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ORIGINAL PAPER

Weak and strong convergence theorems for solving pseudo-monotone variational inequalities with non-Lipschitz mappings

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Dedicated to Professor Le Dung Muu on the occasion of his 70th birthday

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Abstract
The aim of this paper is to study a classical pseudo-monotone and non-Lipschitz continuous variational inequality problem in real Hilbert spaces. Weak and strong convergence theorems are presented under mild conditions. Our methods generalize and extend some related results in the literature and the main advantages of proposed algorithms there is no use of Lipschitz condition of the variational inequality associated mapping. Numerical illustrations in finite and infinite dimensional spaces illustrate the behaviors of the proposed schemes.

Keywords Projection-type method · Variational inequality · Viscosity method · Pseudo-monotone mapping · Non-Lipschitz mapping

Mathematics Subject Classification (2010) 47H09 · 47J20 · 65K15 · 90C25

1 Introduction

The main purpose of this paper to study the classical *variational inequality* of Fichera [13, 14] and Stampacchia [38] (see also Kinderlehrer and Stampacchia [25]) in real Hilbert spaces. Precisely, the classical variational inequality problem (VIP) is of the form: finding a point $x^* \in C$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0 \ \ \forall x \in C, \tag{1}$$

Let us denote by $VI(C, A)$ the solutions set of VIP (1).

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This problem plays an important role as a modeling tool in diverse fields such as in economics, engineering mechanics, transportation, and many more (see for example, [2, 3, 15, 26, 28]). Recently, many iterative methods have been constructed for solving variational inequalities and their related optimization problems (see monographs [12, 28] and references therein).

One of the most popular methods for solving variational inequalities with monotone and Lipschitz continuous mappings is the method proposed by Korpelevich [30] (also independently by Antipin [1]) which is called the extragradient method in the finite dimensional Euclidean space. This method was based on a double-projection method onto the feasible set.

The extragradient method has been studied and extended in infinite-dimensional spaces by many authors (see, e.g., [6–9, 20, 22, 23, 32, 33, 39, 41–44] and the references therein). It is easy to observe that, when the mapping-associated variational inequality is not Lipschitz continuous or the Lipschitz constant of the associated variational inequality mapping is very difficult to compute, it is clear that the extragradient method is not applicable to implement because we can not determine the stepsize.

Khobotov [27] proposed the linesearch for the extragradient method and Marcotte's paper [34] contains its implementation. The first extrapolation method using Armijo-type linesearch was proposed in [29] and the method [19] follows the same approach (see comments of Section 1.3 in [28] and and Section 12.1 in [12]).

This modification in [19, 29] allows convergence without Lipschitz continuity of the mapping-associated variational inequality in finite-dimensional Euclidean space. The algorithm is of the form

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Algorithm 1

Initialization: Given $\gamma > 0, l \in (0,1), \mu \in (0,1)$. Let $x_1 \in C$ be arbitrary

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Iterative Steps: Given the current iterate x_n, calculate x_{n+1} as follows:

Step 1. Compute

$$y_n = P_C(x_n - \lambda_n A x_n)$$

where $\lambda_n := \gamma l^{m_n}$ and m_n is the smallest non-negative integer m satisfying

$$\gamma l^m \|Ax_n - Ay_n\| \le \mu \|x_n - y_n\|.$$

If $x_n = y_n$ then stop and x_n is a solution of VIP. Otherwise
Step 2. Compute

$$x_{n+1} = P_C(x_n - \beta_n A y_n),$$

where

$$\beta_n := \frac{\langle Ay_n, x_n - y_n \rangle}{\|Ay_n\|^2}.$$

Set $n := n+1$ and go to **Step 1**.

Moreover, Algorithm 1 converges under the condition the mapping-associated variational inequality is monotone and continuous on the feasible set in *finite-dimensional spaces*. This brings the following natural question.

Question: Can we obtain convergence result for VIP using a new modification of the extragradient method under a much weaker condition than monotonicity of the cost function?

Our aim in this paper is to answer the above question in the affirmative. Precisely, our contributions in this paper are:

- to construct another modification of extragradient algorithm that converges under a weaker condition in an infinite-dimensional Hilbert space;
- to introduce a modification of extragradient method for solving VIP with uniformly continuous pseudo-monotone mapping in infinite-dimensional real Hilbert spaces;
- to use a different Armijo-type linesearch and obtain convergence results (weak and strong convergence results) when the mapping is pseudo-monotone in the sense of Karamardian [24]
- to compare, using numerical examples, our proposed methods with some methods in the literature. Our numerical analysis (performed both in finite- and infinite-dimensional Hilbert spaces) shows that our methods outperform certain already established methods for solving variational inequality problem with pseudo-monotone mapping in the literature.

We organize the paper as follows: In Sect. 2, we give some definitions and preliminary results to be used in our convergence analysis. In Sect. 3, we deal with analyzing the convergence of the proposed algorithms. Finally, in Sect. 4, several numerical experiments are performed to illustrate the implementation of our proposed algorithms and compare our proposed algorithms with previously known algorithms.

2 Preliminaries

Let C be a non-empty, closed, and convex subset of a real Hilbert space H, $A : H \to H$ is a single-valued mapping, and $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ are the inner product and the norm in H, respectively.

The weak convergence of $\{x_n\}_{n=1}^{\infty}$ to x is denoted by $x_n \rightharpoonup x$ as $n \to \infty$, while the strong convergence of $\{x_n\}_{n=1}^{\infty}$ to x is written as $x_n \to x$ as $n \to \infty$. For each $x, y \in H$ and $\alpha \in \mathbb{R}$, we have

Definition 2.1 Let $T : H \to H$ be a mapping.

1. The mapping T is called L-`Lipschitz continuous` with $L > 0$ if

$$\|Tx - Ty\| \leq L\|x - y\| \quad \forall x, y \in H.$$

if $L = 1$ then the mapping T is called `non-expansive` and if $L \in (0, 1)$, T is called `contraction`.

2. The mapping T is called monotone if

$$\langle Tx - Ty, x - y\rangle \geq 0 \quad \forall x, y \in H.$$

3. The mapping T is called pseudo-monotone if

$$\langle Tx, y - x\rangle \geq 0 \Longrightarrow \langle Ty, y - x\rangle \geq 0 \quad \forall x, y \in H.$$

4. The mapping T is called α-strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \alpha\|x - y\|^2 \quad \forall x, y \in H.$$

5. The mapping T is called sequentially weakly continuous if for each sequence $\{x_n\}$ we have: x_n converges weakly to x implies x_n converges weakly to Tx.

It is easy to see that very monotone mapping is pseudo-monotone but the converse is not true. For example, take $Tx := \frac{1}{1+x}, x > 0$.

For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$ such that $\|x - P_C x\| \leq \|x - y\| \ \forall y \in C$. P_C is called the *metric projection* of H onto C.

Lemma 2.1 [16] *Given $x \in H$ and $z \in C$. Then $z = P_C x \Longleftrightarrow \langle x - z, z - y\rangle \geq 0 \ \forall y \in C$.*

Lemma 2.2 [16] *Let $x \in H$. Then*

i) $\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle \ \forall y \in H$;
ii) $\|P_C x - y\|^2 \leq \|x - y\|^2 - \|x - P_C x\|^2 \ \forall y \in C$;
iii) $\langle (I - P_C)x - (I - P_C)y, x - y\rangle \geq \|(I - P_C)x - (I - P_C)y\|^2 \ \forall y \in H$.

Lemma 2.3 [5] *Given $x \in H$ and $v \in H$, $v \neq 0$ and let $T = \{z \in H : \langle v, z - x\rangle \leq 0\}$. Then, for all $u \in H$, the projection $P_T(u)$ is defined by*

$$P_T(u) = u - \max\left\{0, \frac{\langle v, u - x\rangle}{||v||^2}\right\} v.$$

In particular, if $u \notin T$ then

$$P_T(u) = u - \frac{\langle v, u - x\rangle}{||v||^2} v.$$

Lemma 2.3 gives us an explicit formula to find the projection of any point onto a half-space.

For properties of the metric projection, the interested reader could be referred to Section 3 in [16] and Chapter 4 in [5].

The following Lemmas are useful for the convergence of our proposed methods.

Lemma 2.4 [11] *For $x \in H$ and $\alpha \geq \beta > 0$ the following inequalities hold.*

$$\frac{\|x - P_C(x - \alpha Ax)\|}{\alpha} \leq \frac{\|x - P_C(x - \beta Ax)\|}{\beta},$$
$$\|x - P_C(x - \beta Ax)\| \leq \|x - P_C(x - \alpha Ax)\|.$$

Lemma 2.5 [21] *Let H_1 and H_2 be two real Hilbert spaces. Suppose $A : H_1 \to H_2$ is uniformly continuous on bounded subsets of H_1 and M is a bounded subset of H_1. Then, $A(M)$ is bounded.*

Lemma 2.6 *[[10], Lemma 2.1] Consider the problem $VI(C, A)$ with C being a non-empty, closed, convex subset of a real Hilbert space H and $A : C \to H$ being pseudo-monotone and continuous. Then, x^* is a solution of $VI(C, A)$ if and only if*

$$\langle Ax, x - x^* \rangle \geq 0 \ \ \forall x \in C.$$

Lemma 2.7 [36] *Let C be a non-empty set of H and $\{x_n\}$ be a sequence in H such that the following two conditions hold:*

i) *for every $x \in C$, $\lim_{n\to\infty} \|x_n - x\|$ exists;*
ii) *every sequential weak cluster point of $\{x_n\}$ is in C.*

Then, $\{x_n\}$ converges weakly to a point in C.

