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New strong convergence theorem of the inertial projection and contraction method for variational inequality problems

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Abstract

In this paper, we introduce a new algorithm which combines the inertial projection and contraction method and the viscosity method for solving monotone variational inequality problems in real Hilbert spaces and prove a strong convergence theorem of our proposed algorithm under the standard assumptions imposed on cost operators. Finally, we give some numerical experiments to illustrate the proposed algorithm.

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1 Introduction

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H and $A : H \to H$ be an operator.

The *variational inequality problem* **(VIP)** for A on C is to find a point $x^* \in C$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0, \quad \forall x \in C. \tag{1}$$

Let us denote $VI(C, A)$ by the solution set of the problem **(VIP)** (1). The problem of finding solutions of the problem **(VIP)** (1) is a fundamental problem in optimization theory. Due to this, the problem **(VIP)** (1) has received a lot of attention by many authors. In fact, there are two general approaches to study variational inequality problems, which are the regularized method and the projection method. Based on these directions, many algorithms have been considered and proposed for solving the problem **(VIP)** (1) (see, for example, [13–16, 26, 31, 34, 35, 43, 46, 53–55]).

The basic idea consists of extending the projected gradient method for solving the problem of minimizing $f(x)$ subject to $x \in C$ given by

$$x_{n+1} = P_C(x_n - \alpha_n \nabla f(x_n)), \quad \forall n \geq 0, \tag{2}$$

where $\{\alpha_n\}$ is a positive real sequence satisfying certain conditions and P_C is the metric projection onto C.

For convergence properties of this method for the case in which $f : \mathbb{R}^2 \to \mathbb{R}$ is convex and differentiable function, one may see [1]. An immediate extension of the method (2) to **(VIP)** (1) is the projected gradient method for optimization problems, substituting the operator A for the gradient, so that we generate a sequence $\{x_n\}$ in the following manner:

$$x_{n+1} = P_C(x_n - \alpha_n Ax_n), \quad \forall n \geq 0.$$

However, the convergence of this method requires a slightly strong assumption that the operators are strongly monotone or inverse strongly monotone (see, for example, [51]).

To avoid this strong assumption, Korpelevich [27] introduced the extragradient method **(EGM)** for solving saddle point problems and, after that, this method was further extended to variational inequality problems in both Euclidean spaces and Hilbert spaces. The convergence of the extragradient method only requires that the operator A is monotone and L-Lipschitz continuous. More precisely, the extragradient method is of the form:

$$\begin{cases} y_n = P_C(x_n - \lambda Ax_n), \\ x_{n+1} = P_C(x_n - \lambda Ay_n), \quad \forall n \geq 0, \end{cases} \tag{3}$$

where $\lambda \in (0, 1/L)$ and P_C denotes the metric projection from H onto C.

If the solution set $VI(C, A)$ is nonempty, then the sequence $\{x_n\}$ generated by the process (3) converges weakly to an element in $VI(C, A)$.

In recent years, the **(EGM)** (3) has received great attention by many authors, who improved it in various ways (see, for example, [13–17, 29, 34, 35, 38, 41, 47–49] and the references therein).

In fact, the **(EGM)** (3) needs to calculate two projections onto the closed convex set C in each iteration. If C is a general closed convex set, then this might seriously affect the efficiency of the algorithm.

To our knowledge, there are some methods to overcome this drawback. The first one is the subgradient extragradient method proposed by Censor et al. [14], in which the second projection onto C is replaced by a projection onto a specific constructible half-space. Their algorithm is of the form:

$$\begin{cases} y_n = P_C(x_n - \lambda Ax_n), \\ T_n = \{w \in H | \langle x_n - \lambda Ax_n - y_n, w - y_n \rangle \leq 0\}, \\ x_{n+1} = P_{T_n}(x_n - \lambda Ay_n), \quad \forall n \geq 0, \end{cases}$$

where $\lambda \in (0, 1/L)$.

The second one is the method proposed by Tseng in [49]. Tseng's method is of the form:

$$\begin{cases} y_n = P_C(x_n - \lambda Ax_n), \\ x_{n+1} = y_n - \lambda(Ay_n - Ax_n), \quad \forall n \geq 0, \end{cases}$$

where $\lambda \in (0, 1/L)$. Recently, Tseng's extragradient method for solving **(VIP)** (1) has received great attention by many authors (see, for example, [7, 43, 50] and the references therein).

The third one is the projection and contraction method studied by some authors [25, 42]. The projection and contraction method is of the form:

$$\begin{cases} x_0 \in H, \\ y_n = P_C(x_n - \lambda Ax_n), \\ d(x_n, y_n) = (x_n - y_n) - \lambda(Ax_n - Ay_n), \\ x_{n+1} = x_n - \gamma\beta_n d(x_n, y_n), \quad \forall n \geq 0, \end{cases}$$

where $\gamma \in (0, 2)$, $\lambda \in (0, 1/L)$, and

$$\beta_n := \frac{\phi(w_n, y_n)}{\|d(w_n, y_n)\|^2}, \quad \phi(w_n, y_n) := \langle w_n - y_n, d(w_n, y_n) \rangle, \quad \forall n \geq 0.$$

In recent years, the projection and contraction method has received great attention by many authors, who improved it in various ways (see, for example, [12, 19–21] and the references therein).

