




A new stable collocation method for solving a class of nonlinear fractional delay differential equations

Lei Shi¹  · Zhong Chen¹ · Xiaohua Ding¹ · Qiang Ma¹

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Abstract

In this paper, a stable collocation method for solving the nonlinear fractional delay differential equations is proposed by constructing a new set of multiscale orthonormal bases of $W_{2,0}^1$. Error estimations of approximate solutions are given and the highest convergence order can reach four in the sense of the norm of $W_{2,0}^1$. To overcome the nonlinear condition, we make use of Newton's method to transform the nonlinear equation into a sequence of linear equations. For the linear equations, a rigorous theory is given for obtaining their ε -approximate solutions by solving a system of equations or searching the minimum value. Stability analysis is also obtained. Some examples are discussed to illustrate the efficiency of the proposed method.

Keywords Nonlinear fractional delay differential equations · ε -Approximate solutions · Newton's iterative formula

1 Introduction

Nowadays, fractional differential equations have been a hot topic in the field of differential equations for their widespread applications in many science fields [1, 2]. Among them, fractional delay differential equations begin to arouse attentions of many researchers. These equations have also many applications in various areas such as control theory, biology, and economy [3, 4]. Since some models have a great deal to do with past condition, the insertion of a time delay makes these models more realistic. Therefore, the development of theory and numerical algorithms about fractional delay differential equations is of importance. For example, Hu and Zhao [5] give

✉ Zhong Chen
cz0309@sina.com

Lei Shi
slmath@hit.edu.cn

¹ Department of Mathematics, Harbin Institute of Technology, Weihai, China

the condition of asymptotical stability of nonlinear fractional system with distributed time delay by utilizing the function monotonous properties and the stability theorem of the fractional linear system. Pimenov et al. [6] use a BDF difference scheme based on approximations of Clenshaw and Curtis type to get the numerical solution of the following equation:

$$\begin{cases} D_C^\beta y(t) = f[t, y(t), y(t - \tau)], & 0 \leq t \leq L, \quad 0 < \beta \leq 1, \\ y(t) = \phi(t), & -\tau \leq t \leq 0. \end{cases} \quad (1)$$

For (1), Moghaddam and Mostaghim [4] use the fractional finite difference method to obtain its numerical solutions. Saeed et al. [7] draw on the steps method and Chebyshev wavelet method to solve the nonlinear fractional delay differential equation:

$$\begin{cases} D_C^\alpha y(x) = h(x) + f[y(x), y'(x), y(px - \tau), y'(px - \tau)], & 0 \leq x \leq b, \quad 1 < \alpha \leq 2, \\ y(x) = \phi(x), & -b \leq x \leq 0, \end{cases} \quad (2)$$

and get their approximate solutions. However, to our best knowledge, there are few articles about the study of fractional delay differential equations especially with regard to nonlinear fractional delay differential equations.

Newton's iterative method is a very powerful tool to solve the nonlinear problems. Many researchers have been studying and generalizing Newton's iterative method and they use it to solve nonlinear problems. Deuflhard [8] in his monograph constructs adaptive Newton algorithms for some specific nonlinear problems. Krasnosel'skii et al. [9] in their monograph study the Newton-Kantorovich's method, give a modified Newton-Kantorovich's method and solve the problem of the choice of initial approximations. Xu et al. [10] use quasi-Newton's method to linearize a nonlinear operator equation, and so on. Theoretical analysis shows that the convergence order of Newton's iterative formula is order 2.

Motivated greatly by the abovementioned excellent works, in this paper, we deal with the nonlinear fractional delay differential (2) in the condition $p = 1$, that is the following equation:

$$\begin{cases} D_C^\alpha y(x) = g(x) + f[x, y(x), y'(x), y(x - \tau), y'(x - \tau)], & 0 \leq x \leq 1, \quad 1 < \alpha \leq 2, \\ y(x) = y_0(x), & -\tau \leq x \leq 0, \end{cases} \quad (3)$$

where f has continuous second order partial derivative, $g(x) \in C[0, 1]$ and $y_0(x) \in C^2[-\tau, 0]$ which are known functions. $\tau > 0$ is a constant delay. The fractional derivative is in the sense of Caputo and $y(x) \in C^1[-\tau, 1]$ is the unknown function.

In this article, we develop a stable and effective collocation method to solve (3). The collocation method is one of the most efficient methods for obtaining accurate numerical solutions of differential equations including variable coefficients and nonlinear differential equations [11–14]. The stability of collocation methods has always been an important topic. At present, the definition of its stability is that when there are many collocation nodes, the resulting equations are not ill conditioned and the results are still valid. For a stable collocation method, we can improve the accuracy by increasing the number of approximation terms. Accordingly, it is particularly important to establish a high-precision and stable collocation method.

The choice of bases of a space is important for a collocation method. Approximate solutions with different accuracy can be derived by using different bases for the same equation. For obtaining higher accuracy solution of the nonlinear fractional differential equation, we construct a new set of multiscale orthonormal bases of $W_{2,0}^1$, give error estimations of approximate solutions, and prove in Section 3 that the highest convergence order can reach four in the sense of $W_{2,0}^1$.

Noting that the problem of initial value selection of the Newton’s iterative method has been well solved in [9], so in this paper we transform (3) into a list of linear equations by using Newton’s iterative method. Then a new stable collocation method is proposed to solve these equations. Compared with [15], the final numerical experiments show that our method is better in dealing with this kind of equations.

The remainder of this paper is organized as follows: In Section 2, some relevant definitions and properties of the fractional calculus and the space W_2^1 and $W_{2,0}^1$ are introduced. In Section 3, we construct a set of multiscale orthonormal bases of $W_{2,0}^1$ and give an error estimation of the approximate solution. In Section 4, we construct the ε -approximate solution method and apply this method to solve the linear fractional delay differential equation. In Section 5, we analyze the stability of the ε -approximate solution method. In Section 6, we make use of Newton’s method to transform the nonlinear equation into a sequence of linear equations. In Section 7, we give the algorithm implementation of Newton’s iterative formula for solving the nonlinear fractional delay differential equation. In Section 8, three numerical examples are given to clarify the effectiveness of the algorithm. In the last section, the conclusions are presented.

2 Preliminaries and notations

In this section, some preliminary results about the fractional calculus operators and reproducing kernel spaces are recalled [16–18].

Definition 2.1 The Riemann-Liouville (R-L) fractional integral operator J_0^α is given by

$$J_0^\alpha u(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x - s)^{\alpha-1} u(s) ds, \quad \alpha > 0,$$

where $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$.

Definition 2.2 The Caputo fractional differential operator D_C^α is given by

$$D_C^\alpha u(x) = \frac{1}{\Gamma(n - \alpha)} \int_0^x (x - s)^{n-\alpha-1} u^{(n)}(s) ds, \quad n - 1 < \alpha < n, \quad n = [\alpha].$$

Theorem 2.1 Let $u(x)$ be the exact solution of (3), then $D_C^\alpha u(x) \in C[0, 1]$.

Proof By the assumption, one has

$$|D_C^\alpha u(x)| = |J_0^{2-\alpha} u''(x)| < \infty.$$

Since integrability of a function is equivalent to its absolute integrability, one has

$$\frac{x^{1-\alpha}}{\Gamma(2-\alpha)} \int_0^x |u''(s)| ds \leq \frac{1}{\Gamma(2-\alpha)} \int_0^x (x-s)^{1-\alpha} |u''(s)| ds < \infty.$$

Taking $x = 1$, we obtain $u''(x) \in L^1[0, 1]$. Thus, $u(x)$ and $u'(x)$ are absolutely continuous on $[0, 1]$. Hence, $D_C^\alpha u = g(x) + f[x, u(x), u'(x), u(x - \tau), u'(x - \tau)] \in C[0, 1]$. □

Definition 2.3 Assume that n is a positive integer, then we denote

$$AC^n[0, 1] = \{u(x) | u^{(n-1)}(x) \text{ is absolute continuous in } [0, 1]\}.$$

$$W_2^1 = W_2^1[0, 1] = \{u(x) | u(x) \in AC[0, 1], u' \in L^2[0, 1]\}.$$

The inner product $(\cdot, \cdot)_1$ and the norm $\|\cdot\|_1$ of W_2^1 are given by

$$(u, v)_1 = u(0)v(0) + \int_0^1 u'(x)v'(x) dx, \quad \|u(x)\|_1 = \sqrt{(u(x), u(x))_1}, \quad \forall u, v \in W_2^1.$$

It is easy to see that $W_2^1 \subset C[0, 1]$. Similar to [17], we can prove that W_2^1 is not only a Hilbert space, but also a reproducing kernel space with the reproducing kernel

$$r(x, y) = \begin{cases} 1 + x, & x \leq y, \\ 1 + y, & x > y. \end{cases}$$

Definition 2.4 The inner product space $W_{2,0}^1$ is defined as $W_{2,0}^1 \triangleq W_{2,0}^1[0, 1] = \{u(x) \in W_2^1 | u(0) = 0\}$ with the inner product

$$(u, v)_1 = \int_0^1 u'(x)v'(x) dx, \quad \forall u, v \in W_{2,0}^1.$$

Lemma 2.1

$$\|u(x)\|_C \leq \sqrt{2} \|u(x)\|_1, \quad \forall u(x) \in W_2^1,$$

where $\|u(x)\|_C \triangleq \max_{x \in [0, 1]} |u(x)|$.

Proof For any $u(x) \in W_2^1$, we have

$$\begin{aligned} |u(x)| &= |(u(y), r(x, y))_1| \leq \|u(x)\|_1 \cdot \|r(x, y)\|_1 = \|u(x)\|_1 \cdot \sqrt{r(x, x)} \\ &= \|u(x)\|_1 \sqrt{1+x} \leq \sqrt{2} \|u(x)\|_1. \end{aligned}$$

Thus $\|u(x)\|_C \leq \sqrt{2} \|u(x)\|_1$. □

The following definition will be used in Section 6.

Definition 2.5 The inner product space W_2^α is defined as $W_2^\alpha \triangleq W_2^\alpha[0, 1] = \{J_0^\alpha u(s) | u(s) \in W_{2,0}^1[0, 1]\}$. And the inner product and the norm of W_2^α are given by

$$\begin{aligned} (u, v)_\alpha &= (D_C^\alpha u(s), D_C^\alpha v(s))_1 \\ &= \int_0^1 (D_C^\alpha u)'(s)(D_C^\alpha v)'(s)ds, \quad \|u(x)\|_\alpha = \sqrt{(u(x), u(x))_\alpha}, \end{aligned}$$

where $u, v \in W_2^\alpha$.

Remark 2.1 Similar to [19], one can prove that W_2^α is not only a Hilbert space, but also a reproducing kernel space.

3 Construction of the multiscale orthonormal bases of $W_{2,0}^1$ and error estimations

In this section, we construct the multiscale orthonormal bases of $W_{2,0}^1$ by the famous Legendre multiwavelets, and give error estimations of approximate solutions.