The proof of the following lemma is the same with Lemma 2.3 and was given in [17]. Hence, we state the lemma and omit the proof in real Hilbert spaces.

Lemma 2.8 *Let H be a real Hilbert space and h be a real-valued function on H and define $K := \{x \in H : h(x) \leq 0\}$. If K is nonempty and h is Lipschitz continuous on H with modulus $\theta > 0$, then*

$$dist(x, K) \geq \theta^{-1} \max\{h(x), 0\} \ \ \forall x \in H,$$

where $dist(x, K)$ denotes the distance function from x to K.

Lemma 2.9 [31] *Let $\{a_n\}$ be a sequence of non-negative real numbers such that there exists a subsequence $\{a_{n_j}\}$ of $\{a_n\}$ such that $a_{n_j} < a_{n_j+1}$ for all $j \in \mathbb{N}$. Then there exists a non-decreasing sequence $\{m_k\}$ of $\mathbb{N}$ such that $\lim_{k\to\infty} m_k = \infty$ and the following properties are satisfied by all (sufficiently large) number $k \in \mathbb{N}$:*

$$a_{m_k} \leq a_{m_k+1} \ and a_k \leq a_{m_k+1}.$$

In fact, m_k is the largest number n in the set $\{1, 2, \cdots, k\}$ such that $a_n < a_{n+1}$.

Lemma 2.10 [45] *Let $\{a_n\}$ be a sequence of non-negative real numbers such that:*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n b_n,$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{b_n\}$ is a sequence such that

a) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

b) $\limsup_{n\to\infty} b_n \le 0$.

Then ,$\lim_{n\to\infty} a_n = 0$.

3 Main results

The following conditions are assumed for the convergence of the methods.

Condition 1 The feasible set C is a non-empty, closed, and convex subset of the real Hilbert space H.

Condition 2 The mapping $A : H \to H$ is a pseudo-monotone, uniformly continuous on H and sequentially weakly continuous on C. In finite-dimensional spaces, it suffices to assume that $A : H \to H$ is a continuous pseudo-monotone on H.

Condition 3 The solution set of the VIP (1) is non-empty, that is $VI(C, A) \neq \emptyset$.

3.1 Weak convergence

In this section, we introduce a new algorithm for solving VIP which is constructed based on modified projection-type methods.

Algorithm 2

Initialization: Given $\gamma > 0, l \in (0, 1), \mu \in (0, 1)$. Let $x_1 \in H$ be arbitrary

Iterative Steps: Given the current iterate x_n, calculate x_{n+1} as follows:

Step 1. Compute

$$y_n = P_C(x_n - \lambda_n A x_n),$$

where $\lambda_n := \gamma l^{m_n}$, with m_n is the smallest nonnegative integer m satisfying

$$\gamma l^m \langle Ax_n - Ay_n, x_n - y_n\rangle \le \mu \|x_n - y_n\|^2. \quad (2)$$

Let $x_n = y_n$ or $Ay_n = 0$ then stop and y_n is a solution of VIP. Otherwise

Step 2. Compute

$$x_{n+1} = P_{C_n}(x_n),$$

where

$$C_n := \{x \in H : h_n(x) \le 0\}$$

and

$$h_n(x) = \langle x_n - y_n - \lambda_n(Ax_n - Ay_n), x - y_n\rangle. \quad (3)$$

Set $n := n + 1$ and go to **Step 1**.

Remark 3.1 We note that our Algorithm 3.1 in this paper is proposed in infinite-dimensional real Hilbert spaces while the method proposed by Solodov and Tseng in [40] was done in finite-dimensional spaces. Furthermore, our method is much more general than that of Solodov and Tseng [40] even with a more general cost function than that of Solodov and Tseng [40]. This is confirmed in our numerical examples, where we give examples of variational inequalities with pseudomonotone functions which are not monotone (as assumed in the paper of Solodov and Tseng [40]) even in finite-dimensional spaces.

We start the analysis of the algorithm's convergence by proving the following lemmas

Lemma 3.1 *Assume that Conditions 1–2 hold. The Armijo-line search rule* (2) *is well defined.*

Proof If $x_n \in VI(C, A)$ then $x_n = P_C(x_n - \gamma Ax_n)$ and $m_n = 0$. We consider the situation $x_n \notin VI(C, A)$ and assume the contrary that for all m we have

$$\gamma l^m \langle Ax_n - AP_C(x_n - \gamma l^m Ax_n), x_n - P_C(x_n - \gamma l^m Ax_n)\rangle > \mu \|x_n - P_C(x_n - \gamma l^m Ax_n)\|^2$$

By Cauchy-Schwartz inequality, we have

$$\gamma l^m \|Ax_n - AP_C(x_n - \gamma l^m Ax_n)\| > \mu \|x_n - P_C(x_n - \gamma l^m Ax_n)\|. \tag{4}$$

This implies that

$$\|Ax_n - AP_C(x_n - \gamma l^m Ax_n)\| > \mu \frac{\|x_n - P_C(x_n - \gamma l^m Ax_n)\|}{\gamma l^m}. \tag{5}$$

We consider two possibilities of x_n. First, if $x_n \in C$, then since P_C and A are continuous, we have $\lim_{m\to\infty} \|x_n - P_C(x_n - \gamma l^m Ax_n)\| = 0$. From the uniform continuity of the mapping A on bounded subsets of C, it implies that

$$\lim_{m\to\infty} \|Ax_n - AP_C(x_n - \gamma l^m Ax_n)\| = 0. \tag{6}$$

Combining (5) and (6) we get

$$\lim_{m\to\infty} \frac{\|x_n - P_C(x_n - \gamma l^m Ax_n)\|}{\gamma l^m} = 0. \tag{7}$$

Assume that $z_m = P_C(x_n - \gamma l^m Ax_n)$ we have

$$\langle z_m - x_n + \gamma l^m Ax_n, x - z_m\rangle \geq 0 \;\; \forall x \in C.$$

This implies that

$$\langle \frac{z_m - x_n}{\gamma l^m}, x - z_m\rangle + \langle Ax_n, x - z_m\rangle \geq 0 \;\; \forall x \in C. \tag{8}$$

Taking the limit $m \to \infty$ in (8) and using (7) we obtain

$$\langle Ax_n, x - x_n\rangle \geq 0 \;\; \forall x \in C,$$

which implies that $x_n \in VI(C, A)$ is a contraction.

Now, if $x_n \notin C$, then we have

$$\lim_{m\to\infty} \|x_n - P_C(x_n - \gamma l^m Ax_n)\| = \|x_n - P_C x_n\| > 0. \tag{9}$$

and

$$\lim_{m\to\infty} \gamma l^m \|Ax_n - AP_C(x_n - \gamma l^m Ax_n)\| = 0 \tag{10}$$

Combining (4), (9), and (10), we get a contradiction. □

Remark 3.2 1. In the proof of Lemma 3.1, we do not use the pseudo-monotonicity of A.

2. Now, we show that if $x_n = y_n$ then stop and y_n is a solution of $VI(C, A)$. Indeed, we have $0 < \lambda_n \le \gamma$, which together with Lemma 2.4, we get

$$0 = \frac{\|x_n - y_n\|}{\lambda_n} = \frac{\|x_n - P_C(x_n - \lambda_n Ax_n)\|}{\lambda_n} \ge \frac{\|x_n - P_C(x_n - \gamma Ax_n)\|}{\gamma}.$$

This implies that x_n is a solution of $VI(C, A)$, thus y_n is a solution of $VI(C, A)$.

3. Next, we show that if $Ay_n = 0$ then stop and y_n is a solution of $VI(C, A)$. Indeed, since $y_n \in C$, it is easy to see that if $Ay_n = 0$ then $y_n \in VI(C, A)$.

