




A time two-grid algorithm based on finite difference method for the two-dimensional nonlinear time-fractional mobile/immobile transport model

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Abstract

In this paper, we present a time two-grid algorithm based on the finite difference (FD) method for the two-dimensional nonlinear time-fractional mobile/immobile transport model. We establish the problem as a nonlinear fully discrete FD system, where the time derivative is discretized by the second-order backward difference formula (BDF) scheme, the Caputo fractional derivative is treated by means of L1 discretization formula, and the spatial derivative is approximated by the central difference formula. For solving the nonlinear FD system more efficiently, a time two-grid algorithm is proposed, which consists of two steps: first, the nonlinear FD system on a coarse grid is solved by nonlinear iterations; second, the Newton iteration is utilized to solve the linearized FD system on the fine grid. The stability and convergence in L^2 -norm are obtained for the two-grid FD scheme. Numerical results are consistent with the theoretical analysis. Meanwhile, numerical experiments show that the two-grid FD method is much more efficient than the general FD scheme for solving the nonlinear FD system.

Keywords Two-grid algorithm · Two-dimensional · Nonlinear time-fractional mobile/immobile model · Stability and convergence · Numerical experiments

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1 Introduction

Consider a time two-grid finite difference scheme for the two-dimensional nonlinear time-fractional mobile/immobile transport model [17, 22] as follows:

$$u_t + \frac{\partial^\alpha u}{\partial t^\alpha} - \Delta u + g(u) = f(x, y, t), \quad (x, y, t) \in \Omega \times (0, T], \quad (1)$$

the initial condition is as follows:

$$u(x, y, 0) = u^0(x, y), \quad (x, y) \in \bar{\Omega}, \quad (2)$$

and the boundary conditions are as follows:

$$u(x, y, t) = 0, \quad (x, y, t) \in \partial\Omega \times (0, T], \quad (3)$$

where $\alpha \in (0, 1)$, $\Omega = (0, L_1) \times (0, L_2)$ with boundary $\partial\Omega$, $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the two-dimensional Laplacian operator, $f(x, t)$, $u^0(x)$ are given smooth functions and the nonlinear reaction term $g(u)$ satisfies $|g(u)| \leq \bar{C}|u|$, $|g'(u)| \leq \bar{C}$, \bar{C} is a positive constant [18]. In addition, the Caputo fractional derivative (cf. [24]) is defined by the following:

$$\frac{\partial^\alpha}{\partial t^\alpha} u(x, y, t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{1}{(t-s)^\alpha} \frac{\partial u(x, y, s)}{\partial s} ds. \quad (4)$$

Equation (1) is a fractal mobile/immobile transport model that describes a range of problems, including heat diffusion and the propagation of ocean sound, in physical or mathematical systems with time variables that behave essentially like the diffusion of heat through a solid [1, 22].

To approximate the fractional order mobile/immobile transport equation, the significant progress has been made. For instance, Schumer et al. [28] first considered the fractional mobile/immobile transport model. These transfer equations control the long-term limit of the continuous-time random walk model, which implies the probabilistic interpretation of the mobile/immobile convection-diffusion equation. In most cases, the analytical solutions of some complex models are difficult to be solved; thus, it is necessary to use numerical methods to approximate the analytical solutions. Liu et al. [15] presented effective implicit numerical methods for a class of fractional advection-dispersion models. The stability and convergence of the implicit numerical methods were proved. Liu et al. [17] developed a finite difference method and a meshless method for the fractal mobile/immobile transport model with a fractional time derivative. Zhang et al. [35] proposed a novel implicit numerical method for the time variable fractional order mobile/immobile advection-dispersion model. Additionally, Liu et al. [22] developed a second-order finite difference scheme for the quasilinear time-fractional mobile/immobile equation.

Recently, fractional partial differential equations (FPDEs) have developed more and more rapidly and many good numerical methods have followed, such as finite element methods (FEMs) [8, 10, 18, 36], finite difference methods (FDMs) [2–4, 7, 9, 11, 17, 22, 23, 25, 30, 31], finite volume methods (FVMs) [16, 39], discontinuous

Galerkin methods [6, 32], weak Galerkin methods [37, 38], spectral methods [13, 14], and two-grid methods [12, 19–21].

For solving the nonlinear systems, a two-grid algorithm which is first proposed by Xu [33, 34] has attracted much attention due to its high efficiency. First, for space two-grid algorithm based on the FDMs, Dawson and Wheeler [5] considered a two-grid finite difference scheme for nonlinear parabolic equations. They showed superconvergence of the flux and the pressure in certain discrete H^1 - and L^2 -norms. Rui and Liu [27] presented a two-grid block-centered finite difference method for Darcy-Forchheimer flow in porous media. Built on [5], Li and Rui [12] considered the case of fractional order, and established the error estimates on non-uniform rectangular grid. For a space two-grid algorithm established on the FEMs, Liu et al. solved the nonlinear fourth-order reaction-diffusion problem [19] and the nonlinear time-fractional cable equation [20]. The stability and the priori error estimates were proved. Besides, for the time two-grid algorithm, Liu et al. [21] proposed a time two-mesh algorithm combined with the finite element method for a time-fractional water wave model. However, the time two-grid algorithm based on the finite difference method has not been studied yet. Next, we shall consider that.

It is worth mentioning that compared with the two-grid FEMs [19–21], the two-grid FDMs are relatively simple from the point of view of numerical calculation. This means that our method will be favored by a large number of engineering scientists. Furthermore, for the study of the two-grid finite difference method, our article is different from Ref. [12]. Based on the finite difference method, Li and Rui [12] employed a two-grid algorithm for the spatial direction. However, our article uses a two-grid algorithm for the temporal direction. In addition, through numerical schemes and experiments, we can find that our method has the advantages of the simple numerical calculation and the fast solution of nonlinear equations.

The main purpose of this paper is to establish a time two-grid finite difference scheme for the two-dimensional nonlinear time-fractional mobile/immobile transport model. The time derivative u_t is approximated by the second-order BDF scheme, the Caputo fractional derivative is discretized by the L1 discretization formula, and the central difference formula for the spatial derivative. The time two-grid algorithm is divided into two steps. Firstly, on a coarse mesh, we solve the nonlinear system, and the Lagrangian interpolation formula is applied. Secondly, on a fine mesh, the linearized system is solved. The time two-grid finite difference scheme is stable and convergent with convergence order $O(k_F^{2-\alpha} + k_C^{4-2\alpha} + h_1^2 + h_2^2)$, where k_F and k_C are the fine grid time-step size and the coarse grid time-step size, respectively. Numerical results show that our method is more efficient than general finite difference method in terms of the CPU time.

The structure of the remainder of this paper is constructed as follows. In Section 2, some preliminaries are given. Section 3 devotes to the establishment of the two-grid difference scheme. The stability and convergence of the two-grid FD scheme are considered in Section 4. In Section 5, some numerical results are obtained by utilizing the two-grid FD scheme and the general FD scheme, and some comparisons between two methods are presented. The article ends with a brief conclusions section.