Define the cubic Legendre scaling functions in the interval $[0, 1]$ [20],

$$\begin{aligned} \eta^0(x) &= 1, \\ \eta^1(x) &= \sqrt{3}(2x - 1), \\ \eta^2(x) &= \sqrt{5}(6x^2 - 6x + 1), \\ \eta^3(x) &= \sqrt{7}(20x^3 - 30x^2 + 12x - 1), \end{aligned}$$

and then cubic Legendre multiwavelets are given as

$$\begin{aligned} \phi^0(x) &= \begin{cases} k_1(x) = -\sqrt{\frac{15}{17}}(224x^3 - 216x^2 + 56x - 3), & x \in [0, \frac{1}{2}), \\ k_2(x) = \sqrt{\frac{15}{17}}(224x^3 - 456x^2 + 296x - 61), & x \in [\frac{1}{2}, 1], \end{cases} \\ \phi^1(x) &= \begin{cases} k_3(x) = \sqrt{\frac{1}{21}}(1680x^3 - 1320x^2 + 270x - 11), & x \in [0, \frac{1}{2}), \\ k_4(x) = \sqrt{\frac{1}{21}}(1680x^3 - 3720x^2 + 2670x - 619), & x \in [\frac{1}{2}, 1], \end{cases} \\ \phi^2(x) &= \begin{cases} k_5(x) = -\sqrt{\frac{35}{17}}(256x^3 - 174x^2 + 30x - 1), & x \in [0, \frac{1}{2}), \\ k_6(x) = \sqrt{\frac{35}{17}}(256x^3 - 594x^2 + 450x - 111), & x \in [\frac{1}{2}, 1], \end{cases} \\ \phi^3(x) &= \begin{cases} k_7(x) = \sqrt{\frac{5}{42}}(420x^3 - 246x^2 + 36x - 1), & x \in [0, \frac{1}{2}), \\ k_8(x) = \sqrt{\frac{5}{42}}(420x^3 - 1014x^2 + 804x - 209), & x \in [\frac{1}{2}, 1], \end{cases} \end{aligned}$$

$$\begin{aligned} \phi_{ik}^0(x) &= \sqrt{2}^{i-1} \phi^0(2^{i-1}x - k) = \sqrt{2}^{i-1} \begin{cases} k_1(2^{i-1}x - k), & x \in [\frac{k}{2^{i-1}}, \frac{k+\frac{1}{2}}{2^{i-1}}), \\ k_2(2^{i-1}x - k), & x \in [\frac{k+\frac{1}{2}}{2^{i-1}}, \frac{k+1}{2^{i-1}}], \\ 0, & \text{else,} \end{cases} \\ \phi_{ik}^1(x) &= \sqrt{2}^{i-1} \phi^1(2^{i-1}x - k) = \sqrt{2}^{i-1} \begin{cases} k_3(2^{i-1}x - k), & x \in [\frac{k}{2^{i-1}}, \frac{k+\frac{1}{2}}{2^{i-1}}), \\ k_4(2^{i-1}x - k), & x \in [\frac{k+\frac{1}{2}}{2^{i-1}}, \frac{k+1}{2^{i-1}}], \\ 0, & \text{else,} \end{cases} \\ \phi_{ik}^2(x) &= \sqrt{2}^{i-1} \phi^2(2^{i-1}x - k) = \sqrt{2}^{i-1} \begin{cases} k_5(2^{i-1}x - k), & x \in [\frac{k}{2^{i-1}}, \frac{k+\frac{1}{2}}{2^{i-1}}), \\ k_6(2^{i-1}x - k), & x \in [\frac{k+\frac{1}{2}}{2^{i-1}}, \frac{k+1}{2^{i-1}}], \\ 0, & \text{else,} \end{cases} \\ \phi_{ik}^3(x) &= \sqrt{2}^{i-1} \phi^3(2^{i-1}x - k) = \sqrt{2}^{i-1} \begin{cases} k_7(2^{i-1}x - k), & x \in [\frac{k}{2^{i-1}}, \frac{k+\frac{1}{2}}{2^{i-1}}), \\ k_8(2^{i-1}x - k), & x \in [\frac{k+\frac{1}{2}}{2^{i-1}}, \frac{k+1}{2^{i-1}}], \\ 0, & \text{else,} \end{cases} \end{aligned}$$

where $i = 1, 2, \dots, k = 0, 1, 2, \dots, 2^{i-1} - 1$.

Denote

$$\begin{aligned} \{\varphi_i(x)\}_{i=1}^\infty &\triangleq \{\eta^0(x), \eta^1(x), \eta^2(x), \eta^3(x), \phi^0(x), \phi^1(x), \phi^2(x), \phi^3(x), \phi_{20}^0(x), \\ &\phi_{21}^0(x), \phi_{20}^1(x), \phi_{21}^1(x), \phi_{20}^2(x), \\ &\phi_{21}^2(x), \phi_{20}^3(x), \phi_{21}^3(x), \phi_{30}^0(x), \dots\}. \end{aligned}$$

Lemma 3.1 $\{\varphi_i(x)\}_{i=1}^\infty$ is a set of multiscale orthonormal bases of $L^2[0, 1]$.

Proof For the proof one can refer to [21]. □

Theorem 3.1 $\{\psi_i(x)\}_{i=1}^\infty \triangleq \{J_0^1 \eta^0(x), J_0^1 \eta^1(x), J_0^1 \eta^2(x), J_0^1 \eta^3(x), J_0^1 \phi^0(x), J_0^1 \phi^1(x), J_0^1 \phi^2(x), J_0^1 \phi^3(x), J_0^1 \phi_{20}^0(x), J_0^1 \phi_{21}^0(x), J_0^1 \phi_{20}^1(x), J_0^1 \phi_{21}^1(x), J_0^1 \phi_{20}^2(x), J_0^1 \phi_{21}^2(x), J_0^1 \phi_{20}^3(x), J_0^1 \phi_{21}^3(x), J_0^1 \phi_{30}^0(x), \dots\}$ is a set of orthonormal bases of $W_{2,0}^1[0, 1]$.

Proof Firstly, we show that $\{\psi_i(x)\}_{i=1}^\infty$ is orthonormal in $W_{2,0}^1$.

$$(\psi_i(x), \psi_j(x))_1 = (J_0^1 \varphi_i(x), J_0^1 \varphi_j(x))_1 = (\varphi_i(x), \varphi_j(x))_{L^2} = \delta_{ij},$$

where $\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$. So $\{\psi_i(x)\}_{i=1}^\infty$ is orthonormal in $W_{2,0}^1$.

Next, we prove that $\{\psi_i(x)\}_{i=1}^\infty$ is complete in $W_{2,0}^1$. Let $\xi(x) \in W_{2,0}^1$ and $(\xi(x), \psi_i(x))_1 = 0, i = 1, 2, \dots$. That is,

$$(\xi(x), \psi_i(x))_1 = (\xi(x), J_0^1 \varphi_i(x))_1 = (\xi'(x), \varphi_i(x))_{L^2} = \int_0^1 \xi'(x) \cdot \varphi_i(x) dx = 0. \tag{4}$$

By Lemma 3.1, $\{\varphi_i(x)\}_{i=1}^\infty$ is a set of multiscale orthonormal bases of $L^2[0, 1]$, and we obtain $\xi'(x) = 0$. Noting that $\xi(x) \in W_{2,0}^1$, we have $\xi(0) = 0$. So $\xi(x) =$

$\xi(0) + \int_0^x \xi'(t)dt = 0$. Thus, $\{\psi_i(x)\}_{i=1}^\infty$ is complete in $W_{2,0}^1$ and $\{\psi_i(x)\}_{i=1}^\infty$ is a set of orthonormal bases of $W_{2,0}^1$. □

Remark 3.1 Noting that

$$\begin{aligned}
 J_0^1 \phi_{ik}^m(x) &= \int_0^1 \phi_{ik}^m(x) dx = \begin{cases} 0, & x \in [0, \frac{k}{2^{i-1}}), \\ 2^{-\frac{i-1}{2}} \int_{\frac{k}{2^{i-1}}}^x \phi^m(2^{i-1}t - k) dt, & x \in [\frac{k}{2^{i-1}}, \frac{k+1}{2^{i-1}}], \\ 2^{-\frac{i-1}{2}} \int_{\frac{k}{2^{i-1}}}^{\frac{k+1}{2^{i-1}}} \phi^m(2^{i-1}t - k) dt, & x \in (\frac{k+1}{2^{i-1}}, 1], \end{cases} \\
 &= \begin{cases} 0, & x \in [0, \frac{k}{2^{i-1}}), \\ 2^{-\frac{i-1}{2}} \int_0^{2^{i-1}x-k} \phi^m(t) dt, & x \in [\frac{k}{2^{i-1}}, \frac{k+1}{2^{i-1}}], \\ 2^{-\frac{i-1}{2}} \int_0^1 \phi^m(t) dt, & x \in (\frac{k+1}{2^{i-1}}, 1], \end{cases} \\
 &= \begin{cases} 0, & x \in [0, \frac{k}{2^{i-1}}), \\ 2^{-\frac{i-1}{2}} (J_0^1 \phi^m)(2^{i-1}x - k), & x \in [\frac{k}{2^{i-1}}, \frac{k+1}{2^{i-1}}], \\ 0, & x \in (\frac{k+1}{2^{i-1}}, 1], \end{cases} \\
 &= 2^{-\frac{i-1}{2}} (J_0^1 \phi^m)(2^{i-1}x - k),
 \end{aligned}$$

and we can see that $J_0^1 \phi^m(x)$, $m = 0, 1, 2, 3$ is multiwavelets in $W_{2,0}^1$.

Definition 3.1 Let $\mu(x) \in L^2[0, 1]$. If $\mu(x)$ satisfies the following conditions

$$\int_0^1 \mu(x) \cdot x^j dx = 0, \quad j = 0, 1, 2, \dots, r; \quad \int_0^1 \mu(x) \cdot x^{r+1} dx \neq 0,$$

we call the Vanishing Moment of $\mu(x)$ is $r + 1$.

Property 3.1 The Vanishing Moment of $\phi^i(x)$ is $i + 4$, $i = 0, 1, 2, 3$.

Proof By specific calculation, we can reach that

$$\int_0^1 \phi^i(x) \cdot x^j dx = 0, \quad i = 0, 1, 2, 3; \quad j = 0, 1, \dots, i + 3,$$

and $\int_0^1 \phi^i(x) \cdot x^{i+4} dx \neq 0$, $i = 0, 1, 2, 3$. So the conclusion holds. □

Remark 3.2 According to Property 3.1, we obtain that the Vanishing Moment of $\{\phi^i(x)\}_{i=0}^3$ is at least 4.

Property 3.2 For any $v(x) \in W_{2,0}^1[0, 1]$ and $N \in \mathbb{N}_+$, we have

$$\| v(x) - \sum_{i=1}^N a_i \psi_i(x) \|_1^2 = \sum_{i=N+1}^\infty a_i^2,$$

where $a_i = (v(x), \psi_i(x))_1$.

Proof According to Theorem 3.1, $\{\psi_i(x)\}_{i=1}^\infty$ is a set of orthonormal bases of $W_{2,0}^1[0, 1]$. Hence,

$$v(x) = \sum_{i=1}^\infty a_i \psi_i(x),$$

where $a_i = (v(x), \psi_i(x))_1$. Thus,

$$\begin{aligned} \left\| v(x) - \sum_{i=1}^N a_i \psi_i(x) \right\|_1^2 &= \left\| \sum_{i=1}^\infty a_i \psi_i(x) - \sum_{i=1}^N a_i \psi_i(x) \right\|_1^2 \\ &= \left\| \sum_{i=N+1}^\infty a_i \psi_i(x) \right\|_1^2 = \sum_{i=N+1}^\infty a_i^2. \end{aligned}$$

□

Let $y(x)$ and $y_J(x)$ be an exact solution and an approximate solution of (3), respectively. Denote

$$y(x) = \sum_{i=0}^3 a_i J_0^1 \eta^i(x) + \sum_{i=1}^\infty \sum_{l=0}^3 \sum_{k=0}^{2^{i-1}-1} c_{ik}^l J_0^1 \phi_{ik}^l(x),$$

$$y_J(x) = \sum_{i=0}^3 a_i J_0^1 \eta^i(x) + \sum_{i=1}^J \sum_{l=0}^3 \sum_{k=0}^{2^{i-1}-1} c_{ik}^l J_0^1 \phi_{ik}^l(x),$$

where $a_i = (y(x), J_0^1 \eta^i(x))_1$, $c_{ik}^l = (y(x), J_0^1 \phi_{ik}^l(x))_1$.

