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ORIGINAL PAPER

# Krylov subspace projection method for Sylvester tensor equation with low rank right-hand side

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**Abstract**

Motivated by the effectiveness of Krylov projection methods and the CP decomposition of tensors, which is a low rank decomposition, we propose Arnoldi-based methods (block and global) to solve Sylvester tensor equation with low rank right-hand sides. We apply a standard Krylov subspace method to each coefficient matrix, in order to reduce the main problem to a projected Sylvester tensor equation, which can be solved by a global iterative scheme. We show how to extract approximate solutions via matrix Krylov subspaces basis. Several theoretical results such as expressions of residual and its norm are presented. To show the performance of the proposed approaches, some numerical experiments are given.

**Keywords** Sylvester tensor equation · CP decomposition · Krylov subspace · Block and global Arnoldi

## 1 Introduction

A tensor is a multi-dimensional array, in which the order is the number of dimensions, also known as ways or modes. In the past few years, tensors have attracted significant attention in several applications, such as image processing, machine learning, and

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scientific computing [15, 20]. One of the popular problems in tensor-based modeling is the following equation, known as the Sylvester tensor equation

$$\mathcal{X} \times_1 A^{(1)} + \mathcal{X} \times_2 A^{(2)} + \cdots + \mathcal{X} \times_N A^{(N)} = \mathcal{B}, \tag{1}$$

where the matrices $A^{(n)} \in \mathbb{R}^{I_n \times I_n}$, $n = 1, 2, \cdots, N$, the right-hand side tensor $\mathcal{B} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$ are known, and $\mathcal{X} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$ is the unknown tensor. The product $\times_i$, $i = 1, \ldots, N$, and some notations related to the concept of tensors will be specified in the next section. For simplicity, in the sequel, we define the following linear operator

$$\begin{array}{rcl} \mathcal{M} : \mathbb{R}^{I_1 \times \cdots \times I_N} & \longrightarrow & \mathbb{R}^{I_1 \times \cdots \times I_N} \\ \mathcal{X} & \longmapsto & \mathcal{M}(\mathcal{X}) := \sum_{i=1}^{N} \mathcal{X} \times_i A^{(i)}. \end{array} \tag{2}$$

It is easy to verify that (1) is equivalent to the following linear system of equations

$$\mathbb{A}x = b, \tag{3}$$

with $\mathbb{A} = \sum_{n=1}^{N} I_{I_N} \otimes \cdots \otimes I_{I_{n+1}} \otimes A^{(n)} \otimes I_{I_{n-1}} \otimes \cdots I_{I_1}$, $x = vec(\mathcal{X})$, and $b = vec(\mathcal{B})$, where $\otimes$ denotes the Kronecker product (defined in the next section), $I_{I_n}$ stands for the identity matrix of order $I_n$ and the operator $vec$ rearranges tensor's elements in a vector. It is well known that the Sylvester tensor equation (1) has a unique solution if and only if $\lambda_1 + \lambda_2 + \ldots + \lambda_N \neq 0$, for all $\lambda_i \in \sigma(A^{(i)})$, where $\sigma(A^{(i)})$ denotes the spectrum of $A^{(i)}$ (*Lemma* 4.2 [6]).

Note that the coefficient matrix of the linear system (3) is of order $\prod_{i=1}^{N} I_i$, which may become too large even for moderate values of $I_1, \ldots, I_N$ and solving this linear system can be a real challenge. When $\mathcal{X}$ is a tensor of order two, i.e., a matrix, (1) is reduced to

$$A^{(1)}X + XA^{(2)T} = B, \tag{4}$$

which is exactly the well-known Sylvester matrix equation, it has been widely used in control and communication theory, image restoration, and numerical methods for ordinary differential equations (see [4] and the references therein). Sylvester tensor equation (1) can arise when discretizing high dimensional linear partial differential equations using finite difference or spectral method [19, 21, 24]; a survey of the tensor-structured numerical methods in applications to multidimensional problems in scientific computing is given in [15]. As an example of the applications of (1), one can think of some Laplace-like operator $L$ in an $N-$dimensional domain

$$\begin{aligned} Lu &= f \quad in \quad \Omega \\ u &= 0 \quad on \quad \partial\Omega. \end{aligned}$$

In this paper, we are interested in the case where the discretized right-hand side in (1) is of low rank—this is possible when the function $f$ is sufficiently smooth to be well approximated by a short sum of separable functions (see, e.g., [3, 11]). In general, the right-hand side tensor $\mathcal{B}$ can be approximated by a low rank tensor using CP decomposition [16].

In recent years, various methods have been proposed in order to solve (1). For instance, the tensor format of the GMRES method (GMRES-BTF) has been established by Chen and Lu [6]. In [7], gradient-based iterative algorithms have been proposed for solving (1); they are based on the hierarchical identification principle [9] and tensor arithmetic concepts. Kressner and Tobler proposed a tensor Krylov subspace method to solve (3) when the right-hand side is given in a tensor product structure, i.e., is of rank one. The idea based on applying a standard Krylov subspace method to the coefficient matrices, in order to approximate the solution by a vector of low tensor rank [17]. Ballani and Grasedyck presented an iterative scheme similar to Krylov subspace method to solve (3), relying on truncation operator, whereas the operator is implemented by hierarchical Tucker format [16], to allow applications in high dimensions [2]. Some well-known Krylov subspace methods have been studied in their tensor format by Beik et al. in [1]; the authors have described the tensor format of the full orthogonalization method (FOM-BTF) and conjugate gradient (CG-BTF)–type iterative algorithms. These methods are attractive if the coefficient matrices are not large, since they are based on the use of the tensor Krylov subspace associated to the operator $\mathcal{M}$ defined by (2).

In the concept of Krylov subspace methods, we apply an Arnoldi-based algorithm to the coefficient matrices to get a reduced Sylvester tensor equation which can be solved by a global iterative scheme. The approximate solution is then constructed from the solution of the reduced equation and the Krylov subspaces basis associated to the coefficient matrices.

The rest of this paper is organized as follows. In Section 2, we give notations adopted in this paper and some basic definitions and properties related to tensors. Section 3 is dedicated to theoretical results when the right-hand side tensor in (1) is of rank one. In Section 4, we present our approaches to solve (1) with a right-hand side tensor of a specific rank; we give the block Krylov approach in Section 4.1 and the global Krylov approach in Section 4.2. Some numerical examples are presented in Section 5 to evaluate the performance of our approaches. Finally, in Section 6, we give a brief conclusion and perspectives.

## 2 Notations and preliminary concepts

In this section, we introduce some basic definitions, tensor notations, and common operations related to tensors adopted in this paper (for more details, see [16]). Throughout this paper, vectors (tensors of order one), matrices (tensors of order two), and higher order tensors (order three or higher) are signified by lower-case letter, capital letters, and Euler script letters respectively. A tensor $\mathcal{X}$ is an element in $\mathbb{R}^{I_1\times I_2\times\cdots\times I_N}$, where $I_1, I_2, \cdots, I_N \in \mathbb{N}$; its entries are denoted by $\mathcal{X}_{i_1 i_2\cdots i_N}$, with $i_n \in \{1, \cdots, I_n\}$, for every $1 \le n \le N$. Fibers are the higher order analogue of matrix rows and columns. A fiber is defined by fixing every index but one. The $n$-mode fiber is denoted by $\mathcal{X}_{i_1,\cdots i_{n-1},:,i_{n+1},\cdots,i_N} \in \mathbb{R}^{I_n}$, where the indices $i_j$ are fixed for

$j = 1, \cdots, n-1, n+1, \cdots, N$. The notation $\overline{i_1 \dots i_N}$ corresponds to a multi-index, which is obtained as follows

$$\overline{i_1 \dots i_N} = i_N + (i_{N-1} - 1)I_N + \dots + (i_1 - 1)I_2 \dots I_N.$$