2 Preliminaries

First define the time-step size on the fine grid $k = k_F := T/\mathcal{N}$, $t_n := nk_F (0 \leq n \leq \mathcal{N})$ for positive integer \mathcal{N} , the time domain $(0, T]$ is covered by $\Omega_k = \{t_n | 0 \leq n \leq \mathcal{N}\}$. Similarly, the time-step size on the coarse grid is $k_C := T/N$, $t_n := nk_C (0 \leq n \leq N)$ for positive integer N and satisfies $k_C = Mk_F$ for $M \geq 2$. For any grid function $W_1 = \{w^n | 1 \leq n \leq N\}$ on Ω_k , we define the following:

$$\begin{aligned} \delta_t^{(C)} w^{Mn} &:= \frac{1}{k_C} (w^{Mn} - w^{M(n-1)}), \\ D_t^{(C)} w^{Mn} &:= \frac{1}{2k_C} (3w^{Mn} - 4w^{M(n-1)} + w^{M(n-2)}), \end{aligned}$$

and $W_2 = \{w^n | 1 \leq n \leq \mathcal{N}\}$ on Ω_k , we define the following:

$$\begin{aligned} \delta_t^{(F)} w^n &:= \frac{1}{k_F} (w^n - w^{n-1}), & D_t^{(F)} w^n &:= \frac{1}{2k_F} (3w^n - 4w^{n-1} + w^{n-2}), \\ \delta_t w^n &:= \frac{1}{k} (w^n - w^{n-1}), & D_t w^n &:= \frac{1}{2k} (3w^n - 4w^{n-1} + w^{n-2}). \end{aligned}$$

For two positive integers J_x and J_y , let space-step size $h_1 = L_1/J_x$, $h_2 = L_2/J_y$, $h = \max\{h_1, h_2\}$, we arrive at $x_i = ih_1$, $y_j = jh_2$. Denote $\bar{\Omega}_h = \{(x_i, y_j) | 0 \leq i \leq J_x, 0 \leq j \leq J_y\}$ and $\Omega_h = \bar{\Omega}_h \cap \Omega$, $\partial\Omega_h = \Omega_h \cap \partial\Omega$.

Based on the grid function $W_h = \{w_{ij} | 0 \leq i \leq J_x, 0 \leq j \leq J_y\}$ on Ω_h , we denote the following:

$$\delta_x w_{i+\frac{1}{2},j} := \frac{1}{h_1} (w_{i+1,j} - w_{ij}), \quad \delta_x^2 w_{ij} := \frac{1}{h_1} (\delta_x w_{i+\frac{1}{2},j} - \delta_x w_{i-\frac{1}{2},j}),$$

the notations $\delta_y w_{i+\frac{1}{2},j}$, $\delta_y^2 w_{ij}$ are defined similarly and the discrete Laplace operator $\Delta_h w_{ij} := (\delta_x^2 + \delta_y^2)w_{ij}$.

Also, for any grid function $w, v \in \Omega_h$, we define the following:

$$\langle w, v \rangle := h_1 h_2 \sum_{i=1}^{J_x-1} \sum_{j=1}^{J_y-1} w_{ij} v_{ij}, \quad \|w\| := \sqrt{\langle w, w \rangle}, \quad \|w\|_\infty := \max_{\substack{1 \leq i \leq J_x - 1, \\ 1 \leq j \leq J_y - 1}} |w_{ij}|,$$

$$\|\delta_x w\| := \sqrt{h_1 h_2 \sum_{i=0}^{J_x-1} \sum_{j=1}^{J_y-1} (\delta_x w_{i+\frac{1}{2},j})^2}, \quad \|\delta_y w\| := \sqrt{h_1 h_2 \sum_{i=1}^{J_x-1} \sum_{j=0}^{J_y-1} (\delta_y w_{i,j+\frac{1}{2}})^2}.$$

First, we define the grid functions as follows:

$$U_{ij}^n := u(x_j, y_j, t_n), \quad f_{ij}^n := f(x_j, y_j, t_n), \quad (x_i, y_j) \in \bar{\Omega}_h, \quad 0 \leq n \leq \mathcal{N}.$$

Employing the Taylor series expansion with integral remainder, we arrive at the following lemmas.

Lemma 2.1 [4] Suppose $u(x, y, t) \in C_{x,y}^{4,4}([0, L_1] \times [0, L_2])$, then it holds as follows:

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2}(x_i, y_j, t_n) &= \delta_x^2 U_{ij}^n \\ &\quad - \frac{h_1^2}{6} \int_0^1 \left[\frac{\partial^4 u}{\partial x^4}(x_i + zh_1, y_j, t_n) + \frac{\partial^4 u}{\partial x^4}(x_i - zh_1, y_j, t_n) \right] (1-z)^3 dz, \\ \frac{\partial^2 u}{\partial y^2}(x_i, y_j, t_n) &= \delta_y^2 U_{ij}^n \\ &\quad - \frac{h_2^2}{6} \int_0^1 \left[\frac{\partial^4 u}{\partial y^4}(x_i, y_j + zh_2, t_n) + \frac{\partial^4 u}{\partial y^4}(x_i, y_j - zh_2, t_n) \right] (1-z)^3 dz. \end{aligned}$$

Lemma 2.2 [2, 3] Assume $V(t) \in C^3[t_{n-2}, t_n]$, then we arrive at the following:

$$\begin{aligned} V'(t_n) &- \frac{1}{2k}[3V(t_n) - 4V(t_{n-1}) + V(t_{n-2})] \\ &= -k^2 \int_0^1 [V'''(t_n - kz) - 2V'''(t_n - 2kz)](1-z)^2 dz, \quad n \geq 2, \end{aligned}$$

and

$$V'(t_1) - \frac{1}{k}[V(t_1) - V(t_0)] = k \int_0^1 V''(t_0 + kz)z dz.$$

Lemma 2.3 [26, 30] Assume $V(t) \in C^2[0, t_n]$, it holds as follows:

$$\begin{aligned} \left| \frac{1}{\Gamma(1-\alpha)} \int_0^{t_n} \frac{V'(s)}{(t_n-s)^\alpha} ds - \frac{k^{-1}}{\Gamma(1-\alpha)} \Delta_t^\alpha V(t_n) \right| &\leq \bar{C}k^{2-\alpha}, \quad 0 < \alpha < 1, \\ \Delta_t^\alpha V(t_n) &= a_0 V(t_n) - \sum_{i=1}^{n-1} (a_{n-i-1} - a_{n-i})V(t_i) - a_{n-1}V(t_0), \end{aligned}$$

where $a_l = \int_{t_l}^{t_l+h} z^{-\alpha} dz = \frac{k^{1-\alpha}}{1-\alpha} [(l+1)^{1-\alpha} - l^{1-\alpha}]$, $l \geq 0$.

Remark In this article, \bar{C} denotes a generic positive constant, not necessarily the same at different occurrences, which is independent on the time-step size and the space-step size.

3 Establishment of the two-grid difference scheme

Below, we first formulate the general finite difference scheme for nonlinear problem (1)–(3).

Considering (1) at the point (x_i, y_j, t_n) and using Lemmas 2.1–2.3, we have the following:

$$\delta_t U_{ij}^1 + \frac{k^{-1}}{\Gamma(1-\alpha)} \Delta_t^\alpha U_{ij}^1 - \Delta_h U_{ij}^1 + g(U_{ij}^1) = f_{ij}^1 + (R_1)_{ij}^1, \quad (x_i, y_j) \in \Omega_h, \quad (5)$$

$$D_t U_{ij}^n + \frac{k^{-1}}{\Gamma(1-\alpha)} \Delta_r^\alpha U_{ij}^n - \Delta_h U_{ij}^n + g(U_{ij}^n) = f_{ij}^n + (R_1)_{ij}^n, \quad (6)$$

$$(x_i, y_j) \in \Omega_h, \quad 2 \leq n \leq \mathcal{N},$$

$$U_{ij}^n = 0, \quad (x_i, y_j) \in \partial\Omega_h, \quad 1 \leq n \leq \mathcal{N}, \quad (7)$$

$$U_{ij}^0 = u^0(x_i, y_j), \quad (x_i, y_j) \in \Omega_h, \quad (8)$$

where

$$(R_1)_{ij}^1 = \bar{C}(k + h_1^2 + h_2^2),$$

$$(R_1)_{ij}^n = \bar{C}(k^{2-\alpha} + h_1^2 + h_2^2), \quad 2 \leq n \leq \mathcal{N}. \quad (9)$$

Omitting the truncation errors $(R_1)_{ij}^n$, $1 \leq n \leq \mathcal{N}$, and replacing the function U_{ij}^n with its numerical approximation u_{ij}^n , we construct the following general finite difference scheme as follows:

$$\delta_t u_{ij}^1 + \frac{k^{-1}}{\Gamma(1-\alpha)} \Delta_r^\alpha u_{ij}^1 - \Delta_h u_{ij}^1 + g(u_{ij}^1) = f_{ij}^1, \quad (x_i, y_j) \in \Omega_h, \quad (10)$$

$$D_t u_{ij}^n + \frac{k^{-1}}{\Gamma(1-\alpha)} \Delta_r^\alpha u_{ij}^n - \Delta_h u_{ij}^n + g(u_{ij}^n) = f_{ij}^n, \quad (11)$$

$$(x_i, y_j) \in \Omega_h, \quad 2 \leq n \leq \mathcal{N},$$

$$u_{ij}^n = 0, \quad (x_i, y_j) \in \partial\Omega_h, \quad 1 \leq n \leq \mathcal{N}, \quad (12)$$

$$u_{ij}^0 = u^0(x_i, y_j), \quad (x_i, y_j) \in \Omega_h. \quad (13)$$

Now, we derive the two-grid difference scheme for problem (1)–(3). The process is divided into three steps as follows.