Lemma 3.2 *Let $y(x) \in W_{2,0}^1$, $|y^{(j)}(x)| \leq M, \forall x \in [0, 1]$, for some $j \in \{2, 3, 4, 5\}$. Then $|c_{ik}^l| \leq 2^{-(i-1)(j-1/2)} AM, l = 0, 1, 2, 3$ where A is a constant.*

Proof Without loss of generality, we first consider $|c_{ik}^0|$.

$$|c_{ik}^0| = \left| (y(x), J_0^1 \phi_{ik}^0(x))_1 \right| = \left| \int_0^1 y'(x) \phi_{ik}^0(x) dx \right| = \left| \int_{\frac{k}{2^{i-1}}}^{\frac{k+1}{2^{i-1}}} y'(x) \phi_{ik}^0(x) dx \right|.$$

Expand $y'(x)$ at $x = \frac{k}{2^{i-1}}$:

$$\begin{aligned} y'(x) &= y' \left(\frac{k}{2^{i-1}} \right) + y'' \left(\frac{k}{2^{i-1}} \right) \left(x - \frac{k}{2^{i-1}} \right) + \dots \\ &\quad + \frac{y^{(j)}(\xi)}{(j-1)!} \left(x - \frac{k}{2^{i-1}} \right)^{j-1}, \quad \xi \in \left(\frac{k}{2^{i-1}}, \frac{k+1}{2^{i-1}} \right). \end{aligned}$$

Thus, we have

$$\begin{aligned} |c_{ik}^0| &\leq \left| y' \left(\frac{k}{2^{i-1}} \right) \int_{\frac{k}{2^{i-1}}}^{\frac{k+1}{2^{i-1}}} \phi_{ik}^0(x) dx \right| + \left| y'' \left(\frac{k}{2^{i-1}} \right) \int_{\frac{k}{2^{i-1}}}^{\frac{k+1}{2^{i-1}}} \left(x - \frac{k}{2^{i-1}} \right) \phi_{ik}^0(x) dx \right| \\ &\quad + \dots + \left| \int_{\frac{k}{2^{i-1}}}^{\frac{k+1}{2^{i-1}}} \frac{y^{(j)}(\xi)}{(j-1)!} \left(x - \frac{k}{2^{i-1}} \right)^{j-1} \phi_{ik}^0(x) dx \right|. \end{aligned}$$

The first integral of the above inequality

$$\int_{\frac{k}{2^{i-1}}}^{\frac{k+1}{2^{i-1}}} \phi_{ik}^0(x) dx = \sqrt{2}^{i-1} \int_{\frac{k}{2^{i-1}}}^{\frac{k+1}{2^{i-1}}} \phi^0(2^{i-1}x - k) dx \stackrel{t = 2^{i-1}x - k}{=} \sqrt{2}^{i-1} \int_0^1 \phi^0(t) \left(\frac{1}{2}\right)^{i-1} dt = \left(\frac{1}{\sqrt{2}}\right)^{i-1} \int_0^1 \phi^0(t) dt = 0,$$

by using Property 3.1.

Similarly, one can obtain that

$$\begin{aligned} \int_{\frac{k}{2^{i-1}}}^{\frac{k+1}{2^{i-1}}} \left(x - \frac{k}{2^{i-1}}\right) \phi_{ik}^0(x) dx &= 0, & \int_{\frac{k}{2^{i-1}}}^{\frac{k+1}{2^{i-1}}} \left(x - \frac{k}{2^{i-1}}\right)^2 \phi_{ik}^0(x) dx \\ &= 0, & \int_{\frac{k}{2^{i-1}}}^{\frac{k+1}{2^{i-1}}} \left(x - \frac{k}{2^{i-1}}\right)^3 \phi_{ik}^0(x) dx = 0. \end{aligned}$$

while

$$\begin{aligned} \left| \int_{\frac{k}{2^{i-1}}}^{\frac{k+1}{2^{i-1}}} \frac{y^{(j)}(\xi)}{(j-1)!} \left(x - \frac{k}{2^{i-1}}\right)^{j-1} \phi_{ik}^0(x) dx \right| &\leq \frac{M}{(j-1)!} \int_{\frac{k}{2^{i-1}}}^{\frac{k+1}{2^{i-1}}} \left(x - \frac{k}{2^{i-1}}\right)^{j-1} \left| \phi_{ik}^0(x) \right| dx \\ &= \frac{M}{(j-1)!} \sqrt{2}^{i-1} \int_{\frac{k}{2^{i-1}}}^{\frac{k+1}{2^{i-1}}} \left(x - \frac{k}{2^{i-1}}\right)^{j-1} \left| \phi^0(2^{i-1}x - k) \right| dx \\ &= \frac{M}{(j-1)!} \sqrt{2}^{i-1} \int_0^1 \left(\frac{t}{2^{i-1}}\right)^{j-1} \left| \phi^0(t) \right| \left(\frac{1}{2}\right)^{i-1} dt \quad (t = 2^{i-1}x - k) \\ &= \frac{M}{(j-1)!} \frac{1}{2^{(i-1)(j-1/2)}} \int_0^1 t^{j-1} \left| \phi^0(t) \right| dt \\ &\leq 2^{-(i-1)(j-1/2)} AM, \end{aligned}$$

where $A = \max_{l=0,1,2,3} \left\{ \frac{1}{(j-1)!} \int_0^1 t^{j-1} |\phi^l(t)| dt \right\}$. one has

$$|c_{ik}^0| \leq 2^{-(i-1)(j-1/2)} AM.$$

In the same way, one can derive that $|c_{ik}^l| \leq 2^{-(i-1)(j-1/2)} AM, l = 1, 2, 3$. So the conclusion holds. □

Theorem 3.2 Assume $y(x) \in W_{2,0}^1, |y^{(j)}(x)| \leq M, \forall x \in [0, 1],$ for some $j \in \{2, 3, 4, 5\}$. Then $\|y(x) - y_J(x)\|_1 \leq C \cdot 2^{-(j-1)J}$ where C is a constant.

Proof Using Property 3.2 and Lemma 3.2, we can derive

$$\begin{aligned} \|y(x) - y_J(x)\|_1 &= \left\| \sum_{i=J+1}^{\infty} \sum_{l=0}^3 \sum_{k=0}^{2^{i-1}-1} c_{ik}^l J_0^1 \phi_{ik}^l(x) \right\|_1 = \sqrt{\sum_{i=J+1}^{\infty} \sum_{l=0}^3 \sum_{k=0}^{2^{i-1}-1} (c_{ik}^l)^2} \\ &\leq AM \sqrt{\sum_{i=J+1}^{\infty} \sum_{l=0}^3 \sum_{k=0}^{2^{i-1}-1} (2^{-(i-1)(j-1/2)})^2} \\ &= AM \sqrt{\sum_{i=J+1}^{\infty} 2^{-2(i-1)(j-1)} \cdot 4} \leq C \cdot 2^{-(j-1)J}, \end{aligned}$$

where $C = AM \sqrt{\frac{4}{1-2^{-2(j-1)}}}$. □

4 Linear fractional delay differential equations

In this section, we construct the ε -approximate solution method for solving (3) with f being linear, i.e.,

$$\begin{cases} D_C^\alpha y(x) = h(x) + a(x)y(x) + b(x)y'(x) + c(x)y(x - \tau) + d(x)y'(x - \tau), & 0 \leq x \leq 1, 1 < \alpha \leq 2, \\ y(x) = y_0(x), & -\tau \leq x \leq 0, \end{cases} \tag{5}$$

and analyze the convergence order of the ε -approximate solutions. $a(x), b(x), c(x), d(x), h(x) \in W_2^1[0, 1], y_0(x) \in C^2[-\tau, 0]$ are known functions and $y(x)$ is an unknown function. $\tau > 0$ is a constant delay. Similar to Theorem 2.1, we can prove $D_C^\alpha y(x) \in C[0, 1]$. Here, we further assume $D_C^\alpha y(x) \in W_2^1$.

Let

$$\zeta(x) = \begin{cases} y_0(x), & -\tau \leq x \leq 0, \\ y_0(0) + xy_0'(0), & 0 \leq x \leq 1, \end{cases}$$

and

$$v(x) = y(x) - \zeta(x).$$

Then (5) is equivalently transformed into

$$\begin{cases} D_C^\alpha v = g(x) + a(x)v(x) + b(x)v'(x) + c(x)v(x - \tau) + d(x)v'(x - \tau), & 0 \leq x \leq 1, \\ v(x) = 0, & -\tau \leq x \leq 0, \end{cases} \tag{6}$$

where

$$g(x) = h(x) + a(x)\zeta(x) + b(x)\zeta'(x) + c(x)\zeta(x - \tau) + d(x)\zeta'(x - \tau).$$

Introducing the symbols ${}^{-\tau}D_C^\alpha$ and $J_{-\tau}^\alpha$, we have ${}^{-\tau}D_C^\alpha = D_C^\alpha, J_{-\tau}^\alpha = J_0^\alpha$ due to the fact that ${}^{-\tau}D_C^\alpha v(x) = \frac{1}{\Gamma(2-\alpha)} \int_{-\tau}^x (x-t)^{1-\alpha} v''(t) dt = \frac{1}{\Gamma(2-\alpha)} \int_0^x (x-t)^{1-\alpha} v''(t) dt = D_C^\alpha v(x)$ and $J_{-\tau}^\alpha v(x) = \frac{1}{\Gamma(\alpha)} \int_{-\tau}^x (x-t)^{\alpha-1} v(t) dt = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} v(t) dt = J_0^\alpha v(x)$. So we use the denotation D_C^α and J_0^α below.

Let $\omega(x) = {}^{-\tau}D_C^\alpha v(x) = D_C^\alpha v(x)$. Thus, (6) is transformed into

$$\begin{cases} \omega(x) = g(x) + a(x)J_0^\alpha \omega(x) + b(x)J_0^{\alpha-1} \omega(x) + c(x)J_0^\alpha \omega(x - \tau) + d(x)J_0^{\alpha-1} \omega(x - \tau), & 0 \leq x \leq 1, \\ \omega(x) = 0, & -\tau \leq x \leq 0. \end{cases}$$

And $v(x) = J_{-\tau}^\alpha {}^{-\tau}D_C^\alpha v(x) = J_{-\tau}^\alpha \omega(x) = J_0^\alpha \omega(x)$.

Defining an operator $L : W_{2,0}^1 \rightarrow W_{2,0}^1$,

$$L(\omega(x)) \triangleq \omega(x) - a(x)J_0^\alpha \omega(x) - b(x)J_0^{\alpha-1} \omega(x) - c(x)J_0^\alpha \omega(x - \tau) - d(x)J_0^{\alpha-1} \omega(x - \tau), \tag{7}$$

then (6) is equivalent to

$$L(\omega(x)) = g(x). \tag{8}$$

Therefore, the solution $y(x)$ of (5) can be obtained by

$$y(x) = v(x) + \zeta(x) = J_0^\alpha \omega(x) + \zeta(x). \tag{9}$$

We need the following Lemmas for our aims.

Lemma 4.1 $(J_0^\alpha \omega(x))'_x \in L^1[0, 1]$ for any $\omega(x) \in W_{2,0}^1[0, 1]$ with $\alpha > 0$.

Proof It is easy to see that

$$J_0^\alpha \omega(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \omega(t) dt = \frac{1}{\Gamma(\alpha)} \int_0^x s^{\alpha-1} \omega(x-s) ds.$$

So we can derive

$$(J_0^\alpha \omega)'_x = \frac{1}{\Gamma(\alpha)} \int_0^x s^{\alpha-1} \omega'(x-s) ds = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \omega'(t) dt.$$

Thus,

$$\begin{aligned} \int_0^1 |(J_0^\alpha \omega)'_t| dt &\leq \frac{1}{\Gamma(\alpha)} \int_0^1 \left| \int_0^t (t-s)^{\alpha-1} \omega'(s) ds \right| dt \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^1 \int_0^t (t-s)^{\alpha-1} |\omega'(s)| ds dt \\ &= \frac{1}{\Gamma(\alpha)} \int_0^1 \int_s^1 (t-s)^{\alpha-1} |\omega'(s)| dt ds \\ &= \frac{1}{\Gamma(\alpha+1)} \int_0^1 |\omega'(s)| (1-s)^\alpha ds \\ &\leq \frac{1}{\Gamma(\alpha+1)} \int_0^1 |\omega'(s)| ds < \infty. \end{aligned}$$

This implies $(J_0^\alpha \omega(x))'_x \in L^1[0, 1]$. □

Lemma 4.2 Let $\alpha > 0$. Then

$$J_0^\alpha \omega \in W_{2,0}^1 \text{ and } \|J_0^\alpha \omega\|_1 \leq \sqrt{\tilde{M}_\alpha} \|\omega\|_1,$$

hold for any $\omega(x) \in W^1_{2,0}$ where $\tilde{M}_\alpha = \frac{1}{\alpha^2\Gamma^2(\alpha)}$.