Lemma 3.2 *Assume that Conditions 1–3 hold. Let x^* be a solution of problem (1) and the function h_n be defined by (5). Then $h_n(x^*) \le 0$ and $h_n(x_n) \ge (1-\mu)\|x_n - y_n\|^2$. In particular, if $x_n \neq y_n$ then $h_n(x_n) > 0$.*

Proof Since x^* be a solution of problem (1), using Lemma 2.6 we have

$$\langle Ay_n, x^* - y_n\rangle \le 0. \tag{11}$$

It is implied from (11) and $y_n = P_C(x_n - \lambda_n Ax_n)$ that

$$\begin{aligned} h_n(x^*) &= \langle x_n - y_n - \lambda_n(Ax_n - Ay_n), x^* - y_n\rangle \\ &= \langle x_n - y_n - \lambda_n Ax_n, x^* - y_n\rangle + \lambda_n\langle Ay_n, x^* - y_n\rangle \\ &\le 0. \end{aligned}$$

The first claim of Lemma 3.2 is proven. Now, we prove the second claim. Using (2), we have

$$\begin{aligned} h_n(x_n) &= \langle x_n - y_n - \lambda_n(Ax_n - Ay_n), x_n - y_n\rangle \\ &= \|x_n - y_n\|^2 - \lambda_n\langle Ax_n - Ay_n, x_n - y_n\rangle \\ &\ge \|x_n - y_n\|^2 - \mu\|x_n - y_n\|^2 \\ &= (1-\mu)\|x_n - y_n\|^2. \end{aligned}$$

□

Remark 3.3 Lemma 3.2 implies that $x_n \notin C_n$. According to Lemma 2.3, then x_{n+1} is of the form

$$x_{n+1} = x_n - \frac{\langle x_n - y_n - \lambda_n(Ax_n - Ay_n), x_n - y_n\rangle}{\|x_n - y_n - \lambda_n(Ax_n - Ay_n)\|^2}(x_n - y_n - \lambda_n(Ax_n - Ay_n)).$$

Lemma 3.3 *Assume that Conditions 1–3 hold. Let $\{x_n\}$ be a sequence generated by Algorithm 3.1. If there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to $z \in H$ and $\lim_{k\to\infty}\|x_{n_k} - y_{n_k}\| = 0$ then $z \in VI(C, A)$.*

Proof From $x_{n_k} \rightharpoonup z$, $\lim_{k\to\infty} \|x_{n_k} - y_{n_k}\| = 0$, and $\{y_n\} \subset C$, we get $z \in C$. We have $y_{n_k} = P_C(x_{n_k} - \lambda_{n_k} Ax_{n_k})$ thus,

$$\langle x_{n_k} - \lambda_{n_k} Ax_{n_k} - y_{n_k}, x - y_{n_k} \rangle \leq 0 \ \ \forall x \in C.$$

or equivalently

$$\frac{1}{\lambda_{n_k}} \langle x_{n_k} - y_{n_k}, x - y_{n_k} \rangle \leq \langle Ax_{n_k}, x - y_{n_k} \rangle \ \ \forall x \in C.$$

This implies that

$$\frac{1}{\lambda_{n_k}} \langle x_{n_k} - y_{n_k}, x - y_{n_k} \rangle + \langle Ax_{n_k}, y_{n_k} - x_{n_k} \rangle \leq \langle Ax_{n_k}, x - x_{n_k} \rangle \ \ \forall x \in C. \tag{12}$$

Now, we show that

$$\liminf_{k\to\infty} \langle Ax_{n_k}, x - x_{n_k} \rangle \geq 0. \tag{13}$$

For showing this, we consider two possible cases. Suppose first that $\liminf_{k\to\infty} \lambda_{n_k} > 0$. We have $\{x_{n_k}\}$ is a bounded sequence, A is uniformly continuous on bounded subsets of H. By Lemma 2.6, we get that $\{Ax_{n_k}\}$ is bounded. Taking $k \to \infty$ in (12) since $\|x_{n_k} - y_{n_k}\| \to 0$, we get

$$\liminf_{k\to\infty} \langle Ax_{n_k}, x - x_{n_k} \rangle \geq 0.$$

Now, we assume that $\liminf_{k\to\infty} \lambda_{n_k} = 0$. Assume $z_{n_k} = P_C(x_{n_k} - \lambda_{n_k}.l^{-1} Ax_{n_k})$, we have $\lambda_{n_k} l^{-1} > \lambda_{n_k}$. Applying Lemma 2.4, we obtain

$$\|x_{n_k} - z_{n_k}\| \leq \frac{1}{l} \|x_{n_k} - y_{n_k}\| \to 0 \text{ as} k \to \infty.$$

Consequently, $z_{n_k} \rightharpoonup z \in C$, this implies that $\{z_{n_k}\}$ is bounded, which the uniformly continuity of the mapping A on bounded subsets of H follows that

$$\|Ax_{n_k} - Az_{n_k}\| \to 0 \text{ as} k \to \infty. \tag{14}$$

By the Armijo linesearch rule (2), we must have

$$\lambda_{n_k}.l^{-1} \langle Ax_n - AP_C(x_n - \lambda_{n_k}.l^{-1} Ax_n), x_n - P_C(x_n - \lambda_{n_k}.l^{-1} Ax_n) \rangle > \mu \|x_n - P_C(x_n - \lambda_{n_k}.l^{-1} Ax_n)\|^2$$

By Cauchy-Schwartz inequality, we have

$$\lambda_{n_k}.l^{-1} \|Ax_{n_k} - AP_C(x_{n_k} - \lambda_{n_k} l^{-1} Ax_{n_k})\| > \mu \|x_{n_k} - P_C(x_{n_k} - \lambda_{n_k} l^{-1} Ax_{n_k})\|.$$

That is,

$$\frac{1}{\mu} \|Ax_{n_k} - Az_{n_k}\| > \frac{\|x_{n_k} - z_{n_k}\|}{\lambda_{n_k} l^{-1}}. \tag{15}$$

Combining (14) and (15), we obtain

$$\lim_{k\to\infty} \frac{\|x_{n_k} - z_{n_k}\|}{\lambda_{n_k} l^{-1}} = 0.$$

Furthermore, we have

$$\langle x_{n_k} - \lambda_{n_k} l^{-1} Ax_{n_k} - z_{n_k}, x - z_{n_k} \rangle \leq 0 \ \ \forall x \in C.$$

This implies that

$$\frac{1}{\lambda_{n_k} l^{-1}} \langle x_{n_k} - z_{n_k}, x - z_{n_k} \rangle + \langle Ax_{n_k}, z_{n_k} - x_{n_k} \rangle \leq \langle Ax_{n_k}, x - x_{n_k} \rangle \ \ \forall x \in C. \quad (16)$$

Taking the limit $k \to \infty$ in (16), we get

$$\liminf_{k\to\infty} \langle Ax_{n_k}, x - x_{n_k} \rangle \geq 0.$$

Therefore, the inequality (13) is proven.
On the other hand, we have

$$\langle Ay_{n_k}, x - y_{n_k} \rangle = \langle Ay_{n_k} - Ax_{n_k}, x - x_{n_k} \rangle + \langle Ax_{n_k}, x - x_{n_k} \rangle + \langle Ay_{n_k}, x_{n_k} - y_{n_k} \rangle. \quad (17)$$

Since $\lim_{k\to\infty} \|x_{n_k} - y_{n_k}\| = 0$ and the uniformly continuity of A on H, we get

$$\lim_{k\to\infty} \|Ax_{n_k} - Ay_{n_k}\| = 0,$$

which, together with (13) and (17) implies that

$$\liminf_{k\to\infty} \langle Ay_{n_k}, x - y_{n_k} \rangle \geq 0. \quad (18)$$

Next, we show that $z \in VI(C, A)$. Indeed, we choose a sequence $\{\epsilon_k\}$ of positive numbers decreasing and tending to 0. For each k, we denote by N_k the smallest positive integer such that

$$\langle Ay_{n_j}, x - y_{n_j} \rangle + \epsilon_k \geq 0 \ \ \forall j \geq N_k, \quad (19)$$

where the existence of N_k follows from (18). Since $\{\epsilon_k\}$ is decreasing, it is easy to see that the sequence $\{N_k\}$ is increasing. Furthermore, for each k, since $\{y_{N_k}\} \subset C$ we have $Ay_{N_k} \neq 0$, and setting

$$v_{N_k} = \frac{Ay_{N_k}}{\|Ay_{N_k}\|^2},$$

we have $\langle Ay_{N_k}, v_{N_k} \rangle = 1$ for each k. Now, we can deduce from (19) that for each k

$$\langle Ay_{N_k}, x + \epsilon_k v_{N_k} - y_{N_k} \rangle \geq 0.$$

Since the fact that A is pseudo-monotone, we get

$$\langle A(x + \epsilon_k v_{N_k}), x + \epsilon_k v_{N_k} - y_{N_k} \rangle \geq 0.$$

This implies that

$$\langle Ax, x - y_{N_k} \rangle \geq \langle Ax - A(x + \epsilon_k v_{N_k}), x + \epsilon_k v_{N_k} - y_{N_k} \rangle - \epsilon_k \langle Ax, v_{N_k} \rangle. \quad (20)$$

Now, we show that $\lim_{k\to\infty} \epsilon_k v_{N_k} = 0$. Indeed, since $x_{n_k} \rightharpoonup z$ and $\lim_{k\to\infty} \|x_{n_k} - y_{n_k}\| = 0$,we obtain $y_{N_k} \rightharpoonup z$ as$k \to \infty$. Since A is sequentially weakly continuous on C, $\{Ay_{n_k}\}$ converges weakly to Az. We have that $Az \neq 0$ (otherwise, z is a solution). Since the norm mapping is sequentially weakly lower semicontinuous, we have

$$0 < \|Az\| \leq \liminf_{k\to\infty} \|Ay_{n_k}\|.$$

Since $\{y_{N_k}\} \subset \{y_{n_k}\}$ and $\epsilon_k \to 0$ as $k \to \infty$, we obtain

$$0 \leq \limsup_{k\to\infty} \|\epsilon_k v_{N_k}\| = \limsup_{k\to\infty} \left(\frac{\epsilon_k}{\|Ay_{n_k}\|}\right) \leq \frac{\limsup_{k\to\infty} \epsilon_k}{\liminf_{k\to\infty} \|Ay_{n_k}\|} = 0,$$

which implies that $\lim_{k\to\infty} \epsilon_k v_{N_k} = 0$.