The inner product of two same size tensors $\mathcal{A}, \mathcal{B} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$ is defined by

$$< \mathcal{A}, \mathcal{B} >:= \sum_{i_1=1}^{I_1} \sum_{i_2=1}^{I_2} \cdots \sum_{i_N=1}^{I_N} \mathcal{A}_{i_1 \cdots i_N} \mathcal{B}_{i_1 \cdots i_N},$$

and the norm induced by this inner product is

$$\| \mathcal{A} \| = \sqrt{< \mathcal{A}, \mathcal{A} >}.$$

**Definition 1** [16, 18]
Let $\mathcal{A} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$ be an $N$th-order tensor and $U \in \mathbb{R}^{J \times I_n}$ be a matrix. The n-mode product of $\mathcal{A}$ and $U$, denoted by $\mathcal{A} \times_n U$, is a tensor of size

$$I_1 \times I_2 \times \cdots \times I_{n-1} \times J \times I_{n+1} \times \cdots \times I_N,$$

whose entries are given by

$$(\mathcal{A} \times_n U)_{i_1 \cdots i_{n-1} j i_{n+1} \cdots i_N} = \sum_{i_n=1}^{I_n} \mathcal{A}_{i_1 \cdots i_N} U_{j i_n}.$$

For $\mathcal{X} \in \mathbb{R}^{I_1 \times \cdots I_N}$ and $\{A\}$ a set of matrices $A_n \in \mathbb{R}^{I_n \times I_n}, n = 1, 2, \cdots, N$, their multiplication in all possible modes ($n = 1, 2, ..., N$) is denoted as

$$\mathcal{X} \times \{A\} := \mathcal{X} \times_1 A_1 \times_2 A_2 \cdots \times_N A_N,$$

and

$$\mathcal{X} \times \{A\}^T := \mathcal{X} \times_1 A_1^T \times_2 A_2^T \cdots \times_N A_N^T.$$

**Proposition 1** [16, 18]
*Let $\mathcal{A} \in \mathbb{R}^{I_1 \times \cdots \times I_N}$ be an Nth-order tensor, $U \in \mathbb{R}^{J \times I_m}$, $V \in \mathbb{R}^{K \times I_n}$ and $W \in \mathbb{R}^{I_n \times I_n}$ be three matrices, then for distinct modes in a series of multiplication, the order of the multiplication is irrelevant, i.e.,*

$$\mathcal{A} \times_m U \times_n V = \mathcal{A} \times_n V \times_m U.$$

*If the modes are the same, then*

$$\mathcal{A} \times_n W \times_n V = \mathcal{A} \times_n VW.$$

**Definition 2** [16]
The outer product of two tensors $\mathcal{A} \in \mathbb{R}^{I_1 \times I_2 \times \ldots \times I_N}$ and $\mathcal{B} \in \mathbb{R}^{J_1 \times J_2 \times \cdots \times J_M}$ is a tensor denoted by $\mathcal{A} \circ \mathcal{B} = \mathcal{C} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N \times J_1 \times J_2 \times \cdots \times J_M}$.
Elementwise,

$$\mathcal{C}_{i_1 \ldots i_N j_1 \ldots j_M} = \mathcal{A}_{i_1, \ldots, i_N} \mathcal{B}_{j_1, \ldots, j_M}.$$

If $v_1, v_2, \ldots, v_N$ are N vectors of sizes $I_i,\ i = 1, \ldots, N$, their outer product is an $N$th-order tensor of size $I_1 \times \ldots \times I_N$ and we have

$$v_1 \circ \ldots \circ v_{N\,i_1,\cdots,i_N} = v_1(i_1) \ldots v_n(i_N)$$

**Definition 3** [16]
An $N$th-order tensor $\mathcal{A} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$ is of rank one if it can be written as the outer product of $N$ vectors $v_k \in \mathbb{R}^{I_k},\ k = 1, 2, \cdots, N$, i.e.,

$$\mathcal{A} = v_1 \circ v_2 \circ \cdots \circ v_N.$$

A tensor is of rank $R \in \mathbb{N}$ if it could be written as the sum of $R$ rank one tensors.

**Definition 4** [16, 18]
The Kronecker product of two matrices $A \in \mathbb{R}^{I_1 \times I_2}$ and $B \in \mathbb{R}^{J_1 \times J_2}$ is a matrix of size $I_1 J_1 \times I_1 J_2$ denoted by $A \otimes B$, where

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1I_2}B \\ \vdots & \ddots & \vdots \\ a_{I_1 1}B & \cdots & a_{I_1 I_2}B \end{pmatrix}.$$

The Kronecker product of two tensors $\mathcal{A} \in \mathbb{R}^{I_1 \times \cdots \times I_N}$ and $\mathcal{B} \in \mathbb{R}^{J_1 \times \cdots \times J_N}$ is defined by

$$\mathcal{C} = \mathcal{A} \otimes \mathcal{B} \in R^{I_1 J_1 \times \cdots \times I_N J_N},$$

where

$$\mathcal{C}_{\overline{i_1 j_1}, \cdots, \overline{i_N j_N}} = \mathcal{A}_{i_1, \cdots, i_N} \mathcal{B}_{j_1, \cdots, j_N},$$

for $i_n = 1, \cdots, I_n,\ j_n = 1, \cdots, J_n,\ n = 1, \cdots, N.$

In the following remark, we state the link between Kronecker product of 3 vectors and their outer product

*Remark 1* [8, Page 33]
Let $v_1,\ v_2$, and $v_3$ be 3 vectors of sizes $I_1,\ I_2$, and $I_3$, respectively, we have

$$vec(v_1 \circ v_2 \circ v_3) = v_3 \otimes v_2 \otimes v_1$$

It is easy to verify that the above remark still available for N vectors. This property shows that the idea in Section 3 of the current paper can be considered as a reformulation of the one in [17].

**Proposition 2** [18]
*Let $\mathcal{A} = a_1 \circ a_2 \circ \cdots \circ a_N$ and $\mathcal{B} = b_1 \circ b_2 \circ \cdots \circ b_N$ denote two rank one tensors, then, the Kronecker product $\mathcal{A} \otimes \mathcal{B}$ can be expressed by*

$$\mathcal{A} \otimes \mathcal{B} = (a_1 \otimes b_1) \circ \cdots \circ (a_N \otimes b_N).$$

It is well known that if $A$, $B$, $C$, and $D$ are four matrices, we have

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD).$$

We give an elegant generalization of this property to tensors as follows

**Proposition 3** *Let $\mathcal{A} \in \mathbb{R}^{I_1 \times \cdots \times I_N}$, $\mathcal{B} \in \mathbb{R}^{J_1 \times \cdots \times J_N}$ be two tensors, and $A \in \mathbb{R}^{K_n \times I_n}$ and $B \in \mathbb{R}^{L_n \times J_n}$ be two matrices, then*

$$(\mathcal{A} \otimes \mathcal{B}) \times_n (A \otimes B) = (\mathcal{A} \times_n A) \otimes (\mathcal{B} \times_n B).$$

*Proof*

$$\begin{aligned}
((\mathcal{A} \times_n A) \otimes (\mathcal{B} \times_n B))_{\overline{i_1 j_1} \cdots \overline{k_n l_n} \cdots \overline{i_N j_N}} &= (\mathcal{A} \times_n A)_{i_1 \cdots k_n \cdots i_n} (\mathcal{B} \times_n B)_{j_1 \cdots l_n \cdots j_n} \\
&= \sum_{i_n=1}^{I_n} \mathcal{A}_{i_1 \cdots i_n \cdots i_n} A_{k_n i_n} \sum_{j_n=1}^{J_n} \mathcal{B}_{j_1 \cdots j_n \cdots j_n} B_{l_n j_n} \\
&= \sum_{i_n=1}^{I_n} \sum_{j_n=1}^{J_n} (\mathcal{A} \otimes \mathcal{B})_{\overline{i_1 j_1} \cdots \overline{i_n j_n} \cdots \overline{i_N j_N}} (A \otimes B)_{\overline{k_n l_n}, \overline{i_n j_n}} \\
&= ((\mathcal{A} \otimes \mathcal{B}) \times_n (A \otimes B))_{\overline{i_1 j_1} \cdots \overline{k_n l_n} \cdots \overline{i_N j_N}}.
\end{aligned}$$