Proof We can find that

$$\begin{aligned} \int_0^x (J_0^\alpha \omega)'_t dt &= \frac{1}{\Gamma(\alpha)} \int_0^x \int_0^t (t-s)^{\alpha-1} \omega'(s) ds dt \\ &= \frac{1}{\Gamma(\alpha)} \int_0^x \int_s^x (t-s)^{\alpha-1} \omega'(s) dt ds \\ &= \frac{1}{\Gamma(\alpha)} \int_0^x \omega'(s) \left(\int_s^x (t-s)^{\alpha-1} dt \right) ds \\ &= \frac{1}{\Gamma(\alpha+1)} \int_0^x \omega'(s) (x-s)^\alpha ds, \end{aligned}$$

hold for any $\omega(x) \in W^1_{2,0}$. On the other hand, we obtain

$$J_0^\alpha \omega(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \omega(t) (x-t)^{\alpha-1} dt = \frac{1}{\Gamma(\alpha+1)} \int_0^x \omega'(t) (x-t)^\alpha dt,$$

by using the integration by parts. Hence, we have $J_0^\alpha \omega(x) = \int_0^x (J_0^\alpha \omega)'_t dt$ and $J_0^\alpha \omega(x) \in AC[0, 1]$.

Moreover, we have

$$\begin{aligned} \int_0^1 ((J_0^\alpha \omega(x))'_x)^2 dx &= \frac{1}{\Gamma^2(\alpha)} \int_0^1 \left(\int_0^x (x-t)^{\alpha-1} \omega'(t) dt \right)^2 dx \\ &\leq \frac{1}{\Gamma^2(\alpha)} \int_0^1 \left(\int_0^x (x-t)^{\alpha-1} dt \cdot \int_0^x (x-t)^{\alpha-1} \omega'^2(t) dt \right) dx \\ &= \frac{1}{\Gamma^2(\alpha)} \int_0^1 \frac{x^\alpha}{\alpha} \left(\int_0^x (x-t)^{\alpha-1} \omega'^2(t) dt \right) dx \\ &\leq \frac{1}{\alpha\Gamma^2(\alpha)} \int_0^1 \int_0^x (x-t)^{\alpha-1} \omega'^2(t) dt dx \\ &= \frac{1}{\alpha\Gamma^2(\alpha)} \int_0^1 \int_t^1 (x-t)^{\alpha-1} \omega'^2(t) dx dt \\ &= \frac{1}{\alpha^2\Gamma^2(\alpha)} \int_0^1 \omega'^2(t) (1-t)^\alpha dt \\ &\leq \frac{1}{\alpha^2\Gamma^2(\alpha)} \int_0^1 \omega'^2(t) dt. \end{aligned}$$

Let $\tilde{M}_\alpha = \frac{1}{\alpha^2\Gamma^2(\alpha)}$. Then

$$\int_0^1 ((J_0^\alpha \omega(x))'_x)^2 dx \leq \tilde{M}_\alpha \int_0^1 \omega'^2(t) dt = \tilde{M}_\alpha \|\omega\|_1^2 < \infty,$$

holds. So $(J_0^\alpha \omega(x))'_x \in L^2[0, 1]$. Noting that $J_0^\alpha \omega(0) = 0$, we have $J_0^\alpha \omega(x) \in W^1_{2,0}$ and

$$\|J_0^\alpha \omega\|_1 \leq \sqrt{\tilde{M}_\alpha} \|\omega\|_1. \quad \square$$

Lemma 4.3 *Let $a(x) \in W_2^1[0, 1]$, $\omega(x) \in W_{2,0}^1[0, 1]$, $\alpha > 0$. Then*

$$\|a(x)J_0^\alpha \omega(x)\|_1^2 \leq 4\|a(x)\|_1^2 \left[\frac{1}{\Gamma^2(\alpha + 1)} + \tilde{M}_\alpha \right] \|\omega\|_1^2,$$

holds.

Proof Using Lemmas 2.1 and 4.2, we have

$$\begin{aligned} \|a(x)J_0^\alpha \omega(x)\|_1^2 &= (a(x)J_0^\alpha \omega(x), a(x)J_0^\alpha \omega(x))_1 \\ &= \int_0^1 [(a(x)J_0^\alpha \omega(x))']^2 dx \\ &= \int_0^1 [a'(x)J_0^\alpha \omega(x) + a(x)(J_0^\alpha \omega(x))']^2 dx \\ &\leq 2 \int_0^1 [a'(x)J_0^\alpha \omega(x)]^2 dx + 2 \int_0^1 [a(x)(J_0^\alpha \omega(x))']^2 dx \\ &\leq 2\|J_0^\alpha \omega(x)\|_C^2 \int_0^1 a'^2(x)dx + 2\|a(x)\|_C^2 \int_0^1 [(J_0^\alpha \omega(x))']^2 dx \\ &\leq \frac{2\|a(x)\|_1^2}{\Gamma^2(\alpha)} \left\| \int_0^x (x-t)^{\alpha-1} \omega(t)dt \right\|_C^2 + 4\|a(x)\|_1^2 \tilde{M}_\alpha \|\omega\|_1^2 \\ &\leq \frac{2\|a(x)\|_1^2}{\Gamma^2(\alpha)} \|\omega(x)\|_C^2 \left| \int_0^x (x-t)^{\alpha-1} dt \right|^2 + 4\|a(x)\|_1^2 \tilde{M}_\alpha \|\omega\|_1^2 \\ &= \frac{2\|a(x)\|_1^2}{\Gamma^2(\alpha + 1)} \|\omega(x)\|_C^2 x^{2\alpha} + 4\|a(x)\|_1^2 \tilde{M}_\alpha \|\omega\|_1^2 \\ &\leq \frac{4\|a(x)\|_1^2}{\Gamma^2(\alpha + 1)} \|\omega(x)\|_1^2 + 4\|a(x)\|_1^2 \tilde{M}_\alpha \|\omega\|_1^2 \\ &= 4\|a(x)\|_1^2 \left[\frac{1}{\Gamma^2(\alpha + 1)} + \tilde{M}_\alpha \right] \|\omega\|_1^2. \end{aligned}$$

So the conclusion holds. □

Theorem 4.1 *The operator L defined by (7) is a bounded linear operator from $W_{2,0}^1$ to $W_{2,0}^1$.*

Proof Obviously, L is a linear operator. Noting that $a(x), b(x), c(x), d(x) \in W_2^1[0, 1]$, thus we know $a(x), b(x), c(x), d(x) \in AC[0, 1]$. Denote $M_1 = \max\{\|a(x)\|_1, \|b(x)\|_1, \|c(x)\|_1, \|d(x)\|_1\}$.

Let $\omega(x) \in W_{2,0}^1[0, 1]$, according to Lemma 4.2 and 4.3, and we have $J_0^\alpha \omega \in W_{2,0}^1$, and there exists an \tilde{M}_α such that

$$\|J_0^\alpha \omega\|_1 \leq \sqrt{\tilde{M}_\alpha} \|\omega\|_1,$$

$$\|a(x)J_0^\alpha \omega(x)\|_1^2 \leq 4\|a(x)\|_1^2 \left[\frac{1}{\Gamma^2(\alpha + 1)} + \tilde{M}_\alpha \right] \|\omega\|_1^2 \leq 4M_1^2 \left[\frac{1}{\Gamma^2(\alpha + 1)} + \tilde{M}_\alpha \right] \|\omega\|_1^2,$$

$$\|c(x)J_0^\alpha \omega(x - \tau)\|_1^2 \leq 4\|c(x)\|_1^2 \left[\frac{1}{\Gamma^2(\alpha + 1)} + \tilde{M}_\alpha \right] \|\omega\|_1^2 \leq 4M_1^2 \left[\frac{1}{\Gamma^2(\alpha + 1)} + \tilde{M}_\alpha \right] \|\omega\|_1^2.$$

Similarly, there exists an $\tilde{M}_{\alpha-1}$ such that

$$\|b(x)J_0^{\alpha-1} \omega(x)\|_1^2 \leq 4\|b(x)\|_1^2 \left[\frac{1}{\Gamma^2(\alpha)} + \tilde{M}_{\alpha-1} \right] \|\omega\|_1^2 \leq 4M_1^2 \left[\frac{1}{\Gamma^2(\alpha)} + \tilde{M}_{\alpha-1} \right] \|\omega\|_1^2,$$

$$\|d(x)J_0^{\alpha-1} \omega(x - \tau)\|_1^2 \leq 4\|d(x)\|_1^2 \left[\frac{1}{\Gamma^2(\alpha)} + \tilde{M}_{\alpha-1} \right] \|\omega\|_1^2 \leq 4M_1^2 \left[\frac{1}{\Gamma^2(\alpha)} + \tilde{M}_{\alpha-1} \right] \|\omega\|_1^2.$$

Therefore,

$$\begin{aligned} \|L(\omega(x))\|_1 &= \|\omega(x) - a(x)J_0^\alpha \omega(x) - b(x)J_0^{\alpha-1} \omega(x) - c(x)J_0^\alpha \omega(x - \tau) - d(x)J_0^{\alpha-1} \omega(x - \tau)\|_1 \\ &\leq \|\omega(x)\|_1 + 4M_1 \|\omega\|_1 \sqrt{\frac{1}{\Gamma^2(\alpha + 1)} + \tilde{M}_\alpha} + 4M_1 \|\omega\|_1 \sqrt{\frac{1}{\Gamma^2(\alpha)} + \tilde{M}_{\alpha-1}} \\ &= \left[1 + 4M_1 \sqrt{\frac{1}{\Gamma^2(\alpha + 1)} + \tilde{M}_\alpha} + 4M_1 \sqrt{\frac{1}{\Gamma^2(\alpha)} + \tilde{M}_{\alpha-1}} \right] \|\omega(x)\|_1, \end{aligned}$$

which implies L is a bounded linear operator from $W_{2,0}^1$ to $W_{2,0}^1$. □

Theorem 4.2 *Suppose there exists a unique solution of the equation defined by (8), then L is a bijection from $W_{2,0}^1$ to $L(W_{2,0}^1)$ and $L^{-1} : L(W_{2,0}^1) \rightarrow W_{2,0}^1$ does not only exist but is also bounded.*

Remark 4.1 It is easy to see that L is a bijection from $W_{2,0}^1$ to $L(W_{2,0}^1)$. For the latter part of the conclusion, we can define an operator $J : W_{2,0}^1 \rightarrow W_{2,0}^1$, $J(\omega(x)) = J_0^\alpha \omega(x)$, $\omega(x) \in W_{2,0}^1$. One can prove that the operator J is compact. Since the sum of finite compact operators is still compact and $\{(I - J)\omega(x) | \omega(x) \in W_{2,0}^1\}$ is closed, one can obtain that $L^{-1} : L(W_{2,0}^1) \rightarrow W_{2,0}^1$ is bounded by the Inverse Operator Theorem in the Banach spaces.

Definition 4.1 $y(x)$ is called an ε -approximate solution of (8) if $\|L(y) - g\|_1 < \varepsilon$ for any given $\varepsilon > 0$.