Step I. On the coarse grid, noting that $n = 0, 1, \dots, M, M + 1, \dots, 2M, \dots, NM = \mathcal{N}$, we only calculate (pM) th level, $1 \leq p \leq N$. Similar to the establishment of the general finite difference scheme, the coarse grid difference equations are obtained by the following:

$$\delta_t^{(C)}(u_C)_{ij}^M + \frac{k_C^{-1}}{\Gamma(1-\alpha)} \Delta_r^\alpha (u_C)_{ij}^M - \Delta_h (u_C)_{ij}^M + g((u_C)_{ij}^M) = f_{ij}^M, \quad (14)$$

$$(x_i, y_j) \in \Omega_h,$$

$$D_t^{(C)}(u_C)_{ij}^{pM} + \frac{k_C^{-1}}{\Gamma(1-\alpha)} \Delta_r^\alpha (u_C)_{ij}^{pM} - \Delta_h (u_C)_{ij}^{pM} + g((u_C)_{ij}^{pM}) = f_{ij}^{pM}, \quad (15)$$

$$(x_i, y_j) \in \Omega_h, \quad 2 \leq p \leq N.$$

Step II. Then based on u_C^{pM} obtained in the step I, $1 \leq p \leq N$, we utilize the Lagrange linear interpolation to compute the value of $u_C^{(p-1)M+q}$ on the coarse grid, where $1 \leq q \leq M - 1$. In the time direction, employing the Lagrange interpola-

tion formula between the points $(t_{(p-1)M}, u_C^{(p-1)M})$ and (t_{pM}, u_C^{pM}) , we arrive at the following:

$$\begin{aligned} \mathcal{L}_{u_C}(t_{(p-1)M+q}) &= u_C^{(p-1)M+q} \\ &= \frac{t_{(p-1)M+q} - t_{pM}}{t_{(p-1)M} - t_{pM}} u_C^{(p-1)M} + \frac{t_{(p-1)M+q} - t_{(p-1)M}}{t_{pM} - t_{(p-1)M}} u_C^{pM} \\ &= \left(1 - \frac{q}{M}\right) u_C^{(p-1)M} + \frac{q}{M} u_C^{pM}, \quad 1 \leq p \leq N, \quad 1 \leq q \leq M - 1. \end{aligned} \tag{16}$$

Step III. Finally, based on u_C^n in the step II on the time coarse mesh k_C , the following linear fine grid difference equations on the time fine mesh k_F are given by the following:

$$\begin{aligned} \delta_t^{(F)}(u_F)_{ij}^1 + \frac{k_F^{-1}}{\Gamma(1-\alpha)} \Delta_t^\alpha (u_F)_{ij}^1 - \Delta_h(u_F)_{ij}^1 + g((u_C)_{ij}^1) \\ + g'((u_C)_{ij}^1)[(u_F)_{ij}^1 - (u_C)_{ij}^1] = f_{ij}^1, \quad (x_i, y_j) \in \Omega_h, \end{aligned} \tag{17}$$

$$\begin{aligned} D_t^{(F)}(u_F)_{ij}^n + \frac{k_F^{-1}}{\Gamma(1-\alpha)} \Delta_t^\alpha (u_F)_{ij}^n - \Delta_h(u_F)_{ij}^n + g((u_C)_{ij}^n) \\ + g'((u_C)_{ij}^n)[(u_F)_{ij}^n - (u_C)_{ij}^n] = f_{ij}^n, \quad (x_i, y_j) \in \Omega_h, \quad 2 \leq n \leq \mathcal{N}. \end{aligned} \tag{18}$$

4 Analysis of the two-grid difference scheme

Below, the stability and convergence of the two-grid difference scheme will be derived. For the demand of analysis, we introduce some lemmas as follows:

Lemma 4.1 [4] *For any grid functions $w, v \in W_h$, we obtain the follows:*

$$\begin{aligned} \langle \delta_x^2 w, v \rangle &= -h_1 h_2 \sum_{i=0}^{J_x-1} \sum_{j=1}^{J_y-1} (\delta_x w_{i+\frac{1}{2},j}) (\delta_x v_{i+\frac{1}{2},j}), \\ \langle \delta_y^2 w, v \rangle &= -h_1 h_2 \sum_{i=1}^{J_x-1} \sum_{j=0}^{J_y-1} (\delta_y w_{i,j+\frac{1}{2}}) (\delta_y v_{i,j+\frac{1}{2}}). \end{aligned}$$

Lemma 4.2 [2, 3] *For any $w^n \in W_h$, $1 \leq n \leq N$, it follows that*

$$k \langle \delta_t w^1, w^1 \rangle + k \sum_{n=2}^N \langle D_t w^n, w^n \rangle \geq \frac{3}{4} \|w^N\|^2 - \frac{1}{4} (\|w^{N-1}\|^2 + \|w^1\|^2 + \|w^0\|^2).$$

Lemma 4.3 [30] For any $\tilde{G} = \{\tilde{G}_1, \tilde{G}_2, \tilde{G}_3, \dots\}$ and \tilde{Q} , we have the following:

$$\begin{aligned} & \sum_{n=1}^N \left[a_0 \tilde{G}_n - \sum_{i=1}^{n-1} (a_{n-i-1} - a_{n-i}) \tilde{G}_i - a_{n-1} \tilde{Q} \right] \tilde{G}_n \\ & \geq \frac{T^{-\alpha}}{2} k_\gamma \sum_{n=1}^N \tilde{G}_n^2 - \frac{T^{1-\alpha}}{2(1-\alpha)} \tilde{Q}^2, \quad N \geq 1, \end{aligned}$$

where $0 < \alpha < 1$, $a_l = \frac{k_\gamma^{1-\alpha}}{1-\alpha} [(l+1)^{1-\alpha} - l^{1-\alpha}]$, $l \geq 0$, $\gamma = C$ or F .

Lemma 4.4 [29] (Discrete Gronwall’s inequality) If Z_m is a non-negative real sequence and satisfies such as the follow:

$$Z_m \leq \tilde{A}_m + \sum_{s=0}^{m-1} \tilde{B}_s Z_s, \quad m \geq 0,$$

where \tilde{A}_m is a non-descending and non-negative sequence, $\tilde{B}_s \geq 0$, then, we arrive at the following:

$$Z_m \leq \tilde{A}_m \exp \left(\sum_{s=0}^{m-1} \tilde{B}_s \right), \quad m \geq 0.$$

4.1 Stability

The energy method is applied to establish the stability of the two-grid difference scheme. First, we derive the stability of coarse grid system.

Theorem 4.5 For the coarse grid system (14) and (15), for $1 \leq n \leq \mathcal{N}$ and $k_C = Mk_F$, the following inequality holds:

$$\|u_C^n\| \leq \tilde{C} \left(\|u_C^0\| + k_C \sum_{l=1}^N \|f^{lM}\| \right).$$

Proof The above theorem will be proved by two steps.

