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ORIGINAL PAPER

Modified Newton-DSS method for solving a class of systems of nonlinear equations with complex symmetric Jacobian matrices

Fang Xie[1] · Rong-Fei Lin[2] · Qing-Biao Wu[1]

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Abstract

Double-step scale splitting (DSS) iteration method is proved to be an unconditionally convergent iteration method, which is also efficient and robust for solving a class of large sparse complex symmetric systems of linear equations. In this paper, by making use of the DSS iteration technique as the inner solver to approximately solve the Newton equations, we establish a new modified Newton-DSS method for solving systems of nonlinear equations whose Jacobian matrices are large, sparse, and complex symmetric. Subsequently, we investigate the local and semilocal convergence properties of our method under some proper assumptions. Finally, numerical results on some problems illustrate the superiority of our method over some previous methods.

Keywords Complex nonlinear system · Double-step scale splitting · Modified Newton method · Convergence analysis

✉ Fang Xie
xiefang@zju.edu.cn

✉ Rong-Fei Lin
linrfei@tzc.edu.cn

✉ Qing-Biao Wu
qbwu@zju.edu.cn

[1] School of Mathematical Sciences, Zhejiang University, Hangzhou, 310027 Zhejiang, People's Republic of China

[2] Department of Mathematics, Taizhou University, Linhai, 317000 Zhejiang, People's Republic of China

1 Introduction

We assume that $F : \mathbb{D} \subset \mathbb{C}^n \to \mathbb{C}^n$ is a continuously differentiable mapping defined on an open convex subset of n-dimensional complex linear space $\mathbb{C}^n$ and consider the iteration solution of the large sparse system of nonlinear equations:

$$F(x) = 0. \tag{1.1}$$

The Jacobian matrix of $F(x)$ is large, sparse, and complex symmetric, i.e.,

$$F'(x) = W(x) + iT(x) \tag{1.2}$$

satisfies that matrices $W(x)$ and $T(x)$ are both symmetric and real positive definite, which implies that the complex matrix $F'(x)$ is nonsingular. $i = \sqrt{-1}$ is the imaginary unit. Actually, such nonlinear equations can be derived in many practical cases, such as nonlinear waves, quantum mechanics, chemical oscillations, and turbulence (see [1–4]).

To our knowledge, inexact Newton method [5] is the most classic and popular iteration method for solving the system of nonlinear equations, which can be formulated as:

$$F'(x_k)s_k = -F(x_k) + r_k, \quad \text{with} \quad x_{k+1} := x_k + s_k,$$

where $x_0 \in \mathbb{D}$ is a given initial vector and r_k is a residual yielded by the inner iteration. Obviously, it is the variant of Newton's method where the so-called Newton equation

$$F'(x_k)s_k = -F(x_k)$$

is solved approximately at each iteration. In particular, when the scale of problems is large, linear iterative methods are commonly applied to compute the approximation solution. For example, the Newton-Krylov subspace methods [6], which make use of Krylov subspace methods as inner iterations to solve the Newton equations, have been widely studied and successfully used.

Recently, based on the Hermitian and skew-Hermitian splitting (HSS) iteration method [7] and the special structure of the complex matrix, Bai et al. have proposed a modified Hermitian and skew-Hermitian splitting (MHSS) iteration method [8] and its preconditioned version which is called PMHSS iteration method [9] for the complex linear systems. Because of the elegant properties and high efficiency, the HSS-like iteration methods for complex linear systems have extended in many literatures (see [10–16]). Thereinto, a double-step scale splitting (DSS) iteration scheme [16] was established for solving the complex symmetric linear system

$$Ax = b, \quad A = W + iT \in \mathbb{C}^{n\times n} \text{ and } x, b \in \mathbb{C}^n$$

with matrices W and T both symmetric and real positive definite. It not only is convergent unconditionally but also behaves better than PMHSS iteration method.

By utilizing the HSS iteration method as the inner iteration, Bai and Guo [17] have presented the Newton-HSS method for solving the system of nonlinear equations with non-Hermitian positive definite Jacobian matrices. Meanwhile, the local convergence theorem of Newton-HSS method was proved. Since then, Guo et al. [18]

analyzed the semilocal convergence properties of the above Newton-HSS method. Focusing on the efficiency of the outer iteration method, Wu and Chen [19] have established the modified Newton-HSS method by utilizing the modified Newton method as the outer iteration instead of the Newton method, and given proof of the local and semilocal convergence properties of the method. Subsequently, Chen et al. [20] gave a new convergence theorem of the modified Newton-HSS method under the Hölder continuous condition, which is weaker than the Lipschitz continuous condition. Up to now, there have been a series of literatures showing the feasibleness of the combination of the modified Newton method and other HSS-like iteration methods (see [21–26]).

Nevertheless, when the Jacobian matrices (1.2) are complex, the convergence speed of Newton-HSS method reduces significantly since the resolution of the linear system needs a complex algorithm. In order to overcome this deficiency, Yang et al. [27] and Zhong et al. [28] presented the Newton-MHSS method and the modified Newton-PMHSS method, respectively. Inspired by the above ideas, it is natural to try to combine the DSS iteration method and the modified Newton method as the inner solver and outer solver, respectively. As a result, we construct a modified Newton-DSS method for the systems of nonlinear equations with complex symmetric Jacobian matrices. Under some reasonable assumptions, the local and semilocal convergence theorems of the modified Newton-DSS method are discussed. Finally, we examine the feasibility and efficiency of our method by several numerical examples.

The organization of this paper is as follows. In Section 2, we introduce the modified Newton-DSS method. The local and semilocal convergence properties of the MNDSS method are shown under some suitable assumptions in Sections 3 and 4, respectively. Some numerical results are given in Section 5 to illustrate the advantages of our method compared to the modified Newton-MHSS method even modified Newton-PMHSS method. Finally, in Section 6, we give some brief conclusions.

2 The modified Newton-DSS method

Firstly, let us review some of the standard facts on the double-step scale splitting (DSS) iteration method [15]. Consider the iteration solution of the following linear system

$$Ax = b, \quad A \in \mathbb{C}^{n\times n}, \quad x, b \in \mathbb{C}^n, \tag{2.1}$$

where A is a complex symmetric matrix of the form

$$A = W + iT,$$

and $W, T \in \mathbb{R}^{n\times n}$ are both positive definite and symmetric. Based on the special structure of the coefficient matrix A, Bai et al. in [8] designed a modification of the Hermitian and skew-Hermitian splitting (HSS) iteration method [7] which is called MHSS iteration method. Subsequently, a preconditioned MHSS (PMHSS) iteration method was derived in [9]. In order to improve the convergence rate of the PMHSS method, lots of researchers have developed some efficient iteration methods.

Recently, by picking up the idea of symmetry of the PMHSS method and using the technique of scaling to reconstruct complex linear system, Zheng et al. [16] designed a double-step scale splitting (DSS) iteration method. In this method, by multiplying parameters $(\alpha - i)$ and $(1 - i\alpha)$ for both side of complex linear system (2.1), respectively, we obtain two fixed point equations, i.e.,

$$(\alpha W + T)x = i(W - \alpha T)x + (\alpha - i)b$$

and

$$(\alpha T + W)x = i(\alpha W - T)x + (1 - i\alpha)b$$

Based on the above matrix splitting, there are two iteration formations that we can construct:

$$(\alpha W + T)x_{k+\frac{1}{2}} = i(W - \alpha T)x_k + (\alpha - i)b$$

and

$$(\alpha T + W)x_{k+1} = i(\alpha W - T)x_{k+\frac{1}{2}} + (1 - i\alpha)b$$

Similar to the construction of the PMHSS iteration method, DSS iteration method is proposed by alteranting between above iterations for solving the complex symmetric linear system (2.1). It is described as follows:

Algorithm 1 The DSS iteration method.

Let $x_0 \in \mathbb{C}^n$ be an arbitrary initial guess. For $k = 0, 1, 2, \ldots$ until the sequence of iterates $\{x_k\}_{k=0}^{\infty} \subset \mathbb{C}^n$ converges, compute the next iterate x_{k+1} according to the following procedure:

$$\begin{cases} (\alpha W + T)x_{k+\frac{1}{2}} = i(W - \alpha T)x_k + (\alpha - i)b, \\ (\alpha T + W)x_{k+1} = i(\alpha W - T)x_{k+\frac{1}{2}} + (1 - i\alpha)b, \end{cases} \tag{2.2}$$

where α is a given positive constant.