Theorem 4.3 *For any $\varepsilon > 0$, there exists a positive integer N such that for every fixed $n \geq N$, $\omega_n^*(x) = \sum_{i=0}^n c_i^* \psi_i(x)$ is an ε -approximate solution of (8), where $\{c_i^*\}_{i=0}^n$ satisfies*

$$\|g(x) - \sum_{i=0}^n c_i^* f_i(x)\|_1 = \min_{c_i \in \mathbb{R}} \|g(x) - \sum_{i=0}^n c_i f_i(x)\|_1 \tag{10}$$

and

$$f_i(x) = L(\psi_i(x)), \quad i = 0, 1, 2, 3, \dots .$$

Proof Suppose $\omega(x)$ is the exact solution of (8). By Theorem 3.1, there exists a positive integer N such that for any $n > N$, there exists $\omega_n(x) = \sum_{i=0}^n c_i \psi_i(x)$ such that

$$\|\omega(x) - \omega_n(x)\|_1 = \|\omega(x) - \sum_{i=0}^n c_i \psi_i(x)\|_1 < \frac{\varepsilon}{\|L\|}.$$

So we can derive

$$\|L(\omega(x)) - L(\omega_n(x))\|_1 \leq \|L\| \|\omega(x) - \omega_n(x)\|_1 < \varepsilon.$$

Noting that $L(\omega(x)) = g(x)$, we have $\|g(x) - L(\omega_n(x))\|_1 = \|g(x) - \sum_{i=0}^n c_i f_i(x)\|_1 < \varepsilon$. Thus, we obtain

$$\begin{aligned} \|g(x) - L(\omega_n^*(x))\|_1 &= \|g(x) - L\left(\sum_{i=0}^n c_i^* \psi_i(x)\right)\|_1 = \|g(x) - \sum_{i=0}^n c_i^* f_i(x)\|_1 \\ &= \min_{c_i \in \mathbb{R}} \|g(x) - \sum_{i=0}^n c_i f_i(x)\|_1 < \varepsilon. \end{aligned}$$

That is, $\omega_n^*(x)$ is an ε -approximate solution of (8). □

Next, we find the ε -approximate solution of (8). Denote

$$S(c_1, c_2, \dots, c_n) = \|g(x) - \sum_{i=0}^n c_i f_i(x)\|_1^2. \tag{11}$$

According to the norm definition, we have

$$\begin{aligned} S(c_1, c_2, \dots, c_n) &= (g(x) - \sum_{i=0}^n c_i f_i(x), g(x) - \sum_{j=0}^n c_j f_j(x))_1 \\ &= (g(x), g(x))_1 - 2(g(x), \sum_{i=0}^n c_i f_i(x))_1 + \left(\sum_{i=0}^n c_i f_i(x), \sum_{j=0}^n c_j f_j(x) \right)_1 \\ &= (g(x), g(x))_1 - 2 \sum_{i=0}^n c_i (g(x), f_i(x))_1 + \sum_{i=0}^n \sum_{j=0}^n c_i c_j (f_i(x), f_j(x))_1. \end{aligned} \tag{12}$$

For obtaining the minimum of the $S(c_1, c_2, \dots, c_n)$, we solve the normal equations of (12)

$$\frac{\partial S}{\partial c_k} = -2(g(x), f_k(x))_1 + 2 \sum_{i=0}^n c_i (f_i(x), f_k(x))_1 = 0, \quad k = 0, 1, \dots, n. \tag{13}$$

Theorem 4.4 *The system of normal (13) has a unique solution.*

Proof The system of normal (13) can be rewritten as:

$$\begin{pmatrix} (f_0(x), f_0(x))_1 & (f_0(x), f_1(x))_1 & \cdots & (f_0(x), f_n(x))_1 \\ (f_1(x), f_0(x))_1 & (f_1(x), f_1(x))_1 & \cdots & (f_1(x), f_n(x))_1 \\ \vdots & \vdots & \ddots & \vdots \\ (f_n(x), f_0(x))_1 & (f_n(x), f_1(x))_1 & \cdots & (f_n(x), f_n(x))_1 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} (g(x), f_0(x))_1 \\ (g(x), f_1(x))_1 \\ \vdots \\ (g(x), f_n(x))_1 \end{pmatrix}$$

Denote

$$G = \begin{pmatrix} (f_0(x), f_0(x))_1 & (f_0(x), f_1(x))_1 & \cdots & (f_0(x), f_n(x))_1 \\ (f_1(x), f_0(x))_1 & (f_1(x), f_1(x))_1 & \cdots & (f_1(x), f_n(x))_1 \\ \vdots & \vdots & \ddots & \vdots \\ (f_n(x), f_0(x))_1 & (f_n(x), f_1(x))_1 & \cdots & (f_n(x), f_n(x))_1 \end{pmatrix},$$

then G is a Gram matrix. So G is a nonsingular matrix if and only if $f_0(x), f_1(x), \dots, f_n(x)$ are linearly independent.

We prove that $f_0(x), f_1(x), \dots, f_n(x)$ are linearly independent. Let

$$l_0 f_0(x) + l_1 f_1(x) + \cdots + l_n f_n(x) \equiv 0.$$

We derive $\sum_{i=0}^n l_i L(\psi_i(x)) \equiv 0$, or $L(\sum_{i=0}^n l_i \psi_i(x)) \equiv 0$. Since L is injective, thus $\sum_{i=0}^n l_i \psi_i(x) \equiv 0$. Noting that $\{\psi_i(x)\}_{i=0}^n$ are linearly independent, we obtain $l_i = 0, i = 0, 1, \dots, n$. Therefore, $f_0(x), f_1(x), \dots, f_n(x)$ are linearly independent, G is nonsingular, and the system of normal (13) has a unique solution. \square

Remark 4.2 The unique solution of normal (13) is denoted as $(c_0^*, c_1^*, \dots, c_n^*)$. Similar to [22], we can prove $S(c_1, c_2, \dots, c_n) \geq S(c_0^*, c_1^*, \dots, c_n^*)$. Thus, (10) has a unique solution determined by (13). The desired approximate solution $y^*(x)$ of (5) can be obtained by (9).

Theorem 4.5 Assume that $\omega_n^*(x) = \sum_{i=0}^3 a_i J_0^1 \eta^i(x) + \sum_{i=1}^n \sum_{l=0}^3 \sum_{k=0}^{2^i-1} c_{ik}^l J_0^1 \phi_{ik}^l(x)$ obtained by Theorem 6 is an ε -approximate solution of Eq.(8), $\omega(x)$ is the exact solution of Eq.(8) and $|\omega^{(j)}(x)| \leq M, \forall x \in [0, 1]$, for some $j \in \{2, 3, 4, 5\}$. Then $\|\omega(x) - \omega_n^*(x)\|_1 \leq C \cdot 2^{-(j-1)n}$, where C is a constant.

Proof According to Theorem 3.2, one has $\|\omega(x) - \omega_n(x)\|_1 \leq \bar{C} \cdot 2^{-(j-1)n}$ where \bar{C} is a constant. Thus, one can derive that

$$\begin{aligned} \|\omega(x) - \omega_n^*(x)\|_1 &\leq \|L^{-1}\| \cdot \|L(\omega(x)) - L(\omega_n^*(x))\|_1 = \|L^{-1}\| \cdot \|g(x) - L(\omega_n^*(x))\|_1 \\ &\leq \|L^{-1}\| \|g(x) - L(\omega_n(x))\|_1 = \|L^{-1}\| \|L(\omega(x)) - L(\omega_n(x))\|_1 \\ &\leq \|L^{-1}\| \|L\| \|\omega(x) - \omega_n(x)\|_1 \leq \bar{C} \|L^{-1}\| \|L\| \cdot 2^{-(j-1)n} = C \cdot 2^{-(j-1)n}, \end{aligned}$$

where $C = \bar{C} \|L^{-1}\| \|L\|$. \square

5 Stability analysis

In this section, we consider the stability of our proposed method.

Assume λ is a eigenvalue of the matrix G which is defined in the proof of Theorem 4.4, that is, there exists an $X = [x_0, x_1, \dots, x_n]^T \in \mathbb{R}^{n+1}$, $X \neq 0$, such that $GX = \lambda X$. Hence,

$$\begin{cases} \lambda x_0 = \sum_{j=0}^n (f_0, f_j)_1 \cdot x_j, \\ \vdots \\ \lambda x_i = \sum_{j=0}^n (f_i, f_j)_1 \cdot x_j, \\ \vdots \\ \lambda x_n = \sum_{j=0}^n (f_n, f_j)_1 \cdot x_j. \end{cases}$$

So we derive that

$$\begin{aligned} \lambda x_i &= \sum_{j=0}^n (f_i, f_j)_1 \cdot x_j = \sum_{j=0}^n (L(\psi_i(x)), L(\psi_j(x)))_1 \cdot x_j \\ &= (L(\psi_i(x)), \sum_{j=0}^n L(\psi_j(x)))_1 \cdot x_j = (L(\psi_i(x)), \sum_{j=0}^n L(\psi_j(x)) \cdot x_j)_1, \\ \lambda x_i^2 &= (L(\psi_i(x)) \cdot x_i, \sum_{j=0}^n L(\psi_j(x)) \cdot x_j)_1, \end{aligned}$$

and

$$\lambda \sum_{i=0}^n x_i^2 = \left(\sum_{i=0}^n x_i L(\psi_i(x)), \sum_{j=0}^n x_j L(\psi_j(x)) \right)_1.$$

Thus, we have

$$\lambda \|X\|_{\mathbb{R}^{n+1}}^2 = \left\| L \left(\sum_{i=0}^n x_i \psi_i(x) \right) \right\|_1^2,$$

or

$$\begin{aligned} \lambda &= \frac{\|L(\sum_{i=0}^n x_i \psi_i(x))\|_1^2}{\|X\|_{\mathbb{R}^{n+1}}^2} \\ &= \left\| L \left(\sum_{i=0}^n \frac{x_i}{\|X\|_{\mathbb{R}^{n+1}}} \psi_i(x) \right) \right\|_1^2 \\ &= \left\| L \left(\sum_{i=0}^n y_i \psi_i(x) \right) \right\|_1^2 \quad \left(y_i \triangleq \frac{x_i}{\|X\|_{\mathbb{R}^{n+1}}} \right) \\ &= \|L(u)\|_1^2 \quad \left(u \triangleq \sum_{i=0}^n y_i \psi_i(x) \right). \end{aligned}$$

Noting that $\{\psi_i(x)\}_{i=0}^\infty$ is a set of orthonormal bases of $W_{2,0}^1$, we have

$$\|u\|_1^2 = \left\| \sum_{i=0}^n y_i \psi_i(x) \right\|_1^2 = \sum_{i=0}^n \|y_i \psi_i(x)\|_1^2 = \sum_{i=0}^n |y_i|^2 \cdot \|\psi_i(x)\|_1^2 = \sum_{i=0}^n |y_i|^2 = 1.$$

Hence, we can obtain that

$$\lambda = \|L(u)\|_1^2 \leq \|L\|^2 \cdot \|u\|_1^2 = \|L\|^2.$$

Thus, the largest eigenvalue λ_{max} of G satisfies $\lambda_{max} \leq \|L\|^2$.

On the other hand, we prove that the least eigenvalue of G satisfies $\lambda_{min} \geq \frac{1}{\|L^{-1}\|^2}$.

Otherwise, there exists $\{u_n\}$, $\|u_n\|_1 = 1$ such that $\|L(u_n)\|_1 < \frac{1}{\|L^{-1}\|}$.

Denote $\omega_n = L(u_n)$. Hence, $\|\omega_n\|_1 < \frac{1}{\|L^{-1}\|}$ and $u_n = L^{-1}(\omega_n)$. So we can derive

$$1 = \|u_n\|_1 = \|L^{-1}(\omega_n)\|_1 \leq \|L^{-1}\| \cdot \|\omega_n\|_1 < \|L^{-1}\| \cdot \frac{1}{\|L^{-1}\|} = 1.$$

It is a contradiction. So we obtain $\lambda_{min} \geq \frac{1}{\|L^{-1}\|^2}$.

Therefore, we have

$$1 \leq Cond(G)_2 = \frac{\lambda_{max}}{\lambda_{min}} \leq \frac{\|L\|^2}{\frac{1}{\|L^{-1}\|^2}} = (\|L\| \cdot \|L^{-1}\|)^2.$$

That is, $Cond(G)_2 \leq (\|L\| \cdot \|L^{-1}\|)^2$, which implies that the spectral condition number [23] of the matrix G is bounded, and the algorithm is stable.

6 Nonlinear fractional delay differential equations

In this section, we solve (3) when f is nonlinear. In order to obtain high-accuracy approximate solutions, we employ F-derivative and Newton’s iterative formula.

