ttt}\|_0^2 + \|u_t\|_0^2 + \|v_t\|_0^2\right){\rm ds} + I_6^n + K_6^n.
\end{aligned}\tag{27}$$

Then, multiplying (27) by 2τ and summing from $n = 1, \ldots, m$ $(1 \le m \le N)$, we have the following:

$$\begin{aligned}
c_0\left(\|\xi^m\|_1^2 + \|\gamma^m\|_1^2\right) \le\ & Ch^4\left(\|u_t\|_{L^2(J;H^2(\Omega))}^2 + \|v_t\|_{L^2(J;H^2(\Omega))}^2\right.\\
&\left.+\|u\|_{L^\infty(J;H^2(\Omega))}^2 + \|v\|_{L^\infty(J;H^2(\Omega))}^2\right)\\
&+C\tau^4\left(\|u_{\rm ttt}\|_{L^\infty(J;L^2(\Omega))}^2 + \|v_{\rm ttt}\|_{L^\infty(J;L^2(\Omega))}^2\right)\\
&+3a_{12}^2\tau\sum_{n=1}^m\|\gamma^n\|_0^2 + 3c^2\tau\sum_{n=1}^m\|\xi^n\|_0^2 + 2\tau\sum_{n=1}^m I_6^n\\
&+2\tau\sum_{n=1}^m K_6^n.
\end{aligned}\tag{28}$$

By use of the derivative transfer trick, we can rewrite $2\tau \sum_{n=1}^{m} I_6^n$ as follows:

$$\begin{aligned}
2\tau \sum_{n=1}^{m} I_6^n &= 2\tau \sum_{n=1}^{m} \left\{ D_\tau \left(\nabla \left(\bar{u}^n - u^{n-\frac{1}{2}} \right), \nabla \xi^n \right) - \left(D_\tau \nabla \left(\bar{u}^n - u^{n-\frac{1}{2}} \right), \nabla \xi^{n-1} \right) \right\} \\
&= 2 \sum_{n=1}^{m} \left\{ \left(\nabla \left(\bar{u}^n - u^{n-\frac{1}{2}} \right), \nabla \xi^n \right) - \left(\nabla \left(\bar{u}^{n-1} - u^{n-\frac{3}{2}} \right), \nabla \xi^{n-1} \right) \right\} \\
&\quad -2 \sum_{n=1}^{m} \left\{ \left(\nabla \left(\bar{u}^n - u^{n-\frac{1}{2}} \right) - \nabla \left(\bar{u}^{n-1} - u^{n-\frac{3}{2}} \right), \nabla \xi^{n-1} \right) \right\} \\
&= 2 \left(\nabla \left(\bar{u}^m - u^{m-\frac{1}{2}} \right), \nabla \xi^m \right) \\
&\quad -2 \sum_{n=1}^{m} \left\{ \left(\nabla \left(\bar{u}^n - u^{n-\frac{1}{2}} \right) - \nabla \left(\bar{u}^{n-1} - u^{n-\frac{3}{2}} \right), \nabla \xi^{n-1} \right) \right\}. \qquad (29)
\end{aligned}$$

With Taylor expansion, we have the following:

$$\begin{aligned}
\bar{u}^n - u^{n-\frac{1}{2}} &= \frac{\tau^2}{4} + \frac{1}{4} \left(\int_{t_{n-\frac{1}{2}}}^{t_n} \left(s - t_{n-\frac{1}{2}} \right)^2 u_{\mathrm{ttt}}(s) \mathrm{d}s \right. \\
&\quad \left. + \int_{t_{n-\frac{1}{2}}}^{t_{n-1}} \left(s - t_{n-\frac{1}{2}} \right)^2 u_{\mathrm{ttt}}(s) \mathrm{d}s \right), \qquad (30) \\
\bar{u}^{n-1} - u^{n-\frac{3}{2}} &= \frac{\tau^2}{4} + \frac{1}{4} \left(\int_{t_{n-\frac{3}{2}}}^{t_{n-1}} \left(s - t_{n-\frac{3}{2}} \right)^2 u_{\mathrm{ttt}}(s) \mathrm{d}s \right. \\
&\quad \left. + \int_{t_{n-\frac{3}{2}}}^{t_{n-2}} \left(s - t_{n-\frac{3}{2}} \right)^2 u_{\mathrm{ttt}}(s) \mathrm{d}s \right), \qquad (31) \\
\frac{\tau^2}{4} \left(u_{\mathrm{tt}}^{n-\frac{1}{2}} - u_{\mathrm{tt}}^{n-\frac{3}{2}} \right) &= \frac{\tau^2}{4} \int_{t_{n-\frac{3}{2}}}^{t_{n-\frac{1}{2}}} u_{\mathrm{ttt}}(s) \mathrm{d}s. \qquad (32)
\end{aligned}$$

Then, by Cauchy inequality and Young inequality, we can get the following:

$$\begin{aligned}
&\sum_{n=1}^{m} \left\{ \left(\nabla \left(\bar{u}^n - u^{n-\frac{1}{2}} \right) - \nabla \left(\bar{u}^{n-1} - u^{n-\frac{3}{2}} \right), \nabla \xi^{n-1} \right) \right\} \\
&\leq \sum_{n=1}^{m} \left(C\tau^4 \| \nabla u_{\mathrm{ttt}} \|^2_{L^2([t_{n-1}, t_n]; L^2(\Omega))} + \tau \| \nabla \xi^{n-1} \|_0^2 \right) \\
&\leq C\tau^4 \| \nabla u_{\mathrm{ttt}} \|^2_{L^\infty(J; L^2(\Omega))} + \tau \sum_{n=1}^{m} \| \nabla \xi^{n-1} \|_0^2. \qquad (33)
\end{aligned}$$

So, we have the following:

$$2\tau\sum_{n=1}^{m} I_6^n \leq C\tau^4\left(\frac{1}{\sigma_1}\|\nabla u_{tt}\|^2_{L^\infty(J;L^2(\Omega))}+\|\nabla u_{ttt}\|^2_{L^\infty(J;L^2(\Omega))}\right) +\sigma_1\|\nabla\xi^m\|_0^2+2\tau\sum_{n=1}^{m}\|\nabla\xi^{n-1}\|_0^2. \quad (34)$$

Similarly, we can get the estimate as follows:

$$2\tau\sum_{n=1}^{m} K_6^n \leq C\tau^4\left(\frac{1}{\sigma_2}\|\nabla v_{tt}\|^2_{L^\infty(J;L^2(\Omega))}+\|\nabla v_{ttt}\|^2_{L^\infty(J;L^2(\Omega))}\right)+\sigma_2\|\nabla\gamma^m\|_0^2 +2\tau\sum_{n=1}^{m}\|\nabla\gamma^{n-1}\|_0^2. \quad (35)$$

Substituting (34) and (35) into (28) with $\sigma_i = c_0/2$,

$$\begin{aligned} c_0\left(\|\xi^m\|_1^2+\|\gamma^m\|_1^2\right)/2 \leq\ & Ch^4\left(\|u_t\|^2_{L^2(J;H^2(\Omega))}+\|v_t\|^2_{L^2(J;H^2(\Omega))}\right.\\ &\left.+\|u\|^2_{L^\infty(J;H^2(\Omega))}+\|v\|^2_{L^\infty(J;H^2(\Omega))}\right)\\ &+C\tau^4\left(\|u_{ttt}\|^2_{L^\infty(J;H^1(\Omega))}+\|v_{ttt}\|^2_{L^\infty(J;H^1(\Omega))}\right.\\ &\left.+\|\nabla u_{tt}\|^2_{L^\infty(J;L^2(\Omega))}+\|\nabla v_{tt}\|^2_{L^\infty(J;L^2(\Omega))}\right)\\ &+c_1\tau\sum_{n=1}^{m}\|\gamma^n\|_1^2+c_2\tau\sum_{n=1}^{m}\|\xi^n\|_1^2, \end{aligned} \quad (36)$$

where $c_1 = 3a_{12}^2+2,\ c_2 = 3c^2\tau+2$.