Now, letting $k \to \infty$, then the right hand side of (20) tends to zero by A is uniformly continuous, $\{x_{N_k}\}, \{v_{N_k}\}$ are bounded and $\lim_{k\to\infty} \epsilon_k v_{N_k} = 0$. Thus, we get

$$\liminf_{k\to\infty} \langle Ax, x - y_{N_k}\rangle \geq 0.$$

Hence, for all $x \in C$ we have

$$\langle Ax, x - z\rangle = \lim_{k\to\infty} \langle Ax, x - y_{N_k}\rangle = \liminf_{k\to\infty} \langle Ax, x - y_{N_k}\rangle \geq 0.$$

By Lemma 2.6, we obtain $z \in VI(C, A)$ and the proof is complete. □

Remark 3.4 When the function A is monotone, it is not necessary to impose the sequential weak continuity on A.

Theorem 3.5 *Assume that Conditions 1–3 hold. Then any sequence $\{x_n\}$ generated by Algorithm 3.1 converges weakly to an element of $VI(C, A)$.*

Proof

Claim 1 $\{x_n\}$ is a bounded sequence. Indeed, let $p \in VI(C, A)$ we have

$$\begin{aligned} \|x_{n+1} - p\|^2 = \|P_{C_n} x_n - p\|^2 &\leq \|x_n - p\|^2 - \|P_{C_n} x_n - x_n\|^2 \\ &= \|x_n - p\|^2 - \text{dist}^2(x_n, C_n). \end{aligned} \tag{21}$$

This implies that

$$\|x_{n+1} - p\| \leq \|x_n - p\|.$$

This implies that $\lim_{n\to\infty} \|x_n - p\|$ exists. Thus, the sequence $\{x_n\}$ is bounded and we also have $\{y_n\}$ is bounded.

Claim 2

$$\left[\frac{1}{M}(1-\mu)\|x_n - y_n\|^2\right]^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2,$$

for some $M > 0$. Indeed, since $\{x_n\}, \{y_n\}$ are bounded, thus $\{Ax_n\}, \{Ay_n\}$ are bounded, thus there exists $M > 0$ such that $\|x_n - y_n - \lambda_n(Ax_n - Ay_n)\| \leq M$ for all n. Using this fact, we get for all $u, v \in H$ that

$$\begin{aligned} \|h_n(u) - h_n(v)\| &= \|\langle x_n - y_n - \lambda_n(Ax_n - Ay_n), u - v\rangle\| \\ &\leq \|x_n - y_n - \lambda_n(Ax_n - Ay_n)\| \|u - v\| \\ &\leq M\|u - v\|. \end{aligned}$$

This implies that $h_n(\cdot)$ is M-Lipschitz continuous on H. By Lemma 2.8, we obtain

$$\text{dist}(x_n, C_n) \geq \frac{1}{M} h_n(x_n),$$

which, together with Lemma 3.2, we get

$$\text{dist}(x_n, C_n) \geq \frac{1}{M}(1-\mu)\|x_n - y_n\|^2. \tag{22}$$

Combining (21) and (22), we obtain

$$\|x_{n+1} - p\|^2 \leq \|x_n - z\|^2 - \left[\frac{1}{M}(1-\mu)\|x_n - y_n\|^2\right]^2,$$

which implies Claim 2 is proved.

Claim 3 The sequence $\{x_n\}$ converges weakly to an element of $VI(C, A)$. Indeed, since $\{x_n\}$ is a bounded sequence, there exists the subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to $z \in H$.

According to **Claim 2**, we find

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{23}$$

It is implied from Lemma 3.3 and (23) that $z \in VI(C, A)$.

Therefore, we proved that:

i) For every $p \in VI(C, A)$, $\lim_{n\to\infty} \|x_n - p\|$ exists;
ii) Each sequential weak cluster point of the sequence $\{x_n\}$ is in $VI(C, A)$.

By Lemma 2.7 the sequence $\{x_n\}$ converges weakly to an element of $VI(C, A)$.

□

3.2 Strong convergence

In this section, we introduce an algorithm for strong convergence which is constructed based on viscosity method [35] and modified projection-type methods for solving VIs. In addition, we assume that $f : C \to H$ is a contractive mapping with a coefficient $\rho \in [0, 1)$, and we add the following condition

Condition 4 Let $\{\alpha_n\}$ be a real sequences in $(0, 1)$ such that

$$\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Algorithm 3

Initialization: Given $\gamma > 0, l \in (0, 1), \mu \in (0, 1)$. Let $x_1 \in H$ be arbitrary

Iterative Steps: Given the current iterate x_n, calculate x_{n+1} as follows:

Step 1. Compute

$$y_n = P_C(x_n - \lambda_n Ax_n),$$

where $\lambda_n := \gamma l^{m_n}$, with m_n is the smallest nonnegative integer m satisfying

$$\gamma l^m \langle Ax_n - Ay_n, x_n - y_n \rangle \le \mu \|x_n - y_n\|^2.$$

Step 2. Compute

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) P_{C_n}(x_n),$$

where

$$C_n := \{x \in H : h_n(x) \le 0\}$$

and

$$h_n(x) = \langle x_n - y_n - \lambda_n(Ax_n - Ay_n), x - y_n \rangle.$$

Set $n := n + 1$ and go to **Step 1**.

Theorem 3.6 *Assume that Conditions 1–4 hold. Then any sequence $\{x_n\}$ generated by Algorithm 3.2 converges strongly to $p \in VI(C, A)$, where $p = P_{VI(C,A)} \circ f(p)$.*

Proof

Claim 1 The sequence $\{x_n\}$ is bounded. Indeed, let $z_n = P_{C_n}(x_n)$, according to Claim 1 in Theorem 3.5, we get

$$\|z_n - p\|^2 \le \|x_n - p\|^2 - \left[\frac{1}{M}(1 - \mu)\|x_n - y_n\|^2\right]^2. \tag{24}$$

This implies that

$$\|z_n - p\| \le \|x_n - p\|.$$

Therefore,

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n f(x_n) + (1 - \alpha_n) z_n - p\| \\
&= \|\alpha_n (f(x_n) - p) + (1 - \alpha_n)(z_n - p)\| \\
&\le \alpha_n \|f(x_n) - p\| + (1 - \alpha_n)\|z_n - p\| \\
&\le \alpha_n \|f(x_n) - f(p)\| + \alpha_n \|f(p) - p\| + (1 - \alpha_n)\|z_n - p\| \\
&\le \alpha_n \rho \|x_n - p\| + \alpha_n \|f(p) - p\| + (1 - \alpha_n)\|x_n - p\| \\
&\le [1 - \alpha_n(1 - \rho)]\|x_n - p\| + \alpha_n(1 - \rho)\frac{\|f(p) - p\|}{1 - \rho} \\
&\le \max\{\|x_n - p\|, \frac{\|f(p) - p\|}{1 - \rho}\} \\
&\le ... \le \max\{\|x_1 - p\|, \frac{\|f(p) - p\|}{1 - \rho}\}.
\end{aligned}$$

This implies that the sequence $\{x_n\}$ is bounded. Consequently, $\{f(x_n)\}$, $\{y_n\}$, and $\{z_n\}$ are bounded.