Similar to the linear case, we denote

$$\zeta(x) = \begin{cases} y_0(x), & -\tau \leq x \leq 0, \\ y_0(0) + xy'_0(0), & 0 \leq x \leq 1, \end{cases}$$

and

$$z(x) = y(x) - \zeta(x).$$

Then, (3) can be written as

$$\begin{cases} D_C^\alpha z(x) = g(x) + f[x, (z + \zeta)(x), (z + \zeta)'(x), (z + \zeta)(x - \tau), (z + \zeta)'(x - \tau)], & 0 \leq x \leq 1, \quad 1 < \alpha \leq 2, \\ z(x) = 0, & -\tau \leq x \leq 0. \end{cases}$$

Hence, there is no harm in supposing that the equation to be solved is

$$\begin{cases} D_C^\alpha u(x) = g(x) + f[x, u(x), u'(x), u(x - \tau), u'(x - \tau)], & 0 \leq x \leq 1, \quad 1 < \alpha \leq 2, \\ u(x) = 0, & -\tau \leq x \leq 0. \end{cases} \tag{14}$$

Let u be the exact solution of the above equation and assume that $D_C^\alpha u(x) = \omega(x)$. At the beginning of Section 4, we have proved that $u(x) = J_0^\alpha \omega(x)$ and $\omega \in W_{2,0}^1$. So $u(x) \in W_2^\alpha$.

Define an operator $F : W_2^\alpha[0, 1] \rightarrow W_2^1[0, 1]$,

$$F(u) = D_C^\alpha u - f[x, u(x), u'(x), u(x - \tau), u'(x - \tau)] - g(x), \tag{15}$$

(14) is equivalent to the following equation

$$\begin{cases} F(u) = 0, & 0 \leq x \leq 1, & 1 < \alpha \leq 2, \\ u(x) = 0, & -\tau \leq x \leq 0. \end{cases} \tag{16}$$

Lemma 6.1 Take any $h \in W_2^\alpha$, then $\|h\|_1 \leq \sqrt{\tilde{M}_\alpha} \|h\|_\alpha$ holds, where $\tilde{M}_\alpha = \frac{1}{\alpha^2 \Gamma^2(\alpha)}$.

Proof From the definition of W_2^α , $h = J_0^\alpha h_1$ with $h_1 \in W_{2,0}^1$. By Lemma 4.2, $\|h\|_1 = \|J_0^\alpha h_1\|_1 \leq \sqrt{\tilde{M}_\alpha} \|h_1\|_1 = \sqrt{\tilde{M}_\alpha} \|h\|_\alpha$. □

Theorem 6.1 Suppose that F is defined by (15), then

$$F'(u)h = D_C^\alpha h - \left[\frac{\partial}{\partial u} h + \frac{\partial}{\partial u'} h' + \frac{\partial}{\partial u(x - \tau)} h(x - \tau) + \frac{\partial}{\partial u'(x - \tau)} h'(x - \tau) \right] f[x, u(x), u'(x), u(x - \tau), u'(x - \tau)], \tag{17}$$

where $F'(u)$ refers to the Fréchet derivative [10].

Proof Let $F(u) = D_C^\alpha u - g(x) - f[x, u(x), u'(x), u(x - \tau), u'(x - \tau)] \triangleq F_1(u) - F_2(u)$, here $F_1(u) = D_C^\alpha u - g(x)$, $F_2(u) = f[x, u(x), u'(x), u(x - \tau), u'(x - \tau)]$.

According to the definition of F-derivative and property of Caputo derivatives, we have

$$\lim_{\|h\|_\alpha \rightarrow 0} \frac{1}{\|h\|_\alpha} \|D_C^\alpha(u + h) - g(x) - [D_C^\alpha(u) - g(x)] - D_C^\alpha h\|_1 = 0.$$

So we can conclude that $F'_1(u)h = D_C^\alpha h$, that is $F'_1(u) = D_C^\alpha$. On the other hand,

$$f[x, y, z, p, q] = f[x, y_0, z_0, p_0, q_0] + \left[\frac{\partial}{\partial y}(y - y_0) + \frac{\partial}{\partial z}(z - z_0) + \frac{\partial}{\partial p}(p - p_0) + \frac{\partial}{\partial q}(q - q_0) \right] f[x, y_0, z_0, p_0, q_0] + O(h^2).$$

Substituting $u(x) + h(x)$, $u'(x) + h'(x)$, $u(x - \tau) + h(x - \tau)$, $u'(x - \tau) + h'(x - \tau)$ for y, z, p, q and $u(x)$, $u'(x)$, $u(x - \tau)$, $u'(x - \tau)$ for y_0, z_0, p_0, q_0 in the above equation, we get

$$f[x, u(x) + h(x), u'(x) + h'(x), u(x - \tau) + h(x - \tau), u'(x - \tau) + h'(x - \tau)] = f[x, u(x), u'(x), u(x - \tau), u'(x - \tau)]$$

$$\begin{aligned}
 &+ \left[\frac{\partial}{\partial u} h(x) + \frac{\partial}{\partial u'} h'(x) + \frac{\partial}{\partial u(x-\tau)} h(x-\tau) \right. \\
 &+ \left. \frac{\partial}{\partial u'(x-\tau)} h'(x-\tau) \right] f[x, u(x), u'(x), u(x-\tau), u'(x-\tau)] \\
 &+ O(h^2).
 \end{aligned}$$

Therefore, $\exists M > 0$,

$$\begin{aligned}
 &\frac{1}{\|h\|_\alpha} \|f[x, (u+h)(x), (u+h)'(x), (u+h)(x-\tau), (u+h)'(x-\tau)] \\
 &- f[x, u(x), u'(x), u(x-\tau), u'(x-\tau)] \\
 &- \left[\frac{\partial}{\partial u} h(x) + \frac{\partial}{\partial u'} h'(x) + \frac{\partial}{\partial u(x-\tau)} h(x-\tau) \right. \\
 &+ \left. \frac{\partial}{\partial u'(x-\tau)} h'(x-\tau) \right] f[x, u(x), u'(x), u(x-\tau), u'(x-\tau)] \|_1
 \end{aligned}$$

$\leq M \|h\|_1^2 / \|h\|_\alpha \leq M \tilde{M}_\alpha \|h\|_\alpha^2 / \|h\|_\alpha \leq M \tilde{M}_\alpha \|h\|_\alpha \rightarrow 0, (\|h\|_\alpha \rightarrow 0)$, where Lemma 6.1 is used. Thus,

$$\begin{aligned}
 F'_2(u)h &= \left[\frac{\partial}{\partial u} h(x) + \frac{\partial}{\partial u'} h'(x) + \frac{\partial}{\partial u(x-\tau)} h(x-\tau) \right. \\
 &+ \left. \frac{\partial}{\partial u'(x-\tau)} h'(x-\tau) \right] f[x, u(x), u'(x), u(x-\tau), u'(x-\tau)].
 \end{aligned}$$

Since $F'(u)h = F'_1(u)h + F'_2(u)h$, then the conclusion holds. □

Remark 6.1 An inequality used here is $\|h^2\|_1 \leq 2\sqrt{2}\|h\|_1^2$. In fact, according to the definition of the norm $\|\cdot\|_1$ of W_2^1 and Lemma 2.1, we can obtain

$$\begin{aligned}
 \|h^2\|_1^2 &= \int_0^1 (2h \cdot h')^2 dx = 4 \int_0^1 h^2 \cdot (h')^2 dx \\
 &\leq 4 \|h\|_C^2 \int_0^1 (h')^2 dx = 4 \|h\|_C^2 \|h\|_1^2 \\
 &\leq 4(\sqrt{2}\|h\|_1)^2 \|h\|_1^2 = 8 \|h\|_1^4.
 \end{aligned}$$

Squaring on both sides of the inequality, and the inequality $\|h^2\|_1 \leq 2\sqrt{2}\|h\|_1^2$ holds.

The Newton’s iterative formula for solving $F(u) = 0$ defined by (16) is

$$u_{k+1} = u_k - [F'(u_k)]^{-1} F(u_k), \quad k = 0, 1, 2, \dots, \tag{18}$$

with initial selection $u_0 = 0$, here $F'(u)$ is defined by (17). Equation (18) can be transformed into

$$F'(u_k)(u_{k+1} - u_k) + F(u_k) = 0, \quad k = 0, 1, 2, \dots,$$

which are linear fractional delay differential (5), so we can solve them by the ε -approximate solution method constructed in Section 4.

Remark 6.2 One can refer to [9] about the method of selecting a initial value and the convergence of Newton’s iterative formula.

7 Algorithm implementation of Newton’s iterative formula

In this section, we will concretely show algorithm implementation of Newton’s iterative formula.

Assume

$$\begin{aligned}
 u_{k+1} &= \sum_{i=0}^n c_{k+1,i} J_0^\alpha \psi_i(x), \\
 u_k &= \sum_{i=0}^n c_{k,i} J_0^\alpha \psi_i(x),
 \end{aligned}
 \tag{19}$$

$$u_{k+1} - u_k = \sum_{i=0}^n d_{k,i} J_0^\alpha \psi_i(x),
 \tag{20}$$

where $\{c_{k,i}\}, \{d_{k,i}\}$ are unknown. It is not difficult to see that

$$c_{k+1,i} = c_{k,i} + d_{k,i} \quad (i = 0, 1, 2, \dots, n),
 \tag{21}$$

$$u_{k+1} = u_k + \sum_{i=0}^n d_{k,i} J_0^\alpha \psi_i(x),
 \tag{22}$$

$$F(u_k) = \sum_{i=0}^n c_{k,i} \psi_i(x) - f[x, u_k(x), u'_k(x), u_k(x - \tau), u'_k(x - \tau)] - g(x).
 \tag{23}$$

Using (17) and (20), we have $F'(u_k)(u_{k+1} - u_k)$

$$\begin{aligned}
 &= D_C^\alpha (u_{k+1} - u_k) - \left[\frac{\partial}{\partial u_k} (u_{k+1} - u_k) + \frac{\partial}{\partial u'_k} (u_{k+1} - u_k)' \right. \\
 &\quad + \frac{\partial}{\partial u_k(x - \tau)} (u_{k+1} - u_k)(x - \tau) \\
 &\quad \left. + \frac{\partial}{\partial u'_k(x - \tau)} (u_{k+1} - u_k)'(x - \tau) \right] f[x, u_k, u'_k, u_k(x - \tau), u'_k(x - \tau)] \\
 &= \sum_{i=0}^n d_{k,i} \psi_i(x) - \left[\frac{\partial}{\partial u_k} \sum_{i=0}^n d_{k,i} J_0^\alpha \psi_i(x) + \frac{\partial}{\partial u'_k} \sum_{i=0}^n d_{k,i} J_0^{\alpha-1} \psi_i(x) \right. \\
 &\quad + \frac{\partial}{\partial u_k(x - \tau)} \sum_{i=0}^n d_{k,i} J_0^\alpha \psi_i(x - \tau) \\
 &\quad \left. + \frac{\partial}{\partial u'_k(x - \tau)} \sum_{i=0}^n d_{k,i} J_0^{\alpha-1} \psi_i(x - \tau) \right] f[x, u_k, u'_k, u_k(x - \tau), u'_k(x - \tau)].
 \end{aligned}
 \tag{24}$$

Denote

$$R(x) \triangleq \|F'(u_k)(u_{k+1} - u_k) + F(u_k)\|_1^2, \quad k = 0, 1, 2, \dots
 \tag{25}$$