Thanks to discrete Gronwall's lemma, when $c_0/2 - c_1\tau > 0$ and $c_0/2 - c_2\tau > 0$, we have the following:

$$\begin{aligned} \|\xi^m\|_1^2+\|\gamma^m\|_1^2 \leq\ & Ch^4\left(\|u_t\|^2_{L^2(J;H^2(\Omega))}+\|v_t\|^2_{L^2(J;H^2(\Omega))}\right.\\ &\left.+\|u\|^2_{L^\infty(J;H^2(\Omega))}+\|v\|^2_{L^\infty(J;H^2(\Omega))}\right)\\ &+C\tau^4\left(\|u_{ttt}\|^2_{L^\infty(J;H^1(\Omega))}+\|v_{ttt}\|^2_{L^\infty(J;H^1(\Omega))}\right.\\ &\left.+\|\nabla u_{tt}\|^2_{L^\infty(J;L^2(\Omega))}+\|\nabla v_{tt}\|^2_{L^\infty(J;L^2(\Omega))}\right)\\ \leq\ & C(h^4+\tau^4), \end{aligned}$$

which together with (4) completes the proof. □

3 Superclose estimates for TGMs

3.1 B-E scheme case

We define another bilinear finite element space $\tilde{V}_H^0 \subset \tilde{V}_h^0$ $(h \ll H \ll 1)$ on the coarse grid. Then, TGM for the B-E scheme can be described as follows.

Step 1: On the coarse grid $\mathscr{T}_H$, for $n = 1, \ldots, N$, solve $\left(U_H^n, V_H^n\right) \in \tilde{V}_H^0 \times \tilde{V}_H^0$ for the following nonlinear system, such that

$$\begin{cases} \left(D_\tau U_H^n, \phi_H\right) + d_1 \left(\nabla U_H^n, \nabla \phi_H\right) + a_{11} \left(U_H^n, \phi_H\right) - a_{12} \left(U_H^n, \phi_H\right) = 0, & \forall \phi_H \in \tilde{V}_H^0, \\ \left(D_\tau V_H^n, \psi_H\right) + d_2 \left(\nabla V_H^n, \nabla \psi_H\right) + a_{22} \left(V_H^n, \psi_H\right) = \left(g\left(U_H^n\right), \psi_H\right), & \forall \psi_H \in \tilde{V}_H^0, \\ U_H^0 = R_H u^0, \ V_H^0 = R_H v^0. \end{cases} \tag{37}$$

Step 2: On the fine grid $\mathscr{T}_h$, for $n = 1, \ldots, N$, solve $\left(U_h^n, V_h^n\right) \in \tilde{V}_h^0 \times \tilde{V}_h^0$ for the following linearized system, such that

$$\begin{cases} \left(D_\tau U_h^n, \phi_h\right) + d_1 \left(\nabla U_h^n, \nabla \phi_h\right) + a_{11} \left(U_h^n, \phi_h\right) - a_{12} \left(U_h^n, \phi_h\right) = 0, & \forall \phi_h \in \tilde{V}_h^0, \\ \left(D_\tau V_h^n, \psi_h\right) + d_2 \left(\nabla V_h^n, \nabla \psi_h\right) + a_{22} \left(V_h^n, \psi_h\right) = \left(g\left(U_H^n\right) + g'\left(U_H^n\right)\left(U_h^n - U_H^n\right), \psi_h\right), & \forall \psi_h \in \tilde{V}_h^0, \\ U_h^0 = R_h u^0, \ V_h^0 = R_h v^0. \end{cases} \tag{38}$$

By the similar arguments to Theorem 2, we can easily prove that (38) has a unique solution.

Now, we present the superclose estimate of the above TGM.

Theorem 5 *Let* (u^n, v^n), (U_H^n, V_H^n), *and* (U_h^n, V_h^n) *be the solutions of* (9), (37), *and* (38), *respectively. Then, we have the following:*

$$\|I_H u^n - U_H^n\|_1^2 + \|I_H v^n - V_H^n\|_1^2 \le C(H^4 + \tau^2), \tag{39}$$

$$\|I_h u^n - U_h^n\|_1^2 + \|I_h v^n - V_h^n\|_1^2 \le C(h^4 + H^8 + \tau^2). \tag{40}$$

Proof From Theorem 2, (39) is true obviously. So, we only need to prove (40).

In fact, by Taylor expansion, we have the following:

$$\begin{aligned} g(u^n) &= g\left(U_H^n\right) + g'\left(U_H^n\right)\left(u^n - U_H^n\right) + g''(u^*)\left(u^n - U_H^n\right)^2/2, \quad u^* \\ &= \bar{U}_H^n + \theta\left(\bar{u}^n - \bar{U}_H^n\right) \ (0 < \theta < 1). \end{aligned}$$

Set $u^n - U_h^n = u^n - R_h u^n + R_h u^n - U_h^n := \tilde{\eta}^n + \tilde{\xi}^n$, $v^n - V_h^n = v^n - R_h v^n + R_h v^n - V_h^n := \tilde{\chi}^n + \tilde{\gamma}^n$.

Then, subtracting (38) from (9), and choosing $\phi_h = D_\tau \tilde{\xi}^n$, $\psi_h = D_\tau \tilde{\gamma}^n$. We get that

$$\begin{aligned} &(D_\tau \tilde{\xi}^n, D_\tau \tilde{\xi}^n) + d_1(\nabla \tilde{\xi}^n, \nabla D_\tau \tilde{\xi}^n) + a_{11}(\tilde{\xi}^n, D_\tau \tilde{\xi}^n) \\ = &-d_1(\nabla \tilde{\eta}^n, D_\tau \nabla \tilde{\xi}^n) - (D_\tau \tilde{\eta}^n, D_\tau \tilde{\xi}^n) - a_{11}(\tilde{\eta}^n, D_\tau \tilde{\xi}^n) + a_{12}(\tilde{\gamma}^n + \tilde{\chi}^n, D_\tau \tilde{\xi}^n) \\ &+(r_1^n, D_\tau \tilde{\xi}^n) := \sum_{i=1}^5 A_i, \end{aligned} \tag{41}$$

$$
\begin{aligned}
&(D_\tau\tilde{\gamma}^n, D_\tau\tilde{\gamma}^n) + d_2(\nabla\tilde{\gamma}^n, \nabla D_\tau\tilde{\gamma}^n) + a_{22}(\tilde{\gamma}^n, D_\tau\tilde{\gamma}^n)\\
= &-d_2(\nabla\tilde{\chi}^n, D_\tau\nabla\tilde{\gamma}^n) - (D_\tau\tilde{\chi}^n, D_\tau\tilde{\gamma}^n) - a_{22}(\tilde{\chi}^n, D_\tau\tilde{\gamma}^n)\\
&+\left(g'\left(U_H^n\right)\left(u^n - U_h^n\right), D_\tau\tilde{\gamma}^n\right) + \left(r_2^n, D_\tau\tilde{\gamma}^n\right)\\
&+\left(g''(u^*)\left(u^n - U_H^n\right)^2/2, D_\tau\tilde{\gamma}^n\right) := \sum_{i=1}^{6} B_i.
\end{aligned}
\tag{42}
$$