Since W, T are symmetric and positive definite, and $\alpha \in \mathbb{R}$ is positive, it implies that matrices $\alpha W + T$ and $\alpha T + W$ are both symmetric and positive definite. Therefore, the two subsystems involved in each step of the DSS iteration method (2.2) can be effectively solved exactly by a sparse Cholesky factorization, or inexactly by a preconditioned conjugate gradient (PCG) scheme. Theoretical analysis proved the unconditional convergence of the DSS iteration method and presented two reciprocal optimal iteration parameters. Moreover, the DSS iteration method is superior to the PMHSS iteration method in terms of the iteration counts and CPU time in some numerical examples (for details, see [16]).

After straightforward operations, the above DSS iteration method can be equivalently reformulated as the standard form

$$\begin{aligned} x_{k+1} &= M(\alpha)x_k + N(\alpha)b \\ &= M(\alpha)^{k+1}x_0 + \sum_{j=0}^{k} M(\alpha)^j N(\alpha)b, \quad k = 0, 1, 2, \ldots, \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} M(\alpha) &= (\alpha T + W)^{-1}(\alpha W - T)(\alpha W + T)^{-1}(\alpha T - W), \\ N(\alpha) &= 2\alpha(\alpha T + W)^{-1}(W - iT)(\alpha W + T)^{-1}. \end{aligned}$$

$$\begin{aligned} M(\alpha) &= (\alpha T + W)^{-1}(\alpha W - T)(\alpha W + T)^{-1}(\alpha T - W), \\ N(\alpha) &= 2\alpha(\alpha T + W)^{-1}(W - iT)(\alpha W + T)^{-1}. \end{aligned}$$

Hence, the unconditional convergence property of the DSS iteration method given in [16] can be described as follows.

Theorem 2.1 *Let $A = W + iT \in \mathbb{C}^{n\times n}$ be a nonsingular matrix with W and T both symmetric and positive definite. Let α be a positive constant. Then the spectral radius of the DSS iteration method satisfies*

$$\rho(M(\alpha)) = \max_{\mu \in sp(W^{-1}T)} \left| \frac{(\alpha + \frac{1}{\alpha}) - (\mu + \frac{1}{\mu})}{(\alpha + \frac{1}{\alpha}) + (\mu + \frac{1}{\mu})} \right|$$

where $sp(W^{-1}T)$ denotes the spectrum of the matrix $W^{-1}T$. Consequently,

$$\rho(M(\alpha)) < 1, \quad \text{for } \forall\alpha > 0,$$

so the DSS iteration method converges to the unique solution of the linear system (2.1) *for any initial guess.*

Now, inspired by the MN-HSS method [19] which utilizing the modified Newton iteration

$$\begin{cases} y_k &= x_k - F'(x_k)^{-1}F(x_k), \\ x_{k+1} &= y_k - F'(x_k)^{-1}F(y_k), \end{cases}$$

as the outer iteration, which has R-order of convergence three at least, we can establish a new method named modified Newton-DSS method since the DSS iteration method is applied as the inner iteration. It means that we employ the DSS iteration method to the following linear systems:

$$\begin{aligned} F'(x_k)d_k &= -F(x_k), \quad y_k = x_k + d_k, \\ F'(x_k)h_k &= -F(y_k), \quad x_{k+1} = y_k + h_k, \end{aligned} \tag{2.4}$$

Then, the MN-DSS method for solving the nonlinear sysytem (1.1) with complex symmetric Jacobian matrices is obtained as Algorithm 2 shows.

Algorithm 2 The modified Newton-DSS iteration method.

Let $F : \mathbb{D} \subset \mathbb{C}^n \to \mathbb{C}^n$ be a nonlinear and continuously differentiable function with the positive-definite and complex-symmetric Jacobian matrix $F'(x)$ at any $x \in \mathbb{D}$.

1. Give an initial guess $x_0 \in \mathbb{D}$, positive constants α and tol, and two positive sequence integer $\{l_k\}_{k=0}^{\infty}$ and $\{m_k\}_{k=0}^{\infty}$.
2. For $k = 0, 1, 2, \ldots$ until $\|F(x_k)\| \le tol\|F(x_0)\|$ do:

   2.1 Set $d_{k,0} = h_{k,0} := 0$

   2.2 For $l = 0, 1, \ldots, l_k - 1$, apply the DSS method to the first equation in (2.4):
   $$\begin{cases} (\alpha W(x_k) + T(x_k))d_{k,l+\frac{1}{2}} = i(W(x_k) - \alpha T(x_k))d_{k,l} - (\alpha - i)F(x_k), \\ (\alpha T(x_k) + W(x_k))d_{k,l+1} = i(\alpha W(x_k) - T(x_k))d_{k,l+\frac{1}{2}} - (1 - i\alpha)F(x_k), \end{cases}$$
   and obtain d_{k,l_k} such that
   $$\|F(x_k) + F'(x_k)d_{k,l_k}\| \le \eta_k\|F(x_k)\|, \quad \text{for some } \eta_k \in [0, 1) \tag{2.5}$$

   2.3 Set $y_k = x_k + d_{k,l_k}$.

   2.4 Compute $F(y_k)$.

   2.5 For $m = 0, 1, \ldots, m_k - 1$, apply the DSS method to the second equation in (2.4):
   $$\begin{cases} (\alpha W(x_k) + T(x_k))h_{k,m+\frac{1}{2}} = i(W(x_k) - \alpha T(x_k))h_{k,m} - (\alpha - i)F(y_k), \\ (\alpha T(x_k) + W(x_k))h_{k,m+1} = i(\alpha W(x_k) - T(x_k))h_{k,m+\frac{1}{2}} - (1 - i\alpha)F(y_k), \end{cases}$$
   and obtain h_{k,m_k} such that
   $$\|F(y_k) + F'(x_k)h_{k,m_k}\| \le \tilde{\eta}_k\|F(y_k)\|, \quad \text{for some } \tilde{\eta}_k \in [0, 1) \tag{2.6}$$

   2.6 Set $x_{k+1} = y_k + h_{k,m_k}$.

From the iterative scheme (2.3), the modified Newton-DSS method can be rewritten as following equivalent form after uncomplicated derivations:

$$\begin{cases} y_k \quad = x_k - \sum_{j=0}^{l_k-1} M(\alpha; x_k)^j N(\alpha; x_k)F(x_k), \\ x_{k+1} = y_k - \sum_{j=0}^{m_k-1} M(\alpha; x_k)^j N(\alpha; x_k)F(y_k), \end{cases} \quad k = 0, 1, 2, \ldots \tag{2.7}$$

where

$$\begin{aligned} M(\alpha; x) &= (\alpha T(x) + W(x))^{-1}(\alpha W(x) - T(x))(\alpha W(x) + T(x))^{-1}(\alpha T(x) - W(x)), \\ N(\alpha; x) &= 2\alpha(\alpha T(x) + W(x))^{-1}(W(x) - iT(x))(\alpha W(x) + T(x))^{-1}. \end{aligned}$$

Define matrices $B(\alpha; x)$ and $C(\alpha; x)$ by

$$\begin{aligned} B(\alpha; x) &= \frac{1}{2\alpha}(\alpha W(x) + T(x))(W(x) - iT(x))^{-1}(\alpha T(x) + W(x)), \\ C(\alpha; x) &= \frac{1}{2\alpha}(\alpha W(x) - T(x))(W(x) - iT(x))^{-1}(\alpha T(x) - W(x)). \end{aligned}$$

An easy computation shows that the Jacobian matrix $F'(x)$ possesses a new expression as

$$F'(x) = B(\alpha; x) - C(\alpha; x),$$

and

$$\begin{aligned} M(\alpha; x) &= B(\alpha; x)^{-1}C(\alpha; x), \\ N(\alpha; x) &= B(\alpha; x)^{-1}, \\ F'(x)^{-1} &= (I - M(\alpha; x))^{-1}N(\alpha; x) \end{aligned} \tag{2.8}$$

Due to (2.7) and Neumann Lemma, the modified Newton-DSS method can be equivalently expressed as the following form

$$\begin{cases} y_k &= x_k - (I - M(\alpha; x_k)^{l_k})F'(x_k)^{-1}F(x_k), \\ x_{k+1} &= y_k - (I - M(\alpha; x_k)^{m_k})F'(x_k)^{-1}F(y_k), \end{cases} \quad k = 0, 1, 2, \ldots \tag{2.9}$$

3 Local convergence theorem of the modified Newton-DSS method

In this section, we analyze the local convergence property of the modified Newton-DSS method and prove the local convergence theorem. First of all, we summarize without proofs the relevant definitions and lemmas.