Claim 2

$$\|z_n - x_n\|^2 \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2\alpha_n \langle f(x_n) - p, x_{n+1} - p\rangle.$$

Indeed, we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|\alpha_n(f(x_n) - p) + (1-\alpha_n)(z_n - p)\|^2 \\ &\le (1-\alpha_n)\|z_n - p\|^2 + 2\alpha_n\langle f(x_n) - p, x_{n+1} - p\rangle \\ &\le \|z_n - p\|^2 + 2\alpha_n\langle f(x_n) - p, x_{n+1} - p\rangle. \end{aligned} \tag{25}$$

On the other hand, we have

$$\|z_n - p\|^2 = \|P_{C_n}x_n - p\|^2 \le \|x_n - p\|^2 - \|z_n - x_n\|^2. \tag{26}$$

Substitute (26) into (25), we get

$$\|x_{n+1} - p\|^2 \le \|x_n - p\|^2 - \|z_n - x_n\|^2 + 2\alpha_n\langle f(x_n) - p, x_{n+1} - p\rangle.$$

This implies that

$$\|z_n - x_n\|^2 \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2\alpha_n \langle f(x_n) - p, x_{n+1} - p\rangle.$$

Claim 3

$$(1-\alpha_n)\left[\frac{1}{M}(1-\mu)\|x_n - y_n\|^2\right]^2 \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n\|f(x_n) - p\|^2.$$

Indeed, from the definition of the sequence $\{x_n\}$ and (24) we obtain

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|\alpha_n(f(x_n) - p) + (1-\alpha_n)(z_n - p)\|^2 \\ &= \alpha_n\|f(x_n) - p\|^2 + (1-\alpha_n)\|z_n - p\|^2 - \alpha_n(1-\alpha_n)\|f(x_n) - z_n\|^2 \\ &\le \alpha_n\|f(x_n) - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 - (1-\alpha_n)\left[\frac{1}{L}(1-\mu)\|x_n - y_n\|^2\right]^2 \\ &\le \alpha_n\|f(x_n) - p\|^2 + \|x_n - p\|^2 - (1-\alpha_n)\left[\frac{1}{M}(1-\mu)\|x_n - y_n\|^2\right]^2. \end{aligned}$$

This implies that

$$(1-\alpha_n)\left[\frac{1}{M}(1-\mu)\|x_n - y_n\|^2\right]^2 \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n\|f(x_n) - p\|^2.$$

Claim 4

$$\|x_{n+1} - p\|^2 \le (1-(1-\rho)\alpha_n)\|x_n - p\|^2 + (1-\rho)\alpha_n \frac{2}{1-\rho}\langle f(p) - p, x_{n+1} - p\rangle.$$

Indeed, we have

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &= \|\alpha_n f(x_n)+(1-\alpha_n)z_n-p\|^2\\
&= \|\alpha_n(f(x_n)-f(p))+(1-\alpha_n)(z_n-p)+\alpha_n(f(p)-p)\|^2\\
&\le \|\alpha_n(f(x_n)-f(p))+(1-\alpha_n)(z_n-p)\|^2+2\alpha_n\langle f(p)-p,x_{n+1}-p\rangle\\
&\le \alpha_n\|f(x_n)-f(p)\|^2+(1-\alpha_n)\|z_n-p\|^2+2\alpha_n\langle f(p)-p,x_{n+1}-p\rangle\\
&\le \alpha_n\rho\|x_n-p\|^2+(1-\alpha_n)\|x_n-p\|^2+2\alpha_n\langle f(p)-p,x_{n+1}-p\rangle\\
&= (1-(1-\rho)\alpha_n)\|x_n-p\|^2+(1-\rho)\alpha_n\frac{2}{1-\rho}\langle f(p)-p,x_{n+1}-p\rangle.
\end{aligned}
\tag{27}
$$

Claim 5 The sequence $\{\|x_n-p\|^2\}$ converges to zero. We consider two possible cases on the sequence $\{\|x_n-p\|^2\}$.

Case 1 There exists an $N\in\mathbb{N}$ such that $\|x_{n+1}-p\|^2\le\|x_n-p\|^2$ for all $n\ge N$. This implies that $\lim_{n\to\infty}\|x_n-p\|^2$ exists. It is implied from **Claim 2** that

$$\lim_{n\to\infty}\|x_n-z_n\|=0.$$

Now, according to Claim 3,

$$\lim_{n\to\infty}\|x_n-y_n\|=0. \tag{28}$$

Since the sequence $\{x_n\}$ is bounded, it implies that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that weak convergence to some $z\in C$ such that

$$\limsup_{n\to\infty}\langle f(p)-p,x_n-p\rangle=\lim_{k\to\infty}\langle f(p)-p,x_{n_k}-p\rangle=\langle f(p)-p,z-p\rangle. \tag{29}$$

Since $x_{n_k}\rightharpoonup z$ and (28), it implies from Lemma 3.3 that $z\in VI(C,A)$.
On the other hand,

$$\|x_{n+1}-z_n\|=\alpha_n\|f(x_n)-z_n\|\to 0 \text{ as} n\to\infty.$$

Thus,

$$\|x_{n+1}-x_n\|=\|x_{n+1}-z_n\|+\|x_n-z_n\|\to 0 \text{ as} n\to\infty.$$

Since $p=P_{VI(C,A)}f(p)$ and $x_{n_k}\rightharpoonup z\in VI(C,A)$, using (29), we get

$$\limsup_{n\to\infty}\langle f(p)-p,x_n-p\rangle=\langle f(p)-p,z-p\rangle\le 0.$$

This implies that

$$
\begin{aligned}
\limsup_{n\to\infty}\langle f(p)-p,x_{n+1}-p\rangle \le\ &\limsup_{n\to\infty}\langle f(p)-p,x_{n+1}-x_n\rangle\\
&+\limsup_{n\to\infty}\langle f(p)-p,x_n-p\rangle\le 0,
\end{aligned}
$$

which, together with Claim 4, implies from Lemma 2.10 that

$$x_n\to p \text{ as} n\to\infty.$$

Case 2 There exists a subsequence $\{\|x_{n_j} - p\|^2\}$ of $\{\|x_n - p\|^2\}$ such that $\|x_{n_j} - p\|^2 < \|x_{n_j+1} - p\|^2$ for all $j \in \mathbb{N}$. In this case, it follows Lemma 2.9 that there exists a nondecreasing sequence $\{m_k\}$ of $\mathbb{N}$ such that $\lim_{k\to\infty} m_k = \infty$ and the following inequalities hold for all $k \in \mathbb{N}$:

$$\|x_{m_k} - p\|^2 \le \|x_{m_k+1} - p\|^2 \text{ and} \|x_k - p\|^2 \le \|x_{m_k+1} - p\|^2. \tag{30}$$

According to Claim 2, we have

$$\begin{aligned}\|z_{m_k} - x_{m_k}\|^2 &\le \|x_{m_k} - p\|^2 - \|x_{m_k+1} - p\|^2 + 2\alpha_{m_k}\langle f(x_{m_k}) - p, x_{m_k+1} - p\rangle. \\ &\le \alpha_{m_k}\langle f(x_{m_k}) - p, x_{m_k+1} - p\rangle \\ &\le \alpha_{m_k}\|f(x_{m_k}) - p\|\|x_{m_k+1} - p\| \to 0 \text{ as} k \to \infty.\end{aligned}$$

According to Claim 3, we have

$$\begin{aligned}(1-\alpha_{m_k})\left[\frac{1}{M}(1-\mu)\|x_{m_k} - y_{m_k}\|^2\right]^2 &\le \|x_{m_k} - p\|^2 - \|x_{m_k+1} - p\|^2 + \alpha_{m_k}\|f(x_{m_k}) - p\|^2 \\ &\le \alpha_{m_k}\|f(x_{m_k}) - p\|^2 \to 0 \text{ as} k \to \infty.\end{aligned}$$

Using the same arguments as in the proof of Case 1, we obtain

$$\|x_{m_k+1} - x_{m_k}\| \to 0$$

and

$$\limsup_{k\to\infty}\langle f(p) - p, x_{m_k+1} - p\rangle \le 0.$$

Since (27), we get

$$\begin{aligned}\|x_{m_k+1} - p\|^2 &\le (1 - \alpha_{m_k}(1-\rho))\|x_{m_k} - p\|^2 + 2\alpha_{m_k}\langle f(p) - p, x_{m_k+1} - p\rangle \\ &\le (1 - \alpha_{m_k}(1-\rho))\|x_{m_k+1} - p\|^2 + 2\alpha_{m_k}\langle f(p) - p, x_{m_k+1} - p\rangle.\end{aligned}$$

which, together with (30), implies that

$$\|x_k - p\|^2 \le \|x_{m_k+1} - p\|^2 \le 2\langle f(p) - p, x_{m_k+1} - p\rangle.$$

Therefore, $\limsup_{k\to\infty}\|x_k - p\| \le 0$, that is $x_k \to p$. The proof is completed.

□

Applying Algorithm 3.2 with $f(x) := x_1$ for all $x \in C$, we obtain the following corollary.

Corollary 3.7 *Given* $\gamma > 0, l \in (0,1), \mu \in (0,1)$. *Let* $x_1 \in C$ *be arbitrary. Compute*

$$y_n = P_C(x_n - \lambda_n Ax_n),$$

where λ_n *is chosen to be the largest* $\lambda \in \{\gamma, \gamma l, \gamma l^2, ...\}$ *satisfying*

$$\lambda\langle Ax_n - Ay_n, x_n - y_n\rangle \le \mu\|x_n - y_n\|^2.$$

If $y_n = x_n$ *then stop and* x_n *is the solution of VIP. Otherwise, compute*

$$x_{n+1} = \alpha_n x_1 + (1-\alpha_n)P_C(x_n),$$

where

$$C_n := \{x \in H : h_n(x) \leq 0\}$$

and

$$h_n(x) = \langle x_n - y_n - \lambda_n(Ax_n - Ay_n), x - y_n \rangle.$$

Assume that Conditions 1–4 hold. Then the sequence $\{x_n\}$ *converges strongly to* $p \in VI(C, A)$*, where* $p = P_{VI(C,A)}x_1$.