We write as before, collecting (41)–(42), and we have the following:

$$
\begin{aligned}
\text{LHS} \geq\ &\|D_\tau\tilde{\xi}^n\|_0^2 + d_1\left(\|\nabla\tilde{\xi}^n\|_0^2 - \|\nabla\tilde{\xi}^{n-1}\|_0^2\right)/(2\tau) + a_{11}\left(\|\tilde{\xi}^n\|_0^2 - \|\tilde{\xi}^{n-1}\|_0^2\right)/\tau\\
&+\|D_\tau\tilde{\gamma}^n\|_0^2 + d_2\left(\|\nabla\tilde{\gamma}^n\|_0^2 - \|\nabla\tilde{\gamma}^{n-1}\|_0^2\right)/(2\tau)\\
&+a_{22}\left(\|\tilde{\gamma}^n\|_0^2 - \|\tilde{\gamma}^{n-1}\|_0^2\right)/\tau.
\end{aligned}
\tag{43}
$$

Here, in the same way as the estimates of (16) and (17), we can get the following estimates:

$$
\begin{aligned}
\left|\sum_{i=1}^{5} A_i + \sum_{i=1}^{5} B_i)\right| \leq\ &2a_{12}^2\|\tilde{\gamma}^n\|_0^2 + 2c^2\|\tilde{\xi}^n\|_0^2\\
&+Ch^4\left(\tau^{-1}\int_{t_{n-1}}^{t_n}\left(\|u_t\|_2^2 + \|v_t\|_2^2\right)ds + \|u^n\|_2^2 + \|v^n\|_2^2\right)\\
&+C\tau\left(\|u_{\mathrm{tt}}\|^2_{L^\infty(J;L^2(\Omega))} + \|v_{\mathrm{tt}}\|^2_{L^\infty(J;L^2(\Omega))}\right)\\
&+\|D_\tau\tilde{\xi}^n\|_0^2 + \frac{1}{2}\|D_\tau\tilde{\gamma}^n\|_0^2.
\end{aligned}
\tag{44}
$$

Moreover, due to $H^1 \hookrightarrow L^4$, interpolation theory, and (39), we have the following:

$$
\begin{aligned}
|B_6| &\leq c\left\|u^n - U_H^n\right\|_{0,4}^2\|D_\tau\tilde{\gamma}^n\|_0\\
&\leq c\left(\|u^n - I_H u^n\|_{0,4}^2 + \|I_H u^n - U_H^n\|_{0,4}^2\right)\|D_\tau\tilde{\gamma}^n\|_0\\
&\leq c\left(CH^4\|u^n\|_{2,4}^2 + \|I_H u^n - U_H^n\|_1^2\right)\|D_\tau\tilde{\gamma}^n\|_0 \leq C(H^4 + \tau^2)\|D_\tau\tilde{\gamma}^n\|_0\\
&\leq C(H^8 + \tau^4) + \frac{1}{2}\|D_\tau\tilde{\gamma}^n\|_0^2.
\end{aligned}
\tag{45}
$$

Altogether, we can see the following:

$$
\begin{aligned}
&c_0\left(\|\tilde{\xi}^n\|_1^2 - \|\tilde{\xi}^{n-1}\|_1^2\right)/2\tau + c_0\left(\|\tilde{\gamma}^n\|_1^2 - \|\tilde{\gamma}^{n-1}\|_1^2\right)/2\tau\\
\leq\ &Ch^4\left(\tau^{-1}\int_{t_{n-1}}^{t_n}\left(\|u_t\|_2^2 + \|v_t\|_2^2\right)\mathrm{ds} + \|u^n\|_2^2 + \|v^n\|_2^2\right)\\
&+C\tau^2\left(\|u_{\mathrm{tt}}\|^2_{L^\infty(J;L^2(\Omega))} + \|v_{\mathrm{tt}}\|^2_{L^\infty(J;L^2(\Omega))}\right) + 2a_{12}^2\|\tilde{\gamma}^n\|_0^2 + 2c^2\|\tilde{\xi}^n\|_0^2\\
&+C(H^8 + \tau^4).
\end{aligned}
\tag{46}
$$

Multiplying (46) by 2τ and after integration, we have the following:

$$c_0\left(\|\tilde{\xi}^m\|_1^2+\|\tilde{\gamma}^m\|_1^2\right)\le Ch^4\left(\|u_t\|^2_{L^2(J;H^2(\Omega))}+\|v_t\|^2_{L^2(J;H^2(\Omega))}\right.$$
$$\left.+\|u\|^2_{L^\infty(J;H^2(\Omega))}+\|v\|^2_{L^\infty(J;H^2(\Omega))}\right)+CH^8$$
$$+C\tau^2\left(\|u_{\rm tt}\|^2_{L^\infty(J;L^2(\Omega))}+\|v_{\rm tt}\|^2_{L^\infty(J;L^2(\Omega))}\right)$$
$$+4a_{12}^2\tau\sum_{n=1}^{m}\|\tilde{\gamma}^m\|_1^2+4c^2\tau\sum_{n=1}^{m}\|\tilde{\xi}^m\|_1^2.$$

So, by discrete Gronwall's lemma, when $c_0-4a_{12}^2\tau>0$ and $c_0-4c^2\tau>0$, we have the following:

$$\|\tilde{\xi}^m\|_1^2+\|\tilde{\gamma}^m\|_1^2\le Ch^4\left(\|u_t\|^2_{L^2(J;H^2(\Omega))}+\|v_t\|^2_{L^2(J;H^2(\Omega))}+\|u\|^2_{L^\infty(J;H^2(\Omega))}\right.$$
$$\left.+\|v\|^2_{L^\infty(J;H^2(\Omega))}\right)+CH^8$$
$$+C\tau^2\left(\|u_{\rm tt}\|^2_{L^\infty(J;L^2(\Omega))}+\|v_{\rm tt}\|^2_{L^\infty(J;L^2(\Omega))}\right)$$
$$\le C\left(h^4+H^8+\tau^2\right),$$

which together with (4) completes the proof. □

3.2 C-N scheme case

We first establish TGM for the C-N scheme of the problem (19) as follows.

Step 1: On the coarse grid $\mathscr{T}_H$, for $n=1,\ldots,N$, solve the following nonlinear system for $\left(U_H^n,V_H^n\right)\in\tilde{V}_H^0\times\tilde{V}_H^0$, such that

$$\begin{cases}\left(D_\tau U_H^n,\phi_H\right)+d_1\left(\nabla\bar{U}_H^n,\nabla\phi_H\right)+a_{11}\left(\bar{U}_H^n,\phi_H\right)-a_{12}\left(\bar{U}_H^n,\phi_H\right)=0, & \forall\phi_H\in\tilde{V}_H^0,\\ \left(D_\tau V_H^n,\psi_H\right)+d_2\left(\nabla\bar{V}_H^n,\nabla\psi_H\right)+a_{22}\left(\bar{V}_H^n,\psi_H\right)=\left(g\left(\bar{U}_H^n\right),\psi_H\right), & \forall\psi_H\in\tilde{V}_H^0,\\ U_H^0=R_Hu^0,\ V_H^0=R_Hv^0.\end{cases}\tag{47}$$

Step 2: On the fine grid $\mathscr{T}_h$, for $n=1,\ldots,N$, solve the following linear system for $(U_h^n,V_h^n)\in\tilde{V}_h^0\times\tilde{V}_h^0$, such that

$$\begin{cases}\left(D_\tau U_h^n,\phi_h\right)+d_1\left(\nabla\bar{U}_h^n,\nabla\phi_h\right)+a_{11}\left(\bar{U}_h^n,\phi_h\right)-a_{12}\left(\bar{U}_h^n,\phi_h\right)=0, & \forall\phi_h\in\tilde{V}_h^0,\\ \left(D_\tau V_h^n,\psi_h\right)+d_2\left(\nabla\bar{V}_h^n,\nabla\psi_h\right)+a_{22}\left(\bar{V}_h^n,\psi_h\right)=\left(g\left(\bar{U}_H^n\right)+g'\left(\bar{U}_H^n\right)\left(\bar{U}_h^n-\bar{U}_H^n\right),\psi_h\right), & \forall\psi_h\in\tilde{V}_h^0,\\ U_h^0=R_hu^0,\ V_h^0=R_hv^0.\end{cases}\tag{48}$$

Similar to the proof of Theorem 3, we see that (48) has a unique solution.