□

It is easy to verify the following result

**Proposition 4** *Let $\mathcal{A} = a_1 \circ a_2 \circ \cdots \circ a_N$ a rank one tensor and $\{V\}$ a set of $N$ matrices $\{V_1, V_2, \ldots, V_N\}$, then we have*

$$\mathcal{A} \times \{V\} = V_1 a_1 \circ \ldots \circ V_N a_N.$$

**Definition 5** (CP decomposition [16]) Let $\mathcal{A} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$ be an $N$th-order tensor. The CP decomposition of $\mathcal{A}$ is

$$\mathcal{A} = \sum_{r=1}^{R} a_r^{(1)} \circ a_r^{(2)} \cdots \circ a_r^{(N)},$$

where $a_r^{(k)}$ are vectors of size $I_k$ with $1 \leq k \leq N$. If we define $A_n = \left[a_1^{(n)} a_2^{(n)} \cdots a_R^{(n)}\right]$ for $n \in \{1, \cdots, N\}$, the CP decomposition can be symbolically written as

$$\mathcal{A} = A_1 \circ A_2 \circ \cdots \circ A_N,$$

the matrices $A_n \in \mathbb{R}^{I_n \times R}$ are called factor matrices. Often, the vectors $a_r^{(n)}$ are chosen such that $\|a_r^{(n)}\| = 1$. In this case, the CP decomposition is written as

$$\mathcal{A} = \sum_{r=1}^{R} \lambda_r a_r^{(1)} \circ a_r^{(2)} \cdots \circ a_r^{(N)},$$

where $\lambda_r$ is a scalar that compensates for the magnitudes of vectors $a_r^{(n)}$.

In the following, we denote by $m = (m_1, m_2, \cdots, m_N)$ a multi-index that represents all the Krylov subspaces dimensions, $e_{m_i} = [0, \ldots, 0, 1]^T \in \mathbb{R}^{m_i}$, $e_1^{(m_i)} = [1, 0, \ldots, 0]^T \in \mathbb{R}^{m_i}$, $E_{m_i} = [0_R, \ldots, 0_R, I_R]^T \in \mathbb{R}^{Rm_i \times R}$, $E_1^{(m_i)} = [I_R, 0_R \ldots, 0_R]^T \in \mathbb{R}^{Rm_i \times R}$ and $e_r^{(R)} \in \mathbb{R}^R$ the $r$th vector from the canonical basis of $\mathbb{R}^R$, where $0_R$ and $I_R$ correspond to the square matrix full of zeros and the identity matrix of size $R$ respectively.

## 3 Rank one right-hand side tensor

In this section, we assume that the right-hand side tensor in (1) is of rank one; i.e., it can be written as follows

$$\mathcal{B} = b_1 \circ b_2 \circ \cdots \circ b_N$$

where $b_i \in \mathbb{R}^{I_i}$, $i = 1, \ldots, N$. Applying the Arnoldi algorithm (section 6.3 in [23]) to the pairs $(A^{(i)}, b_i)$, $i = 1, \ldots, N$, leads to the following relations, for $i = 1, \ldots, N$,

$$A^{(i)} V_{m_i} = V_{m_i} H_{m_i} + h_{m_i+1} v_{m_i+1} e_{m_i}^T, \tag{5}$$

and

$$V_{m_i}^T A^{(i)} V_{m_i} = H_{m_i}, \tag{6}$$

where the first vector of each basis $V_{m_i}$ is exactly the normalized vector

$$v_1^{(i)} = \frac{b_i}{\|b_i\|}.$$

We consider the following approximate solution of (1)

$$\mathcal{X}_m = \mathcal{Y}_m \times \{V_m\},$$

where $\mathcal{Y}_m \in \mathbb{R}^{m_1 \times \cdots \times m_N}$ and $\{V_m\}$ is the set of matrices $\{V_{m_1}, \cdots, V_{m_N}\}$.

The associated residual tensor is given by

$$\begin{aligned}\mathcal{R}_m &= \mathcal{B} - \mathcal{M}(\mathcal{X}_m)\\ &= \mathcal{B} - \sum_{i=1}^{N} \mathcal{Y}_m \times_1 V_{m_1} \cdots \times_i A^{(i)} V_{m_i} \cdots \times_N V_{m_N}.\end{aligned}$$

We consider the Petrov-Galerkin condition on its tensor format as follows

$$\mathcal{R}_m \times \{V_m\}^T = 0,$$

hence

$$\begin{aligned}0 &= \mathcal{B} \times \{V_m\}^T - \sum_{i=1}^{N} \mathcal{Y}_m \times_1 V_{m_1} \cdots \times_i A^{(i)} V_{m_i} \cdots \times_N V_{m_N} \times_1 V_{m_1}^T \cdots \times_N V_{m_N}^T\\ &= V_{m_1}^T b_1 \circ V_{m_2}^T b_2 \circ \cdots \circ V_{m_N}^T b_N - \sum_{i=1}^{N} \mathcal{Y}_m \times_i H_{m_i}\\ &= \beta e_1^{(m_1)} \circ e_1^{(m_2)} \circ \cdots \circ e_1^{(m_N)} - \sum_{i=1}^{N} \mathcal{Y}_m \times_i H_{m_i}\end{aligned}$$

where $\beta = \prod_{i=1}^{N} \|b_i\|$. Thus, the reduced Sylvester tensor equation is given as follows

$$\sum_{i=1}^{N} \mathcal{Y}_m \times_i H_{m_i} = \beta \mathcal{E}_m, \tag{7}$$

where $\mathcal{E}_m = e_1^{(m_1)} \circ e_1^{(m_2)} \circ \cdots \circ e_1^{(m_N)}$. The following proposition gives the associated residual tensor

**Proposition 5** *Let $\mathcal{Y}_m$ be the solution of (7) and for $i = 1, \ldots, N$, $V_{m_i}$ are the basis obtained by applying Arnoldi algorithm to the pairs $(A^{(i)}, b_i)$, then*

$$\mathcal{R}_m = -\sum_{i=1}^{N} h_{m_i+1} \mathcal{Y}_m \times_1 V_{m_1} \cdots \times_i v_{m_i+1} e_{m_i}^T \cdots \times_N V_{m_N}.$$

*Proof* We have

$$\begin{aligned}\mathcal{R}_m &= \mathcal{B} - \mathcal{M}(\mathcal{X}_m) \\ &= \mathcal{B} - \sum_{i=1}^{N} \mathcal{Y}_m \times_1 V_{m_1} \cdots \times_i A^{(i)} V_{m_i} \cdots \times_N V_{m_N}.\end{aligned}$$

Using the relations (5) and the expression of the right-hand side $\mathcal{B}$, we obtain

$$\begin{aligned}\mathcal{R}_m \quad &= b_1 \circ b_2 \circ \cdots \circ b_N - \sum_{i=1}^{N} \mathcal{Y}_m \times_i H_{m_i} \times \{V_m\} \\ &- \sum_{i=1}^{N} h_{m_i+1} \mathcal{Y}_m \times_1 V_{m_1} \cdots \times_i v_{m_i+1} e_{m_i}^T \cdots \times_N V_{m_N}.\end{aligned}$$

Taking in consideration the fact that $b_i = \|b_i\| v_1^{(i)} = \|b_i\| V_{m_i} e_1^{(m_i)}$ and using the proposition (4), it results

$$\begin{aligned}\mathcal{R}_m &= \left( \beta e_1^{(m_1)} \circ e_1^{(m_2)} \circ \cdots \circ e_1^{(m_N)} - \sum_{i=1}^{N} \mathcal{Y}_m \times_i H_{m_i} \right) \times \{V_m\} \\ &- \sum_{i=1}^{N} h_{m_i+1} \mathcal{Y}_m \times_1 V_{m_1} \cdots \times_i v_{m_i+1} e_{m_i}^T \cdots \times_N V_{m_N}.\end{aligned}$$