Definition 3.1 A nonlinear mapping $F : \mathbb{D} \subset \mathbb{C}^n \to \mathbb{C}^n$ is Gateaux differentiable (or G-differentiable) at an interior point x of $\mathbb{D}$ if there exists a linear operator $A \in L(\mathbb{R}^n, \mathbb{R}^m)$ such that

$$\lim_{t \to 0} \frac{1}{t}\|F(x + th) - F(x) - tAh\| = 0$$

for any $h \in \mathbb{C}^n$. Moreover, $F : \mathbb{D} \subset \mathbb{C}^n \to \mathbb{C}^n$ is said to be G-differentiable on an open set $\mathbb{D}_0 \subset \mathbb{D}$ if it is G-differentiable at any point in $\mathbb{D}_0$

Lemma 3.1 *(Neumann Lemma) Let $A \in L(\mathbb{R}^n)$ satisfy $\|A\| < 1$. Then $(I - A)^{-1}$ exists and*

$$(I - A)^{-1} = \lim_{k \to \infty} \sum_{i=0}^{k} A^i.$$

Lemma 3.2 *(Banach Lemma) Let $A, B \in \mathbb{C}^{n \times n}$ satisfy $\|I - BA\| < 1$. Then the matrices A, B are nonsingular. Moreover,*

$$\|A^{-1}\| \le \frac{\|B\|}{1 - \|I - BA\|}, \quad \|B^{-1}\| \le \frac{\|A\|}{1 - \|I - BA\|},$$

and

$$\|A^{-1} - B\| \le \frac{\|B\|\|I - BA\|}{1 - \|I - BA\|}, \quad \|A - B^{-1}\| \le \frac{\|A\|\|I - BA\|}{1 - \|I - BA\|}$$

Assume that $F : \mathbb{D} \subset \mathbb{C}^n \to \mathbb{C}^n$ is G-differentiable on an open neighborhood $N_0 \subset \mathbb{D}$ of a point $x_* \in \mathbb{D}$ at which the Jacobian matrix $F'(x)$ is continuous, positive definite, complex symmetric and $F(x_*) = 0$. Let us split $F'(x)$ into the form $F'(x) = W(x) + iT(x)$, where $W(x)$ and $T(x)$ are both real positive definite and symmetric matrices for any $x \in \mathbb{D}$, respectively. Denote with $\mathbb{N}(x_*, r)$ an open ball centered at x_* with radius $r > 0$.

Assumption 3.1 *For all $x \in \mathbb{N}(x_*, r) \subset \mathbb{N}_0$, suppose that the following conditions hold. (THE BOUNDED CONDITION) there exist positive constants β, γ and δ such that*

$$\max\{\|W(x_*)\|, \|T(x_*)\|\} \le \beta, \quad \|F'(x_*)^{-1}\| \le \gamma.$$

(THE LIPSCHITZ CONDITION) there exist nonnegative constants L_w and L_t such that

$$\begin{aligned} \|W(x) - W(x_*)\| &\le L_w \|x - x_*\|, \\ \|T(x) - T(x_*)\| &\le L_t \|x - x_*\|. \end{aligned}$$

In the following, we will prove the local convergence of our method.

Lemma 3.3 *If $r \in (0, \frac{1}{\gamma L})$ and* Assumption 3.1 *holds, then $F'(x)^{-1}$ exists for any $x \in \mathbb{N}(x_*, r) \subset \mathbb{N}_0$. Moreover, the following inequalities hold with $L := L_w + L_t$ for all $x, y \in \mathbb{N}(x_*, r)$:*

$$\begin{aligned} &\|F'(x) - F'(x_*)\| \le L\|x - x_*\|, \\ &\|F'(x)^{-1}\| \le \frac{\gamma}{1 - \gamma L\|x - x_*\|}, \\ &\|F(y)\| \le \frac{L}{2}\|y - x_*\|^2 + 2\beta\|y - x_*\|, \\ &\|y - x_* - F'(x)^{-1}F(y)\| \le \frac{\gamma}{1 - \gamma L\|x - x_*\|}\left(\frac{L}{2}\|y - x_*\| + L\|x - x_*\|\right)\|y - x_*\|. \end{aligned}$$

Proof From the Lipschitz condition, it is directly implied that

$$\begin{aligned} \|F'(x) - F'(x_*)\| &= \|W(x) + iT(x) - W(x_*) - iT(x_*)\| \\ &\le \|W(x) - W(x_*)\| + \|i(T(x) - T(x_*))\| \\ &\le (L_w + L_t)\|x - x_*\| = L\|x - x_*\|. \end{aligned}$$

Moreover, the condition $r \in (0, 1/\gamma L)$ suggests that

$$\|F'(x_*)^{-1}(F'(x_*) - F'(x))\| \le \|F'(x_*)^{-1}\|\|F'(x_*) - F'(x)\| \le \gamma L\|x - x_*\| < 1.$$

It follows from Lemma 3.2 that $F'(x)^{-1}$ exists, and

$$\|F'(x)^{-1}\| \le \frac{\|F'(x_*)^{-1}\|}{1 - \|F'(x_*)^{-1}(F'(x_*) - F'(x))\|} \le \frac{\gamma}{1 - \gamma L\|x - x_*\|}.$$

In addition, since the definition of integral shows

$$\begin{aligned} F(y) &= F(y) - F(x_*) - F'(x_*)(y - x_*) + F'(x_*)(y - x_*) \\ &= \int_0^1 (F'(x_* + t(y - x_*)) - F'(x_*))dt(y - x_*) + F'(x_*)(y - x_*), \end{aligned}$$

and the bounded condition results in

$$\|F'(x_*)\| = \|W(x_*) + iT(x_*)\| \leq \|W(x_*)\| + \|T(x_*)\| \leq 2\beta,$$

we obtain

$$\begin{aligned} \|F(y)\| &\leq \|\int_0^1 (F'(x_* + t(y - x_*)) - F'(x_*))dt(y - x_*)\| + \|F'(x_*)(y - x_*)\| \\ &\leq \|y - x_*\| \int_0^1 \|(F'(x_* + t(y - x_*)) - F'(x_*))\|dt + \|F'(x_*)(y - x_*)\| \\ &\leq \|y - x_*\| \int_0^1 \|Lt(y - x_*)\|dt + \|F'(x_*)(y - x_*)\| \\ &\leq \frac{L}{2}\|y - x_*\|^2 + 2\beta\|y - x_*\|. \end{aligned}$$

Clearly, it holds that

$$\begin{aligned} y - x_* - F'(x)^{-1}F(y) &= -F'(x)^{-1}(F(y) - F(x_*) - F'(x)(y - x_*)) \\ &= -F'(x)^{-1}(F(y) - F(x_*) - F'(x_*)(y - x_*)) \\ &\quad +F'(x)^{-1}(F'(x) - F'(x_*))(y - x_*) \\ &= -F'(x)^{-1}\int_0^1 (F'(x_* + t(y - x_*)) - F'(x_*))dt(y - x_*) \\ &\quad +F'(x)^{-1}(F'(x) - F'(x_*))(y - x_*). \end{aligned}$$

Hence,

$$\begin{aligned} &\|y - x_* - F'(x)^{-1}F(y)\| \\ &\leq \| - F'(x)^{-1}\|(\int_0^1 \|F'(x_* + t(y - x_*)) - F'(x_*)\|dt + \|(F'(x) - F'(x_*)\|)\|y - x_*\| \\ &\leq \frac{\gamma}{1 - \gamma L\|x - x_*\|}(\frac{L}{2}\|y - x_*\| + L\|x - x_*\|)\|y - x_*\|. \end{aligned}$$

The proof of Lemma 3.3 is completed. □

Lemma 3.4 *Under the conditions of* Lemma 3.3, *let* $r \in (0, r_0)$ *and define* $r_0 := \min\{r_1, r_2\}$, *where* r_1 *is the minimal positive solution of the quadratic equation*

$$\frac{2(\alpha + 1)^2\gamma^2(L^2x^2 + 2(1 + \beta\gamma)\beta Lx)}{2\alpha(1 - \gamma Lx) - (\alpha + 1)^2\gamma^2(L^2x^2 + 2(1 + \beta\gamma)\beta Lx)} = \tau\theta.$$

and

$$r_2 = \frac{1 - 2\beta\gamma[(\tau + 1)\theta]^u}{3\gamma L},$$

with $u = \min\{l_*, m_*\}$, $l_* = \liminf_{k\to\infty} l_k$, $m_* = \liminf_{k\to\infty} m_k$, *and the constant* u *satisfies*

$$u > \lfloor -\frac{\ln(2\beta\gamma)}{\ln((\tau+1)\theta)} \rfloor,$$

where the symbol $\lfloor\bullet\rfloor$ *is used to denote the smallest integer no less than the corresponding real number,* $\tau \in (0, \frac{1-\theta}{\theta})$ *a prescribed positive constant and*

$$\theta \equiv \theta(\alpha; x_*) = \|M(\alpha; x_*)\| \le \frac{\sqrt{\alpha^2 + \xi_{\max}^2}}{\alpha + \lambda_{\min}} \equiv \sigma(\alpha; x_*)$$
$$\textit{with } \lambda_{\min} = \min_{\lambda_j \in sp(H)} \{\lambda_j\}, \quad \xi_{\max} = \max_{i\xi_j \in sp(S)} \{|\xi_j|\}$$