4 Numerical illustrations

Some numerical implementations of our proposed methods in this paper are provided in this section. We give the test examples both in finite-dimensional and infinite-dimensional Hilbert spaces and give numerical comparisons in all cases.

In the first two examples, we consider test examples in finite dimensional and implement our proposed Algorithm 3.1. We compare our method with Algorithm 1 of Iusem [19] (Iusem Alg. 1.1).

Example 4.1 Let us consider VIP (1) with

$$A(x) = \begin{bmatrix} (x_1^2 + (x_2 - 1)^2)(1 + x_2) \\ \\ -x_1^3 - x_1(x_2 - 1)^2 \end{bmatrix}$$

and

$$C := \{x \in \mathbb{R}^2 : -10 \leq x_i \leq 10, i = 1, 2\}.$$

This VIP has unique solution $x^* = (0, -1)^T$. It is easy to see that A is not a monotone map on C. However, using the Monte Carlo approach (see [18]), it can be shown that A is pseudo-monotone on C. Let x_1 be the initial point be randomly generated vector in C, $l = 0.1$, $\gamma = 2$. We terminate the iterations if $\|x_n - y_n\|_2 \leq \varepsilon$ with $\varepsilon = 10^{-3}$, $\|.\|_2$ is the Euclidean norm on $\mathbb{R}^2$. The results are listed in Table 1 and Figs. 1, 2, 3, and 4 below. We consider different values of μ (Table 2).

Example 4.2 Consider VIP (1) with

$$A(x) = \begin{bmatrix} 0.5x_1x_2 - 2x_2 - 10^7 \\ \\ -4x_1 + 0.1x_2^2 - 10^7 \end{bmatrix}$$

and

$$C := \{x \in \mathbb{R}^2 : (x_1 - 2)^2 + (x_2 - 2)^2 \leq 1\}.$$

Then A is not monotone on C but pseudo-monotone (see [18]). Furthermore, the VIP (1) has a unique solution $x^* = (2.707, 2.707)^T$. Take $l = 0.1$, $\gamma = 3$, and $\mu = 0.2$. We terminate the iterations if $\|x_n - y_n\|_2 \leq \varepsilon$ with $\varepsilon = 10^{-2}$. The results are listed in Table 3 and Figs. 5, 6, 7, and 8 below. We consider different choices of initial point x_1 in C.

Table 1 Example 1 comparison: proposed alg. 3.2 vs Iusem alg. 1.1

	Proposed alg. 3.2			Iusem alg. 1.1		
μ	Iter.	CPU time)	Sol.	Iter.	CPU (time)	Sol.
0.1	52	1.1396×10^{-3}	$(-0.0002, -1.0008)^T$	5025	9.6265×10^{-2}	$(0.1560, 0.9780)^T$
0.5	29	1.0681×10^{-3}	$(-0.0008, -1.0000)^T$	1556	2.5867×10^{-2}	$(-0.0122, -1.0025)^T$
0.7	29	8.5953×10^{-4}	$(-0.0008, -1.0000)^T$	967	1.7481×10^{-2}	$(-0.0033, -0.9878)^T$
0.9	35	7.1767×10^{-4}	$(0.0000, -0.9992)^T$	53	1.2833×10^{-3}	$(-0.0004, -0.9992)^T$

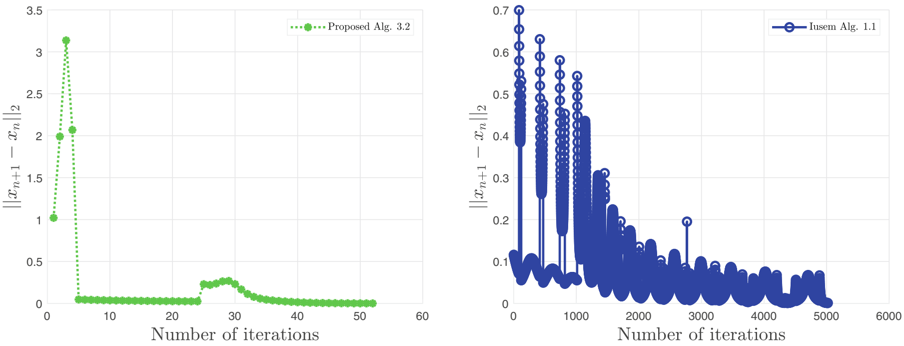

Fig. 1 Example 1 comparison with $\mu = 0.1$

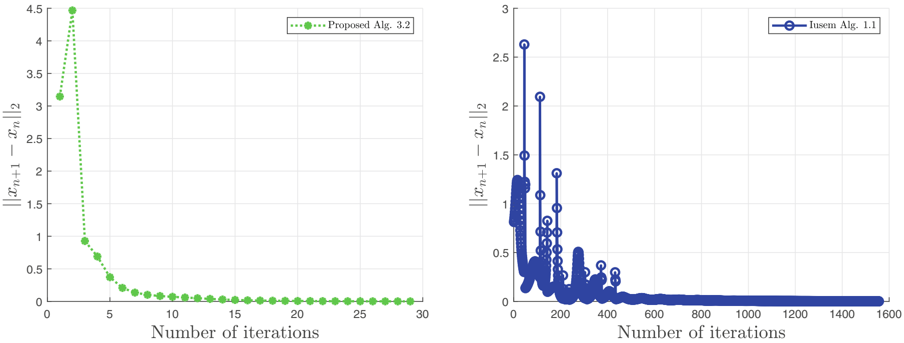

Fig. 2 Example 1 comparison with $\mu = 0.5$

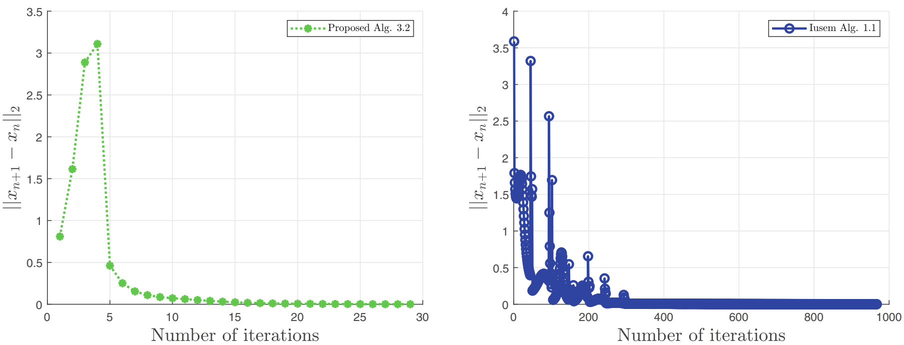

Fig. 3 Example 1 comparison with $\mu = 0.7$

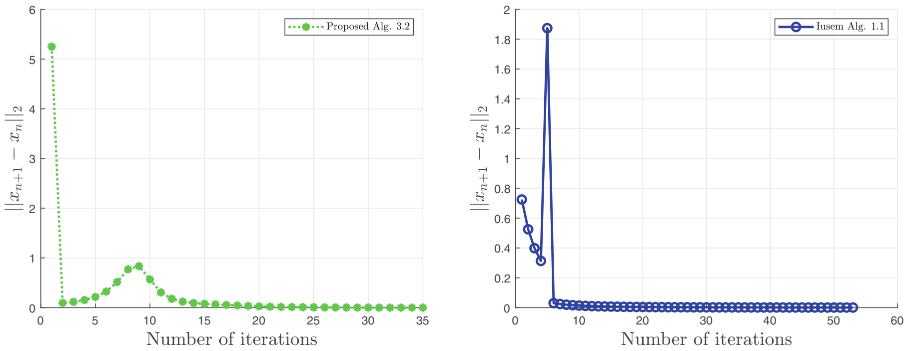

Fig. 4 Example 1 comparison with $\mu = 0.9$

Next, we give the following two examples in infinite-dimensional spaces to illustrate our proposed Algorithm 2. Here, we compare our proposed Algorithm 2 with the method proposed by Vuong and Shehu in [46] with $\alpha_n = \frac{1}{n+1}$.