Now, we present the superclose estimates of the above TGM of (47)–(48).

Theorem 6 *Let* (u^n, v^n), $\left(U_H^n, V_H^n\right)$, *and* $\left(U_h^n, V_h^n\right)$ *be the solutions of* (21), (47), *and* (48), *respectively. Then, we have the following:*

$$\left\|I_H u^n - U_H^n\right\|_1^2 + \left\|I_H v^n - V_H^n\right\|_1^2 \le C(H^4 + \tau^4), \tag{49}$$

$$\left\|I_h u^n - U_h^n\right\|_1^2 + \left\|I_h v^n - V_h^n\right\|_1^2 \le C(h^4 + H^8 + \tau^4). \tag{50}$$

Proof From Theorem 4, (49) is true obviously. So, we only need to prove (50).

In fact, by Taylor expansion, we have the following:

$$\begin{aligned} g(\bar{u}^n) &= g\left(\bar{U}_H^n\right) + g'\left(\bar{U}_H^n\right)\left(\bar{u}^n - \bar{U}_H^n\right) + g''(u^*)\left(\bar{u}^n - \bar{U}_H^n\right)^2/2, \quad u^* \\ &= \bar{U}_H^n + \theta\left(\bar{u}^n - \bar{U}_H^n\right) \ (0 < \theta < 1). \end{aligned}$$

Let $u^n - U_h^n = u^n - R_h u^n + R_h u^n - U_h^n := \tilde{\eta}^n + \tilde{\xi}^n$, $v^n - V_h^n = v^n - R_h v^n + R_h v^n - V_h^n := \tilde{\chi}^n + \tilde{\gamma}^n$.

Then, subtracting (48) from (19) and choosing $\phi_h = D_\tau \tilde{\xi}^n$, $\psi_h = D_\tau \tilde{\gamma}^n$, we can get error equations at once:

$$\begin{aligned} &(D_\tau \tilde{\xi}^n, D_\tau \tilde{\xi}^n) + d_1(\nabla \bar{\tilde{\xi}}^n, \nabla D_\tau \tilde{\xi}^n) + a_{11}(\bar{\tilde{\xi}}^n, D_\tau \tilde{\xi}^n) \\ = &-(D_\tau \tilde{\eta}^n, D_\tau \tilde{\xi}^n) - d_1(\nabla \bar{\tilde{\eta}}^n, \nabla D_\tau \tilde{\xi}^n) - a_{11}(\bar{\tilde{\eta}}^n, D_\tau \tilde{\xi}^n) \\ &+ a_{12}(\bar{\tilde{\chi}}^n + \bar{\tilde{\gamma}}^n, D_\tau \tilde{\xi}^n) + \left(r_1^{n-\frac{1}{2}}, D_\tau \tilde{\xi}^n\right) - a_{11}\left(\bar{u}^n - u^{n-\frac{1}{2}}, D_\tau \tilde{\xi}^n\right) \\ &+ a_{12}\left(\bar{v}^n - v^{n-\frac{1}{2}}, D_\tau \tilde{\xi}^n\right) - d_1\left(\nabla \bar{u}^n - \nabla u^{n-\frac{1}{2}}, \nabla D_\tau \tilde{\xi}^n\right) := \sum_{i=1}^{8} J_i^n, \end{aligned} \tag{51}$$

$$\begin{aligned} &(D_\tau \tilde{\gamma}^n, D_\tau \tilde{\gamma}^n) + d_2(\nabla \bar{\tilde{\gamma}}^n, \nabla D_\tau \tilde{\gamma}^n) + a_{22}(\bar{\tilde{\gamma}}^n, D_\tau \tilde{\gamma}^n) \\ = &-(D_\tau \tilde{\chi}^n, D_\tau \tilde{\gamma}^n) - d_2(\nabla \bar{\tilde{\chi}}^n, \nabla D_\tau \tilde{\gamma}^n) - a_{22}(\bar{\tilde{\chi}}^n, D_\tau \tilde{\gamma}^n) \\ &+ \left(r_2^{n-\frac{1}{2}}, D_\tau \tilde{\gamma}^n\right) - d_2\left(\nabla \bar{v}^n - \nabla v^{n-\frac{1}{2}}, \nabla D_\tau \tilde{\gamma}^n\right) - a_{22}\left(\bar{v}^n - v^{n-\frac{1}{2}}, D_\tau \tilde{\gamma}^n\right) \\ &+ \left(g\left(u^{n-\frac{1}{2}}\right) - g(\bar{u}^n), D_\tau \tilde{\gamma}^n\right) + \left(g'\left(\bar{U}_H^n\right)\left(\bar{u}^n - \bar{U}_h^n\right), D_\tau \tilde{\gamma}^n\right) \\ &+ \left(g''(u^*)\left(\bar{u}^n - \bar{U}_H^n\right)^2/2, D_\tau \tilde{\gamma}^n\right) := \sum_{i=1}^{9} G_i^n. \end{aligned} \tag{52}$$

Then, similar to the estimates of I_i^n, K_i^n $(i = 1, \ldots, 5, 7, 8)$ in Theorem 4, we can easily get the following:

$$\begin{aligned} \left|\sum_{i=1}^{7} J_i^n + \sum_{i=1}^{6} G_i^n\right| \le\ & \frac{3a_{12}^2}{2}\left(\|\tilde{\gamma}^n\|_0^2 + \|\tilde{\gamma}^{n-1}\|_0^2\right) + \frac{7c^2}{4}\left(\|\tilde{\xi}^n\|_0^2 + \|\tilde{\xi}^{n-1}\|_0^2\right) \\ &+ Ch^4\left(\tau^{-1}\int_{t_{n-1}}^{t_n}\left(\|u_t\|_2^2 + \|v_t\|_2^2\right)ds + \|\bar{u}^n\|_2^2 + \|\bar{v}^n\|_2^2\right) \\ &+ C\tau^3\left(\int_{t_{n-1}}^{t_n}\left(\|u_{ttt}\|_0^2 + \|v_{ttt}\|_0^2 + \|u_t\|_0^2 + \|v_t\|_0^2\right)\mathrm{ds}\right. \\ &+ \|D_\tau \tilde{\xi}^n\|_0^2 + \frac{5}{7}\|D_\tau \tilde{\gamma}^n\|_0^2 \end{aligned} \tag{53}$$