In addition, we utilize the notation

$$g(t; v) = \frac{2\gamma}{1 - \gamma L t}(Lt + \beta[(\tau+1)\theta]^v).$$

Then, for any $x \in \mathbb{N}(x_*, r) \subset \mathbb{N}_0$, $t \in (0, r)$ *and* $v > u$, *it holds that*

$$\|M(\alpha; x)\| \le (\tau+1)\theta < 1,$$
$$g(t; v) < g(r_0; u) < 1.$$

Proof According to the bounded condition, the equality (2.8) and the fact

$$\|M(\alpha; x_*)\| \le \sigma(\alpha; x_*) < 1$$

under some moderate conditions, it holds that

$$\begin{aligned}\|B(\alpha; x_*)^{-1}\| &= \|(I - M(\alpha; x_*))F'(x_*)^{-1}\| \\ &\le \|I - M(\alpha; x_*)\|\|F'(x_*)^{-1}\| \\ &\le (1 + \|M(\alpha; x_*)\|)\|F'(x_*)^{-1}\| \le 2\gamma.\end{aligned}$$

Firstly, from the bounded condition, we have

$$\|\alpha W(x_*) + T_(x_*)\| \le \alpha\|W(x_*)\| + \|T(x_*)\| \le (\alpha+1)\beta$$

It follows from the Assumption 3.1 that we can further obtain

$$\begin{aligned}&\|(\alpha W(x) + T(x)) - (\alpha W(x_*) + T(x_*))\| \\ &\le \alpha\|W(x) - W(x_*)\| + \|T(x) - T(x_*)\| \\ &\le (\alpha L_w + L_t)\|x - x_*\|.\end{aligned}$$

Moreover, we have

$$\|(\alpha T(x) + W(x)) - (\alpha T(x_*) + W(x_*))\| \le (\alpha L_t + L_w)\|x - x_*\|,$$

and

$$\begin{aligned}\|\alpha T(x) + W(x)\| &\le \|(\alpha T(x) + W(x)) - (\alpha T(x_*) + W(x_*))\| + \|\alpha T(x_*) + W(x_*)\| \\ &\le (\alpha L_t + L_w)\|x - x_*\| + (\alpha+1)\beta.\end{aligned}$$

Since

$$\|(W(x_*) - iT(x_*))^{-1}\| = \|\overline{F'(x_*)}^{-1}\| = \|F'(x_*)^{-1}\| \le \gamma,$$

we see at once that

$$\begin{aligned}
\|(W(x)-iT(x))^{-1}\| &\le \frac{\|(W(x_*)-iT(x_*))^{-1}\|}{1-\|I-(W(x_*)-iT(x_*))^{-1}(W(x)-iT(x))\|}\\
&\le \frac{\|(W(x_*)-iT(x_*))^{-1}\|}{1-\|(W(x_*)-iT(x_*))^{-1}\|\|(W(x_*)-iT(x_*))-(W(x)-iT(x))\|}\\
&\le \frac{\gamma}{1-\gamma(\|W(x)-W(x_*)\|+\|T(x)-T(x_*)\|)}\\
&\le \frac{\gamma}{1-\gamma L\|x-x_*\|},
\end{aligned}$$

provided r is small enough such that $\gamma L\|x-x_*\| < 1$ which resulting in $\|I-(W(x_*)-iT(x_*))^{-1}(W(x)-iT(x))\| < 1$. It follows immediately that

$$\begin{aligned}
&\|(W(x)-iT(x))^{-1}-(W(x_*)-iT(x_*))^{-1}\|\\
&\le \|(W(x_*)-iT(x_*))^{-1}\|\|(W(x_*)-iT(x_*))-(W(x)-iT(x))\|\|(W(x)-iT(x))^{-1}\|\\
&\le \frac{\gamma^2}{1-\gamma L\|x-x_*\|}(\|W(x)-W(x_*)\|+\|T(x)-T(x_*)\|)\\
&\le \frac{\gamma^2 L\|x-x_*\|}{1-\gamma L\|x-x_*\|}.
\end{aligned}$$

On account of the above proof, we can easily get

$$\begin{aligned}
&2\alpha\|B(\alpha;x)-B(\alpha;x_*)\|\\
&=\|(\alpha W(x)+T(x))(W(x)-iT(x))^{-1}(\alpha T(x)+W(x))\\
&-(\alpha W(x_*)+T(x_*))(W(x_*)-iT(x_*))^{-1}(\alpha T(x_*)+W(x_*))\|\\
&\le \|[(\alpha W(x)+T(x))-(\alpha W(x_*)+T(x_*))](W(x)-iT(x))^{-1}(\alpha T(x)+W(x))\\
&+(\alpha W(x_*)+T(x_*))[(W(x)-iT(x))^{-1}-(W(x_*)-iT(x_*))^{-1}](\alpha T(x)+W(x))\\
&+(\alpha W(x_*)+T(x_*))(W(x_*)-iT(x_*))^{-1}[(\alpha T(x)+W(x))-(\alpha T(x_*)+W(x_*))]\\
&\le (\alpha L_w+L_t)\|x-x_*\|\frac{\gamma}{1-\gamma L\|x-x_*\|}[(\alpha L_t+L_w)\|x-x_*\|+(\alpha+1)\beta]\\
&+[(\alpha+1)\beta]\frac{\gamma^2 L\|x-x_*\|}{1-\gamma L\|x-x_*\|}[(\alpha L_t+L_w)\|x-x_*\|+(\alpha+1)\beta]\\
&+[(\alpha+1)\beta]\gamma(\alpha L_t+L_w)\|x-x_*\|\\
&\le \frac{\frac{1}{2}(\alpha+1)^2\gamma L^2\|x-x_*\|^2+(\alpha+1)^2(1+\beta\gamma)\beta\gamma L\|x-x_*\|}{1-\gamma L\|x-x_*\|}.
\end{aligned}$$

Hence,

$$\|B(\alpha;x)-B(\alpha;x_*)\| \le \frac{(\alpha+1)^2\gamma(L^2\|x-x_*\|^2+2(1+\beta\gamma)\beta L\|x-x_*\|)}{4\alpha(1-\gamma L\|x-x_*\|)}. \tag{3.1}$$

Likewise, we have

$$\|C(\alpha;x)-C(\alpha;x_*)\| \le \frac{(\alpha+1)^2\gamma(L^2\|x-x_*\|^2+2(1+\beta\gamma)\beta L\|x-x_*\|)}{4\alpha(1-\gamma L\|x-x_*\|)}. \tag{3.2}$$

Consequently, by making use of the Banach lemma, i.e., Lemma 3.2, it holds that

$$\begin{aligned}\|B(\alpha;x)^{-1}\| &\le \frac{\|B(\alpha;x_*)^{-1}\|}{1-\|I-B(\alpha;x_*)^{-1}B(\alpha;x)\|}\\ &\le \frac{\|B(\alpha;x_*)^{-1}\|}{1-\|B(\alpha;x_*)^{-1}\|\|B(\alpha;x_*)-B(\alpha;x)\|}\\ &\le \frac{4\alpha\gamma(1-\gamma L\|x-x_*\|)}{2\alpha(1-\gamma L\|x-x_*\|)-(\alpha+1)^2\gamma^2(L^2\|x-x_*\|^2+2(1+\beta\gamma)\beta L\|x-x_*\|)}\end{aligned} \tag{3.3}$$

for all $x \in \mathbb{N}(x_*, r)$, provided r is small enough such that

$$(\alpha+1)^2\gamma^2(L^2\|x-x_*\|^2+2(1+\beta\gamma)\beta L\|x-x_*\|) < 2\alpha(1-\gamma L\|x-x_*\|)$$

which resulting in $\|I - B(\alpha; x_*)^{-1}B(\alpha; x)\| < 1$. From (2.8) we immediately get the equality

$$\begin{aligned}&M(\alpha;x)-M(\alpha;x_*)\\ &= B(\alpha;x)^{-1}C(\alpha;x)-B(\alpha;x_*)^{-1}C(\alpha;x_*)\\ &= B(\alpha;x)^{-1}((C(\alpha;x)-C(\alpha;x_*))-(B(\alpha;x)-B(\alpha;x_*))M(\alpha;x_*)).\end{aligned}$$