Invoking (7), the result achieved. □

**Theorem 1** *Let $\mathcal{R}_m$ be the corresponding residual, then*

$$\|\mathcal{R}_m\| = \left( \sum_{i=1}^{N} |h_{m_i+1}|^2 \|\mathcal{Y}_m \times_i e_{m_i}^T\|^2 \right)^{1/2}.$$

*where $\mathcal{Y}_m$ is the solution of (7) and $e_{m_i} = [0, \ldots, 0, 1]^T \in \mathbb{R}^{m_i}$.*

*Proof*

$$
\begin{aligned}
\|\mathcal{R}_m\|^2 &= \langle \mathcal{R}_m, \mathcal{R}_m \rangle \\
&= \sum_{i=1}^{N} |h_{m_i+1}|^2 \|\mathcal{Y}_m \times_1 V_{m_1} \cdots \times_i v_{m_i+1} e_{m_i}^T \cdots \times_N V_{m_N}\|^2 \\
&= \sum_{i=1}^{N} |h_{m_i+1}|^2 \langle \mathcal{Y}_m \times_i e_{m_i} e_{m_i}^T, \mathcal{Y}_m \rangle \\
&= \sum_{i=1}^{N} |h_{m_i+1}|^2 \langle \mathcal{Y}_m \times_i e_{m_i}^T, \mathcal{Y}_m \times_i e_{m_i}^T \rangle \\
&= \sum_{i=1}^{N} |h_{m_i+1}|^2 \|\mathcal{Y}_m \times_i e_{m_i}^T\|^2.
\end{aligned}
$$

□

## 4 Rank R right-hand side tensor

Inspired by the work addressed the Sylvester matrix equation in [10, 12], and the fact that an *N*th-order tensor can be decomposable using the CP decomposition mentioned in the definition 5, we propose two approaches to extract approximate solutions to (1) with low rank right-hand sides; by this means, we assume in the following that the right-hand side is of rank $R$, i.e.,

$$\mathcal{B} = \sum_{r=1}^{R} b_1^{(r)} \circ \cdots \circ b_N^{(r)},$$

where $b_i^{(r)} \in \mathbb{R}^{I_i}$, for $i \in \{1, \ldots, N\}$, and $r \in \{1, \ldots, R\}$. We set for $i = 1, 2, \cdots, N$

$$B^{(i)} = \left[ b_i^{(1)}, b_i^{(2)}, \cdots, b_i^{(R)} \right].$$

Straightforward computations show that the right-hand side tensor can also be written as follows

$$\mathcal{B} = \mathcal{I}_R \times_1 B^{(1)} \cdots \times_N B^{(N)},$$

where $\mathcal{I}_R$, called identity tensor, is the *N*th-order tensor of size $R \times \ldots \times R$ with ones along the super-diagonal. In the following two paragraphs, we will show how to extract approximate solutions to (1), via block and global Krylov methods (for more details about the block and global Arnoldi algorithms, we refer the reader to section 6.1 in [23] and [13] respectively).

### 4.1 Block Krylov approach

Let $\mathbb{U}_{m_i} = \left[ U_1^{(i)}, \ldots, U_{m_i}^{(i)} \right]$ and $\mathbb{H}_{m_i}$ be the matrices obtained by applying block Arnoldi algorithm to the pairs $(A^{(i)}, B^{(i)})$, $\quad i = 1, 2, \ldots, N$, starting with $U_1^{(i)} = Q^{(i)}$, where $Q^{(i)}$ is obtained from the QR factorisation of $B^{(i)}$, i.e., $B^{(i)} = Q^{(i)} R_i$, then the following relation holds, for $i = 1, \ldots, N$,

$$A^{(i)} \mathbb{U}_{m_i} = \mathbb{U}_{m_i} \mathbb{H}_{m_i} + U_{m_i+1} H_{m_i+1, m_i} E_{m_i}^T. \tag{8}$$

An approximate solution is given by

$$\mathcal{X}_m = \mathcal{Y}_m \times \{\mathbb{U}_m\},$$

where $\mathcal{Y}_m \in \mathbb{R}^{Rm_1 \times \cdots \times Rm_N}$ and $\{\mathbb{U}_m\} = \{\mathbb{U}_{m_1}, \ldots, \mathbb{U}_{m_N}\}$.

We consider the following Petrov-Galerkin condition of orthogonality

$$\mathcal{R}_m \times \{\mathbb{U}_m\}^T = 0, \tag{9}$$

with $\mathcal{R}_m$ the residual tensor, which is given by

$$\begin{aligned} \mathcal{R}_m &= \mathcal{B} - \mathcal{M}(\mathcal{X}_m) \\ &= \mathcal{B} - \sum_{i=1}^{N} \mathcal{Y}_m \times \{\mathbb{U}_m\} \times_i A^{(i)} \\ &= \mathcal{B} - \sum_{i=1}^{N} \mathcal{Y}_m \times_1 \mathbb{U}_{m_1} \cdots \times_i A^{(i)} \mathbb{U}_{m_i} \cdots \times_N \mathbb{U}_{m_N}. \end{aligned}$$

Using the relation (8) and the condition (9), we obtain

$$0 = \mathcal{B} \times \{\mathbb{U}_m\}^T - \sum_{i=1}^{N} \mathcal{Y}_m \times_i \mathbb{H}_{m_i}.$$

Using the expression of the right-hand side tensor $\mathcal{B}$ and the fact that $U_1^{(i)}$, $i = 1, \ldots, N$, are obtained from QR factorizations of $B^{(i)}$, $i = 1, \ldots, N$, we have

$$\begin{aligned} \mathcal{B} \times \{\mathbb{U}_m\}^T &= \mathcal{I}_R \times_1 \mathbb{U}_{m_1}^T B^{(1)} \cdots \times_N \mathbb{U}_{m_N}^T B^{(N)} \\ &= \mathcal{I}_R \times_1 \mathbb{U}_{m_1}^T U_1^{(m_1)} R^{(1)} \cdots \times_N \mathbb{U}_{m_N}^T U_1^{(m_N)} R^{(N)} \\ &= \mathcal{I}_R \times_1 E_1^{(m_1)} R^{(1)} \cdots \times_N E_1^{(m_N)} R^{(N)} \\ &= \mathcal{I}_R \times_1 \tilde{R}^{(1)} \cdots \times_N \tilde{R}^{(N)}, \end{aligned}$$

where $\tilde{R}^{(i)} = E_1^{(m_i)} R^{(i)}$.

Then, the low dimensional Sylvester tensor equation is given as follows

$$\sum_{i=1}^{N} \mathcal{Y}_m \times_i \mathbb{H}_{m_i} = \mathcal{B}_m, \tag{10}$$

where $\mathcal{B}_m = \mathcal{I}_R \times_1 \tilde{R}^{(1)} \cdots \times_N \tilde{R}^{(N)}$.