Note that the three methods above need only to calculate one projection onto C in each iteration. This may increase the performance of the algorithms.

Now, let us mention an inertial-type algorithm which is based upon a discrete version of a second-order dissipative dynamical system [4, 5] and it can be regarded as a procedure of speeding up the convergence properties (see [3, 33, 39]).

In 2001, Alvarez and Attouch [3] applied the inertial technique to obtain an inertial proximal method for solving the problem of finding zero of a maximal monotone operator, which is as follows: for any $x_{n-1}, x_n \in H$ and two parameters $\theta_n \in [0, 1)$, $\lambda_n > 0$, find $x_{n+1} \in H$ such that

$$0 \in \lambda_n A(x_{n+1}) + x_{n+1} - x_n - \theta_n(x_n - x_{n-1}), \quad \forall n \geq 0,$$

which can be written equivalently as follows:

$$x_{n+1} = J_{\lambda_n}^A(x_n + \theta_n(x_n - x_{n-1})), \quad \forall n \geq 0,$$

where $J_{\lambda_n}^A$ is the resolvent of A with parameter λ_n and the inertia is induced by the term $\theta_n(x_n - x_{n-1})$.

Recently, the inertial methods have been studied by several authors (see [2, 3, 6–11, 18, 22, 32, 36, 37, 44, 45]).

In 2015, Bot and Csetnek [11] introduced the so-called inertial hybrid proximal hybrid proximal-extragradient algorithm, which combines the inertial-type algorithm and the hybrid proximal-extragradient for a maximally monotone operator.

Very recently, Dong et al. [22] proposed an algorithm as a combination between inertial projection and contraction method (shortly, IPCM) and inertial method. The IPCM is of the form

$$\begin{cases} x_0, x_1 \in H, \\ w_n = x_n + \alpha_n(x_n - x_{n-1}), \\ y_n = P_C(w_n - \lambda A w_n), \\ d(w_n, y_n) = (w_n - y_n) - \lambda(A w_n - A y_n), \\ x_{n+1} = w_n - \gamma \beta_n d(w_n, y_n), \quad \forall n \geq 0, \end{cases}$$

where $\gamma \in (0, 2)$, $\lambda \in (0, 1/L)$, and

$$\beta_n := \begin{cases} \dfrac{\phi(w_n, y_n)}{\|d(w_n, y_n)\|^2}, & \text{if } d(w_n, y_n) \neq 0 \\ 0, & \text{if } d(w_n, y_n) = 0, \end{cases}$$

where $\phi(w_n, y_n) := \langle w_n - y_n, d(w_n, y_n) \rangle$. Under appropriate conditions, they proved that the sequence $\{x_n\}$ converges weakly to an element of $VI(C, A)$.

Motivated and inspired by the works in the literature. In this paper, we study strong convergence of the algorithm for solving classical variational inequalities problem with Lipschitz continuous and monotone mapping in real Hilbert spaces. The algorithm is inspired by the inertial projection and contraction method and the viscosity method. Under several appropriate conditions imposed on parameters, we will prove that the proposed algorithm converges strongly to some point in $VI(C, A)$. Finally, we present several numerical experiments to support convergence of theorems. The numerical illustrations show that the speed of the proposed algorithm with inertial effects is better than the original algorithm without inertial effects.

This paper is organized as follows: In Section 2, we recall some definitions and preliminary results for further use. Section 3 deals with analyzing the convergence of the proposed algorithm. Finally, in Section 4, we perform several numerical examples to illustrate the computational performance of our algorithm.

2 Preliminaries

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. The weak convergence of $\{x_n\}$ to x is denoted by $x_n \rightharpoonup x$ as $n \to \infty$, while the strong convergence of $\{x_n\}$ to x is written as $x_n \to x$ as $n \to \infty$.

For all $x, y \in H$, we have

$$\|x + y\|^2 = \|x\|^2 + 2\langle x, y \rangle + \|y\|^2$$

and

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle. \tag{4}$$

For all $x \in H$, there exists the unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C.$$

P_C is called the *metric projection* of H onto C. It is known that P_C is nonexpansive.

Lemma 1 [23] *Let C be a nonempty closed convex subset of a real Hilbert space H. For any $x \in H$ and $z \in C$, we have*

$$z = P_C x \iff \langle x - z, z - y \rangle \geq 0, \quad \forall y \in C.$$

Lemma 2 [23] *Let C be a closed and convex subset in a real Hilbert space H and let $x \in H$. Then we have the following:*

(1) $\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y \rangle$ *for all $y \in H$;*
(2) $\|P_C x - y\|^2 \leq \|x - y\|^2 - \|x - P_C x\|^2$ *for all $y \in C$.*

For some properties of the metric projection, refer to Section 3 in [23].