Based on (3.1), (3.2), and (3.3), we can easily obtain that

$$\begin{aligned}&\|M(\alpha;x)-M(\alpha;x_*)\|\\ &\le \|B(\alpha;x)^{-1}\|(\|C(\alpha;x)-C(\alpha;x_*)\|+\|B(\alpha;x)-B(\alpha;x_*)\|\|M(\alpha;x_*)\|)\\ &\le \frac{2(\alpha+1)^2\gamma^2(L^2\|x-x_*\|^2+2(1+\beta\gamma)\beta L\|x-x_*\|)}{2\alpha(1-\gamma L\|x-x_*\|)-(\alpha+1)^2\gamma^2(L^2\|x-x_*\|^2+2(1+\beta\gamma)\beta L\|x-x_*\|)}.\end{aligned}$$

Meanwhile, $r < r_1$ implies that

$$\frac{2(\alpha+1)^2\gamma^2(L^2\|x-x_*\|^2+2(1+\beta\gamma)\beta L\|x-x_*\|)}{2\alpha(1-\gamma L\|x-x_*\|)-(\alpha+1)^2\gamma^2(L^2\|x-x_*\|^2+2(1+\beta\gamma)\beta L\|x-x_*\|)} < \tau\theta.$$

Hence

$$\|M(\alpha;x)\| \le \|M(\alpha;x)-M(\alpha;x_*)\|+\|M(\alpha;x_*)\| \le (\tau+1)\theta.$$

Furthermore, since $t \in (0, r)$ and $r < r_2$, it is obvious that

$$g(t;v) = \frac{2\gamma}{1-\gamma Lt}(Lt+\beta[(\tau+1)\theta]^v) < g(r_0;u) < 1.$$

□

Theorem 3.1 *Under the conditions of* Lemma 3.3 *and* 3.4, *then for any* $x_0 \in \mathbb{N}(x_*, r)$ *and any sequences* $\{l_k\}_{k=0}^{\infty}$, $\{m_k\}_{k=0}^{\infty}$ *of positive integers, the iteration sequence* $\{x_k\}_{k=0}^{\infty}$ *generated by the modified Newton-DSS method is well-defined and converges to* x_*. *Moreover, it holds that*

$$\limsup_{k\to\infty} \|x_k - x_*\|^{\frac{1}{k}} \le g(r_0;u)^2.$$

Proof From Lemma 3.3, Lemma 3.4, and (2.7), we can easily obtain that

$$\begin{aligned}
\|y_k - x_*\| &= \|x_k - x_* - (I - M(\alpha; x_k)^{l_k})F'(x_k)^{-1}F(x_k)\| \\
&\leq \|x_k - x_* - F'(x_k)^{-1}F(x_k)\| + \|M(\alpha; x_k)^{l_k}\|\|F'(x_k)^{-1}F(x_k)\| \\
&\leq \frac{\gamma}{1-\gamma L\|x - x_*\|}\frac{3L}{2}\|x_k - x_*\|^2 \\
&+\frac{\gamma[(\tau+1)\theta]^{l_k}}{1-\gamma L\|x - x_*\|}(\frac{L}{2}\|x_k - x_*\|^2 + 2\beta\|x_k - x_*\|) \\
&= \frac{(3+[(\tau+1)\theta]^{l_k})\gamma L}{2(1-\gamma L\|x_k - x_*\|)}\|x_k - x_*\|^2 + \frac{2\beta\gamma[(\tau+1)\theta]^{l_k}}{1-\gamma L\|x_k - x_*\|}\|x_k - x_*\| \\
&\leq \frac{2\gamma}{1-\gamma L\|x - x_*\|}(L\|x_k - x_*\| + \beta[(\tau+1)\theta]^{l_k})\|x_k - x_*\| \\
&= g(\|x_k - x_*\|; l_k)\|x_k - x_*\| \\
&< g(r_0; u)\|x_k - x_*\| < \|x_k - x_*\|
\end{aligned}$$

and

$$\begin{aligned}
\|x_{k+1} - x_*\| &= \|y_k - x_* - (I - M(\alpha; x_k)^{m_k})F'(x_k)^{-1}F(y_k)\| \\
&\leq \|y_k - x_* - F'(x_k)^{-1}F(y_k)\| + \|M(\alpha; x_k)^{m_k}\|\|F'(x_k)^{-1}F(y_k)\| \\
&\leq \frac{\gamma}{1-\gamma L\|x - x_*\|}(\frac{L}{2}\|y_k - x_*\| + L\|x_k - x_*\|)\|y_k - x_*\| \\
&+\frac{\gamma[(\tau+1)\theta]^{m_k}}{1-\gamma L\|x - x_*\|}(\frac{L}{2}\|y_k - x_*\|^2 + 2\beta\|y_k - x_*\|) \\
&\leq (\frac{\gamma L}{1-\gamma L\|x_k - x_*\|}(\frac{1+[(\tau+1)\theta]^{m_k}}{2}\|y_k - x_*\| + \|x_k - x_*\|) \\
&+\frac{2\beta\gamma[(\tau+1)\theta]^{m_k}}{1-\gamma L\|x_k - x_*\|})\|y_k - x_*\| \\
&\leq \frac{2\gamma g(\|x_k - x_*\|; l_k)}{1-\gamma L\|x_k - x_*\|}(\frac{1+g(\|x_k - x_*\|; l_k)}{2}L\|x_k - x_*\| \\
&+\beta[(\tau+1)\theta]^{m_k})\|x_k - x_*\| \\
&< \frac{2\gamma g(\|x_k - x_*\|; l_k)}{1-\gamma L\|x_k - x_*\|}(L\|x_k - x_*\| + \beta[(\tau+1)\theta]^{m_k})\|x_k - x_*\| \\
&= g(\|x_k - x_*\|; l_k)g(\|x_k - x_*\|; m_k)\|x_k - x_*\| \\
&< g(\|x_k - x_*\|; u)^2\|x_k - x_*\| \\
&< g(r_0; u)^2\|x_k - x_*\| < \|x_k - x_*\|.
\end{aligned}$$

We can further prove that $\{x_k\}_{k=0}^{\infty} \subset \mathbb{N}(x_*, r)$ converges to x_* by induction. In fact, for $k = 0$, we can obtain $\|x_0 - x_*\| < r < r_0$ and

$$\|x_1 - x_*\| < g(\|x_0 - x_*\|; u)^2\|x_0 - x_*\| < \|x_0 - x_*\| < r,$$

as $x_0 \in \mathbb{N}(x_*, r)$. Hence we have $x_1 \in \mathbb{N}(x_*, r)$. Now, suppose that $x_n \in \mathbb{N}(x_*, r)$ for some positive integer $k = n$, then we can straightforwardly deduce the estimate

$$\begin{aligned}\|x_{n+1} - x_*\| &< g(\|x_n - x_*\|; u)^2 \|x_n - x_*\| \\ &< g(r_0; u)^2 \|x_n - x_*\| < g(r_0; u)^{2(n+1)} \|x_o - x_*\| < r,\end{aligned}$$

which shows that $x_{n+1} \in \mathbb{N}(x_*, r)$ for $k = n + 1$. Moreover, $x_{n+1} \to x_*$ when $n \to \infty$.

The proof of theorem is completed. □

4 Semilocal convergence theorem of the modified Newton-DSS method

In this section, we prove a Kantorovich-type semilocal convergence for the modified Newton-DSS method by utilizing the major function. That is, if we impose some conditions on the initial vector x_0 but do not require knowledge of the existence of a solution, the exact solution x_* of the nonlinear system must exist in some neighborhood of x_0.

Assume that $F : \mathbb{D} \subset \mathbb{C}^n \to \mathbb{C}^n$ is G-differentiable on an open neighborhood $N_0 \subset \mathbb{D}$ of a point $x_0 \in \mathbb{D}$ at which the Jacobian matrix $F'(x)$ is continuous, positive definite and complex symmetric. Suppose $F'(x) = W(x) + iT(x)$, where $W(x)$ and $T(x)$ are both real positive definite and symmetric matrices for any $x \in \mathbb{D}$, respectively. Denote with $\mathbb{N}(x_0, r)$ an open ball centered at x_0 with radius $r > 0$.