(I). In the first step, we estimate the norm $\|u_C^{pM}\|$ for $p = 1, 2, \dots, N$. Taking the inner products of (14) and (15) with $k_C u_C^M$ and $k_C u_C^{pM}$, respectively, we arrive at the following:

$$\begin{aligned} & k_C \langle \delta_t^{(C)} u_C^M, u_C^M \rangle + \frac{1}{\Gamma(1-\alpha)} \langle \Delta_t^\alpha u_C^M, u_C^M \rangle - k_C \langle \Delta_h u_C^M, u_C^M \rangle \\ & + k_C \langle g(u_C^M), u_C^M \rangle = k_C \langle f^M, u_C^M \rangle. \end{aligned} \tag{19}$$

$$\begin{aligned} & k_C \langle D_t^{(C)} u_C^{pM}, u_C^{pM} \rangle + \frac{1}{\Gamma(1-\alpha)} \langle \Delta_t^\alpha u_C^{pM}, u_C^{pM} \rangle - k_C \langle \Delta_h u_C^{pM}, u_C^{pM} \rangle \\ & + k_C \langle g(u_C^{pM}), u_C^{pM} \rangle = k_C \langle f^{pM}, u_C^{pM} \rangle, \quad 2 \leq p \leq N. \end{aligned} \tag{20}$$

Then, for $N \geq 1$, we readily obtain the following:

$$\begin{aligned}
 &k_C \langle \delta_t^{(C)} u_C^M, u_C^M \rangle + k_C \sum_{p=2}^N \langle D_t^{(C)} u_C^{pM}, u_C^{pM} \rangle + \frac{1}{\Gamma(1-\alpha)} \sum_{p=1}^N \langle \Delta_t^\alpha u_C^{pM}, u_C^{pM} \rangle \\
 &- k_C \sum_{p=1}^N \langle \Delta_h u_C^{pM}, u_C^{pM} \rangle + k_C \sum_{p=1}^N \langle g(u_C^{pM}), u_C^{pM} \rangle = k_C \sum_{p=1}^N \langle f^{pM}, u_C^{pM} \rangle.
 \end{aligned} \tag{21}$$

Next, we shall estimate each term of (21). Firstly, from Lemmas 4.1–4.3 and using the Cauchy-Schwarz inequality, we have the following:

$$\begin{aligned}
 &k_C \langle \delta_t^{(C)} u_C^{pM}, u_C^{pM} \rangle + k_C \sum_{p=2}^N \langle D_t^{(C)} u_C^{pM}, u_C^{pM} \rangle \\
 &\geq \frac{3}{4} \|u_C^{NM}\|^2 - \frac{1}{4} (\|u_C^{(N-1)M}\|^2 + \|u_C^M\|^2 + \|u_C^0\|^2),
 \end{aligned} \tag{22}$$

$$\frac{1}{\Gamma(1-\alpha)} \sum_{p=1}^N \langle \Delta_t^\alpha u_C^{pM}, u_C^{pM} \rangle \geq \frac{T^{-\alpha}}{2\Gamma(1-\alpha)} k_C \sum_{p=1}^N \|u_C^{pM}\|^2 - \frac{T^{1-\alpha}}{2\Gamma(2-\alpha)} \|u_C^0\|^2, \tag{23}$$

$$-k_C \sum_{p=1}^N \langle \Delta_h u_C^{pM}, u_C^{pM} \rangle = -k_C \sum_{p=1}^N (\|\delta_x u_C^{pM}\|^2 + \|\delta_y u_C^{pM}\|^2) \leq 0. \tag{24}$$

In addition, we also have the following:

$$\langle g(u_C^{pM}), u_C^{pM} \rangle \leq \|g(u_C^{pM})\| \|u_C^{pM}\| \leq \bar{C} \|u_C^{pM}\|^2, \tag{25}$$

and

$$\langle f^{pM}, u_C^{pM} \rangle \leq \|f^{pM}\| \|u_C^{pM}\|. \tag{26}$$

Substituting (22)–(26) into (21), then it holds that as follows:

$$\begin{aligned}
 &\frac{3}{4} \|u_C^{NM}\|^2 - \frac{1}{4} (\|u_C^{(N-1)M}\|^2 + \|u_C^M\|^2 + \|u_C^0\|^2) - \frac{T^{1-\alpha}}{2\Gamma(2-\alpha)} \|u_C^0\|^2 \\
 &\leq \bar{C} k_C \sum_{p=1}^N \|u_C^{pM}\|^2 + k_C \sum_{p=1}^N \|f^{pM}\| \|u_C^{pM}\|.
 \end{aligned} \tag{27}$$

Taking the appropriate \tilde{s} such that $\|u_C^{\tilde{s}M}\| = \max_{0 \leq p \leq N} \|u_C^{pM}\|$, we obtain the following:

$$\begin{aligned}
 &\frac{1}{2} \|u_C^{NM}\| \leq \frac{1}{2} \|u_C^{\tilde{s}M}\| \leq \frac{1}{4} (\|u_C^M\| + \|u_C^0\|) \\
 &+ \frac{T^{1-\alpha}}{2\Gamma(2-\alpha)} \|u_C^0\| + \bar{C} k_C \sum_{p=1}^N \|u_C^{pM}\| + k_C \sum_{p=1}^N \|f^{pM}\|.
 \end{aligned} \tag{28}$$

When $k_C \leq 1/(4\bar{C})$, following from Lemma 4.4, inequality (28) becomes the following:

$$\begin{aligned}
 &\|u_C^{NM}\| \leq \bar{C}(T) \exp\{Nk_C\} \left(\|u_C^M\| + \|u_C^0\| + k_C \sum_{p=1}^N \|f^{pM}\| \right) \\
 &\leq \bar{C} \left(\|u_C^M\| + \|u_C^0\| + k_C \sum_{p=1}^N \|f^{pM}\| \right).
 \end{aligned} \tag{29}$$

Similarly, using (19), we have the following:

$$\|u_C^M\| \leq \bar{C}(T)(\|u_C^0\| + k_C \|f^M\|). \tag{30}$$

Thus, for any $1 \leq p \leq N$, combining (29) with (30), we obtain the following:

$$\|u_C^{pM}\| \leq \bar{C} \left(\|u_C^0\| + k_C \sum_{l=1}^N \|f^{lM}\| \right). \tag{31}$$

(II). In the second step, we estimate the norm $\|u_C^{(p-1)M+q}\|$ for $1 \leq p \leq N$ and $1 \leq q \leq M - 1$.

Noting that (16), it follows that

$$\begin{aligned} \|u_C^{(p-1)M+q}\| &= \left\| \left(1 - \frac{q}{M}\right) u_C^{(p-1)M} + \frac{q}{M} u_C^{pM} \right\| \\ &\leq \left(1 - \frac{q}{M}\right) \|u_C^{(p-1)M}\| + \frac{q}{M} \|u_C^{pM}\| \leq \bar{C} \left(\|u_C^0\| + k_C \sum_{l=1}^N \|f^{lM}\| \right). \end{aligned} \tag{32}$$

Using (31) and (32), we finish the proof.

Below, we construct the stability of fine grid system. □

Theorem 4.6 *For the fine grid system (17) and (18), for $1 \leq n \leq \mathcal{N}$, it holds the following:*

$$\|u_F^n\| \leq \bar{C}(\|u_C^0\| + \|u_F^0\| + \max_{1 \leq l \leq \mathcal{N}} \|f^l\|).$$

Proof We take the inner products of (17) and (18) with $k_F u_F^1$ and $k_F u_F^n$, respectively, then,

$$\begin{aligned} k_F \langle \delta_t^{(F)} u_F^1, u_F^1 \rangle + \frac{1}{\Gamma(1-\alpha)} \langle \Delta_t^\alpha u_F^1, u_F^1 \rangle - k_F \langle \Delta_h u_F^1, u_F^1 \rangle \\ + k_F \langle g(u_C^1) + g'(u_C^1)(u_F^1 - u_C^1), u_F^1 \rangle = k_F \langle f^1, u_F^1 \rangle. \end{aligned} \tag{33}$$

$$\begin{aligned} k_F \langle D_t^{(F)} u_F^n, u_F^n \rangle + \frac{1}{\Gamma(1-\alpha)} \langle \Delta_t^\alpha u_F^n, u_F^n \rangle - k_F \langle \Delta_h u_F^n, u_F^n \rangle \\ + k_F \langle g(u_C^n) + g'(u_C^n)(u_F^n - u_C^n), u_F^n \rangle = k_F \langle f^n, u_F^n \rangle, \quad 2 \leq n \leq \mathcal{N}. \end{aligned} \tag{34}$$