**Proposition 6** *Let $\mathcal{Y}_m$ be the solution of* (10)*, the associated residual tensor to the approximate solution $\mathcal{X}_m = \mathcal{Y}_m \times \{\mathbb{U}_m\}$ is*

$$\mathcal{R}_m = -\sum_{i=1}^{N} \mathcal{Y}_m \times_1 \mathbb{U}_{m_1} \cdots \times_i U_{m_i+1} H_{m_i+1,m_i} E_{m_i}^T \cdots \times_N \mathbb{U}_{m_N}.$$

*Proof* We have

$$\mathcal{R}_m = \mathcal{B} - \sum_{i=1}^{N} \mathcal{Y}_m \times_1 \mathbb{U}_{m_1} \cdots \times_i A^{(i)}\mathbb{U}_{m_i} \cdots \times_N \mathbb{U}_{m_N}.$$

Since $U_1^{(i)}, i = 1, \dots, N$, are obtained from QR factorizations of $B^{(i)}, i = 1, \dots, N$, the right-hand side tensor $\mathcal{B}$ can be written as follows

$$\begin{aligned}
\mathcal{B} &= \mathcal{I}_R \times_1 B^{(1)} \cdots \times_N B^{(N)} \\
&= \mathcal{I}_R \times_1 U_1^{(1)} R_1 \cdots \times_N U_1^{(N)} R_N \\
&= \mathcal{I}_R \times_1 \mathbb{U}_{m_1} E_1^{(m_1)} R_1 \cdots \times_N \mathbb{U}_{m_N} E_1^{(m_N)} R_N \\
&= \mathcal{I}_R \times_1 \tilde{R}^{(1)} \cdots \times_N \tilde{R}^{(N)} \times \{\mathbb{U}_m\} \\
&= \mathcal{B}_m \times \{\mathbb{U}_m\}.
\end{aligned}$$

Using the expression below and the relation (11), we obtain

$$\begin{aligned}
\mathcal{R}_m = &\left(\mathcal{B}_m - \sum_{i=1}^{N} \mathcal{Y}_m \times_i \mathbb{H}_{m_i}\right) \times \{\mathbb{U}_m\} \\
&- \sum_{i=1}^{N} \mathcal{Y}_m \times_1 \mathbb{U}_{m_1} \cdots \times_i U_{m_i+1} H_{m_i+1,m_i} E_{m_i}^T \cdots \times_N \mathbb{U}_{m_N}.
\end{aligned}$$

Invoking (13), the result in the proposition achieved. □

The following theorem can be established in the same way as theorem 1 in Section 2

**Theorem 2** *Let $\mathcal{R}_m$ be the corresponding residual to the approximate solution obtained by the block Arnoldi approach, then*

$$\|\mathcal{R}_m\| = \left(\sum_{i=1}^{N} \|\mathcal{Y}_m \times_i H_{m_i+1,m_i} E_{m_i}^T\|^2\right)^{1/2}.$$

*where $\mathcal{Y}_m$ is the solution of* (10) *and* $E_{m_i} = [0_R, \dots, 0_R, I_R]^T \in \mathbb{R}^{Rm_i \times R}$.

The block Arnoldi algorithm for Sylvester tensor equation is summarized in algorithm 1. By the end of this section, we point out that the solution of (10) is of size $Rm_1 \times Rm_2 \times \dots Rm_N$, which may become large even for moderate values of $R$ and $m_i$. Solving (10) required in step (3), Algorithm (1), can be then challenging; by this means, we propose the method in the following section.

**Algorithm 1**

1: **Input:** Coefficient matrices $A^{(i)}$, $i = 1, \ldots, N$, and the right hand side in low rank representation $\mathcal{B} = \left[B^{(1)}, \ldots, B^{(N)}\right]$.

2: **Output:** An approximate solution, $\mathcal{X}_m$, to (1).

3: Choose a tolerance $\epsilon > 0$, integer parameters $k_i'$, $i = 1, \ldots, N$, set for $i = 1, \ldots, N$, $k_i = 0$, $m_i = k_i'$.

4: For $i = 1, \ldots, N$ do

For $j = k_i + 1, \ldots, k_i + k_i'$, construct the orthonormal basis $\left[V_{k_i+1}, \ldots, V_{k_i+k_i'}\right]$ and the matrix $H_{m_i}$ by block Arnoldi algorithm.

5: Solve the low dimensional equation $\sum_{i=1}^{N} \mathcal{Y}_m \times_i H_{m_i} = \beta \mathcal{E}_m$

6: Compute the upper bound for the residual norm $r_m^2 = \sum_{i=1}^{N} \|\mathcal{Y}_m \times_i H_{m_i+1,m_i} E_{m_i}^T\|^2$.

7: If $r_m > \epsilon$, set for $i = 1, \ldots, N$, $k_i = k_i + k_i'$, $m_i = k_i + k_i'$ and go to step 2.

8: The approximate solution is given by $\mathcal{X}_m = \mathcal{Y}_m \times_1 \mathbb{V}_{m_1} \ldots \times_N \mathbb{V}_{m_N}$

### 4.2 Global Krylov approach

The previous block Krylov approach reduced (1) to a low dimensional Sylvester tensor equation, where the solution is of size $Rm_1 \times Rm_2 \times \ldots Rm_N$, while the global Krylov approach constructs an approximate solution from an $m_1 \times \ldots \times m_N$ tensor.

Let $\mathbb{V}_{m_i} = \left[V_1^{(i)}, \cdots, V_{m_i}^{(i)}\right]$ be the matrices obtained by applying the global Arnoldi algorithm to the pairs $(A^{(i)}, B^{(i)})$, $i = 1, 2, \ldots, N$, starting with $V_1^{(i)} = B^{(i)}/\|B^{(i)}\|$, then the following relations hold, for $i = 1, \ldots, N$,

$$A^{(i)}\mathbb{V}_{m_i} = \mathbb{V}_{m_i}\left(H_{m_i} \otimes I_R\right) + h_{m_i+1,m_i} V_{m_i+1} E_{m_i}^T. \tag{11}$$

$$A^{(i)}\mathbb{V}_{m_i} = \mathbb{V}_{m_i+1}\left(\tilde{H}_{m_i} \otimes I_R\right). \tag{12}$$

An approximate solution is given by

$$\mathcal{X}_m = (\mathcal{Y}_m \otimes \mathcal{I}_R) \times \{\mathbb{V}_m\},$$

where $\mathcal{Y}_m$ is the $m_1 \times \cdots \times m_N$ tensor, satisfying the low dimensional Sylvester tensor equation

$$\sum_{i=1}^{N} \mathcal{Y}_m \times_i H_{m_i} = \beta \mathcal{E}_m, \tag{13}$$

with $\mathcal{E}_m = e_1^{(m_1)} \circ \cdots \circ e_1^{(m_N)}$ and $\beta = \prod_{i=1}^{N} \|B^{(i)}\|$.

**Proposition 7** *Let $\mathcal{X}_m$ be the approximate solution obtained by the global Arnoldi approach and $\mathcal{R}_m$ be the corresponding residual, then*

$$\mathcal{R}_m = -\sum_{i=1}^{N} h_{m_i+1,m_i} \left(\mathcal{Y}_m \times_i e_{m_i}^T\right) \otimes \mathcal{I}_R \times_1 \mathbb{V}_{m_1} \cdots \times_i V_{m_i+1} \cdots \times_N \mathbb{V}_{m_N}. \quad (14)$$

*where $\mathcal{Y}_m$ is the solution of* (13) *and* $e_{m_i} = [0, \ldots, 0, 1]^T \in \mathbb{R}^{m_i}$.

*Proof* We have

$$\begin{aligned}\mathcal{R}_m &= \mathcal{B} - \mathcal{M}(\mathcal{X}_m)\\ &= \mathcal{B} - \sum_{i=1}^{N} (\mathcal{Y}_m \otimes \mathcal{I}_R) \times \{\mathbb{V}_m\} \times_i A^{(i)}\\ &= \mathcal{B} - \sum_{i=1}^{N} (\mathcal{Y}_m \otimes \mathcal{I}_R) \times_i \left(H_{m_i} \otimes I_R\right) \times \{\mathbb{V}_m\}\\ &\quad - \sum_{i=1}^{N} h_{m_i+1,m_i} (\mathcal{Y}_m \otimes \mathcal{I}_R) \times_1 \mathbb{V}_{m_1} \cdots \times_i V_{m_i+1} E_{m_i}^T \cdots \times_N \mathbb{V}_{m_N}.\end{aligned}$$

Using the proposition (3) and the fact that $E_{m_i} = e_{m_i} \otimes I_R$, we obtain

$$\begin{aligned}\mathcal{R}_m &= \mathcal{B} - \sum_{i=1}^{N} \left(\mathcal{Y}_m \times_i H_{m_i}\right) \otimes \mathcal{I}_R \times \{\mathbb{V}_m\}\\ &\quad - \sum_{i=1}^{N} h_{m_i+1,m_i} \left(\mathcal{Y}_m \times_i e_{m_i}^T\right) \otimes \mathcal{I}_R \times_1 \mathbb{V}_{m_1} \cdots \times_i V_{m_i+1} \cdots \times_N \mathbb{V}_{m_N}.\end{aligned}$$

Since the first blocks in each basis are taken from the right-hand side tensor $\mathcal{B}$, we have

$$\begin{aligned}\mathcal{B} &= \sum_{r=1}^{R} b_1^{(r)} \circ \cdots \circ b_N^{(r)}\\ &= \sum_{r=1}^{R} \|B^{(1)}\| V_{m_1}^{(1)}(:, r) \circ \cdots \circ \|B^{(N)}\| V_{m_N}^{(1)}(:, r)\\ &= \beta \sum_{r=1}^{R} E_1^{(m_1)} e_r^{(R)} \circ \cdots \circ E_1^{(m_N)} e_r^{(R)} \times \{\mathbb{V}_m\}.\end{aligned}$$

where $\beta = \prod_{i=1}^{N} \|B^{(i)}\|$.