Example 4.3 Consider $H := L^2([0, 1])$ with inner product $\langle x, y\rangle := \int_0^1 x(t)y(t)dt$ and norm $\|x\|_2 := (\int_0^1 |x(t)|^2 dt)^{\frac{1}{2}}$. Suppose $C := \{x \in H : \|x\|_2 \leq 2\}$. Let $g : C \to \mathbb{R}$ be defined by

$$g(u) := \frac{1}{1 + \|u\|_2^2}.$$

Observe that g is L_g-Lipchitz continuous with $L_g = \frac{16}{25}$ and $\frac{1}{5} \leq g(u) \leq 1, \forall u \in C$. Define the Volterra integral mapping $F : L^2([0, 1]) \to L^2([0, 1])$ by

$$F(u)(t) := \int_0^t u(s)ds, \forall u \in L^2([0, 1]), t \in [0, 1].$$

Then F is bounded linear monotone (see Exercise 20.12 of [4]). Now, define $A : C \to L^2([0, 1])$ by

$$A(u)(t) := g(u)F(u)(t), \forall u \in C, t \in [0, 1].$$

Table 2 Example 1: comparison of the inner loop to obtain λ_n

μ	Proposed alg. 3.2 iter.	Iusem alg. 1.1 iter.
0.1	77	17853
0.5	28	3099
0.7	30	2071
0.9	26	322

Table 3 Example 2 Comparison: proposed alg. 3.2 vs Iusem alg. 1.1

	Proposed alg. 3.2			Iusem alg. 1.1		
x_1	Iter.	CPU (time)	Sol.	Iter.	CPU (time)	Sol.
$(1.5, 1.7)^T$	4	6.0743×10^{-3}	$(2.7071, 2.7069)^T$	9952	0.24550	$(2.7000, 2.7141)^T$
$(2, 3)^T$	3	3.2088×10^{-3}	$(2.7071, 2.7072)^T$	10000	0.24496	$(2.7000, 2.7141)^T$
$(2, 1)^T$	3	2.7252×10^{-3}	$(2.7070, 2.7071)^T$	10000	0.24191	$(2.7141, 2.7000)^T$
$(2.7, 2.6)^T$	3	3.9589×10^{-3}	$(2.7074, 2.7070)^T$	9802	0.24557	$(2.7141, 2.7000)^T$

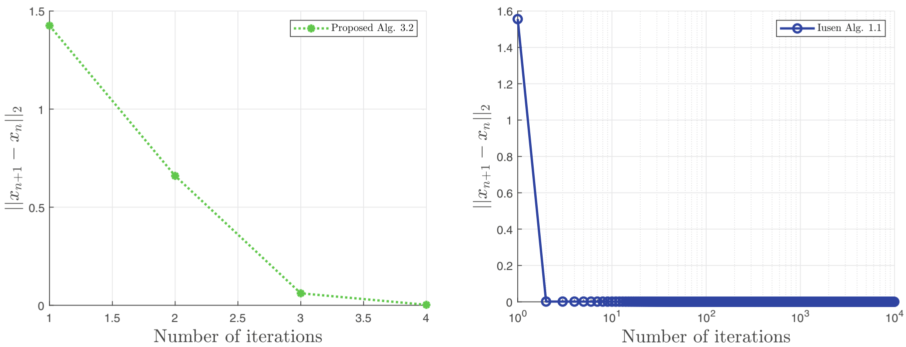

Fig. 5 Example 2 comparison with $x_1 = (1.5, 1.7)^T$

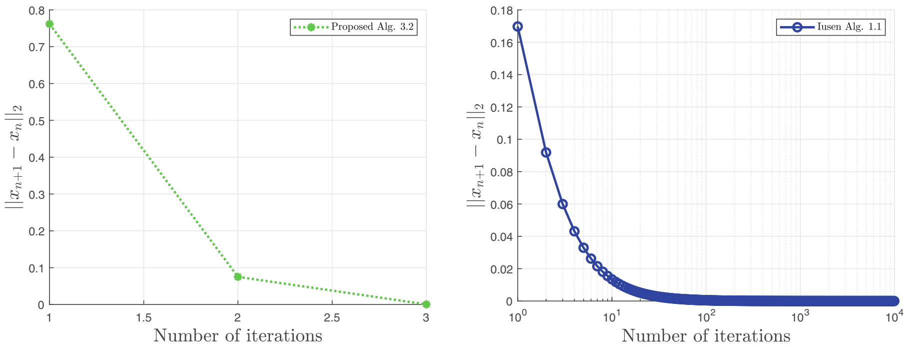

Fig. 6 Example 2 comparison with $x_1 = (2, 3)^T$

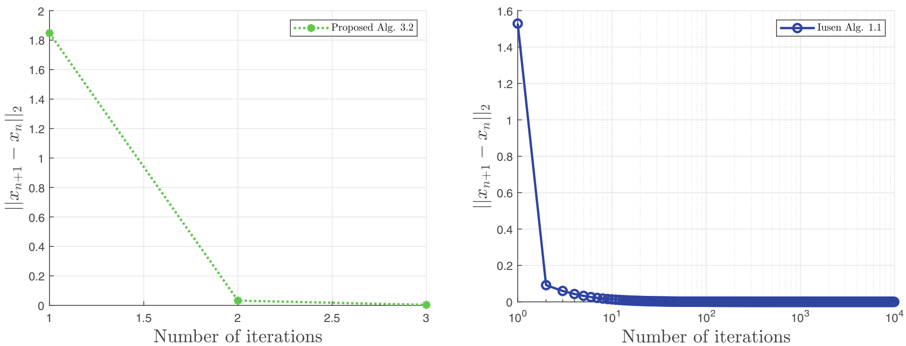

Fig. 7 Example 2 comparison with $x_1 = (2, 1)^T$

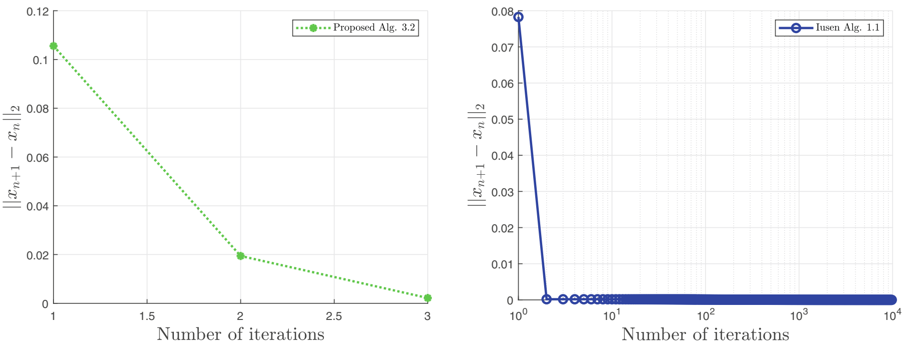

Fig. 8 Example 2 comparison with $x_1 = (2.7, 2.6)^T$

As given in [37], A is pseudo-monotone mapping but not monotone since

$$\langle Av - Au, v - u\rangle = -\frac{3}{10} < 0$$

with $v = 1$ and $u = 2$.

Take $l = 0.015$, $\gamma = 3$ and $\mu = 0.1$ (Table 4). We terminate the iterations if $\|x_n - P_C(x_n - A(x_n))\|_2 \leq \varepsilon$ with $\varepsilon = 10^{-2}$. The results are listed in Table 5 and Figs. 9, 10 ,and 11 below. We consider different choices of initial point x_1 in C (Table 6).

Example 4.4 Take

$$H := L^2([0, 1]) \quad \text{and} \quad C := \{x \in H : \|x\|_2 \leq 2\}.$$

Define $A : L^2([0, 1]) \to L^2([0, 1])$ by

$$A(u)(t) := e^{-\|u\|_2} \int_0^t u(s)ds, \forall u \in L^2([0, 1]), t \in [0, 1].$$

It can also be shown that A is pseudo-monotone but not monotone on H.

Table 4 Example 2: comparison of the inner loop to obtain λ_n

Initial point x_1	Proposed alg. 3.2 iter.	Iusem alg. 1.1 iter.
$(1.5, 1.7)^T$	5	19902
$(2, 3)^T$	3	19998
$(2, 1)^T$	3	19998
$(2.7, 2.6)^T$	3	19602

Table 5 Example 3 comparison: proposed alg. 3.3 vs Vuong and Shehu alg.