Definition 1 Let $T : H \to H$ be an operator. Then

(1) T is said to be *L-Lipschitz continuous* with $L > 0$ if

$$\|Tx - Ty\| \leq L\|x - y\|, \quad \forall x, y \in H;$$

(2) T is said to be *monotone* if

$$\langle Tx - Ty, x - y \rangle \geq 0, \quad \forall x, y \in H.$$

Lemma 3 [30] *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that there exists a subsequence $\{a_{n_j}\}$ of $\{a_n\}$ such that $a_{n_j} < a_{n_j+1}$ for each $j \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\}$ of $\mathbb{N}$ such that $\lim_{k \to \infty} m_k = \infty$ and the following properties are satisfied by all (sufficiently large) number $k \in \mathbb{N}$:*

$$a_{m_k} \leq a_{m_k+1}, \quad a_k \leq a_{m_k+1}.$$

In fact, m_k is the largest number n in the set $\{1, 2, \cdots, k\}$ such that $a_n < a_{n+1}$.

Lemma 4 *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that:*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n b_n, \quad \forall n \geq 0,$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{b_n\}$ are sequences such that

(a) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(b) $\limsup_{n \to \infty} b_n \leq 0$.
      Then $\lim_{n \to \infty} a_n = 0$.

Remark 1 Lemma 4 was shown and used by several authors. For detail proofs, see Liu [28] and Xu [52]. Furthermore, a variant of Lemma 4 has already been used by Reich in [40].

Lemma 5 [26] *Let $A : H \to H$ be a monotone and L-Lipschitz continuous mapping on C. Let $S = P_C(I - \mu A)$, where $\mu > 0$. If $\{x_n\}$ is a sequence in H satisfying $x_n \rightharpoonup q$ and $x_n - Sx_n \to 0$, then $q \in VI(C, A) = Fix(S)$.*

3 Main results

In this section, we assume that $A : H \to H$ is monotone and Lipschitz continuous on H with the constant L, $VI(C, A) \neq \emptyset$ and $f : H \to H$ be a contraction mapping with contraction parameter $\kappa \in [0, 1)$.

Now, we introduce the following algorithm:

Algorithm 1 (The inertial projection and contraction method for the problem **(VIP)** (1)).

Initialization: Let $\lambda \in (0, 1/L)$ and $\alpha > 0$. Choose two positive sequences $\{\epsilon_n\} \subset [0, \infty)$, $\{\beta_n\} \subset (0, 1)$ satisfying

$$\lim_{n \to \infty} \beta_n = 0, \quad \sum_{n=1}^{\infty} \beta_n = \infty, \tag{5}$$

$$\sum_{n=1}^{\infty} \epsilon_n < \infty, \quad \epsilon_n = \circ(\beta_n). \tag{6}$$

Let $x_0, x_1 \in H$ be arbitrary.
Step 1. Given the iterates x_{n-1} and x_n ($n \geq 1$), choose α_n such that

$$\alpha_n = \begin{cases} \min\left\{\alpha, \dfrac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\} & \text{if } x_n \neq x_{n-1}, \\ \alpha & \text{otherwise.} \end{cases} \tag{7}$$

Set $w_n = x_n + \alpha_n(x_n - x_{n-1})$ and compute

$$y_n = P_C(w_n - \lambda A w_n).$$

If $y_n = w_n$ then stop and y_n is a solution of the problem **(VIP)** (1). Otherwise, go to **Step 2**.
Step 2. Compute

$$x_{n+1} = \beta_n f(x_n) + (1 - \beta_n)(w_n - \theta_n d_n),$$

where

$$d_n := w_n - y_n - \lambda(A w_n - A y_n)$$

and

$$\theta_n := (1 - \lambda L)\frac{\|w_n - y_n\|^2}{\|d_n\|^2}.$$

Set $n := n + 1$ and go to **Step 1**.

Lemma 6 *If $y_n = w_n$ or $d_n = 0$ in Algorithm 1, then $y_n \in VI(C, A)$.*

Proof Since A is L-Lipschitz continuous, we have

$$
\begin{aligned}
\|d_n\| &= \|w_n - y_n - \lambda[Aw_n - Ay_n]\| \\
&\geq \|w_n - y_n\| - \lambda\|Aw_n - Ay_n\| \\
&\geq \|w_n - y_n\| - \lambda L\|w_n - y_n\| \\
&= (1 - \lambda L)\|w_n - y_n\|.
\end{aligned}
\tag{8}
$$

It is also easy to see that

$$
\|d_n\| \leq (1 + \lambda L)\|w_n - y_n\|.
\tag{9}
$$

Combining (8) and (9), we get

$$
(1 - \lambda L)\|w_n - y_n\| \leq \|d_n\| \leq (1 + \lambda L)\|w_n - y_n\|
\tag{10}
$$

and so $w_n = y_n$ if and only if $d(w_n, y_n) = 0$. Therefore, if $w_n = y_n$ or $d_n = 0$, then $w_n = y_n$ and we get

$$
y_n = P_C(y_n - \lambda Ay_n).
$$

This implies that $y_n \in VI(C, A)$. This completes the proof. □

Theorem 1 *The sequence $\{x_n\}$ generated by Algorithm 1 converges strongly to an element $p \in VI(C, A)$, where $p = P_{VI(C,A)} \circ f(p)$.*