Assumption 4.1 *Let $x_0 \in \mathbb{C}^n$ and suppose that the following conditions hold. (THE BOUNDED CONDITION) there exist positive constants β, γ and ϵ such that*

$$\max\{\|W(x_0)\|, \|T(x_0)\|\} \le \beta, \quad \|F'(x_0)^{-1}\| \le \gamma, \quad \|F(x_0)\| \le \epsilon.$$

(THE LIPSCHITZ CONDITION) there exist nonnegative constants L_h and L_s such that for all $x, y \in \mathbb{N}(x_0, r) \subset \mathbb{N}_0$,

$$\begin{aligned}\|W(x) - W(y)\| &\le L_w \|x - y\|, \\ \|T(x) - T(y)\| &\le L_t \|x - y\|.\end{aligned}$$

From Assumption 4.1, Banach's Lemma, and the integral mean-value theorem, let $L := L_w + L_t$ and we can easily get Lemma 4.1 without detailed proof as follows.

Lemma 4.1 *Under* Assumption 4.1*, for all $x, y \in \mathbb{N}(x_0, r)$, if $r \in (0, \frac{1}{\gamma L})$, then $F'(x)^{-1}$ exists. And we have the following inequalities:*

$$\begin{aligned}
&\|F'(x) - F'(y)\| \leq L\|x - y\|,\\
&\|F'(x)\| \leq L\|x - x_0\| + 2\beta,\\
&\|F(x) - F(y) - F'(y)(x - y)\| \leq \frac{L}{2}\|x - y\|^2,\\
&\|F'(x)^{-1}\| \leq \frac{\gamma}{1 - \gamma L\|x - x_0\|}.
\end{aligned}$$

Define

$$a = L\gamma(1+\eta), \quad b = 1 - \eta, \quad c = 2\gamma\epsilon, \quad \text{where } \eta = \max_k\{\max\{\eta_k, \tilde{\eta}_k\}\} < 1.$$

The iterative sequences $\{t_k\}$, $\{s_k\}$ are generated by the following formulas

$$\begin{cases} t_0 &= 0,\\ s_k &= t_k - \frac{g(t_k)}{h(t_k)},\\ t_{k+1} &= s_k - \frac{g(s_k)}{h(t_k)}, \end{cases} \tag{4.1}$$

where

$$\begin{cases} g(t) = \frac{1}{2}at^2 - bt + c,\\ h(t) = at - 1. \end{cases}$$

Now, we claim that the sequences $\{t_k\}$, $\{s_k\}$ converge monotone increasingly to some number as shown by the following lemma.

Lemma 4.2 *Assume that the constants satisfy*

$$\gamma^2\epsilon L \leq \frac{(1-\eta)^2}{4(1+\eta)}.$$

Denote $t_* = \frac{b - \sqrt{b^2 - 2ac}}{a}$, then the the sequences $\{t_k\}$, $\{s_k\}$, generated by the formulas (4.1), increase and converge to t_*. Moreover,

$$\begin{aligned}
&0 \leq t_k < s_k < t_{k+1} < t_*,\\
&t_{k+1} - s_k < s_k - t_k.
\end{aligned}$$

Proof Details see Lemma 4.2 and Lemma 4.3 in [19]. □

Theorem 4.1 *Under the assumptions of* Lemmas 4.1 *and* 4.2*, define $r := \min(r_1, r_2)$, where r_1 is the minimal positive solution of the quadratic equation*

$$\frac{2(\alpha+1)^2\gamma^2(L^2x^2 + 2(1+\beta\gamma)\beta Lx)}{2\alpha(1-\gamma Lx) - (\alpha+1)^2\gamma^2(L^2x^2 + 2(1+\beta\gamma)\beta Lx)} = \tau\theta.$$

and

$$r_2 = \frac{b - \sqrt{b^2 - 2ac}}{a},$$

and define $u = \min\{l_*, m_*\}$, *with* $l_* = \liminf_{k\to\infty} l_k$, $m_* = \liminf_{k\to\infty} m_k$, *and the constant* u *satisfies*

$$u > \lfloor \frac{\ln \eta}{\ln((\tau + 1)\theta)} \rfloor,$$

where the symbol $\lfloor \bullet \rfloor$ *is used to denote the smallest integer no less than the corresponding real number,* $\tau \in (0, \frac{1-\theta}{\theta})$ *a prescribed positive constant and*

$$\theta \equiv \theta(\alpha; x_0) = \|M(\alpha; x_0)\| < 1$$

Then the iteration sequence $\{x_k\}_{k=0}^{\infty}$ *generated by the modified Newton-DSS method is well-defined and converges to* x_*, *which satisfies* $F(x_*) = 0$.

Proof Firstly, analysis similar to that in the proof of Lemma 3.4 shows the estimate about the iterative matrix $M(\alpha; x)$ of the linear solver: if $x \in \mathbb{N}(x_0, r)$, then

$$\|M(\alpha; x)\| \le (\tau + 1)\theta < 1.$$

Now we prove following inequalities by induction

$$\begin{cases} \|x_k - x_0\| \le t_k - t_0 \\ \|F(x_k)\| \le \frac{1-\gamma L t_k}{(1+\eta)\gamma}(s_k - t_k) \\ \|y_k - x_k\| \le s_k - t_k \\ \|F(y_k)\| \le \frac{1-\gamma L t_k}{(1+\eta)\gamma}(t_{k+1} - s_k) \\ \|x_{k+1} - y_k\| \le t_{k+1} - s_k \end{cases} \tag{4.2}$$

Since

$$\begin{aligned}
&\|x_0 - x_0\| = 0 \le t_0 - t_0,\\
&\|F(x_0)\| \le \epsilon \le \frac{2\gamma\epsilon}{\gamma(1+\eta)} = \frac{1 - \gamma L t_0}{(1+\eta)}(s_0 - t_0),\\
&\|y_0 - x_0\| = \|I - M(\alpha; x_0)^{l_0}\|\|F'(x_0)^{-1}F(x_0)\| \le (1 + \theta^{l_0})\gamma\epsilon < 2\gamma\epsilon = s_0,\\
&\|F(y_0)\| \le \|F(y_0) - F(x_0) - F'(x_0)(y_0 - x_0)\| + \|F(x_0) + F'(x_0)(y_0 - x_0)\|\\
&\le \frac{L}{2}\|y_0 - x_0\|^2 + \eta\|F(x_0)\| \le \frac{L}{2}s_0^2 + \eta\epsilon \le \frac{1 - \gamma L t_0}{(1+\eta)\gamma}(t_1 - s_0),\\
&\|x_1 - y_0\| \le \|I - M(\alpha; x_0)^{m_0}\|\|F'(x_0)^{-1}F(y_0)\|\\
&\le (1 + \theta^{m_0})\|F'(x_0)^{-1}\|\|F(y_0)\| < (1+\eta)\gamma\|F(y_0)\| \le t_1 - s_0,
\end{aligned}$$

the inequalities (4.2) are correct for $k = 0$. Suppose that (4.2) holds for all nonnegative integers less than k. We need to prove that it holds for k. For the first inequality in (4.2), we have

$$\|x_k - x_0\| \le \|x_k - y_{k-1}\| + \|y_{k-1} - x_{k-1}\| + \|x_{k-1} - x_0\| \le t_k - t_0 < t_* < r_2.$$

Since $x_{k-1}, y_{k-1} \in \mathbb{N}(x_0, r)$ and by the inequality(2.4) and the inequalities in Lemma 4.1, we have

$$\begin{aligned}
(1+\eta)\gamma\|F(x_k)\| &\leq (1+\eta)\gamma\|F(x_k)-F(y_{k-1})-F'(x_{k-1})(x_k-y_{k-1})\| \\
&\quad +(1+\eta)\gamma\|F(y_{k-1})+F'(x_{k-1})(x_k-y_{k-1})\| \\
&\leq \frac{(1+\eta)\gamma L}{2}\|x_k-y_{k-1}\|^2+\eta(1+\eta)\gamma\|F(y_{k-1})\| \\
&\leq \frac{(1+\eta)\gamma L}{2}(t_k-s_{k-1})^2+\eta(1-\gamma L t_{k-1})(t_k-s_{k-1}) \\
&= g(t_k)-g(s_{k-1})+b(t_k-s_{k-1})-as_{k-1}(t_k-s_{k-1}) \\
&\quad +\eta(1-\gamma L t_{k-1})(t_k-s_{k-1}) \\
&= g(t_k)-g(s_{k-1})+(1-\gamma L(1+\eta)s_{k-1}-\eta\gamma L t_{k-1})\frac{g(s_{k-1})}{-h(tk-1)} \\
&= g(t_k)+\frac{(1+\eta)\gamma L s_{k-1}-\gamma L t_{k-1}}{h(t_{k-1})}g(s_{k-1}) \\
&< g(t_k)=-h(t_k)(s_k-t_k)<(1-\gamma L t_k)(s_k-t_k).
\end{aligned}$$