	Proposed alg. 3.3		Vuong and Shehu alg.	
x_1	Iter.	CPU (time)	Iter.	CPU (Time)
$\frac{\sin(t)}{6}$	22	8.2281×10^{-3}	1693	0.12414
$\frac{5}{29}t$	29	1.1487×10^{-2}	1907	0.12442
$\frac{\cos(t)}{7}$	33	1.3024×10^{-2}	2656	0.18212

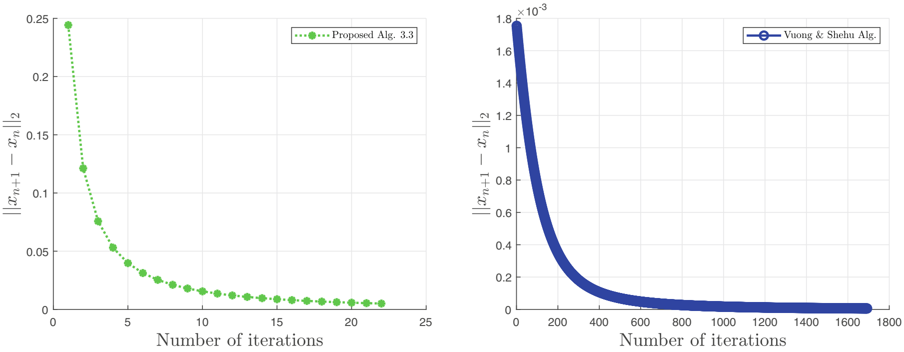

Fig. 9 Example 3 comparison with $x_1 = \frac{\sin(t)}{6}$

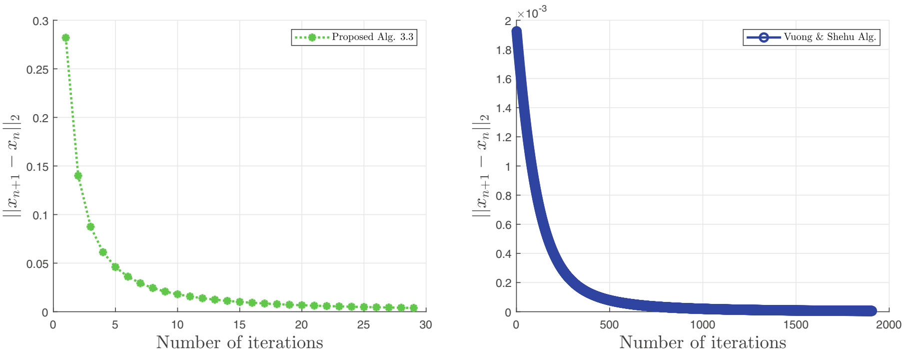

Fig. 10 Example 3 comparison with $x_1 = \frac{5}{29}t$

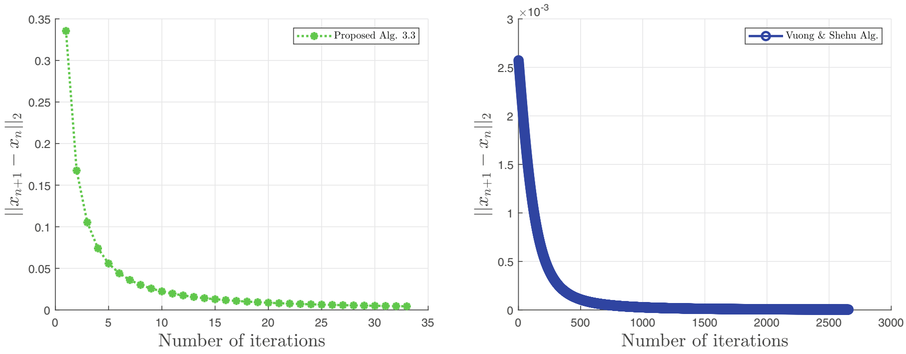

Fig. 11 Example 3 comparison with $x_1 = \frac{\cos(t)}{7}$

Table 6 Example 3: Comparison of the inner loop to obtain λ_n

Initial point x_1	Proposed alg. 3.3 iter.	Vuong and Shehu alg. iter.
$\frac{\sin(t)}{6}$	21	1692
$\frac{5}{29}t$	28	1906
$\frac{\cos(t)}{7}$	32	2655

Table 7 Example 4 comparison: proposed alg. 3.3 vs Vuong and Shehu alg.

	Proposed alg. 3.3		Vuong and Shehu alg.	
x_1	Iter.	CPU (time)	Iter.	CPU (time)
$\frac{\sin(t)}{6}$	29	8.9724×10^{-3}	1693	0.11536
$\frac{5}{29}t$	38	1.1887×10^{-2}	1907	0.11993
$\frac{\cos(t)}{7}$	41	1.4232×10^{-2}	2656	0.16087

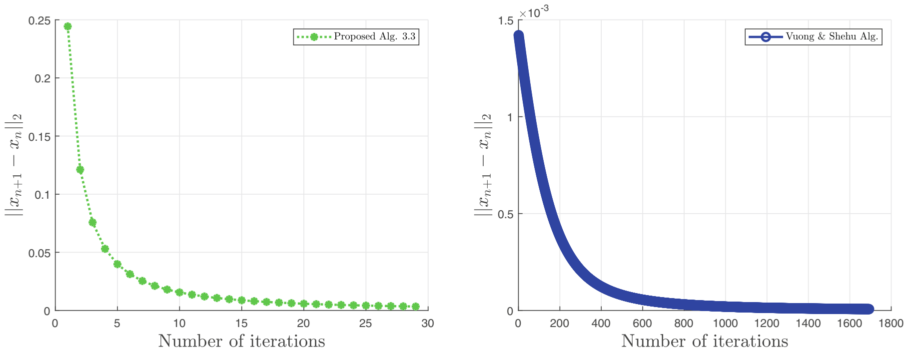

Fig. 12 Example 4 comparison with $x_1 = \frac{\sin(t)}{6}$

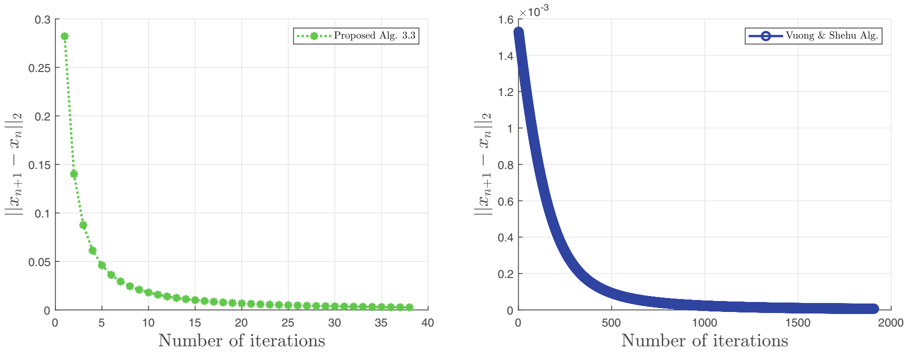

Fig. 13 Example 4 comparison with $x_1 = \frac{5}{29}t$

Take $l = 0.015$, $\gamma = 4$, and $\mu = 0.1$. We terminate the iterations if $\|x_n - P_C(x_n - A(x_n))\|_2 \leq \varepsilon$ with $\varepsilon = 10^{-2}$. The results are listed in Table 7 and Figs. 12, 13, and 14 below. We consider different choices of initial point x_1 in C.

Remark 4.5

1. Our proposed algorithms are efficient and easy to implement evident from many examples provided above.
2. We observe that the choices of initial point x_1 and μ have no significant effect on the number of iterations and the CPU time required to reach the stopping criterion. See all the examples above.
3. Clearly from the numerical examples presented above, our proposed algorithms outperformed Algorithm 1 proposed by Iusem both in the number of iterations and CPU time required to reach the stopping criterion. The same observation is seen when compared with the algorithm proposed by Vuong and Shehu in [46].
4. Furthermore, comparison of our proposed algorithms 3.2 and 3.3 are made with both algorithms proposed by Iusem and Vuong and Shehu using the inner loop

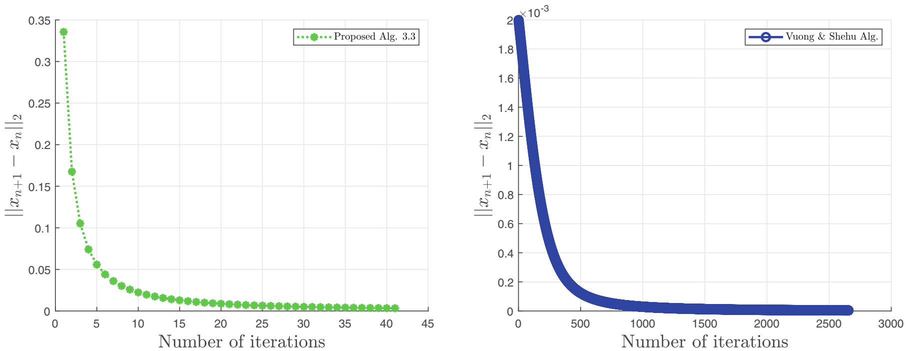

Fig. 14 Example 4 comparison with $x_1 = \frac{\cos(t)}{7}$

Table 8 Example 4: comparison of the inner loop to obtain λ_n

Initial point x_1	Proposed alg. 3.3 iter.	Vuong and Shehu alg. iter.
$\frac{\sin(t)}{6}$	28	1692
$\frac{5}{29}t$	37	1906
$\frac{\cos(t)}{7}$	40	2655

to obtain λ_n, see Tables 2, 4, 6, and 8. Again, we could observe great advantages of our proposed algorithms over others.

5 Conclusions

We obtain weak and strong convergence of two projection-type methods for solving VIP under pseudo-monotonicity and non-Lipschitz continuity of the VI-associated mapping A. These two properties emphasize the applicability and advantages over several existing results in the literature. Numerical experiments performed in both finite- and infinite-dimensional spaces real Hilbert spaces show that our proposed methods outperform some already known methods for solving VIP in the literature.

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