Proof **Claim 1.** Let $z_n = w_n - \theta_n d_n$. Now, we show that

$$
\|z_n - p\|^2 \leq \|w_n - p\|^2 - \|z_n - w_n\|^2.
\tag{11}
$$

Indeed, we have

$$
\begin{aligned}
&\langle w_n - p, d_n \rangle \\
&= \langle w_n - y_n, d_n \rangle + \langle y_n - p, d_n \rangle \\
&= \langle w_n - y_n, w_n - y_n - \lambda(Aw_n - Ay_n) \rangle + \langle y_n - p, w_n - y_n - \lambda(Aw_n - Ay_n) \rangle \\
&= \|w_n - y_n\|^2 - \langle w_n - y_n, \lambda(Aw_n - Ay_n) \rangle + \langle y_n - p, w_n - y_n - \lambda(Aw_n - Ay_n) \rangle \\
&\geq \|w_n - y_n\|^2 - \lambda L\|w_n - y_n\|^2 + \langle y_n - p, w_n - y_n - \lambda(Aw_n - Ay_n) \rangle. \quad (12)
\end{aligned}
$$

Note that, from $y_n = P_C(w_n - \lambda Aw_n)$, it follows that

$$
\langle y_n - w_n + \lambda Aw_n, y_n - p \rangle \leq 0.
\tag{13}
$$

By the monotonicity of A and $p \in VI(C, A)$, we have

$$
\langle Ay_n, y_n - p \rangle \geq \langle Ap, y_n - p \rangle \geq 0.
\tag{14}
$$

Thus, from (13) and (14), it follows that

$$
\langle w_n - y_n - \lambda_n(Aw_n - Ay_n), y_n - p \rangle \geq 0.
\tag{15}
$$

Combining (12) and (15), we obtain

$$
\langle w_n - p, d_n \rangle \geq (1 - \lambda L)\|w_n - y_n\|^2.
\tag{16}
$$

On the other hand, we have

$$
\begin{aligned}
\|z_n - p\|^2 &= \|w_n - \theta_n d_n - p\|^2 \\
&= \|w_n - p\|^2 - 2\theta_n \langle d_n, w_n - p \rangle + \theta_n^2 \|d_n\|^2.
\end{aligned}
\tag{17}
$$

Combining (16) and (17), we get

$$\|z_n - p\|^2 \leq \|w_n - p\|^2 - 2\theta_n(1 - \lambda L)\|w_n - y_n\|^2 + \theta_n^2\|d_n\|^2. \tag{18}$$

On the other hand, since $\theta_n = (1 - \lambda L)\dfrac{\|w_n - y_n\|^2}{\|d_n\|^2}$, it follows that

$$\|w_n - y_n\|^2 = \frac{\theta_n\|d_n\|^2}{1 - \lambda L}. \tag{19}$$

Substituting (19) into (18), we obtain

$$\begin{aligned}
\|z_n - p\|^2 &\leq \|w_n - p\|^2 - 2\theta_n^2 \cdot \|d_n\|^2 + \theta_n^2\|d_n\|^2 \\
&= \|w_n - p\|^2 - \|\theta_n \cdot d_n\|^2.
\end{aligned} \tag{20}$$

By the definition of the sequence $\{z_n\}$, we have $z_n = w_n - \theta_n d_n$ and so

$$\theta_n d_n = w_n - z_n, \tag{21}$$

which implies, from (20) and (21), that

$$\|z_n - p\|^2 = \|w_n - p\|^2 - \|w_n - z_n\|^2.$$

Claim 2. The sequence $\{x_n\}$ is bounded. By Claim 1, we have

$$\|z_n - p\| \leq \|w_n - p\|. \tag{22}$$

From the definition of w_n, we get

$$\begin{aligned}
\|w_n - p\| &= \|x_n + \alpha_n(x_n - x_{n-1}) - p\| \\
&\leq \|x_n - p\| + \alpha_n\|x_n - x_{n-1}\| \\
&= \|x_n - p\| + \beta_n \cdot \frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\|.
\end{aligned} \tag{23}$$

From (5), (6), and (7), we have $\dfrac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\| \to 0$; hence, there exists a constant $M_1 > 0$ such that

$$\frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\| \leq M_1 \ \forall n. \tag{24}$$

Combining (22), (23), and (24), we obtain

$$\|z_n - p\| \leq \|w_n - p\| \leq \|x_n - p\| + \beta_n M_1. \tag{25}$$

Since we have $x_{n+1} = \beta_n f(x_n) + (1 - \beta_n)z_n$, it follows that

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\beta_n f(x_n) + (1 - \beta_n)z_n - p\| \\
&= \|\beta_n(f(x_n) - p) + (1 - \beta_n)(z_n - p)\| \\
&\leq \beta_n\|f(x_n) - p\| + (1 - \beta_n)\|z_n - p\| \\
&\leq \beta_n\|f(x_n) - f(p)\| + \beta_n\|f(p) - p\| + (1 - \beta_n)\|z_n - p\| \\
&\leq \beta_n\kappa\|x_n - p\| + \beta_n\|f(p) - p\| + (1 - \beta_n)\|z_n - p\|. \tag{26}
\end{aligned}$$

Substituting (25) into (26), we obtain

$$
\begin{aligned}
\|x_{n+1} - p\| &\leq (1 - (1 - \kappa)\beta_n)\|x_n - p\| + \beta_n M_1 + \beta_n \|f(p) - p\|. \\
&= (1 - (1 - \kappa)\beta_n)\|x_n - p\| + (1 - \kappa)\beta_n \frac{M_1 + \|f(p) - p\|}{1 - \kappa} \\
&\leq \max\left\{\|x_n - p\|, \frac{M_1 + \|f(p) - p\|}{1 - \kappa}\right\} \\
&\leq \cdots \\
&\leq \max\left\{\|x_0 - p\|, \frac{M_1 + \|f(p) - p\|}{1 - \kappa}\right\}.
\end{aligned}
$$

This implies $\{x_n\}$ is bounded. Also, we see that $\{z_n\}$, $\{f(x_n)\}$, and $\{w_n\}$ are bounded. This completes the proof.