Then, summing for n from 2 to \mathcal{N} and adding (33), we arrive at the following:

$$\begin{aligned} k_F \langle \delta_t^{(F)} u_F^1, u_F^1 \rangle + k_F \sum_{n=2}^{\mathcal{N}} \langle D_t^{(F)} u_F^n, u_F^n \rangle \\ + \frac{1}{\Gamma(1-\alpha)} \sum_{n=1}^{\mathcal{N}} \langle \Delta_t^\alpha u_F^n, u_F^n \rangle - k_F \sum_{n=1}^{\mathcal{N}} \langle \Delta_h u_F^n, u_F^n \rangle \\ = -k_F \sum_{n=1}^{\mathcal{N}} \langle g(u_C^n) + g'(u_C^n)(u_F^n - u_C^n), u_F^n \rangle + k_F \sum_{n=1}^{\mathcal{N}} \langle f^n, u_F^n \rangle. \end{aligned} \tag{35}$$

Analogous to the process of (22)–(24), using Lemmas 4.1–4.3 and the Cauchy-Schwarz inequality, we obtain the following:

$$\begin{aligned} \frac{3}{4} \|u_F^{\mathcal{N}}\|^2 - \frac{1}{4} (\|u_F^{\mathcal{N}-1}\|^2 + \|u_F^1\|^2 + \|u_F^0\|^2) - \frac{T^{1-\alpha}}{2\Gamma(2-\alpha)} \|u_F^0\|^2 \\ \leq -k_F \sum_{n=1}^{\mathcal{N}} \langle g(u_C^n) + g'(u_C^n)(u_F^n - u_C^n), u_F^n \rangle + k_F \sum_{n=1}^{\mathcal{N}} \langle f^n, u_F^n \rangle \\ \leq k_F \sum_{n=1}^{\mathcal{N}} |\langle g(u_C^n) + g'(u_C^n)(u_F^n - u_C^n), u_F^n \rangle| + k_F \sum_{n=1}^{\mathcal{N}} \|f^n\| \|u_F^n\|, \end{aligned} \tag{36}$$

moreover, we have the following:

$$\begin{aligned}
 & | \langle g(u_C^n) + g'(u_C^n)(u_F^n - u_C^n), u_F^n \rangle | \leq \bar{C} (\|u_C^n\| \|u_F^n\|) \\
 & + \|g'(u_C^n)\|_\infty (\|u_F^n\| + \|u_C^n\|) \|u_F^n\| \leq \bar{C} (\|u_C^n\| \|u_F^n\| + \|u_F^n\|^2).
 \end{aligned}
 \tag{37}$$

Choosing the suitable m such that $\|u_F^m\| = \max_{0 \leq n \leq \mathcal{N}} \|u_F^n\|$, and substituting (37) into (36), we get the following:

$$\begin{aligned}
 \frac{1}{2} \|u_F^{\mathcal{N}}\| & \leq \frac{1}{2} \|u_F^m\| \leq \frac{1}{4} (\|u_F^1\| + \|u_F^0\|) + \frac{T^{1-\alpha}}{2\Gamma(2-\alpha)} \|u_F^0\| \\
 & + \bar{C} k_F \sum_{n=1}^{\mathcal{N}} (\|u_F^n\| + \|u_C^n\|) + k_F \sum_{n=1}^{\mathcal{N}} \|f^n\|.
 \end{aligned}
 \tag{38}$$

When $k_F \leq 1/(6\bar{C})$, from Lemma 4.4, inequality (38) turns into the following:

$$\begin{aligned}
 \|u_F^{\mathcal{N}}\| & \leq \bar{C}(T) \exp\{\mathcal{N}k_F\} \left(\|u_F^1\| + \|u_F^0\| + k_F \sum_{n=1}^{\mathcal{N}} \|u_C^n\| + k_F \sum_{n=1}^{\mathcal{N}} \|f^n\| \right) \\
 & \leq \bar{C} \left(\|u_F^1\| + \|u_F^0\| + k_F \sum_{n=1}^{\mathcal{N}} \|u_C^n\| + k_F \sum_{p=1}^{\mathcal{N}} \|f^p\| \right).
 \end{aligned}
 \tag{39}$$

Also, following from (33), we get the following:

$$\|u_F^1\| \leq \bar{C} (\|u_F^0\| + k_F \|u_C^1\| + k_F \|f^1\|).
 \tag{40}$$

Combining the above two inequalities and employing Theorem 4.5, it follows that

$$\|u_F^{\mathcal{N}}\| \leq \bar{C} \left(\|u_C^0\| + \|u_F^0\| + k_C \sum_{l=1}^{\mathcal{N}} \|f^{lM}\| + k_F \sum_{n=1}^{\mathcal{N}} \|f^n\| \right).
 \tag{41}$$

This completes the proof. □

4.2 Convergence

The convergence of the two-grid difference scheme is considered by the energy method. Firstly, we shall derive the convergence of the coarse grid system as follows.

Let,

$$(e_C)_{ij}^n = U_{ij}^n - (u_C)_{ij}^n, \quad (x_i, y_j) \in \bar{\Omega}_h, \quad 0 \leq n \leq \mathcal{N}.$$

Subtracting (14)–(15), (12)–(13) from (5)–(8), respectively, we get the following error equations:

$$\begin{aligned}
 \delta_t^{(C)} (e_C)_{ij}^M + \frac{k_C^{-1}}{\Gamma(1-\alpha)} \Delta_t^\alpha (e_C)_{ij}^M - \Delta_h (e_C)_{ij}^M + g(U_{ij}^M) - g((u_C)_{ij}^M) \\
 = (R_1)_{ij}^M, \quad (x_i, y_j) \in \Omega_h,
 \end{aligned}
 \tag{42}$$

$$\begin{aligned}
 D_t^{(C)} (e_C)_{ij}^{pM} + \frac{k_C^{-1}}{\Gamma(1-\alpha)} \Delta_t^\alpha (e_C)_{ij}^{pM} - \Delta_h (e_C)_{ij}^{pM} + g(U_{ij}^{pM}) - g((u_C)_{ij}^{pM}) \\
 = (R_1)_{ij}^{pM}, \quad (x_i, y_j) \in \Omega_h, \quad 2 \leq p \leq N,
 \end{aligned}
 \tag{43}$$

$$(e_C)_{ij}^n = 0, \quad (x_i, y_j) \in \partial\Omega_h, \quad 1 \leq n \leq \mathcal{N},
 \tag{44}$$

$$(e_C)_{ij}^0 = 0, \quad (x_i, y_j) \in \Omega_h.
 \tag{45}$$

Theorem 4.7 Suppose $u(x, y, t) \in C_{x,y,t}^{4,4,3}([0, L_1] \times [0, L_2] \times (0, T])$ be the solution of (5)–(6) and u_C^n be the solution of (14)–(16), for $1 \leq n \leq \mathcal{N}$, it holds the following:

$$\max_{1 \leq n \leq \mathcal{N}} \|U^n - u_C^n\| \leq \bar{C}(k_C^{2-\alpha} + h_1^2 + h_2^2).$$

Proof The proof consists of two steps:

(I). Taking the inner products of (42) and (43) with $k_C e_C^M$ and $k_C e_C^{pM}$, respectively, summing up for p from 2 to N with (43) and adding (42), then we obtain the following:

$$\begin{aligned} & k_C \langle \delta_t^{(C)} e_C^M, e_C^M \rangle + k_C \sum_{p=2}^N \langle D_t^{(C)} e_C^{pM}, e_C^{pM} \rangle + \frac{1}{\Gamma(1-\alpha)} \sum_{p=1}^N \langle \Delta_t^\alpha e_C^{pM}, e_C^{pM} \rangle \\ & - k_C \sum_{p=1}^N \langle \Delta_h e_C^{pM}, e_C^{pM} \rangle = -k_C \sum_{p=1}^N \langle g(U^{pM}) - g(u_C^{pM}), e_C^{pM} \rangle \\ & + k_C \sum_{p=1}^N \langle R_1^{pM}, e_C^{pM} \rangle, \quad 1 \leq p \leq N. \end{aligned} \tag{46}$$