Thanks to the Lipschitz continuous property of $g(\cdot)$, G_8^n can be bounded,

$$
\begin{aligned}
\left|G_8^n\right| &\le c\left\|\bar{u}^n-\bar{U}_h^n\right\|_0 \|D_\tau\tilde{\gamma}^n\|_0 \le \frac{7c^2}{2}\left(\left\|\bar{\tilde{\eta}}^n\right\|_0^2+\left\|\bar{\tilde{\xi}}^n\right\|_0^2\right)+\frac{1}{7}\|D_\tau\tilde{\gamma}^n\|_0^2 \\
&\le Ch^4\|\bar{u}^n\|_2^2+\frac{7c^2}{4}\left(\left\|\tilde{\xi}^n\right\|_0^2+\left\|\tilde{\xi}^{n-1}\right\|_0^2\right)+\frac{1}{7}\|D_\tau\tilde{\gamma}^n\|_0^2. \qquad (54)
\end{aligned}
$$

Moreover, with the help of $H^1 \hookrightarrow L^4$, interpolation theory, and (49), we can get the following:

$$
\begin{aligned}
|G_9^n| &\le C\left\|\bar{u}^n-\bar{U}_H^n\right\|_{0,4}^2 \|D_\tau\tilde{\gamma}^n\|_0 \\
&\le C\left(\|\bar{u}^n-I_H\bar{u}^n\|_{0,4}^2+\|I_H\bar{u}^n-\bar{U}_H^n\|_{0,4}^2\right)\|D_\tau\tilde{\gamma}^n\|_0 \\
&\le C\left(H^4\|\bar{u}^n\|_{2,4}^2+\|I_H\bar{u}^n-\bar{U}_H^n\|_1^2\right)\|D_\tau\tilde{\gamma}^n\|_0 \le C(H^4+\tau^2)\|D_\tau\tilde{\gamma}^n\|_0 \\
&\le C(H^8+\tau^4)+\frac{1}{7}\|D_\tau\tilde{\gamma}^n\|_0^2. \qquad (55)
\end{aligned}
$$

Then, substituting (53)–(55) into (51)–(52), we have that

$$
\begin{aligned}
& c_0\left(\|\tilde{\xi}^n\|_1^2-\|\tilde{\xi}^{n-1}\|_1^2\right)/2\tau+c_0\left(\|\tilde{\gamma}^n\|_1^2-\|\tilde{\gamma}^{n-1}\|_1^2\right)/2\tau \\
\le\ & \frac{3a_{12}^2}{2}\left(\|\tilde{\gamma}^n\|_0^2+\|\tilde{\gamma}^{n-1}\|_0^2\right)+\frac{7c^2}{2}\left(\|\tilde{\xi}^n\|_0^2+\|\tilde{\xi}^{n-1}\|_0^2\right)+CH^8 \\
& +Ch^4\left(\tau^{-1}\int_{t_{n-1}}^{t_n}\left(\|u_t\|_2^2+\|v_t\|_2^2\right)\mathrm{ds}+\|\bar{u}^n\|_2^2+\|\bar{v}^n\|_2^2\right) \\
& +C\tau^3\int_{t_{n-1}}^{t_n}\left(\|u_{\mathrm{ttt}}\|_0^2+\|v_{\mathrm{ttt}}\|_0^2+\|u_t\|_0^2+\|v_t\|_0^2\right)\mathrm{ds}+J_8^n+G_7^n. \qquad (56)
\end{aligned}
$$

Multiplying (56) by 2τ and summing from $n=1,\ldots,m$ $(1\le m\le N)$, we have the following:

$$
\begin{aligned}
c_0\left(\|\tilde{\xi}^m\|_1^2+\|\tilde{\gamma}^m\|_1^2\right) \le\ & Ch^4\left(\|u_t\|_{L^2(J;H^2(\Omega))}^2+\|v_t\|_{L^2(J;H^2(\Omega))}^2+\|u\|_{L^\infty(J;H^2(\Omega))}^2\right. \\
& \left.+\|v\|_{L^\infty(J;H^2(\Omega))}^2\right)+CH^8 \\
& +C\tau^4\left(\|u_{\mathrm{tt}}\|_{L^\infty(J;L^2(\Omega))}^2+\|v_{\mathrm{tt}}\|_{L^\infty(J;L^2(\Omega))}^2\right) \\
& +3a_{12}^2\tau\sum_{n=1}^m\|\tilde{\gamma}^n\|_1^2+7c^2\tau\sum_{n=1}^m\|\tilde{\xi}^n\|_1^2 \\
& +2\tau\sum_{n=1}^m J_8^n+2\tau\sum_{n=1}^m G_7^n. \qquad (57)
\end{aligned}
$$

Similar to the estimate of $2\tau \sum_{n=1}^{m} I_6^n$ and $2\tau \sum_{n=1}^{m} K_6^n$, we can obtain the following:

$$\begin{aligned}\left|2\tau\sum_{n=1}^{m} J_8^n + 2\tau\sum_{n=1}^{m} G_7^n\right| \le\; & C\tau^4\left(\frac{1}{\sigma_3}\|\nabla u_{\mathrm{tt}}\|^2_{L^\infty(J;L^2(\Omega))} + \|\nabla u_{\mathrm{ttt}}\|^2_{L^\infty(J;L^2(\Omega))}\right.\\ & \left.+\frac{1}{\sigma_4}\|\nabla v_{\mathrm{tt}}\|^2_{L^\infty(J;L^2(\Omega))} + \|\nabla v_{\mathrm{ttt}}\|^2_{L^\infty(J;L^2(\Omega))}\right)\\ & +\sigma_3\|\nabla\tilde{\xi}^m\|^2 + \sigma_4\|\nabla\tilde{\gamma}^m\|^2 + 2\tau\sum_{n=1}^{m}\|\nabla\tilde{\xi}^{n-1}\|_0^2\\ & +2\tau\sum_{n=1}^{m}\|\nabla\tilde{\gamma}^{n-1}\|_0^2. \end{aligned}\tag{58}$$

So, substituting (58) into (57) with $\sigma_3 = \sigma_4 = c_0/2$, we have as follows:

$$\begin{aligned} c_0\left(\|\tilde{\xi}^m\|_1^2 + \|\tilde{\gamma}^m\|_1^2\right) \le\; & Ch^4\left(\|u_t\|^2_{L^2(J;H^2(\Omega))} + \|v_t\|^2_{L^2(J;H^2(\Omega))} + \|u\|^2_{L^\infty(J;H^2(\Omega))}\right.\\ & \left.+\|v\|^2_{L^\infty(J;H^2(\Omega))}\right) + CH^8\\ & +C\tau^4\left(\|u_{\mathrm{tt}}\|^2_{L^\infty(J;L^2(\Omega))} + \|v_{\mathrm{tt}}\|^2_{L^\infty(J;L^2(\Omega))}\right)\\ & +c_1\tau\sum_{n=1}^{m}\|\tilde{\gamma}^n\|_1^2 + c_2\tau\sum_{n=1}^{m}\|\tilde{\xi}^n\|_1^2, \end{aligned}\tag{59}$$

where $c_1 = 3a_{12}^2 + 2,\ c_2 = 7c^2 + 2$.