Claim 3.

$$
(1 - \beta_n)\|z_n - w_n\|^2 \leq \|x_{n+1} - p\|^2 - \|x_n - p\|^2 + \beta_n M_4
$$

for some $M_4 > 0$. Indeed, we get

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \beta_n \|f(x_n) - p\|^2 + (1 - \beta_n)\|z_n - p\|^2 \\
&\leq \beta_n(\|f(x_n) - f(p)\| + \|f(p) - p\|)^2 + \|z_n - p\|^2 \\
&\leq \beta_n(\kappa\|x_n - p\| + \|f(p) - p\|)^2 + (1 - \beta_n)\|z_n - p\|^2 \\
&\leq \beta_n(\|x_n - p\| + \|f(p) - p\|)^2 + (1 - \beta_n)\|z_n - p\|^2 \\
&= \beta_n\|x_n - p\|^2 + \beta_n(2\|x_n - p\| \cdot \|f(p) - p\| + \|f(p) - p\|^2) \\
&\quad + (1 - \beta_n)\|z_n - p\|^2 \\
&\leq \beta_n\|x_n - p\|^2 + (1 - \beta_n)\|z_n - p\|^2 + \beta_n M_2 \qquad (27)
\end{aligned}
$$

for some $M_2 > 0$. Substituting (11) into (27), we get

$$
\|x_{n+1} - p\|^2 \leq \beta_n\|x_n - p\|^2 + (1 - \beta_n)\|w_n - p\|^2 - (1 - \beta_n)\|z_n - w_n\|^2 + \beta_n M_2, \tag{28}
$$

which implies from (25) that

$$
\begin{aligned}
\|w_n - p\|^2 &\leq (\|x_n - p\| + \beta_n M_1)^2 \\
&= \|x_n - p\|^2 + \beta_n(2M_1\|x_n - p\| + \beta_n M_1^2) \\
&\leq \|x_n - p\|^2 + \beta_n M_3 \qquad (29)
\end{aligned}
$$

for some $M_3 > 0$. Combining (28) and (29), we obtain

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \beta_n\|x_n - p\|^2 + (1 - \beta_n)\|x_n - p\|^2 + \beta_n M_3 \\
&\quad - (1 - \beta_n)\|z_n - w_n\|^2 + \beta_n M_2 \\
&= \|x_n - p\|^2 + \beta_n M_3 - (1 - \beta_n)\|z_n - w_n\|^2 + \beta_n M_2.
\end{aligned}
$$

This implies that

$$
(1 - \beta_n)\|z_n - w_n\|^2 \leq \|x_{n+1} - p\|^2 - \|x_n - p\|^2 + \beta_n M_4,
$$

where $M_4 := M_2 + M_3$.

Claim 4.

$$\|w_n - y_n\|^2 \le \frac{(1 + \lambda L)^2}{(1 - \lambda L)^2} \|z_n - w_n\|^2.$$

Indeed, we have

$$\|w_n - y_n\|^2 = \frac{\theta_n \|d_n\|^2}{1 - \lambda L} = \frac{\|\theta_n d_n\|^2}{(1 - \lambda L)\theta_n} = \frac{\|z_n - w_n\|^2}{(1 - \lambda L)\theta_n}. \tag{30}$$

It follows from (10) that

$$\frac{\|w_n - y_n\|}{\|d_n\|} \ge \frac{1}{1 + \lambda L}.$$

Therefore, we have

$$\theta_n = (1 - \lambda L)\frac{\|w_n - y_n\|^2}{\|d_n\|^2} \ge \frac{1 - \lambda L}{(1 + \lambda L)^2},$$

that is,

$$\frac{1}{\theta_n} \le \frac{(1 + \lambda L)^2}{1 - \lambda L}. \tag{31}$$

Combining (30) and (31), we obtain

$$\|w_n - y_n\|^2 \le \frac{(1 + \lambda L)^2}{(1 - \lambda L)^2} \|z_n - w_n\|^2.$$

Claim 5.

$$\|x_{n+1} - p\|^2$$
$$\le (1 - (1 - \kappa)\beta_n)\|x_n - p\|^2$$
$$+ (1 - \kappa)\beta_n \cdot \left[\frac{2}{1 - \kappa}\langle f(p) - p, x_{n+1} - p \rangle + \frac{3M}{1 - \kappa} \cdot \frac{\alpha_n}{\beta_n} \cdot \|x_n - x_{n-1}\| \right]$$

for some $M > 0$. Indeed, we have

$$\|w_n - p\|^2$$
$$= \|x_n + \alpha_n(x_n - x_{n-1}) - p\|^2$$
$$= \|x_n - p\|^2 + 2\alpha_n\langle x_n - p, x_n - x_{n-1} \rangle + \alpha_n^2 \|x_n - x_{n-1}\|^2$$
$$\le \|x_n - p\|^2 + 2\alpha_n\|x_n - p\|\|x_n - x_{n-1}\| + \alpha_n^2 \|x_n - x_{n-1}\|^2. \tag{32}$$