Using Lemmas 4.1–4.3 and the Cauchy-Schwarz inequality, we arrive at the following:

$$\begin{aligned} & \frac{3}{4} \|e_C^{NM}\|^2 - \frac{1}{4} (\|e_C^{(N-1)M}\|^2 + \|e_C^M\|^2 + \|e_C^0\|^2) - \frac{T^{1-\alpha}}{2\Gamma(2-\alpha)} \|e_C^0\|^2 \\ & \leq k_C \sum_{p=1}^N \|g(U^{pM}) - g(u_C^{pM})\| \|e_C^{pM}\| + k_C \sum_{p=1}^N \|R_1^{pM}\| \|e_C^{pM}\| \\ & \leq \bar{C} k_C \sum_{p=1}^N \|e_C^{pM}\|^2 + k_C \sum_{p=1}^N \|R_1^{pM}\| \|e_C^{pM}\|. \end{aligned} \tag{47}$$

Taking a suitable \tilde{l} such that $\|e_C^{\tilde{l}M}\| = \max_{0 \leq p \leq N} \|e_C^{pM}\|$, then it holds the following:

$$\begin{aligned} \frac{1}{2} \|e_C^{NM}\| & \leq \frac{1}{2} \|e_C^{\tilde{l}M}\| \leq \frac{1}{4} (\|e_C^M\| + \|e_C^0\|) \\ & + \frac{T^{1-\alpha}}{2\Gamma(2-\alpha)} \|e_C^0\| + \bar{C} k_C \sum_{p=1}^N \|e_C^{pM}\| + k_C \sum_{p=1}^N \|R_1^{pM}\|. \end{aligned} \tag{48}$$

Utilizing Lemma 4.4 and noting that (45), we obtain the following:

$$\begin{aligned} \|e_C^{NM}\| & \leq \bar{C}(T) \exp\{\text{Nk}_C\} \left(\|e_C^M\| + \|e_C^0\| + k_C \sum_{p=1}^N \|R_1^{pM}\| \right) \\ & \leq \bar{C} \left(\|e_C^M\| + \|e_C^0\| + k_C \|R_1^M\| + k_C \sum_{p=2}^N \|R_1^{pM}\| \right) \\ & \leq \bar{C}(T) (\|e_C^M\| + k_C^2 + k_C^{2-\alpha} + h_1^2 + h_2^2). \end{aligned} \tag{49}$$

From (42), then we have the following:

$$\|e_C^M\| \leq \bar{C}(T) (\|e_C^0\| + k_C \|R_1^M\|) \leq \bar{C}(k_C^2 + h_1^2 + h_2^2). \tag{50}$$

Utilizing (49) and (50), it follows that

$$\|e_C^{pM}\| \leq \bar{C}(k_C^{2-\alpha} + h_1^2 + h_2^2), \quad 1 \leq p \leq N. \tag{51}$$

(II). For $1 \leq p \leq N$ and $1 \leq q \leq M - 1$, we use the Lagrange’s interpolation formula, then,

$$U^{(p-1)M+q} = (1 - \frac{q}{M})U^{(p-1)M} + \frac{q}{M}U^{pM} + \frac{U''(\xi)}{2}(t_{(p-1)M+q} - t_{(p-1)M})(t_{(p-1)M+q} - t_{pM}). \tag{52}$$

Subtracting (15) from (52), we have the following:

$$e_C^{(p-1)M+q} = (1 - \frac{q}{M})e_C^{(p-1)M} + \frac{q}{M}e_C^{pM} + \frac{U''(\xi)}{2}(t_{(p-1)M+q} - t_{(p-1)M})(t_{(p-1)M+q} - t_{pM}),$$

then, using the triangle inequality and (51), we get the following:

$$\|e_C^{(p-1)M+q}\| \leq (1 - \frac{q}{M})\|e_C^{(p-1)M}\| + \frac{q}{M}\|e_C^{pM}\| + \frac{\|U''(\xi)\|_\infty}{2}k_C^2 \leq \bar{C}(k_C^{2-\alpha} + h_1^2 + h_2^2), \quad 1 \leq p \leq N, \quad 1 \leq q \leq M - 1. \tag{53}$$

Using (51) and (53), the proof is completed.

Secondly, the convergence of the fine grid system will be derived.

Let,

$$(e_F)_{ij}^n = U_{ij}^n - (u_F)_{ij}^n, \quad (x_i, y_j) \in \bar{\Omega}_h, \quad 0 \leq n \leq \mathcal{N}.$$

Subtracting (17)–(18), (12)–(13) from (5)–(8), respectively, the error equations are obtained by the following:

$$\delta_t^{(F)}(e_F)_{ij}^1 + \frac{k_F^{-1}}{\Gamma(1-\alpha)}\Delta_t^\alpha(e_F)_{ij}^1 - \Delta_h(e_F)_{ij}^1 + g(U_{ij}^1) - g((u_C)_{ij}^1) - g'((u_C)_{ij}^1)[((u_F)_{ij}^1) - ((u_C)_{ij}^1)] = (R_1)_{ij}^1, \quad (x_i, y_j) \in \Omega_h, \tag{54}$$

$$D_t^{(F)}(e_F)_{ij}^n + \frac{k_F^{-1}}{\Gamma(1-\alpha)}\Delta_t^\alpha(e_F)_{ij}^n - \Delta_h(e_F)_{ij}^n + g(U_{ij}^n) - g((u_C)_{ij}^n) - g'((u_C)_{ij}^n)[((u_F)_{ij}^n) - ((u_C)_{ij}^n)] = (R_1)_{ij}^n, \tag{55}$$

$$(x_i, y_j) \in \Omega_h, \quad 2 \leq n \leq \mathcal{N},$$

$$(e_F)_{ij}^n = 0, \quad (x_i, y_j) \in \partial\Omega_h, \quad 1 \leq n \leq \mathcal{N}, \tag{56}$$

$$(e_F)_{ij}^0 = 0, \quad (x_i, y_j) \in \Omega_h. \tag{57}$$

□

Theorem 4.8 Let $u(x, y, t) \in C_{x,y,t}^{4,4,3}([0, L_1] \times [0, L_2] \times (0, T])$ be the solution of (5)–(6) and u_F^n be the solution of (17)–(18), for $1 \leq n \leq \mathcal{N}$, we have the following:

$$\max_{1 \leq n \leq \mathcal{N}} \|U^n - u_F^n\| \leq \bar{C}(k_F^{2-\alpha} + k_C^{4-2\alpha} + h_1^2 + h_2^2).$$

Proof We take the inner products of (42) and (43) with $k_F e_F^1$ and $k_F e_F^n$, respectively, for $\mathcal{N} \geq 1$, then we arrive at the following:

$$\begin{aligned} &k_F \langle \delta_t^{(F)} e_F^1, e_F^1 \rangle + k_F \sum_{n=2}^{\mathcal{N}} \langle D_t^{(F)} e_F^n, e_C^n \rangle \\ &+ \frac{1}{\Gamma(1-\alpha)} \sum_{n=1}^{\mathcal{N}} \langle \Delta_t^\alpha e_F^n, e_F^n \rangle - k_F \sum_{n=1}^{\mathcal{N}} \langle \Delta_h e_F^n, e_F^n \rangle \\ &= -k_F \sum_{n=1}^{\mathcal{N}} \langle g(U^n) - g(u_C^n) - g'(u_C^n)(u_F^n - u_C^n), e_F^n \rangle + k_F \sum_{n=1}^{\mathcal{N}} \langle R_1^n, e_F^n \rangle. \end{aligned} \tag{58}$$