Using the fact that $E_1^{(m_i)} e_r^{(R)} = e_r^{(Rm_i)} = e_1^{(m_i)} \otimes e_r^{(R)}$ and the proposition (2), we obtain

$$\begin{aligned}\mathcal{B} &= \beta \sum_{r=1}^{R} (e_1^{(m_1)} \circ \cdots \circ e_1^{(m_N)}) \otimes (e_r^{(R)} \circ \cdots \circ e_r^{(R)}) \times \{\mathbb{V}_m\}\\ &= \beta (e_1^{(m_1)} \circ \cdots \circ e_1^{(m_N)}) \otimes \sum_{r=1}^{R} (e_r^{(R)} \circ \cdots \circ e_r^{(R)}) \times \{\mathbb{V}_m\},\end{aligned}$$

then

$$\mathcal{B} = \beta \mathcal{E}_m \otimes \mathcal{I}_R \times \{ \mathbb{V}_m \}. \tag{15}$$

Now, we obtain

$$\begin{aligned} \mathcal{R}_m &= \left( \beta \mathcal{E}_m - \sum_{i=1}^{N} \mathcal{Y}_m \times_i H_{m_i} \right) \otimes \mathcal{I}_R \times \{ \mathbb{V}_m \} \\ &\quad - \sum_{i=1}^{N} h_{m_i+1,m_i} \left( \mathcal{Y}_m \otimes \mathcal{I}_R \right) \times_1 \mathbb{V}_{m_1} \cdots \times_i V_{m_i+1} E_{m_i}^T \cdots \times_N \mathbb{V}_{m_N}. \end{aligned}$$

Invoking (13) the result achieved. □

**Lemma 1** *Let $\mathbb{V}_{m_i}$ be the basis generated by the global Arnoldi algorithm applied to the pairs $(A^{(i)}, B^{(i)})$ and $\mathcal{X} \in \mathbb{R}^{J_1 \times \cdots \times J_N}$ with $J_i = Rm_i$ and $\mathcal{Z} \in \mathbb{R}^{K_1 \times \cdots \times K_N}$ with $K_i = m_i$. Then*

$$\| \mathcal{X} \times_i \mathbb{V}_{m_i} \| \leq \| \mathcal{X} \|, \tag{16}$$

$$\| (\mathcal{Z} \otimes \mathcal{I}_R) \times_i \mathbb{V}_{m_i} \| = \| \mathcal{Z} \|. \tag{17}$$

*Proof* We set $\mathcal{P} = \mathcal{X} \times_i \mathbb{V}_{m_i}$, the $i$th mode fiber of the tensor $\mathcal{P}$ can be written as follows

$$\begin{aligned} \mathcal{P}_{(j_1,\ldots,:,\ldots,j_N)} &= \mathbb{V}_{m_i} \mathcal{X}_{(j_1,\ldots,:,\ldots,j_N)} \\ &= \sum_{k=1}^{m_i} V_k \left[ \mathcal{X}_{(j_1,\ldots,(k-1)R+1,\ldots,j_N)}, \ldots, \mathcal{X}_{(j_1,\ldots,kR+1,\ldots,j_N)} \right]^T. \end{aligned}$$

As the norm of a tensor can be expressed as the sum of the norm of all its $i$-mode fibers, it results that

$$\begin{aligned} \|\mathcal{P}\|^2 &= \sum_{j_1,\ldots,j_{n-1},j_{n+1},\ldots,j_N} \| \mathcal{P}_{(j_1,\ldots,:,\ldots,j_n)} \|^2 \\ &= \sum_{j_1,\ldots,j_{n-1},j_{n+1},\ldots,j_N} \| \sum_{k=1}^{m_i} V_k \left[ \mathcal{X}_{(j_1,\ldots,(k-1)R+1,\ldots,j_n)}, \ldots, \mathcal{X}_{(j_1,\ldots,kR+1,\ldots,j_N)} \right]^T \|^2 \end{aligned}$$

Since $\mathbb{V}_{m_i} = \left[ V_1, \ldots, V_{m_i} \right]$ is orthonormal, we obtain

$$\begin{aligned} \|\mathcal{P}\|^2 &\leq \sum_{j_1,\ldots,j_{n-1},j_{n+1},\ldots,j_N} \sum_{k=1}^{m_i} \| \left[ \mathcal{X}_{(j_1,\ldots,(k-1)R+1,\ldots,j_n)}, \ldots, \mathcal{X}_{(j_1,\ldots,kR+1,\ldots,j_N)} \right]^T \|^2 \\ &\leq \|\mathcal{X}\|^2, \end{aligned}$$

therefore (16) is achieved.

We set $\mathcal{Q} = (\mathcal{Z} \otimes \mathcal{I}_R) \times_i \mathbb{V}_{m_i}$, if $\mathcal{Q}_{(\overline{r_1 j_1},\ldots,:,\ldots,\overline{r_N j_N})}$, $\mathcal{Z}_{(j_1,\ldots,:,\ldots,j_N)}$ and $\mathcal{I}_{R\,(r_1,\ldots,:,\ldots,r_N)}$ denotes the $i$th-mode fibers of $\mathcal{Q}$, $\mathcal{Z}$, and $\mathcal{I}_R$, respectively, we have

$$\begin{aligned} \mathcal{Q}_{(\overline{r_1 j_1},\ldots,:,\ldots,\overline{r_N j_N})} &= \mathbb{V}_{m_i} \left( \mathcal{Z}_{(j_1,\ldots,:,\ldots,j_N)} \otimes \mathcal{I}_{R\,(r_1,\ldots,:,\ldots,r_N)} \right) \\ &= \sum_{k=1}^{m_i} V_k \mathcal{Z}_{(j_1,\ldots,k,\ldots,j_N)} \otimes \mathcal{I}_{R\,(r_1,\ldots,:,\ldots,r_N)} \\ &= \sum_{k=1}^{m_i} \mathcal{Z}_{(j_1,\ldots,k,\ldots,j_N)} V_k \otimes \mathcal{I}_{R\,(r_1,\ldots,:,\ldots,r_N)}. \end{aligned}$$

As $\mathbb{V}_{m_i} = \left[V_1, \ldots, V_{m_i}\right]$ is orthonormal, we obtain

$$\begin{aligned} \|\mathcal{Q}_{(\overline{r_1 j_1}, \ldots, :, \ldots, \overline{r_N j_N})}\|^2 &= \sum_{k=1}^{m_i} |\mathcal{Z}_{(j_1, \ldots, k, \ldots, j_N)}|^2 \|V_k \otimes \mathcal{I}_{R(r_1, \ldots, :, \ldots, r_N)}\|^2 \\ &= \|\mathcal{Z}_{(j_1, \ldots, k, \ldots, j_N)}\|^2, \end{aligned}$$

then (17) is achieved. □

In the following theorem, we give an upper bound for the residual norm

**Theorem 3** *Let $\mathcal{X}_m$ be the approximate solution obtained by the global Arnoldi approach and $\mathcal{R}_m$ be the corresponding residual, then*

$$\|\mathcal{R}_m\| \leq \left(\sum_{i=1}^{N} |h_{m_i+1, m_i}|^2 \|\mathcal{Y}_m \times_i e_{m_i}^T\|^2\right)^{1/2}, \tag{18}$$