Using (4), we have

$$\|x_{n+1} - p\|^2$$
$$= \|\beta_n f(x_n) + (1 - \beta_n)z_n - p\|^2$$
$$= \|\beta_n(f(x_n) - f(p)) + (1 - \beta_n)(z_n - p) + \beta_n(f(p) - p)\|^2$$
$$\le \|\beta_n(f(x_n) - f(p)) + (1 - \beta_n)(z_n - p)\|^2 + 2\beta_n\langle f(p) - p, x_{n+1} - p \rangle$$
$$\le \beta_n\|f(x_n) - f(p)\|^2 + (1 - \beta_n)\|z_n - p\|^2 + 2\beta_n\langle f(p) - p, x_{n+1} - p \rangle$$
$$\le \beta_n\kappa^2\|x_n - p\|^2 + (1 - \beta_n)\|z_n - p\|^2 + 2\beta_n\langle f(p) - p, x_{n+1} - p \rangle$$
$$\le \beta_n\kappa\|x_n - p\|^2 + (1 - \beta_n)\|z_n - p\|^2 + 2\beta_n\langle f(p) - p, x_{n+1} - p \rangle$$
$$\le \beta_n\kappa\|x_n - p\|^2 + (1 - \beta_n)\|w_n - p\|^2 + 2\beta_n\langle f(p) - p, x_{n+1} - p \rangle. \tag{33}$$

Substituting (32) into (33), we have

$$\|x_{n+1} - p\|^2$$
$$\leq (1 - (1-\kappa)\beta_n)\|x_n - p\|^2 + 2\alpha_n\|x_n - p\|\|x_n - x_{n-1}\|$$
$$\quad + \alpha_n^2\|x_n - x_{n-1}\|^2 + 2\beta_n\langle f(p) - p, x_{n+1} - p\rangle$$
$$= (1 - (1-\kappa)\beta_n)\|x_n - p\|^2 + (1-\kappa)\beta_n \cdot \frac{2}{1-\kappa}\langle f(p) - p, x_{n+1} - p\rangle$$
$$\quad + \alpha_n\|x_n - x_{n-1}\|(2\|x_n - p\| + \alpha_n\|x_n - x_{n-1}\|)$$
$$\leq (1 - (1-\kappa)\beta_n)\|x_n - p\|^2 + (1-\kappa)\beta_n \cdot \frac{2}{1-\kappa}\langle f(p) - p, x_{n+1} - p\rangle$$
$$\quad + \alpha_n\|x_n - x_{n-1}\|(2\|x_n - p\| + \alpha\|x_n - x_{n-1}\|)$$
$$\leq (1 - (1-\kappa)\beta_n)\|x_n - p\|^2$$
$$\quad + (1-\kappa)\beta_n \cdot \frac{2}{1-\kappa}\langle f(p) - p, x_{n+1} - p\rangle + 3M\alpha_n\|x_n - x_{n-1}\|$$
$$\leq (1 - (1-\kappa)\beta_n)\|x_n - p\|^2$$
$$\quad + (1-\kappa)\beta_n \cdot \left[\frac{2}{1-\kappa}\langle f(p) - p, x_{n+1} - p\rangle + \frac{3M}{1-\kappa} \cdot \frac{\alpha_n}{\beta_n} \cdot \|x_n - x_{n-1}\|\right],$$

where $M := \sup_{n\in\mathbb{N}}\{\|x_n - p\|, \alpha\|x_n - x_{n-1}\|\} > 0$.

Claim 6. Finally, we show that $\{\|x_n - p\|^2\}$ converges to zero by considering two possible cases on the sequence $\{\|x_n - p\|^2\}$.

Case 1. There exists $N \in \mathbb{N}$ such that $\|x_{n+1} - p\|^2 \leq \|x_n - p\|^2$ for each $n \geq N$. This implies that $\lim_{n\to\infty}\|x_n - p\|$ exists and, according to Claim 3, we get

$$\lim_{n\to\infty}\|z_n - w_n\| = 0. \tag{34}$$

Now, we show that, as $n \to \infty$,

$$\|x_{n+1} - x_n\| \to 0, \quad \|y_n - w_n\| \to 0. \tag{35}$$

Indeed, we have

$$\|x_{n+1} - z_n\| = \beta_n\|z_n - f(x_n)\| \to 0, \quad \|x_n - w_n\| = \alpha_n\|x_n - x_{n-1}\| = \beta_n \cdot \frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\| \to 0. \tag{36}$$

This implies, from (34) and (36), that

$$\|x_{n+1} - x_n\| \leq \|x_{n+1} - z_n\| + \|z_n - w_n\| + \|w_n - x_n\| \to 0.$$