It follows that

$$\|F(x_k)\| \leq \frac{(1-\gamma L t_k)}{(1+\eta)\gamma}(s_k-t_k).$$

and then

$$\begin{aligned}
\|y_k-x_k\| &\leq \|I-M(\alpha;x_k)^{l_k}\|\|F'(x_k)^{-1}F(x_k)\| \\
&\leq (1+((1_\tau)\theta)^{l_k})\|F'(x_k)^{-1}\|\|F(x_k)\| \\
&\leq (1+\eta)\frac{\gamma}{1-\gamma L t_k}\|F(x_k)\| \\
&\leq s_k-t_k.
\end{aligned}$$

Likewise, we can prove that

$$\|F(y_k)\| \leq \frac{1-\gamma L t_k}{(1+\eta)\gamma}(t_{k+1}-s_k)$$

and

$$\|x_{k+1}-y_k\| \leq t_{k+1}-s_k.$$

Hence, the inequalities (4.2) hold for all k. Since the sequences $\{t_k\}$, $\{s_k\}$ converge to t_* and

$$\|x_{k+1}-x_0\| \leq \|x_{k+1}-y_k\|+\|y_k-x_k\|+\|x_k-x_0\| \leq t_{k+1}-t_0 < t_* < r_2,$$

the sequence $\{x_k\}$ also converges, to say x_*. Since $\|M(\alpha;x_*)\|<1$, we have $F(x_*)=0$ from the iteration (2.9).

The proof of theorem is completed. □

5 Numerical examples

In this section, we present two examples which are of the complex nonlinear system of the form (1.1) with its Jacobian matrix that has the form (1.2). By making use of these examples, we illustrate the efficiency of our modified Newton-DSS method (MNDSS) compared with that of the modified Newton-MHSS method (MNMHSS) and the modified Newton-PMHSS method (MNPMHSS), in the sense of both the number of iteration steps (denoted as “IT”) and the elapsed CPU time in seconds (denoted as “CPU time”). The optimal parameters used in actual computations are obtained experimentally by minimizing the corresponding iteration steps and error relative to the exact solution. The experimental results point to the conclusion that MNDSS method outperforms both MNMHSS method and MNPMHSS method.

Example 1 We consider the following nonlinear equations [27]:

$$\begin{cases} u_t - (\alpha_1 + i\beta_1)(u_{xx} + u_{yy}) + \varrho u = -(\alpha_2 + i\beta_2)u^{\frac{4}{3}}, & \text{in } (0,1] \times \Omega, \\ u(0,x,y) = u_0(x,y), & \text{in } \Omega, \\ u(t,x,y) = 0, & \text{on } (0,1] \times \partial\Omega, \end{cases}$$

where $\Omega = [0,1] \times [0,1]$, $\partial\Omega$ is the boundary of Ω. The coefficients $\alpha_1 = \beta_1 = 1$, $\alpha_2 = \beta_2 = 1$ and ϱ is a positive constant used to control the magnitude of the reaction term. By discretizing this equation with centered finite difference scheme on the equidistant discretization grid $\Delta t = h = 1/(N+1)$, at each temporal step of the implicit scheme, we can obtain the system of nonlinear equations $F(x) = 0$ with following form:

$$F(u) = Mu + (\alpha_2 + i\beta_2)h\Delta t\Psi)(u) = 0,$$

where

$$M = h(1 + \varrho\Delta t)I_n + (\alpha_1 + i\beta_1)\frac{\Delta t}{h}(A_N \otimes I_N + I_N \otimes A_N),$$
$$\Psi(u) = (u_1^{\frac{4}{3}}, u_2^{\frac{4}{3}}, \ldots, u_n^{\frac{4}{3}})^T,$$

with $A_N = \text{tridiag}(-1, 2, -1)$ and $n = N \times N$. Here $\otimes$ is the Kronecker product symbol.

In our computations, we choose the initial guess to be $u_0 = 1$, the stopping criterion for the outer Newton iteration is set to be

$$\frac{\|F(u_k)\|}{\|F(u_0)\|} \leq 10^{-6},$$

Table 1 The experimentally optimal values α for MNMHSS method

N	$\varrho = 1$			$\varrho = 10$			$\varrho = 200$		
	$\eta = 0.1$	$\eta = 0.2$	$\eta = 0.4$	$\eta = 0.1$	$\eta = 0.2$	$\eta = 0.4$	$\eta = 0.1$	$\eta = 0.2$	$\eta = 0.4$
2^5	0.45	0.46	0.44	0.47	0.48	0.47	0.79	0.74	0.78
2^6	0.28	0.29	0.27	0.28	0.29	0.29	0.45	0.44	0.43
2^7	0.18	0.18	0.18	0.18	0.18	0.18	0.25	0.25	0.25

Table 2 The experimentally optimal values α for MNPMHSS method

N	$\varrho = 1$			$\varrho = 10$			$\varrho = 200$		
	$\eta = 0.1$	$\eta = 0.2$	$\eta = 0.4$	$\eta = 0.1$	$\eta = 0.2$	$\eta = 0.4$	$\eta = 0.1$	$\eta = 0.2$	$\eta = 0.4$
2^5	0.90	0.90	0.90	0.85	0.84	0.84	0.66	0.65	0.50
2^6	0.82	0.81	0.80	0.78	0.77	0.76	0.76	0.70	0.55
2^7	0.66	0.70	0.70	0.61	0.67	0.68	0.80	0.72	0.55

and the prescribed tolerance η_k and $\tilde{\eta}_k$ for controlling the accuracy of the iteration methods are set to be the same value η.

Firstly, we finished some upfront work for the parameter α. There are some theoretical method about how to select the optimal parameter α. It is a interesting topic in the future research. Here, we decide the choice based on experiment in Tables 1, 2, and 3, in which we present the experimentally optimal parameter α for MNMHSS, MNPMHSS and MNDSS methods, respectively. Subsequently, we adopt these experimentally optimal parameters α for the three methods to solve the nonlinear equation.

In Tables 4, 5, 6, 7, and 8, we display the experimental results about the modified Newton method incorporated with MHSS, PMHSS, and DSS, corresponding to the scale of the problem $N = 2^5, 2^6, 2^7$, the inner tolerance $\eta = 0.1, 0.2, 0.4$ and the problem parameter $\varrho = 1, 10, 200$, respectively. From these tables, we can easily observe that all these iteration methods can compute an approximate solution of the system of nonlinear equations. In particular, the modified Newton-DSS method remarkably outperforms the modified Newton-MHSS even modified Newton-PMHSS methods from the point of view of number of iterations and CPU time. Here, the number of outer iteration and the total numbers of inner iterations are denoted with Outer IT and Inner IT.

Moreover, we see that the Inner ITs for the MNDSS method almost remain constant with problem size, which means the extensibility as the MNPMHSS method possesses. Actually, the CPU time and Inner IT are both almost half of the MNPMHSS methods. Even though the two methods can deal with the problems more efficiently than the MNMHSS method, the MNDSS method considerably works better than the MNPMHSS method, both from aspects of iteration counts and CPU time.

Table 3 The experimentally optimal values α for MNDSS method

N	$\varrho = 1$			$\varrho = 10$			$\varrho = 200$		
	$\eta = 0.1$	$\eta = 0.2$	$\eta = 0.4$	$\eta = 0.1$	$\eta = 0.2$	$\eta = 0.4$	$\eta = 0.1$	$\eta = 0.2$	$\eta = 0.4$
2^5	1.8	1.8	1.8	2.1	2.0	2.0	4.6	4.7	4.7
2^6	2.3	2.4	2.3	2.0	2.5	2.5	4.8	4.8	4.4
2^7	3.2	3.2	3.2	3.3	3.3	3.3	5.2	5.1	5.2

Table 4 Numerical results for the three methods with $\eta = 0.1,\ N = 2^5$ for Example 1

ϱ	Method	Error	CPU time (s)	Outer IT	Inner IT
1	MNMHSS	7.2790E−07	0.318	3	88
	MNPMHSS	7.2882E−08	0.187	3	24
	MNDSS	3.7506E−07	0.096	3	6
10	MNMHSS	7.4263E−07	0.250	3	84
	MNPMHSS	8.7206E−08	0.107	3	24
	MNDSS	8.0884E−08	0.059	2	8
200	MNMHSS	5.3905E−07	0.192	3	56
	MNPMHSS	1.8192E−07	0.127	3	33
	MNDSS	1.6813E−07	0.098	3	18