Then, by discrete Gronwall's lemma, when $c_0/2 - c_1\tau > 0$ and $c_0/2 - c_2\tau > 0$, we have the following:

$$\begin{aligned} \|\tilde{\xi}^m\|_1^2 + \|\tilde{\gamma}^m\|_1^2 \le\; & Ch^4\left(\|u_t\|^2_{L^2(J;H^2(\Omega))} + \|v_t\|^2_{L^2(J;H^2(\Omega))} + \|u\|^2_{L^\infty(J;H^2(\Omega))}\right.\\ & \left.+\|v\|^2_{L^\infty(J;H^2(\Omega))}\right) + CH^8\\ & +C\tau^4\left(\|u_{\mathrm{tt}}\|^2_{L^\infty(J;L^2(\Omega))} + \|v_{\mathrm{tt}}\|^2_{L^\infty(J;L^2(\Omega))}\right)\\ \le\; & C(h^4 + H^8 + \tau^4), \end{aligned}$$

which together with (4) completes the proof. □

4 Global superconvergence analysis of TGM

Now, we start to derive the superconvergence results by applying the interpolated postprocessing operator Π_{2h}^2 (see [24]), satisfying the following:

$$\begin{cases} \Pi_{2h}^2 I_h w = \Pi_{2h}^2 w, \\ \left\|\Pi_{2h}^2 w - w\right\|_1 \le Ch^2\|w\|_3, \quad w \in H^3(\Omega), \\ \left\|\Pi_{2h}^2 w\right\|_1 \le C\|w\|_1, \quad \forall w \in S_2^h, \end{cases}\tag{60}$$

where S_2^h is the biquadratic finite element space.

Theorem 7 *Under the assumption of* (8)*, we have the following:*

$$\left\|u^n - \Pi_{2h}^2 U_h^n\right\|_1^2 + \left\|v^n - \Pi_{2h}^2 V_h^n\right\|_1^2 \le \begin{cases} C(h^4 + H^8 + \tau^2), \ \textit{for B-E scheme} \ (a), \\ C(h^4 + H^8 + \tau^4), \ \textit{for C-N scheme} \ (b). \end{cases} \tag{61}$$

Proof We start to prove (61(a)), and (61(b)) can be treated in the same way. As usual, we shall write the error as a sum of two terms: $u - \Pi_{2h}^2 U_h = u - \Pi_{2h}^2 I_h u + \Pi_{2h}^2 I_h u - \Pi_{2h}^2 U_h$, then by use of (60), we have the following:

$$\left\|u - \Pi_{2h}^2 I_h u\right\|_1^2 = \left\|u - \Pi_{2h}^2 u\right\|_1^2 \le \mathrm{Ch}^4 \|u\|_3.$$

Again applying (60), with the help of Theorem 3, we obtain the following:

$$\begin{aligned}\left\|\Pi_{2h}^2 U_h - \Pi_{2h}^2 I_h u\right\|_1^2 &= \left\|\Pi_{2h}^2 (U_h - I_h u)\right\|_1^2 \le \mathrm{C}\left\|I_h u - U_h\right\|_1^2 \\ &\le C(h^4 + H^8 + \tau^2).\end{aligned}$$

Together with the estimates above, it is easy to see that

$$\left\|u - \Pi_{2h}^2 U_h\right\|_1^2 = \left\|u - \Pi_{2h}^2 I_h u\right\|_1^2 + \left\|\Pi_{2h}^2 I_h u - \Pi_{2h}^2 U_h\right\|_1^2 \le C(h^4 + H^8 + \tau^2).$$

Similarly, we can get desired result of v as follows:

$$\left\|v - \Pi_{2h}^2 V_h\right\|_1^2 \le C(h^4 + H^8 + \tau^2).$$

This completes the proof. □

Remark 1 In Theorem 2, if we use R_h alone, how to construct an interpolated post-processing operator Π_{2h}^2 to satisfy $\Pi_{2h}^2 R_h u = \Pi_{2h}^2 u$ is still an open problem. In addition, if we only use the operator I_h and the estimation $(\nabla(\phi - I_h\phi), \nabla\phi_h) = O(h^2)\|\phi\|_3\|\phi_h\|_1$ proved in [23], it will result in the following:

$$\begin{aligned}&\left\|U_h^n - I_h u^n\right\|_1^2 + \left\|V_h^n - I_h v^n\right\|_1^2 \\ &\le Ch^4\left(\|u_t\|_{L^2(J;H^3(\Omega))}^2 + \|v_t\|_{L^2(J;H^3(\Omega))}^2 + \|u\|_{L^\infty(J;H^2(\Omega))}^2 + \|v\|_{L^\infty(J;H^2(\Omega))}^2\right) \\ &\quad + C\tau^2\left(\|u_{\mathrm{tt}}\|_{L^\infty(J;L^2(\Omega))}^2 + \|v_{\mathrm{tt}}\|_{L^\infty(J;L^2(\Omega))}^2\right).\end{aligned} \tag{62}$$

Obviously, the requirement of $u_t, v_t \in L^2(J; H^3(\Omega))$ in (62) as well as [25] is higher than that $u_t, v_t \in L^2(J; H^2(\Omega))$ in Theorem 2. This is the main reason why we use the combination technique in our work.

On the other hand, in the proof of Theorem 4, the derivative transfer trick is crucial to estimate J_6^n and K_6^n. Otherwise, how to get the superclose estimate of order $O(h^2 + H^4 + \tau^2)$ in H^1-norm is also an open problem.

Remark 2 Our analysis presented herein are also valid to some other popular finite elements:

(i) For the conforming linear triangular element space $\tilde{V}_h^0$ [23], we have the following:

$$(\nabla(u - I_h u), \nabla v) \leq Ch^2||u||_3||v||_1, \ u \in H^3(\Omega) \cap H_0^1(\Omega), \ v \in \tilde{V}_h^0. \quad (63)$$

So, we can get the results of Theorems 2, 4–6 by use of (63) and applying the combination technique as our paper.

(ii) For the nonconforming elements Q_1^{rot} [26, 27] on square mesh, EQ_1^{rot} [28, 29] and $\text{CNQ}_1^{\text{rot}}$ [30] on rectangular mesh, there holds for $u \in H^3(\Omega) \bigcap H_0^1(\Omega)$ as follows:

$$(\nabla_h(u - \Pi_h u), \nabla_h v) \leq \begin{cases} 0, \ v \in \tilde{V}_h^0, & \text{for } Q_1^{\text{rot}} \ \textit{and} \ \text{EQ}_1^{\text{rot}} \text{ elements}, \\ Ch^2||u||_3||v||_1, \ v \in \tilde{V}_h^0, & \text{for } \text{CNQ}_1^{\text{rot}} \text{ element}, \end{cases} \quad (64)$$

$$\left| \sum_{K \in \mathscr{T}_h} \int_{\partial K} \frac{\partial u}{\partial n} \text{vds} \right| = O(h^2)\|u\|_3\|v\|_h, \ v \in \tilde{V}_h^0, \quad (65)$$

where Π_h is the corresponding interpolator over $\tilde{V}_h^0$, ∇_h denotes the piecewise gradient operator, and $\|.\|_h = \left(\sum_K |.|_{1,K}^2\right)^{\frac{1}{2}}$ is the norm on $\tilde{V}_h^0$. So, we can also get the results of Theorems 2, 4–6 through (64)–(65).

(iii) For the quasi-Wilson element [31] on rectangular mesh, the modified quasi-Wilson element [32] on arbitrary quadrilateral mesh and the quasi-Carey element [33] on triangular mesh, since their consistency error estimations can reach order of $O(h^2)$ when the exact solution (u, v) belongs to $H_0^1(\Omega) \bigcap H^3(\Omega)$, it can be proved that Theorems 2, 4–6 are also valid to these finite elements.

However, for the rectangular Wilson element [34] and the triangular Carey element [35], how to get the desired results of our work still remains open, for their consistency error estimations only can reach order of $O(h)$.