*where $\mathcal{Y}_m$ is the solution of (13) and $e_{m_i} = [0, \ldots, 0, 1]^T \in \mathbb{R}^{m_i}$.*

*Proof* We have

$$\begin{aligned} \mathcal{R}_m &= \mathcal{B} - \sum_{i=1}^{N} (\mathcal{Y}_m \otimes \mathcal{I}_R) \times \{\mathbb{V}_m\} \times_i A^{(i)} \\ &= \mathcal{B} - \sum_{i=1}^{N} (\mathcal{Y}_m \otimes \mathcal{I}_R) \times_1 \mathbb{V}_{m_1} \ldots \times_i A^{(i)} \mathbb{V}_{m_i} \ldots \times_N \mathbb{V}_{m_N}. \end{aligned}$$

Using the relations (12) and (15), we obtain

$$\mathcal{R}_m = \beta \mathcal{E}_m \otimes \mathcal{I}_R \times \{\mathbb{V}_m\} - \sum_{i=1}^{N} \left(\mathcal{Y}_m \times_i \tilde{H}_{m_i}\right) \otimes \mathcal{I}_R \times_1 \mathbb{V}_{m_1} \ldots \times_i \mathbb{V}_{m_i+1} \ldots \times_N \mathbb{V}_{m_N}.$$

Let the tensor $\mathcal{Z}_m^{(i)} \in \mathbb{R}^{(m_1+1) \times \cdots \times (m_N+1)}$, for $i = 1, \ldots, N$, defined by

$$\begin{cases} \mathcal{Z}_m^{(i)}{}_{j_1, \ldots, j_N} = \left(\mathcal{Y}_m \times_i H_{m_i}\right)_{j_1, \ldots, j_N}, & j_k \in \{1, \ldots, m_k\}, \ (k = 1, \ldots, N) \\ \mathcal{Z}_m^{(i)}{}_{j_1, \ldots, j_N} = h_{m_i+1, m_i} \mathcal{Y}_m \times_i e_{m_i}^T, & j_k = 1, \ (k = 1, \ldots, N) \\ \mathcal{Z}_m^{(i)}{}_{j_1, \ldots, j_N} = 0, & j_l = 1, (l \neq k) \end{cases}$$

and the tensor $\mathcal{F}_m \in \mathbb{R}^{(m_1+1) \times \cdots \times (m_N+1)}$, defined by

$$\begin{cases} \mathcal{F}_{m\, j_1, \ldots, j_N} = \mathcal{E}_m, & j_k \in \{1, \ldots, m_k\}, \ (k = 1, \ldots, N) \\ \mathcal{F}_{m\, j_1, \ldots, j_N} = 0, & j_k = 1, \ (k = 1, \ldots, N) \end{cases}$$

Then

$$\mathcal{R}_m = \beta \mathcal{F}_m \otimes \mathcal{I}_R \times \{\mathbb{V}_{m+1}\} - \sum_{i=1}^{N} \mathcal{Z}_m^{(i)} \otimes \mathcal{I}_R \times_1 \mathbb{V}_{m_1+1} \ldots \times_i \mathbb{V}_{m_i+1} \ldots \times_N \mathbb{V}_{m_N+1}.$$

By setting $\tilde{\mathcal{Z}}_m = \sum_{i=1}^{N} \mathcal{Z}_m^{(i)}$, we obtain

$$\mathcal{R}_m = \left(\beta \mathcal{F}_m - \tilde{\mathcal{Z}}_m\right) \otimes \mathcal{I}_R \times \{\mathbb{V}_{m+1}\}.$$

We set $\mathcal{Z}_m^{(0)} = \beta\mathcal{F}_m - \tilde{\mathcal{Z}}_m$, we have

$$\begin{cases} \mathcal{Z}_m^{(0)}{}_{j_1,\ldots,j_N} = 0 & j_k \in \{1,\ldots,m_k\},\ (k=1,\ldots,N) \\ \mathcal{Z}_m^{(0)}{}_{j_1,\ldots,j_N} = -h_{m_i+1,m_i}\mathcal{Y}_m \times_i e_{m_i}^T, & j_k = 1,\ (k=1,\ldots,N) \end{cases}$$

and

$$\mathcal{R}_m = \mathcal{Z}_m^{(0)} \circledast \mathcal{I}_R \times \{\mathbb{V}_{m+1}\}.$$

By applying the relation (16) of lemma 1 $(N-1)$ times to $\|\mathcal{R}_m\|$, we obtain

$$\|\mathcal{R}_m\|^2 \leq \|\mathcal{Z}_m^{(0)} \circledast \mathcal{I}_R \times_1 \mathbb{V}_{m_1+1}\|^2.$$

Then, the relation (17) of lemma 1 leads to

$$\begin{aligned} \|\mathcal{R}_m\|^2 &\leq \|\mathcal{Z}_m^{(0)}\|^2 \\ &\leq \sum_{i=1}^{N} |h_{m_i+1,m_i}|^2 \|\mathcal{Y}_m \times_i e_{m_i}^T\|^2. \end{aligned}$$

□

The Global Arnoldi algorithm for Sylvester tensor equation is summarized as follows

**Algorithm 2**

1: **Input:** Coefficient matrices $A^{(i)}, i = 1,\ldots,N$, and the right hand side in low rank representation $\mathcal{B} = \left[B^{(1)},\ldots,B^{(N)}\right]$.

2: **Output:** An approximate solution, $\mathcal{X}_m$, to (1).

3: Choose a tolerance $\epsilon > 0$, integer parameters $k_i'$, $i = 1,\ldots,N$, set for $i = 1,\ldots,N,\ k_i = 0,\ m_i = k_i'$.

4: For $i = 1,\ldots,N$ do
For $j = k_i+1,\ldots,k_i+k_i'$, construct the orthonormal basis $\left[V_{k_i+1},\ldots,V_{k_i+k_i'}\right]$ and the matrix $H_{m_i}$ by global Arnoldi algorithm.

5: Solve the low dimensional equation $\sum_{i=1}^{N} \mathcal{Y}_m \times_i H_{m_i} = \beta\mathcal{E}_m$

6: Compute the upper bound for the residual norm $r_m^2 = \sum_{i=1}^{N} |h_{m_i+1,m_i}|^2 \|\mathcal{Y}_m \times_i e_{m_i}^T\|^2$.

7: If $r_m > \epsilon$, set for $i = 1,\ldots,N$, $k_i = k_i + k_i',\ m_i = k_i + k_i'$ and go to step 2.

8: The approximate solution is given by $\mathcal{X}_m = (\mathcal{Y}_m \circledast \mathcal{I}_R) \times_1 \mathbb{V}_{m_1} \ldots \times_N \mathbb{V}_{m_N}$

### 4.3 Complexity consideration

In this section, we present the required number of operations to apply the global or block Arnoldi algorithm to the coefficient matrices. For sake of simplicity, we consider the 3-mode case, i.e., $\mathcal{B} \in \mathbb{R}^{n\times n\times n}$ and $A^{(i)} \in \mathbb{R}^{n\times n}$, $i = 1,2,3$. The associated number of operations in the Arnoldi process while computing $A^{(i)}V_j^{(i)}$ is determined by

$$(2n_z(A^{(i)}) - 1)R,$$

where $n_z(A^{(i)})$ refer to the number of non-zero entries of the matrix $A^{(i)}$. So for 3 matrices and after $m$ iterations, the total cost is

$$mR[2(nz(A^{(1)}) + nz(A^{(2)}) + nz(A^{(3)})) - 3],$$

which is $3mR(2n^2 - 1)$ in the worst case when $A^{(i)},\ i = 1,2,3$, are full matrices.