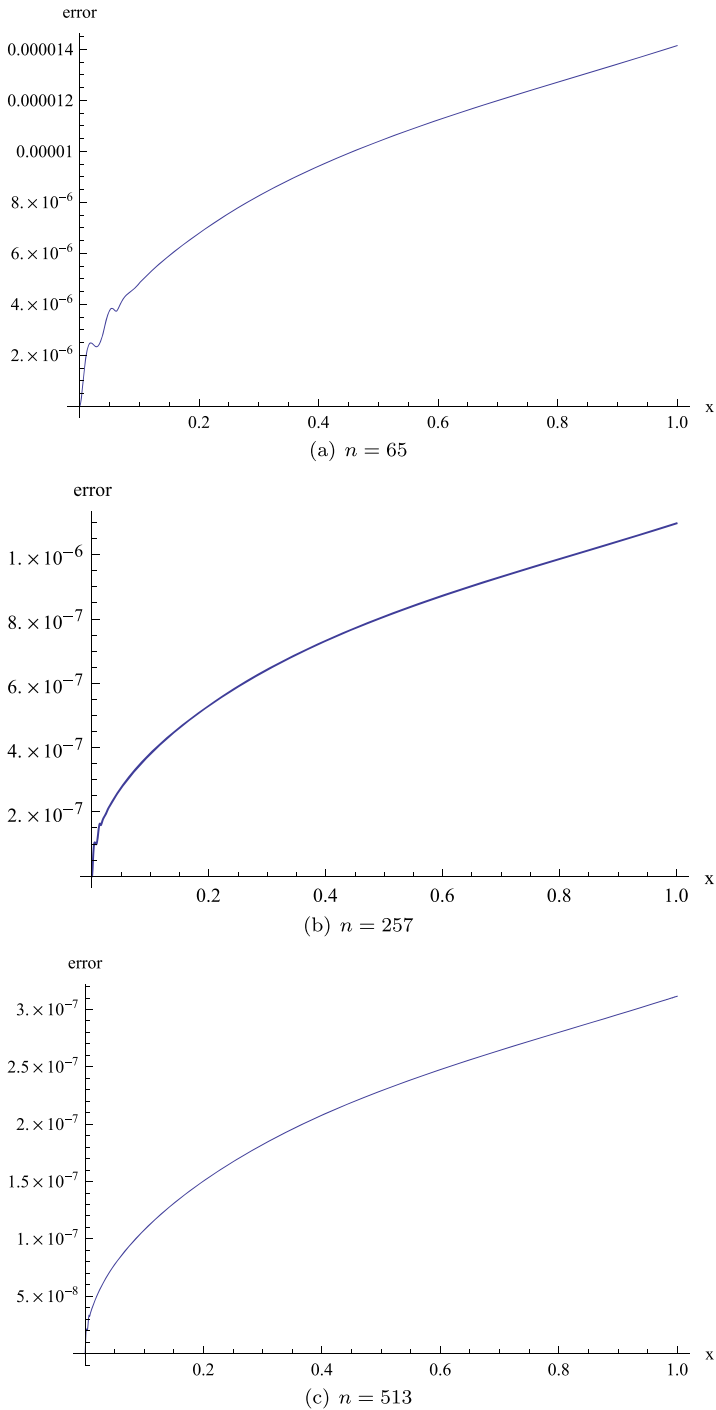


Fig. 1 The relative error curve with $n = 65, 257, 513$ when the exact solution is $x^{2.3}$ for Example 8.1

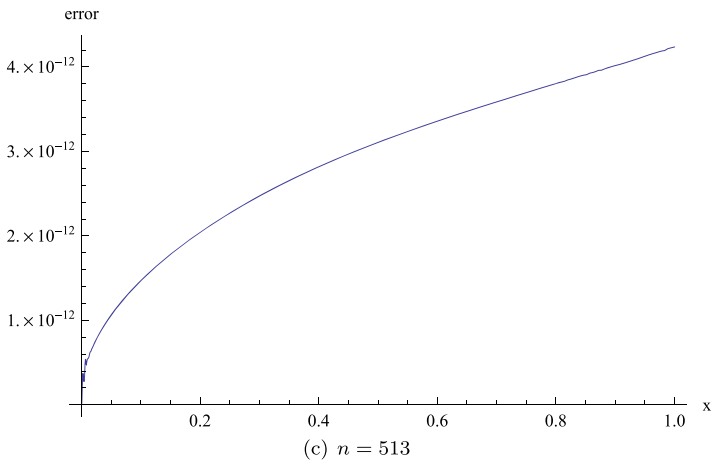
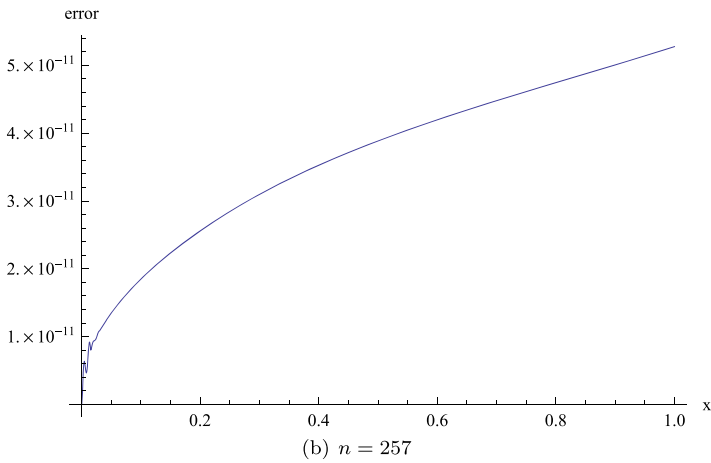
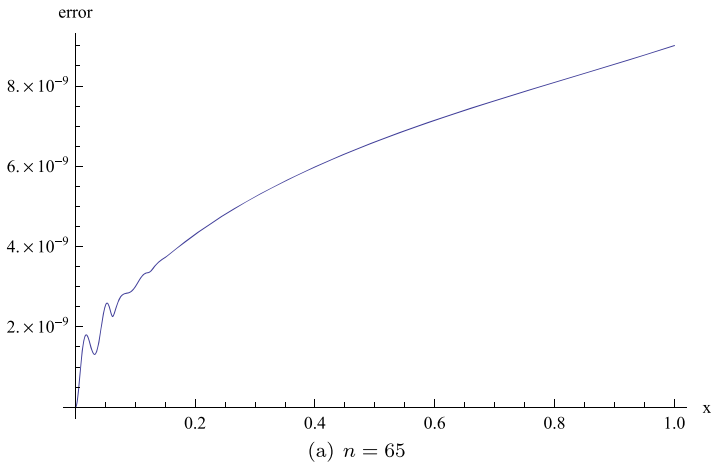


Fig. 2 The relative error curve with $n = 65, 257, 513$ when the exact solution is $x^{4.1}$ for Example 8.1

Table 1 The comparison of infinity-norm of the relative errors at discrete points and estimates of the convergence order on the interval [0,1] between the exact solutions $x^{2.3}$ and $x^{4.1}$ for Example 8.1

n	$x^{2.3}$	ECO	$x^{4.1}$	ECO
17	0.0001744		1.36584×10^{-6}	
33	0.0000513	1.7648	1.18324×10^{-7}	3.52898
65	0.0000142	1.85935	9.00737×10^{-9}	3.71549
129	3.9095×10^{-6}	1.85585	6.77928×10^{-10}	3.73190
257	1.0975×10^{-6}	1.83273	5.27700×10^{-11}	3.68334
513	3.1161×10^{-7}	1.81643	4.21943×10^{-12}	3.66446

Substituting the above equations (19), (23), and (24) into the (25), we solve the following equations to obtain $\{d_{k,i}\}$:

$$\frac{\partial R}{\partial d_{k,i}} = 0, \quad i = 0, 1, 2, \dots, n. \tag{26}$$

To sum up, the algorithm is as follows:

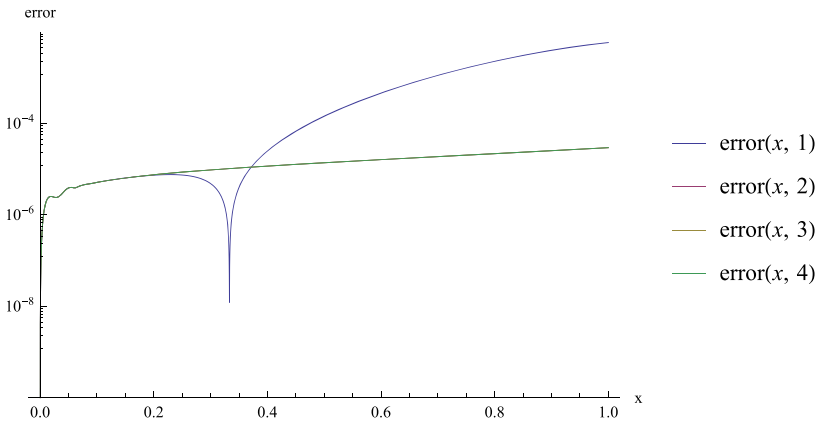
- Step 1 Homogenize the equation as (14).
- Step 2 Input $\{\psi_i(x)\}$, and compute $\{J_0^\beta \psi_i(x)\}$, for any $\beta > 0$.
- Step 3 Input α . Input the number of approximate items n . Set $c_{0,i} = 0$.
- Step 4 Begin the iterative procedure. From $k = 0$, do
 - (1) Compute u_k by (19).
 - (2) Compute the $F(u_k), F'(u_k)(u_{k+1} - u_k)$ according to (23) and (24).
 - (3) Compute $R(x)$ by (25), solve the equations (26), and obtain the $\{d_{k,i}\}$.
 - (4) Compute $c_{k+1,i} = c_{k,i} + d_{k,i}$ ($i = 0, 1, 2, \dots, n$).
 - (5) If $\|u_{k+1} - u_k\|_C \geq \varepsilon$, go to (1); or else go to Step 5.
- Step 5 Output the last approximate solution $u_{k+1} = \sum_{i=0}^n c_{k+1,i} J_0^\alpha \psi_i(x)$.

Using the software Mathematica, we can get the approximate solution $u_{k+1}(x)$. We don't need too much iterations to reach desired approximate solutions because of high efficiency of Newton's iterative method.

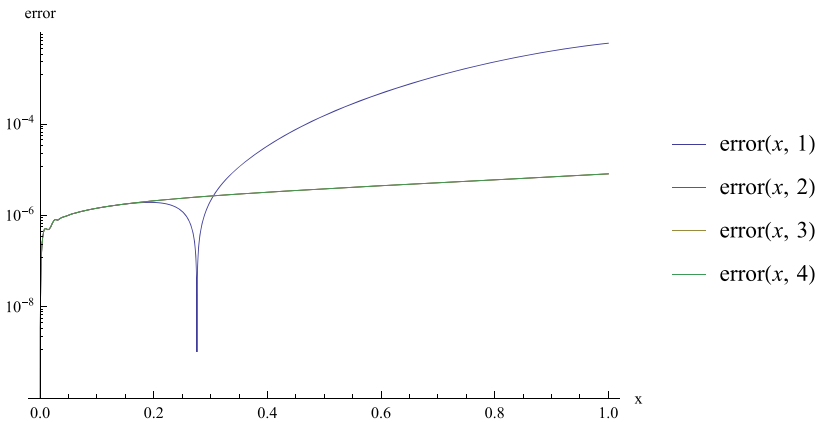
Remark 7.1 The algorithm can be extended to the case of $m - 1 < \alpha \leq m$ ($m \in \mathbf{N}_+$) for the fractional delay differential (3).