From (34) and Claim 4, we get

$$\lim_{n\to\infty}\|y_n - w_n\| = 0. \tag{37}$$

Since the sequence $\{x_n\}$ is bounded, it follows that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging weakly to a point $z \in H$ such that

$$\limsup_{n\to\infty}\langle f(p) - p, x_n - p\rangle = \lim_{k\to\infty}\langle f(p) - p, x_{n_k} - p\rangle = \langle f(p) - p, z - p\rangle, \tag{38}$$

which implies, from (34), that

$$w_{n_k} \rightharpoonup z. \tag{39}$$

From (37), (39), and Lemma 5, we have $z \in VI(C, A)$. Also, from (38) and the definition of $p = P_{VI(C,A)} \circ f(p)$, we have

$$\limsup_{n \to \infty} \langle f(p) - p, x_n - p \rangle = \langle f(p) - p, z - p \rangle \leq 0. \tag{40}$$

Combining (35) and (40), we have

$$\limsup_{n \to \infty} \langle f(p) - p, x_{n+1} - p \rangle = \limsup_{n \to \infty} \langle f(p) - p, x_n - p \rangle$$
$$= \langle f(p) - p, z - p \rangle$$
$$\leq 0.$$

Using Lemma 4 and Claim 5, we get $\lim_{n \to \infty} \|x_n - p\| = 0$.

Case 2. There exists a subsequence $\{\|x_{n_j} - p\|^2\}$ of $\{\|x_n - p\|^2\}$ such that

$$\|x_{n_j} - p\|^2 < \|x_{n_j+1} - p\|^2, \quad \forall j \geq 1.$$

In this case, it follows from Lemma 3 that there exists a nondecreasing sequence $\{m_k\}$ of $\mathbb{N}$ such that $\lim_{k \to \infty} m_k = \infty$ and the following inequalities hold: for each $k \geq 1$,

$$\|x_{m_k} - p\|^2 \leq \|x_{m_k+1} - p\|^2, \quad \|x_k - p\|^2 \leq \|x_{m_k} - p\|^2. \tag{41}$$

According to Claim 3, we have

$$(1 - \beta_{m_k}) \|z_{m_k} - w_{m_k}\|^2 \leq \|x_{m_k} - p\|^2 - \|x_{m_k+1} - p\|^2 + \beta_{m_k} M_4$$
$$\leq \beta_{m_k} M_4.$$

Therefore, we obtain

$$\lim_{k \to \infty} \|z_{m_k} - w_{m_k}\| = 0.$$

Using the same arguments as in the proof of Case 1, we obtain

$$\|x_{m_k+1} - x_{m_k}\| \to 0$$

and

$$\limsup_{k \to \infty} \langle f(p) - p, x_{m_k+1} - p \rangle \leq 0.$$

According to Claim 5, we have

$$\|x_{m_k+1} - p\|^2$$
$$\leq (1 - (1 - \kappa)\beta_{m_k}) \|x_{m_k} - p\|^2$$
$$\quad + (1 - \kappa)\beta_{m_k} \cdot \left[\frac{2}{1 - \kappa} \langle f(p) - p, x_{m_k+1} - p \rangle + \frac{3M}{1 - \kappa} \cdot \frac{\alpha_{m_k}}{\beta_{m_k}} \cdot \|x_{m_k} - x_{m_k-1}\| \right]. \tag{42}$$

From (41) and (42), we obtain

$$\|x_{m_k+1} - p\|^2$$
$$\leq (1 - (1 - \kappa)\beta_{m_k}) \|x_{m_k+1} - p\|^2$$
$$\quad + (1 - \kappa)\beta_{m_k} \cdot \left[\frac{2}{1 - \kappa} \langle f(p) - p, x_{m_k+1} - p \rangle + \frac{3M}{1 - \kappa} \cdot \frac{\alpha_{m_k}}{\beta_{m_k}} \cdot \|x_{m_k} - x_{m_k-1}\| \right].$$

Table 1 Comparison of three algorithms in Example 1

	$x_0 = \frac{1}{2}t^2$		$x_0 = e^{-t}$		$x_0 = (t^3 + 1)e^{-t}$	
	Sec.	Iter.	Sec.	Iter.	Sec.	Iter.
Algorithm KR	11.4953	28321	40.0575	83,227	44.5052	95,145
Algorithm M	4.2687	28321	17.1946	83,227	20.3869	95,145
iPCM	3.0231	14161	9.4175	41,614	10.8513	47,573

Thus, we have

$$\|x_{m_k+1} - p\|^2 \leq \frac{2}{1 - \kappa} \langle f(p) - p, x_{m_k+1} - p \rangle + \frac{3M}{1 - \kappa} \cdot \frac{\alpha_{m_k}}{\beta_{m_k}} \cdot \|x_{m_k} - x_{m_k-1}\|.$$