5 Numerical experiment

In this section, we present numerical example to demonstrate the theoretical analysis. Setting the domain $\Omega = [0, 1] \times [0, 1]$, and the finial time $T = 1$. Then, we consider the following problem:

$$\begin{cases} u_t - d_1 \Delta u + a_{11} u - a_{12} v = f_1, & (\mathbf{x}, t) \in \Omega \times J, \\ v_t - d_2 \Delta v + a_{22} v - u^2/(1 + u^2) = f_2, & (\mathbf{x}, t) \in \Omega \times J, \\ u(\mathbf{x}, t) = v(\mathbf{x}, t) = 0, & (\mathbf{x}, t) \in \Omega \times J, \\ u(\mathbf{x}, 0) = u_0(\mathbf{x}) \ , \ \ v(\mathbf{x}, 0) = v_0(\mathbf{x}), & \mathbf{x} \in \Omega. \end{cases} \quad (66)$$

Table 1 The errors at $t = 0.1$ (B-E scheme)

H	h	$\left\|I_h u^n - \tilde{U}_h^n\right\|_1$	Order	$\left\|u^n - I_{2h}^2 \tilde{U}_h^n\right\|_1$	Order	$\left\|I_h v^n - \tilde{V}_h^n\right\|_1$	Order	$\left\|v^n - I_{2h}^2 \tilde{V}_h^n\right\|_1$	Order
1/2	1/4	1.1537e-01	–	1.1745e-01	–	1.1623e-01	–	1.1818e-01	–
1/4	1/16	7.3659e-03	1.9846	7.3719e-03	1.9969	7.4573e-03	1.9811	7.4629e-03	1.9925
1/6	1/36	1.4565e-03	1.9987	1.4568e-03	1.9995	1.4750e-03	1.9983	1.4753e-03	1.9991
1/8	1/64	4.6094e-04	1.9997	4.6096e-04	1.9999	4.6684e-04	1.9995	4.6686e-04	1.9997

Table 2 The errors at $t = 0.5$ (B-E scheme)

H	h	$\left\| I_h u^n - \tilde{U}_h^n \right\|_1$	Order	$\left\| u^n - I_{2h}^2 \tilde{U}_h^n \right\|_1$	Order	$\left\| I_h v^n - \tilde{V}_h^n \right\|_1$	Order	$\left\| v^n - I_{2h}^2 \tilde{V}_h^n \right\|_1$	Order
1/2	1/4	6.4549e-02	–	6.7523e-02	–	6.5456e-02	–	6.8269e-02	–
1/4	1/16	4.2210e-03	1.9674	4.2301e-03	1.9983	4.2920e-03	1.9654	4.3005e-03	1.9943
1/6	1/36	8.3575e-04	1.9971	8.3611e-04	1.9992	8.4993e-04	1.9969	8.5026e-04	1.9989
1/8	1/64	2.6455e-04	1.9993	2.6458e-04	1.9998	2.6906e-04	1.9991	2.6909e-04	1.9996

Table 3 The errors at $t = 1.0$ (B-E scheme)

H	h	$\left\|I_h u^n - \tilde{U}_h^n\right\|_1$	Order	$\left\|u^n - I_{2h}^2 \tilde{U}_h^n\right\|_1$	Order	$\left\|I_h v^n - \tilde{V}_h^n\right\|_1$	Order	$\left\|v^n - I_{2h}^2 \tilde{V}_h^n\right\|_1$	Order
1/2	1/4	3.9185e-02	–	4.0960e-02	–	4.0008e-02	–	4.1676e-02	–
1/4	1/16	2.5601e-03	1.9675	2.5656e-03	1.9984	2.6096e-03	1.9692	2.6147e-03	1.9973
1/6	1/36	5.0688e-04	1.9971	5.0709e-04	1.9993	5.1646e-04	1.9976	5.1666e-04	1.9996
1/8	1/64	1.6044e-04	1.9993	1.6046e-04	1.9998	1.6346e-04	1.9995	1.6347e-04	2.0000

Table 4 The errors at $t = 0.1$ (C-N scheme)

H	h	$\left\|I_h u^n - \tilde{U}_h^n\right\|_1$	Order	$\left\|u^n - I_{2h}^2 \tilde{U}_h^n\right\|_1$	Order	$\left\|I_h v^n - \tilde{V}_h^n\right\|_1$	Order	$\left\|v^n - I_{2h}^2 \tilde{V}_h^n\right\|_1$	Order
1/2	1/4	1.0724e-01	–	1.1017e-01	–	1.0816e-01	–	1.1095e-01	–
1/4	1/16	7.2674e-03	1.9416	7.2740e-03	1.9604	7.3596e-03	1.9387	7.3657e-03	1.9565
1/6	1/36	1.4413e-03	1.9950	1.4416e-03	1.9959	1.4599e-03	1.9948	1.4601e-03	1.9956
1/8	1/64	4.5636e-04	1.9988	4.5638e-04	1.9990	4.6228e-04	1.9987	4.6230e-04	1.9989

Table 5 The errors at $t = 0.5$ (C-N scheme)

H	h	$\left\|I_h u^n - \tilde{U}_h^n\right\|_1$	Order	$\left\|u^n - I_{2h}^2 \tilde{U}_h^n\right\|_1$	Order	$\left\|I_h v^n - \tilde{V}_h^n\right\|_1$	Order	$\left\|v^n - I_{2h}^2 \tilde{V}_h^n\right\|_1$	Order
1/2	1/4	6.3680e-02	–	6.6795e-02	–	6.4594e-02	–	6.7542e-02	–
1/4	1/16	4.1645e-03	1.9673	4.1741e-03	2.0001	4.2360e-03	1.9653	4.2450e-03	1.9960
1/6	1/36	8.2456e-04	1.9971	8.2494e-04	1.9994	8.3886e-04	1.9969	8.3920e-04	1.9990
1/8	1/64	2.6100e-04	1.9993	2.6104e-04	1.9998	2.6556e-04	1.9991	2.6559e-04	1.9996

Table 6 The errors at $t = 1.0$ (C-N scheme)

H	h	$\left\Vert I_h u^n - \tilde{U}_h^n \right\Vert_1$	Order	$\left\Vert u^n - I_{2h}^2 \tilde{U}_h^n \right\Vert_1$	Order	$\left\Vert I_h v^n - \tilde{V}_h^n \right\Vert_1$	Order	$\left\Vert v^n - I_{2h}^2 \tilde{V}_h^n \right\Vert_1$	Order
1/2	1/4	3.8637e-02	–	4.0524e-02	–	3.8637e-02	–	4.1241e-02	–
1/4	1/16	2.5258e-03	1.9676	2.5317e-03	2.0003	2.5758e-03	1.9693	2.5812e-03	1.9990
1/6	1/36	5.0009e-04	1.9971	5.0032e-04	1.9994	5.0976e-04	1.9976	5.0997e-04	1.9997
1/8	1/64	1.5892e-04	1.9993	1.5832e-04	1.9999	1.6134e-04	1.9995	1.6136e-04	2.0000

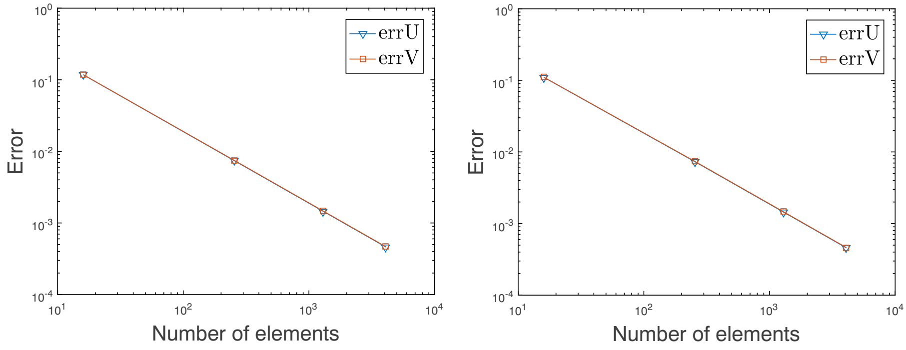

Fig. 1 Error reduction results of u and v at $t = 0.1$ for B-E scheme(left) and C-N scheme (right)