Noticing that

$$\begin{aligned} &g(U^n) - g(u_C^n) - g'(u_C^n)(u_F^n - u_C^n) \\ &= g'(u_C^n)(U^n - u_C^n) + \frac{1}{2}g''(\theta^n)(U^n - u_C^n)^2 - g'(u_C^n)(u_F^n - u_C^n) \\ &= g'(u_C^n)e_F^n + \frac{1}{2}g''(\theta^n)(e_C^n)^2, \end{aligned}$$

therefore,

$$\begin{aligned} &-k_F \sum_{n=1}^{\mathcal{N}} \langle g(U^n) - g(u_C^n) - g'(u_C^n)(u_F^n - u_C^n), e_F^n \rangle \\ &\leq \bar{C} (\|e_F^n\|^2 + \|e_C^n\|^2 \|e_F^n\|). \end{aligned} \tag{59}$$

By Lemmas 4.1–4.3 and Cauchy-Schwarz inequality, it follows that

$$\begin{aligned} &\frac{3}{4} \|e_F^{\mathcal{N}}\|^2 - \frac{1}{4} (\|e_F^{\mathcal{N}-1}\|^2 + \|e_F^1\|^2 + \|e_F^0\|^2) - \frac{T^{1-\alpha}}{2\Gamma(2-\alpha)} \|e_F^0\|^2 \\ &\leq \bar{C} k_F \sum_{n=1}^{\mathcal{N}} (\|e_F^n\|^2 + \|e_C^n\|^2 \|e_F^n\|) + k_F \sum_{n=1}^{\mathcal{N}} \|R_1^n\| \|e_F^n\|, \end{aligned} \tag{60}$$

Taking m so that $\|e_F^m\| = \max_{0 \leq n \leq \mathcal{N}} \|e_F^n\|$, and noting that (57), we have the following:

$$\frac{1}{2} \|e_F^{\mathcal{N}}\| \leq \frac{1}{2} \|e_F^m\| \leq \frac{1}{4} \|e_F^1\| + \bar{C} k_F \sum_{n=1}^{\mathcal{N}} (\|e_F^n\| + \|e_C^n\|^2) + k_F \sum_{n=1}^{\mathcal{N}} \|R_1^n\|. \tag{61}$$

Utilizing Lemma 4.4, we get the following:

$$\|e_F^{\mathcal{N}}\| \leq \bar{C} \exp\{\mathcal{N} k_F\} \left(\|e_F^1\| + k_F \sum_{n=1}^{\mathcal{N}} \|e_C^n\|^2 + k_F \sum_{n=1}^{\mathcal{N}} \|R_1^n\| \right). \tag{62}$$

From (54), using the same way, we have the following:

$$\|e_F^1\| \leq \bar{C}(T) (\|e_F^0\| + k_F \|e_C^1\|^2 + k_F \|R_1^1\|). \tag{63}$$

Substituting (9) and (63) into (62), and combining Theorem 4.7, it holds the following:

$$\|e_F^{\mathcal{N}}\| \leq \bar{C} (k_F^{2-\alpha} + k_C^{4-2\alpha} + h_1^2 + h_2^2). \tag{64}$$

We finish the proof. □

5 Numerical experiment

In this section, we set $L_1 = L_2 = T = 1$. All experiments are performed on a Windows 10 (64 bit) PC-Inter(R) Core(TM) i7-8750H CPU 2.20 GHz, 8.0 GB of RAM using MATLAB R2014b.

The two-grid difference (TGD) scheme (14)–(18) and the general finite difference (GFD) scheme (10)–(13) are used to simulate the problem (1)–(3). To show the effectiveness of the two-grid algorithm, we perform two numerical examples. Let u_{TGD} and u_{GFD} be the numerical solutions, respectively.

Table 1 The L^2 -errors, convergence rates and CPU time (seconds) with $h = 1/100$ and $M = 3$

α	k_C	k_F	E_{GFD}	$\text{rate}_{\text{GFD}}^t$	CPU(s)	E_{TGD}	$\text{rate}_{\text{TGD}}^t$	CPU(s)
0.25	1/4	1/12	1.696213e-3	–	132.72	1.696341e-3	–	86.17
	1/8	1/24	5.169795e-4	1.714	251.62	5.169807e-4	1.714	169.23
	1/16	1/48	1.640700e-4	1.656	480.42	1.640700e-4	1.656	322.63
0.75	1/4	1/12	3.618949e-3	–	125.91	3.674818e-3	–	83.11
	1/8	1/24	1.515827e-3	1.255	242.38	1.271433e-3	1.271	161.31
	1/16	1/48	6.623354e-4	1.194	460.25	6.630738e-4	1.199	310.53

Denote,

$$E_{\text{TGD}}(h, k) = \max_{1 \leq n \leq \mathcal{N}} \|U^n - u_F^n\|,$$

$$\text{rate}_{\text{TGD}}^t = \log_2 \left(\frac{E_{\text{TGD}}(h, 2k)}{E_{\text{TGD}}(h, k)} \right), \quad \text{rate}_{\text{TGD}}^x = \log_2 \left(\frac{E_{\text{TGD}}(2h, k)}{E_{\text{TGD}}(h, k)} \right),$$

the notations $E_{\text{GFD}}(h, k)$, $\text{rate}_{\text{GFD}}^t$, and $\text{rate}_{\text{GFD}}^x$ are defined similarly.

Example 1 In the first example, the nonlinear term is $g(u) = u^3$ and the inhomogeneous term is as follows:

$$f(x, y, t) = [(2 + \alpha) \left(1 + \frac{\Gamma(2+\alpha)}{2} t^{1-\alpha} \right) + 2\pi^2 t] t^{1+\alpha} \sin \pi x \sin \pi y + (t^{2+\alpha} \sin \pi x \sin \pi y)^3.$$

The corresponding exact solution is as follows:

$$u(x, y, t) = t^{2+\alpha} \sin \pi x \sin \pi y.$$

When the spatial step $J_x = J_y = 100$ is fixed, in Tables 1 and 2, we present the L^2 -errors, the temporal convergence orders and the CPU time for $\alpha = 0.25, 0.75$ by using the two-grid difference scheme (14)–(18) and the general finite difference scheme (10)–(13), respectively. The numerical results show that two difference schemes are convergent with the order $2 - \alpha$ in the time direction. Moreover, combining Tables 1 and 2, it can be seen that the two-grid difference scheme will save more time than the general finite difference scheme as M increases. Fixed the time-step

Table 2 The L^2 -errors, convergence rates and CPU time (seconds) with $h = 1/100$ and $M = 5$

α	k_C	k_F	E_{GFD}	$\text{rate}_{\text{GFD}}^t$	CPU(s)	E_{TGD}	$\text{rate}_{\text{TGD}}^t$	CPU(s)
0.25	1/2	1/10	2.278547e-3	–	112.67	2.281892e-3	–	59.28
	1/4	1/20	7.141716e-4	1.674	207.24	7.142017e-4	1.676	106.27
	1/8	1/40	2.249906e-4	1.666	391.88	2.249923e-4	1.666	199.63
0.75	1/2	1/10	4.573776e-3	–	105.81	5.428355e-3	–	57.72
	1/4	1/20	1.899303e-3	1.268	198.29	1.967804e-3	1.464	103.21
	1/8	1/40	8.190562e-4	1.213	389.10	8.269552e-4	1.251	191.31

Table 3 The L^2 -errors and convergence rates with $k_C = 1/256$ and $k_F = 1/1024$

α	h	E_{GFD}	$\text{rate}_{\text{GFD}}^x$	E_{TGD}	$\text{rate}_{\text{TGD}}^x$
0.20	1/4	4.114949e-2	–	4.114945e-2	–
	1/8	1.019620e-2	2.013	1.019618e-2	2.013
	1/16	2.542821e-3	2.003	2.542802e-3	2.003
	1/32	6.353620e-4	2.001	6.353432e-4	2.001
	1/64	1.588730e-4	2.000	1.588543e-4	2.000
0.50	1/4	3.989599e-2	–	3.989596e-2	–
	1/8	9.892164e-3	2.012	9.892145e-3	2.012
	1/16	2.468250e-3	2.003	2.468233e-3	2.003
	1/32	6.176084e-4	1.999	6.175918e-4	1.999
	1/64	1.552887e-4	1.992	1.552722e-4	1.992
0.80	1/4	3.833725e-2	–	3.833723e-2	–
	1/8	9.526970e-3	2.009	9.526959e-3	2.009
	1/16	2.392026e-3	1.994	2.392016e-3	1.994
	1/32	6.129641e-4	1.964	6.129538e-4	1.964
	1/64	1.684969e-4	1.863	1.684867e-4	1.863

$\mathcal{N} = 1024$, the results in Table 3 reflect a convergence rate in space ≈ 2 for two difference schemes. The results in the time-space direction are in good agreement with the theoretical analysis.