## 5 Numerical examples

In this section, we present three numerical examples to show the effectiveness of our approaches for solving (1), with large-scale coefficient matrices (Example 1). The low dimensional Sylvester tensor (10) and (13) will be solved by the GLS-BTF algorithm given in [14] when the size of the reduced equation is small, or by the recursive algorithm presented in [5]. The numerical results were performed on a 2.7-GHz Intel Core i5 and 8 Go 1600-MHz DDR3 with Matlab R2016a. In all the examples, the right-hand side tensor is either constructed randomly or constructed so that the exact solution $\mathcal{X}^*$ is given. Note that each cycle corresponds to $k' = 5$ iterations in all examples. The used stopping criterion is

$$\|\mathcal{R}_m\| < \epsilon,$$

where $\epsilon$ is a given tolerance and $\mathcal{R}_m$ is the $m$th residual associated with the approximated solution $\mathcal{X}_m$.

### 5.1 Example 1

We point out that this section is restricted to the special case of the Sylvester tensor equation with $N = 3$, i.e.,

$$\mathcal{X} \times_1 A^{(1)} + \mathcal{X} \times_2 A^{(2)} + \mathcal{X} \times_N A^{(3)} = \mathcal{B}.$$

In this first example, the coefficient matrices are taken from [25, Example 35.1], and have the same size $n$. They are generated by the Matlab-commands `eye` and `rand` as follows

$$A^{(i)} = \mathtt{eye}(n) + \frac{0.5}{sqrt(n)}\,\mathtt{rand}(n) \quad \textit{for } i = 1, 2, 3,$$

and the right-hand side tensor $\mathcal{B}$ is chosen so that the exact solution $\mathcal{X}^*$ is a random tensor of rank $r$, i.e.,

$$\mathcal{X}^* = \sum_{k=1}^{r} x_1^{(k)} \circ x_2^{(k)} \circ x_3^{(k)},$$

where $x_i^{(k)} = \mathtt{rand}(n, 1)$, for $i = 1, 2, 3$, and $k = 1, \ldots, r$. Notice, in this case, that straightforward computations show that the right-hand side tensor is of rank $R = 3r$. We point out that the Krylov subspace dimensions are chosen to be the same, i.e., $m_1 = m_2 = m_3 = m$, since the coefficient matrices are the same. We take $n$ in the set $\{500, 10,000\}$, and the tolerance $\epsilon$ in the set $\{10^{-12}, 10^{-9}\}$ respectively. The numerical results are reported in Table 1. We point out that the CPU time does not cover the construction of the approximate solution; it covers only the construction of the Krylov subspaces basis and the solution of the reduced Sylvester tensor equation. For $n = 500$, we gave the exact error and the residual norm, where for $n = 10,000$, we gave only the residual norm due to the time needed to construct the whole approximate solution tensor. The larger CPU time needed for the block method is caused by the computational expenses of the block Arnoldi algorithm and to the computation of the solution of the reduced Sylvester tensor equation of order $mR \times mR \times mR$ for

**Table 1** Example 1

| | $n$ | $R$ | Cycles | $\|\mathcal{R}_m\|$ | $\|\mathcal{X}^* - \mathcal{X}_m\|$ | CPU time |
|---|---|---|---|---|---|---|
| Block | 500 | 3 | 2 | $1.13 \cdot 10^{-12}$ | $3.47 \cdot 10^{-9}$ | 2.03 |
| | | 9 | 2 | $2.83 \cdot 10^{-12}$ | $7.80 \cdot 10^{-9}$ | 30.6 |
| | 10,000 | 3 | 2 | $2.58 \cdot 10^{-11}$ | – | 4.04 |
| | | 9 | 2 | $2.13 \cdot 10^{-10}$ | – | 54.5 |
| Global | 500 | 3 | 3 | $9.40 \cdot 10^{-12}$ | $6.86 \cdot 10^{-9}$ | 1.59 |
| | | 9 | 3 | $5.24 \cdot 10^{-11}$ | $3.56 \cdot 10^{-8}$ | 2.45 |
| | 10,000 | 3 | 3 | $1.73 \cdot 10^{-9}$ | – | 3.97 |
| | | 3 | 9 | $9.02 \cdot 10^{-9}$ | – | 5.41 |

increasing $m$; for this reason, we will compute the following examples using only the global method.

### 5.2 Example 2

In this example, the coefficient matrices $A^{(i)}, i = 1, \ldots, N$, are obtained from the 5-point discretization of the Poisson equation on an $n_0$−by−$n_0$ mesh, with homogeneous Dirichlet boundary conditions. We use the following Matlab command to generate $A^{(i)}, i = 1, \ldots, N$,

$$A^{(i)} = \texttt{gallery}('poisson', n_0)$$

$A^{(i)}$ are then of size $n_0^2 \times n_0^2$. For this experiment, we take $n$ in the set $\{400, 121, 100\}$ and $N$ in the set $\{3, 4\}$, the tolerance $\epsilon$ is set to $10^{-6}$. We construct the right-hand side tensor so that all the components of the exact solution $\mathcal{X}^*$ are one (the numerical examples are reported in Table 2). We run the same example with a random rank $R$ right-hand side tensor (see Table 3).

### 5.3 Example 3

Here, we keep the same data as the previous example, except for the coefficient matrices $A^{(i)}, \quad i = 1, \ldots, N$, which are obtained from the discretization of the operator

$$Lu := \Delta u - f_1(x, y)\frac{\partial u}{\partial x} + f_2(x, y)\frac{\partial u}{\partial y} + g(x, y),$$

**Table 2** $\mathcal{X}^* = ones(n, n, n)$ for $N = 3, n = 400$ and $\mathcal{X}^* = ones(n, n, n, n)$ for $N = 4, n = 100$

| Matrices | $N$ | $n$ | Cycles | $\|\mathcal{R}_m\|$ | $\|\mathcal{X}^* - \mathcal{X}_m\|$ | CPU time |
|---|---|---|---|---|---|---|
| Example 2 | 3 | 400 | 8 | $4.410^{-8}$ | $1.510^{-8}$ | 4.01 |
| | 4 | 100 | 5 | $210^{-10}$ | $7.4210^{-11}$ | 3.44 |
| Example 3 | 3 | 400 | 15 | $5.510^{-6}$ | $1.1510^{-8}$ | 26.6 |
| | 4 | 100 | 7 | $8.9310^{-6}$ | $2.7810^{-8}$ | 24.9 |

**Table 3** The right-hand side $\mathcal{B}$ is a random tensor of rank $R = 5$

| Matrices | $N$ | $n$ | Cycles | $\|\mathcal{R}_m\|$ | CPU time |
|---|---|---|---|---|---|
| Example 2 | 3 | 400 | 14 | $2.8210^{-6}$ | 14.5 |
| | 4 | 121 | 8 | $5.710^{-6}$ | 26.54 |
| Example 3 | 3 | 400 | 16 | $1.4510^{-6}$ | 56.0 |
| | 4 | 121 | 9 | $7.210^{-6}$ | 36.09 |

on the unit square $[0, 1] \times [0, 1]$ with homogeneous Dirichlet boundary conditions. The number of inner grid points in each direction is $n_0$ for the operator $L$. The dimensions of the matrices $A^{(i)}, i = 1, \ldots, N$, are $I_i = n_0^2$. The discretization of the operator $L$ yields matrices extracted from the Lyapack packag [22], using the command `fdm` and denoted as

$$A^{(i)} = fdm(n_0, f_1(x, y), f_2(x, y), g(x, y)),$$

with $f_1(x, y) = e^{xy}, \quad f_2(x, y) = sin(xy), \quad g(x, y) = y^2 - x^2$. The numerical results for this example are reported in Tables 2 and 3.

## 6 Conclusion

In this paper, we have proposed new approaches to extract approximate solutions to (1) with low rank right-hand sides. The first approach is based on the use of the block Arnoldi algorithm for the coefficient matrices in (1), which leads to the reduced Sylvester tensor (10). The second approach is based on the use of the global Arnoldi algorithm in order to obtain the low dimensional Sylvester tensor (13). We gave the expressions of the residuals and the residual norms for each approach. Numerical examples show that our approaches lead to satisfactory results when applied to (1). Combining our approaches with some tensors decompositions in order to work with full rank right-hand sides is a future project.

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