8 Numerical examples

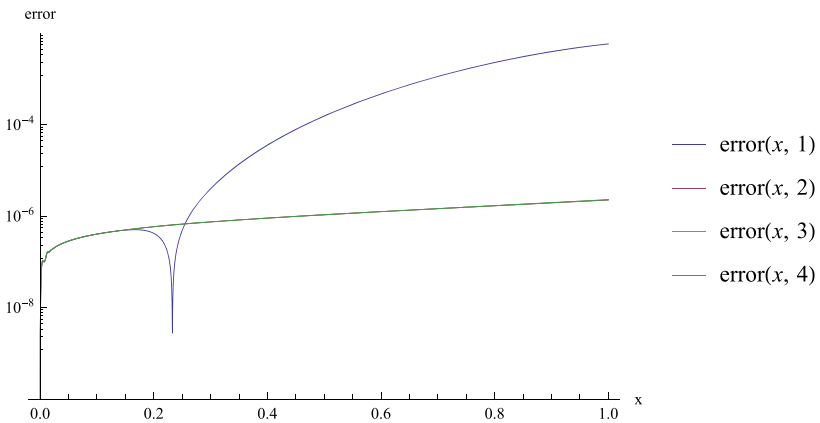
In this section, the algorithm presented above is applied to solve linear and nonlinear fractional delay differential equations. Three examples are considered to illustrate the



(a) $n = 65$



(b) $n = 129$



(c) $n = 257$

Fig. 3 The relative error curve with $n = 65, 129, 257$ after 1, 2, 3, and 4 iterations when the exact solution is $x^{2.3}$ for Example 8.2

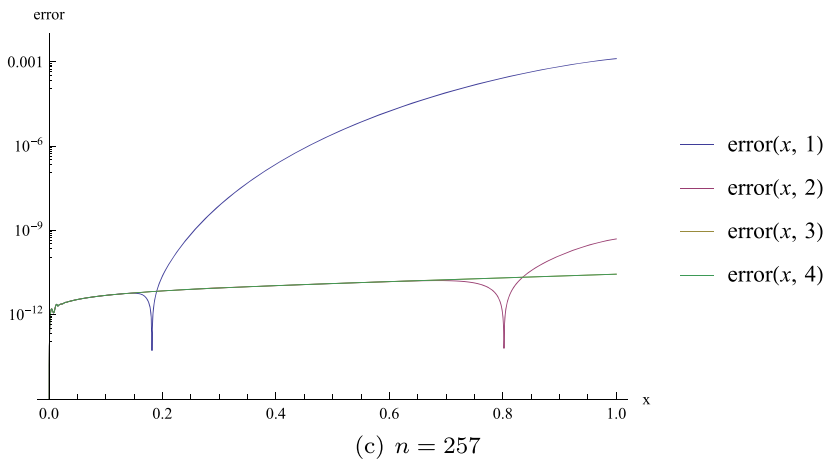
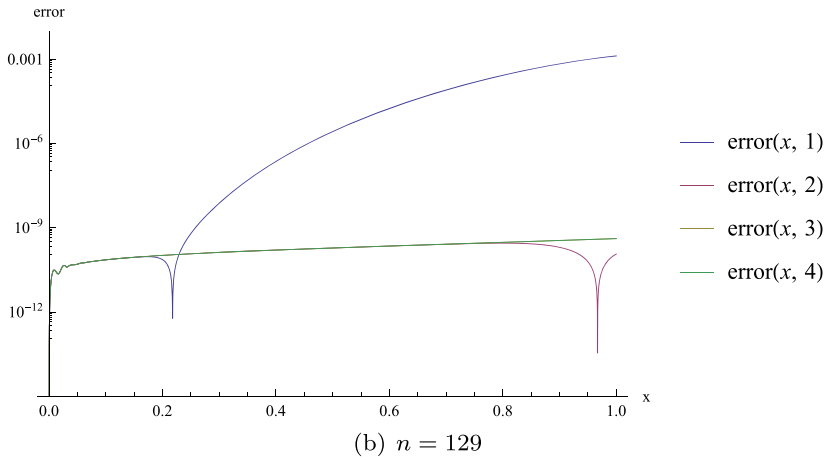
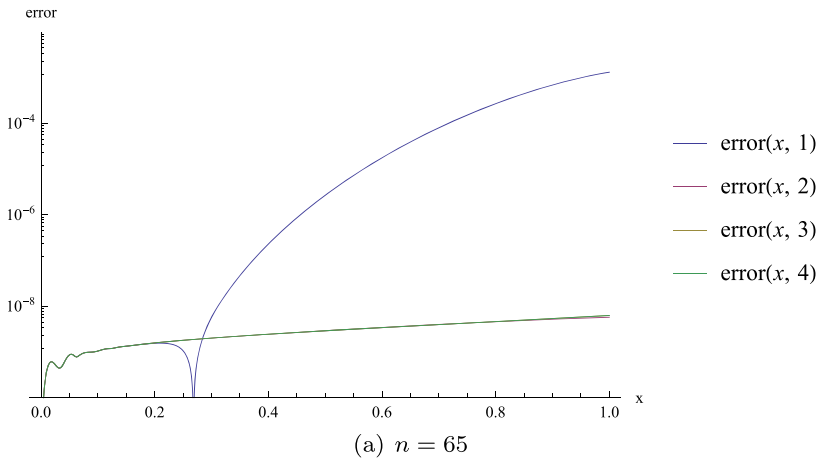


Fig. 4 The relative error curve with $n = 65, 129, 257$ after 1, 2, 3, and 4 iterations when the exact solution is $x^{4.3}$ for Example 8.2

Table 2 The comparison of infinity-norm of the relative errors at discrete points and estimates of the convergence order on the interval [0,1] between the exact solutions $x^{2.3}$ and $x^{4.3}$ for Example 8.2

n	$x^{2.3}$	ECO	$x^{4.3}$	ECO
9	0.001229870		0.000017226	
17	0.000355909	1.78893	1.23037×10^{-6}	3.80745
33	0.000105201	1.75837	9.30212×10^{-8}	3.72539
65	0.000029364	1.84102	6.26881×10^{-9}	3.89129
129	8.15525×10^{-6}	1.84825	4.12792×10^{-10}	3.92471
257	2.29599×10^{-6}	1.82861	2.79184×10^{-11}	3.88613

efficiency of the suggested algorithm. The relative error here is defined as $R_n = \frac{u-u_n}{\|u\|_C}$ with the exact solution u and an approximate solution u_n obtained by Theorem 4.3. Basing on the numerical results, we adopt the formula $\log_2[R_n/R_{2n-1}]$ to estimate the convergence order.

Example 8.1 Consider the linear fractional differential equation with delay

$$\begin{cases} D_C^{1.5}y(x) = xy(x) + \frac{x-1}{5}y'(x) - y(x - 0.2) + \frac{x}{3}y'(x - 0.2) + g(x), & x \in [0, 1], \\ y(x) = 0, & -0.2 \leq x \leq 0, \end{cases}$$

where $g(x)$ is chosen such that the exact solution is $x^{2.3}$ or $x^{4.1}$. When the exact solution is $x^{2.3}$, we solve the equation with $n = 65, 257, 513$, respectively, and obtain the corresponding approximate solutions. The relative errors between the exact solution and the approximate solutions are displayed in Fig. 1. When the exact solution of this example is chosen to be $x^{4.1}$ which is of higher smoothness, the relative errors are displayed in Fig. 2. One can see that the relative errors are decreased with the increase of n . The results show that the error is decreased fast when the smoothness of solution is increased. In Table 1, we provide some numerical results illustrating the fact that the convergence order can be improved when smoothness of the solution is improved. Even if the solution of the equation is less smooth, the computing error is still acceptable for the engineering. Therefore, the algorithm is stable, reliable, and adaptive.

Example 8.2 Let us consider the nonlinear fractional differential equation with delay

$$\begin{cases} D_C^{1.5}y(x) = (x + 1)y(x) + \frac{x-1}{2}y(x)y(x - 0.1) + g(x), & x \in [0, 1], \\ y(x) = 0, & -0.1 \leq x \leq 0, \end{cases}$$

where $g(x)$ is chosen such that the exact solution of this example is $x^{2.3}$ or $x^{4.3}$. The relative errors between the exact solution and the approximate solutions with different times of iteration are displayed in Figs. 3 and 4 with $n = 65, 129, 257$ in

Fig. 5 The relative error curve with $n = 65, 257, 513, 1025$ for Example 8.3

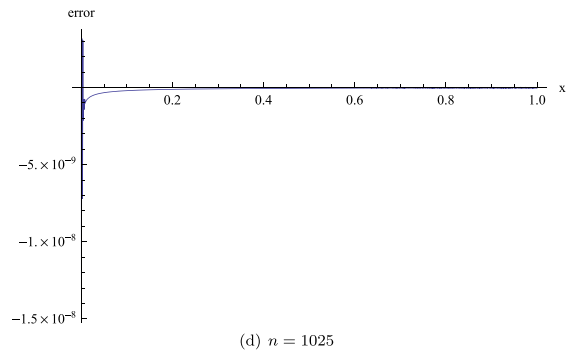
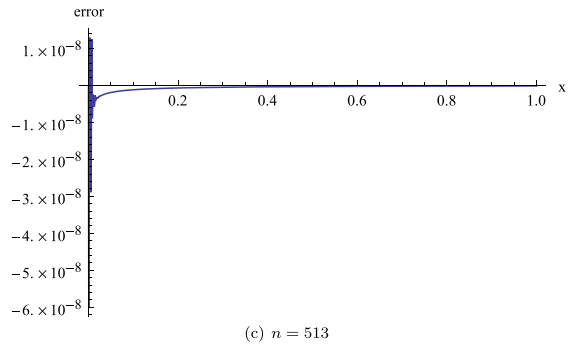
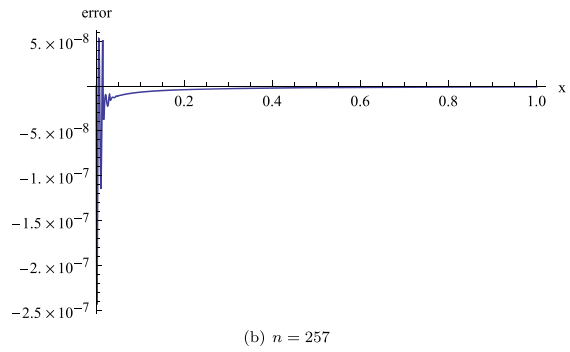
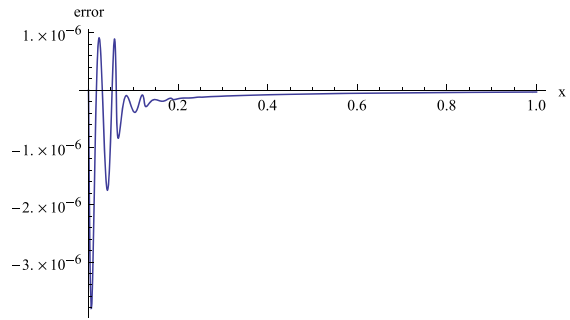


Table 3 The comparison of infinity-norm of the absolute errors at discrete points with the backward difference method on the interval [0,1] for Example 8.3

Step h	Difference method	n	Present method
1/10	0.0491843	9	0.0002360
1/20	0.0276172	17	0.0000604
1/40	0.0146507	33	0.0000150
1/80	0.00756493	65	3.80892×10^{-6}
1/160	0.00385284	129	4.4984×10^{-7}
1/320	0.00194804	257	2.4383×10^{-7}
1/640	0.000980855	513	6.1313×10^{-8}

both cases, respectively. The semi-log plots in error are displayed in Figs. 3 and 4. In Table 2, we compare the infinity-norm of relative errors at the discrete points and estimates of convergence order for both solution cases. The results show that our method is still valid, stable, and adaptive when solving nonlinear problems.

Example 8.3 Consider the linear fractional differential equation with delay [15]

$$\begin{cases} D_C^{0.5} y(x) = y(x - 1) - y(x) + 2x - 1 + \frac{\Gamma(3)}{\Gamma(5/2)} x^{3/2}, & x \in [0, 1], \\ y(x) = x^2, & -1 \leq x \leq 0. \end{cases}$$

The exact solution of this example is x^2 . The relative errors between the exact solution and the approximate solutions are displayed in Fig. 5 with $n = 65, 257, 513, 1025$, respectively. One can see that the relative errors are decreased with the increase of n . We compare the infinity-norm of absolute errors at the discrete points with that from Ref. [15] in Table 3. The numerical results show that our method has higher accuracy in this example and the computing errors may be acceptable for engineering.

9 Conclusion

In this paper, we construct a new stable collocation method for solving a kind of non-linear fractional delay differential equations. More suitable multiscale orthonormal bases of $W_{2,0}^1$ are constructed, error estimations of approximate solutions are given, and the highest convergence order can reach four in the sense of the norm of W_2^1 . Newton’s iterative formula is used to linearize the nonlinear equation, and for the obtained linear equations we develop an ε -approximate solution method based on multiscale orthonormal bases to solve them. A concrete algorithm implementation is given. Numerical examples show that compared with [15], the presented method is more accurate in dealing with this kind of equations.

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