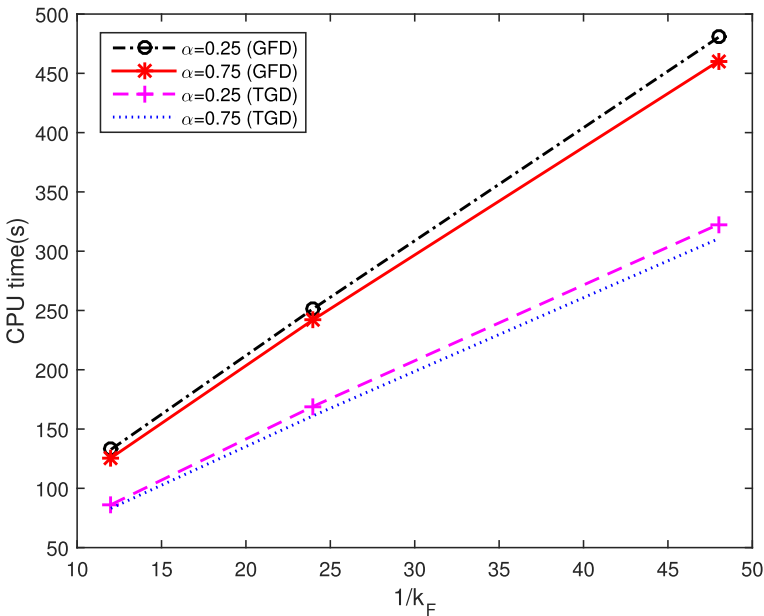


Fig. 1 The comparison of two methods for CPU time with $h = 1/100$ and $M = 3$

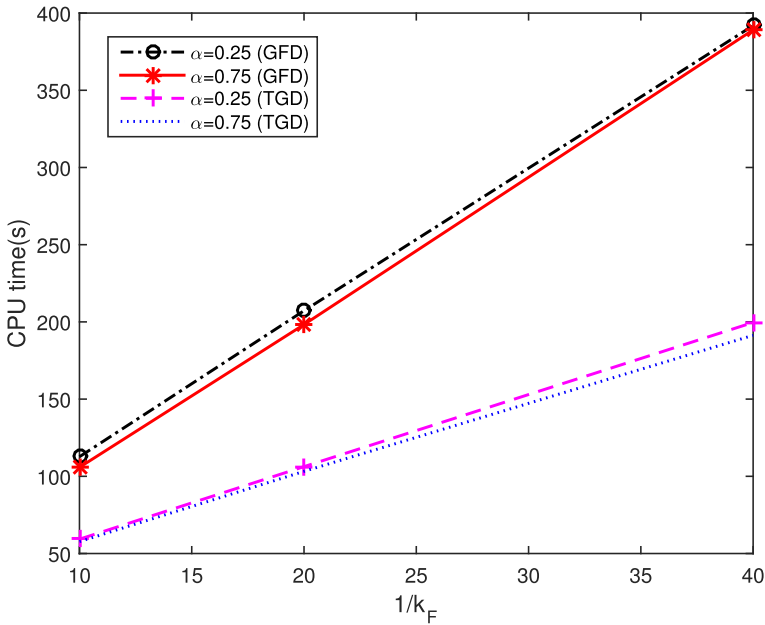


Fig. 2 The comparison of two methods for CPU time with $h = 1/100$ and $M = 5$

Figures 1 and 2 give the comparison of the two-grid difference and the general finite difference scheme, respectively, and show intuitively that given the same accuracy the two-grid algorithm is much more efficient than the general finite difference method in the temporal direction.

Example 2 In the second example, the nonlinear term is $g(u) = u^3$ and the exact solution is given as follows:

$$u(x, y, t) = (1 + 4t^3) \sin \pi x \sin \pi y.$$

Table 4 The L^2 -errors, convergence rates and CPU time (seconds) with $h = 1/100$ and $M = 2$

α	k_C	k_F	E_{GFD}	$\text{rate}_{\text{GFD}}^t$	CPU(s)	E_{TGD}	$\text{rate}_{\text{TGD}}^t$	CPU(s)
0.25	1/4	1/8	6.119585e-3	–	290.91	6.474947e-3	–	204.86
	1/8	1/16	1.663753e-3	1.879	455.07	1.728943e-3	1.905	301.13
	1/16	1/32	5.147191e-4	1.693	707.64	5.189988e-4	1.736	495.18
0.50	1/4	1/8	8.810485e-3	–	285.67	9.273411e-3	–	202.12
	1/8	1/16	2.840304e-3	1.633	437.77	2.952957e-3	1.651	296.21
	1/16	1/32	9.893523e-4	1.521	702.43	9.971202e-4	1.566	488.93
0.75	1/4	1/8	1.635303e-2	–	280.28	1.835445e-2	–	198.17
	1/8	1/16	6.567541e-3	1.316	427.03	6.734695e-3	1.446	289.91
	1/16	1/32	2.711785e-3	1.276	690.12	2.723183e-3	1.306	479.98

Table 5 The L^2 -errors and convergence rates with $k_C = 1/256$ and $k_F = 1/1024$

α	h	E_{GFD}	$\text{rate}_{\text{GFD}}^x$	E_{TGD}	$\text{rate}_{\text{TGD}}^x$
0.25	1/4	7.357866e-2	–	7.357928e-2	–
	1/8	1.897176e-2	1.955	1.897233e-2	1.955
	1/16	4.778252e-3	1.989	4.778809e-3	1.989
	1/32	1.196736e-3	1.997	1.197283e-3	1.997
0.50	1/4	7.362154e-2	–	7.362216e-2	–
	1/8	1.898381e-2	1.955	1.898436e-2	1.955
	1/16	4.783566e-3	1.987	4.784111e-3	1.988
	1/32	1.200354e-3	1.995	1.200892e-3	1.994
0.75	1/4	7.390821e-2	–	7.390870e-2	–
	1/8	1.907500e-2	1.954	1.907544e-2	1.954
	1/16	4.825988e-3	1.982	4.826421e-3	1.983
	1/32	1.230567e-3	1.971	1.230968e-3	1.971

The initial condition is $u^0(x, y) = \sin \pi x \sin \pi y$ and the corresponding source term is as follows:

$$f(x, y, t) = \left(12t^2 + 2\pi^2(1 + 4t^3) + \frac{24t^{3-\alpha}}{\Gamma(4-\alpha)} \right) \sin \pi x \sin \pi y + ((1 + 4t^3) \sin \pi x \sin \pi y)^3.$$

In Tables 4 and 5, we give the numerical results with the initial condition $u^0(x, y) \neq 0$ for $\alpha = 0.25, 0.50$, and 0.75 by utilizing the two-grid difference scheme (14)–(18) and the general finite difference scheme (10)–(13), respectively. The numerical results demonstrate that the two-grid difference method is more efficient than the general difference method. Furthermore, the results also show that the convergence orders in the space-time direction are approximately 2 and $2 - \alpha$, respectively.

6 Conclusions

In this article, a time two-grid algorithm based on the finite difference method is proposed and analyzed for the two-dimensional nonlinear time-fractional mobile/immobile transport model. The two-grid algorithm mainly includes two computational steps: first, on the coarse grid a nonlinear system is solved, then a linearized system is solved on the fine grid. The stability and convergence of the two-grid difference scheme are proved. The proposed two-grid algorithm is implemented, and the numerical results demonstrate that the two-grid algorithm is a more efficient way to solve the nonlinear time-fractional mobile/immobile transport problem.

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Compliance with Ethical Standards

Conflict of interest The authors declare that they have no conflict of interest.

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