Table 5 Numerical results for the three methods with $\eta = 0.2,\ N = 2^5$ for Example 1

ϱ	Method	Error	CPU time (s)	Outer IT	Inner IT
1	MNMHSS	4.9280E−08	0.364	5	109
	MNPMHSS	7.4718E−08	0.149	4	24
	MNDSS	3.7506E−07	0.074	3	6
10	MNMHSS	5.0143E−08	0.343	5	104
	MNPMHSS	9.3472E−08	0.126	4	24
	MNDSS	3.6583E−08	0.088	4	8
200	MNMHSS	7.1307E−07	0.214	4	55
	MNPMHSS	7.0172E−07	0.136	4	30
	MNDSS	9.9273E−07	0.109	4	16

Table 6 Numerical results for the three methods with $\eta = 0.4,\ N = 2^5$ for Example 1

ϱ	Method	Error	CPU time (s)	Outer IT	Inner IT
1	MNMHSS	9.7372E−07	0.348	7	86
	MNPMHSS	7.3461E−08	0.155	6	24
	MNDSS	3.7506E−07	0.070	3	6
10	MNMHSS	9.5369E−07	0.339	7	82
	MNPMHSS	9.2401E−08	0.157	6	24
	MNDSS	3.6583E−08	0.089	4	8
200	MNMHSS	8.4941E−07	0.279	7	54
	MNPMHSS	8.2641E−07	0.169	6	30
	MNDSS	9.9273E−07	0.108	4	16

Table 7 Numerical results for the three methods with $\eta = 0.4,\ N = 2^6$ for Example 1

ϱ	Method	Error	CPU time (s)	Outer IT	Inner IT
1	MNMHSS	9.1057E−07	3.408	7	133
	MNPMHSS	1.1226E−07	1.562	6	24
	MNDSS	5.8268E−07	0.999	4	8
10	MNMHSS	2.3714E−07	3.814	8	144
	MNPMHSS	1.4335E−07	1.556	6	24
	MNDSS	7.3948E−08	0.211	5	10
200	MNMHSS	9.6074E−07	2.997	7	86
	MNPMHSS	7.8701E−07	1.607	6	29
	MNDSS	7.5427E−07	1.288	5	18

Example 2 The second test is for the complex nonlinear Helmholtz equation:

$$-\Delta u + \sigma_1 u + i\sigma_2 u = -e^u,$$

with σ_1 and σ_2 being real coefficient functions. Here, u subjects to homogeneous Dirichlet boundary conditions in the square $\Omega = [0, 1] \times [0, 1]$. We discretize the problem with finite differences on a $N \times N$ grid with mesh size $h = 1/(N+1)$. This leads to a system of nonlinear equations $F(x) = 0$ with following form:

$$F(x) = Mx + \Phi(x) = 0,$$

where

$$\begin{aligned} M &= (K + \sigma_1 I) + i\sigma_2 I, \\ \Phi(x) &= (e^{x_1}, e^{x_2}, \ldots, e^{x_n})^T \end{aligned}$$

Table 8 Numerical results for the three methods with $\eta = 0.4,\ N = 2^7$ for Example 1

ϱ	Method	Error	CPU time (s)	Outer IT	Inner IT
1	MNMHSS	2.3578E−07	180.937	8	222
	MNPMHSS	2.6894E−07	101.909	6	24
	MNDSS	2.3891E−07	54.442	6	12
10	MNMHSS	2.3230E−07	200.675	8	218
	MNPMHSS	3.2313E−07	114.263	6	24
	MNDSS	3.9163E−07	80.279	6	12
200	MNMHSS	1.9379E−07	209.818	8	162
	MNPMHSS	9.0820E−07	100.215	6	28
	MNDSS	2.4539E−07	66.009	5	20

Table 9 The experimentally optimal values α

N	MNMHSS			MNPMHSS			MNDSS		
	$\eta = 0.1$	$\eta = 0.2$	$\eta = 0.4$	$\eta = 0.1$	$\eta = 0.2$	$\eta = 0.4$	$\eta = 0.1$	$\eta = 0.2$	$\eta = 0.4$
30	553	557	557	1.81	1.79	1.79	4.2	4.0	4.1
60	775	781	788	1.26	1.28	1.37	5.4	5.1	5.3
90	899	890	907	1.11	1.12	1.15	8.3	7.7	7.7

Table 10 Numerical results for the three methods with the scale $N = 30$ for Example 2

η	Method	Error	CPU time (s)	Outer IT	Inner IT
0.1	MNMHSS	4.6502E−07	0.217	3	30
	MNPMHSS	4.6503E−07	0.091	3	30
	MNDSS	2.5060E−08	0.071	3	16
0.2	MNMHSS	1.7833E−07	0.122	4	32
	MNPMHSS	1.7834E−07	0.115	4	32
	MNDSS	1.8580E−08	0.091	4	16
0.4	MNMHSS	1.7833E−07	0.188	8	32
	MNPMHSS	1.7834E−07	0.171	8	32
	MNDSS	1.5430E−07	0.112	7	14

Table 11 Numerical results for the three methods with the scale $N = 60$ for Example 2

η	Method	error	CPU time (s)	Outer IT	Inner IT
0.1	MNMHSS	4.4020E−07	0.894	3	31
	MNPMHSS	6.9827E−07	0.765	3	30
	MNDSS	1.9119E−07	0.656	3	19
	MNMHSS	2.7986E−07	1.168	4	32
0.2	MNPMHSS	2.7986E−07	0.917	4	32
	MNDSS	1.7124E−07	0.811	4	20
	MNMHSS	1.7806E−07	2.121	8	33
0.4	MNPMHSS	7.0880E−07	1.359	7	30
	MNDSS	5.8649E−07	0.938	5	17

Table 12 Numerical results for the three methods with the scale $N = 90$ for Example 2

η	Method	Error	CPU time (s)	Outer IT	Inner IT
0.1	MNMHSS	2.8124E−07	4.536	3	33
	MNPMHSS	2.8126E−07	3.395	3	33
	MNDSS	2.2823E−07	3.216	3	28
0.2	MNMHSS	4.3020E−07	5.190	4	32
	MNPMHSS	4.3023E−07	4.115	4	32
	MNDSS	3.1132E−07	3.925	4	27
0.4	MNMHSS	1.8412E−07	8.882	7	34
	MNPMHSS	2.8390E−07	6.399	7	33
	MNDSS	8.9406E−07	5.568	6	24

with the matrix $K \in \mathbb{R}^{n\times n}$ possessing the tensor-product form

$$K = I \otimes B_N + B_N \otimes I \quad \text{and} \quad B_N = \frac{1}{h^2}\text{tridiag}(-1, 2, -1) \in \mathbb{R}^{N\times N}.$$

For the numerical tests, we set $\sigma_1 = 100$ and $\sigma_2 = 1000$. In addition, initial guess is chosen as $x_0 = 0$ and the iteration is terminated once the current x_k satisfies

$$\frac{\|F(u_k)\|}{\|F(u_0)\|} \leq 10^{-6},$$

The prescribed tolerances $\eta_k = \tilde{\eta}_k \equiv \eta = 0.1, 0.2, 0.4$ and the scale of problem $N = 30, 60, 90$, respectively. Now we solve the nonlinear problem by MNMHSS, MNPMHSS and MNDSS methods and show the experimental results. They are compared in elapsed CPU times, the number of outer iterations and inner iterations.

We reselect the experimental optimal parameters α for three iteration methods (see Table 9). The numerical results are displayed in Tables 10, 11, and 12. From these tables, we can find the same conclusion as the previous instance: MNDSS and MNPMHSS are far superior to MNMHSS method while our method performs more efficiently than MNPMHSS method in the sense of CPU time and the number of iterations.

6 Conclusion

In this paper, by utilizing the double-step scale splitting (DSS) iteration method as inner iteration and employing the modified Newton method as outer iteration, we have established a modified Newton-DSS (MNDSS) method for the solution of non-linear complex systems, especially whose Jacobian matrices are large, sparse and complex symmetric. There are many feasible techniques, such as MHSS, PMHSS and some deformed methods for complex symmetric linear systems. Thereinto, DSS method is competitive with the result that the combination of modified Newton method and DSS method, i.e., MNDSS method can work better. We have also proved the local and semilocal convergence theorems of the modified Newton-DSS method. At last, the numerical experiments with experimental choices for parameters demonstrate its effectiveness. Actually, in the whole example section, we find that it does take much time to determine the experimental optimal parameters α. In our future studies, we are to rise to the challenges to the complicated headache, i.e., how to choose the optimal parameters.

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