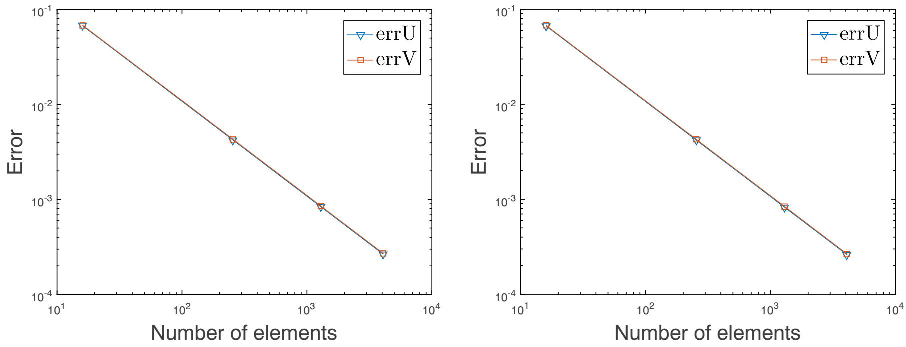

Fig. 2 Error reduction results of u and v at $t = 0.5$ for B-E scheme(left) and C-N scheme (right)

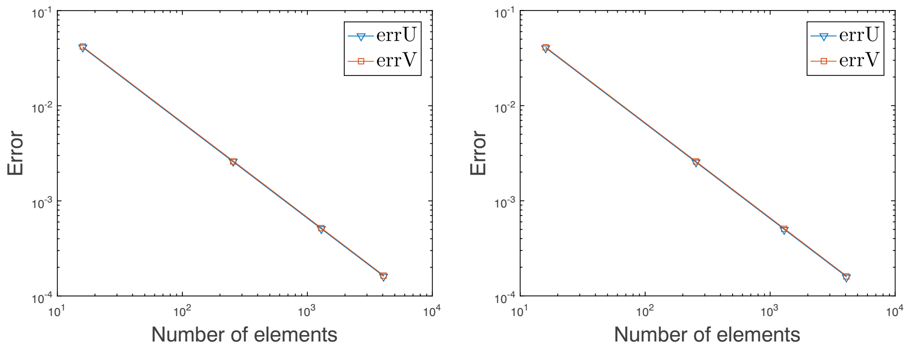

Fig. 3 Error reduction results of u and v at $t = 1$ for B-E scheme(left) and C-N scheme (right)

Table 7 Errors and CPU cost of the Galerkin FEM and the TGM (B-E scheme)

t_n	$\left\|u^n - I_{2h}^2 \tilde{U}_h^n\right\|_1$	$\left\|v^n - I_{2h}^2 \tilde{V}_h^n\right\|_1$	CPU time (s) (Galerkin FEM)	$\left\|u^n - I_{2h}^2 \tilde{U}_h^n\right\|_1$	$\left\|v^n - I_{2h}^2 \tilde{V}_h^n\right\|_1$	CPU time (s) (TGM)
0.1	7.3715e-03	7.4505e-03	21.9	7.3719e-03	7.4629e-03	11.9
0.2	5.8580e-03	5.9655e-03	47.7	5.8585e-03	5.9716e-03	23.5
0.3	5.1876e-03	5.2924e-03	83.1	5.1876e-03	5.2872e-03	42.9
0.4	4.6783e-03	4.7756e-03	99.9	4.6778e-03	4.7609e-03	53.5
0.5	4.2310e-03	4.3218e-03	124.4	4.2301e-03	4.3005e-03	64.9
0.6	4.7970e-03	4.8831e-03	145.0	4.7959e-03	4.8578e-03	75.3
0.7	4.3407e-03	4.4226e-03	166.3	4.3395e-03	4.3956e-03	87.2
0.8	3.1346e-03	3.2126e-03	187.9	3.1333e-03	3.1857e-03	98.8
0.9	2.8365e-03	2.9113e-03	202.1	2.8353e-03	2.8856e-03	109.1
1	2.5668e-03	2.6386e-03	221.0	2.5656e-03	2.6147e-03	117.3

where f_1, f_2, u_0, and v_0 are computed from the exact solution as follows:

$$(u(x, y, t), v(x, y, t)) = (e^{-t}\sin(\pi x)\sin(\pi y), e^{-t}\sin(\pi x)\sin(\pi y)).$$

In order to confirm the superclose and superconvergence orders in Theorems 2, 4–6, we choose $H^2 = h$ and use Newton iterations on coarse mesh in our computation. It can be seen from Tables 1, 2, 3, 4, 5, and 6 that $\left\|I_h u^n - \tilde{U}_h^n\right\|_1$, $\left\|u^n - I_{2h}^2 \tilde{U}_h^n\right\|_1$, $\left\|I_h v^n - \tilde{V}_h^n\right\|_1$, and $\left\|v^n - I_{2h}^2 \tilde{V}_h^n\right\|_1$ are convergent

Table 8 Errors and CPU cost of the Galerkin FEM and the TGM (C-N scheme)

t_n	$\left\|u^n - I_{2h}^2 \tilde{U}_h^n\right\|_1$	$\left\|v^n - I_{2h}^2 \tilde{V}_h^n\right\|_1$	CPU time (s) (Galerkin FEM)	$\left\|u^n - I_{2h}^2 \tilde{U}_h^n\right\|_1$	$\left\|v^n - I_{2h}^2 \tilde{V}_h^n\right\|_1$	CPU time (s) (TGM)
0.1	1.4415e-03	1.4573e-03	23.4	1.4416e-03	1.4601e-03	11.9
0.2	1.1426e-03	1.1642e-03	45.9	1.1427e-03	1.1660e-03	23.4
0.3	1.0116e-03	1.0327e-03	68.5	1.0117e-03	1.0321e-03	34.8
0.4	9.1232e-04	9.3186e-04	91.7	9.1224e-04	9.2922e-04	46.0
0.5	8.2510e-04	8.4331e-04	114.0	8.2494e-04	8.3920e-04	57.4
0.6	7.7106e-03	7.7279e-03	135.4	7.7104e-03	7.7229e-03	68.3
0.7	6.9769e-03	6.9933e-03	158.3	6.9766e-03	6.9879e-03	79.6
0.8	6.1129e-04	6.2694e-04	181.7	6.1103e-04	6.2141e-04	91.9
0.9	5.5317e-04	5.6817e-04	204.2	5.5290e-04	5.6283e-04	103.5
1	5.0057e-04	5.1496e-04	226.9	5.0032e-04	5.0997e-04	114.8

at order of $O(h^2)$ for B-E and C-N schemes, respectively, which coincide with our theoretical analysis. At the same time, we present the error reduction results at $t = 0.1, 0.5$, and 1 in Figs. 1, 2, and 3, respectively,where errU stands for $\|u^n - I_{2h}^2 \tilde{U}_h^n\|_1$ and errV stands for $\|v^n - I_{2h}^2 \tilde{V}_h^n\|_1$.

On the other hand, we also compare the CPU cost of the Galerkin FEMs to the TGMs for B-E scheme in Table 7 with the same partition ($h = 1/16$) and for C-N scheme in Table 8 with the same partition ($h = 1/36$) on a different time level. We can see that the TGMs take almost half as much CPU time as the Galerkin FEMs. Therefore, the proposed TGMs are very efficient algorithms.

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