Therefore, we have

$$\limsup_{k \to \infty} \|x_{m_k+1} - p\| \leq 0. \tag{43}$$

Combining (41) and (43), we have $\limsup_{k \to \infty} \|x_k - p\| \leq 0$, that is, $x_k \to p$. This completes the proof. □
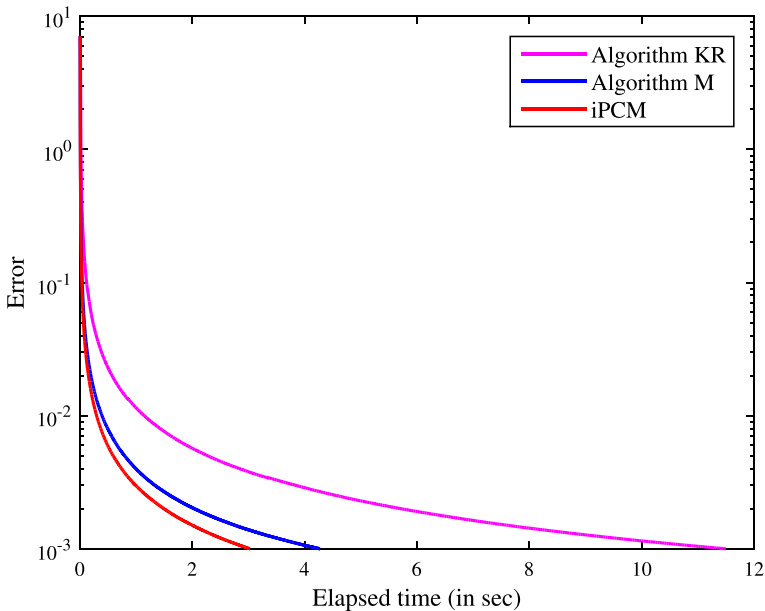

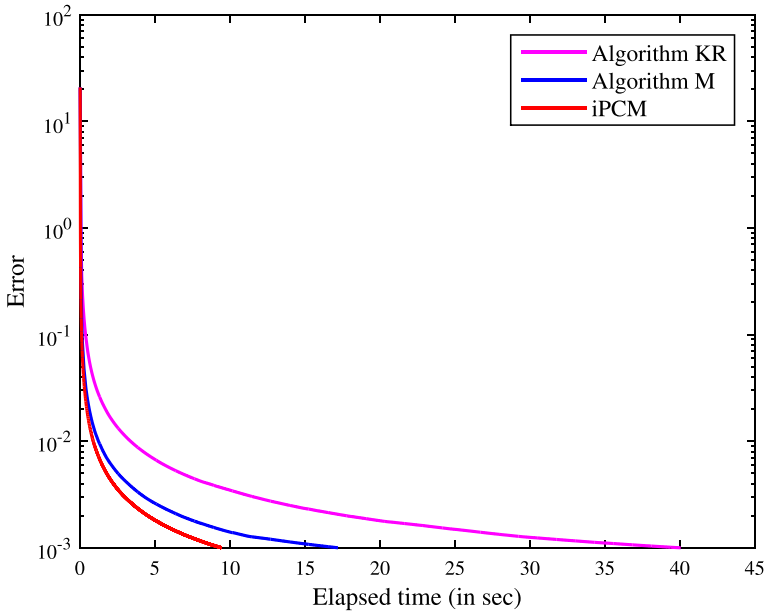**Fig. 1** Comparison of three algorithms in Example 1 with $x_0 = \frac{1}{2}t^2$

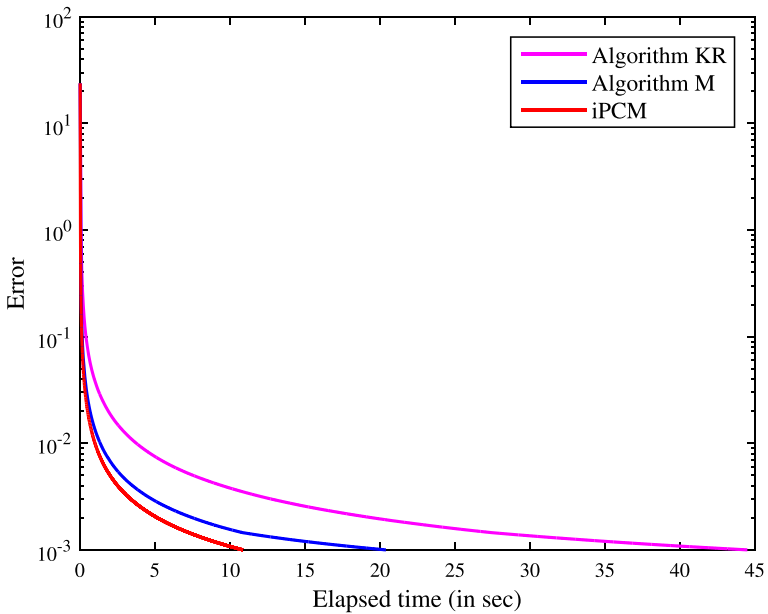**Fig. 2** Comparison of three algorithms in Example 1 with $x_0 = e^{-t}$

Fig. 3 Comparison of three algorithms in Example 1 with $x_0 = (t^3 + 1)e^{-t}$

Table 2 Comparison of two algorithms with different m

	$m = 10$		$m = 50$		$m = 80$		$m = 150$	
	Sec.	Iter.	Sec.	Iter.	Sec.	Iter.	Sec.	Iter.
iPCM	0.5140	84	0.5626	75	3.4844	203	0.6347	57
PCM	0.6219	114	2.6014	376	10.2969	615	5.1199	487

4 Numerical illustrations

In this section, we provide two numerical examples to test the proposed algorithms. All the codes were written in Matlab (R2015a) and run on PC with Intel(R) Core(TM) i3-370M Processor 2.40 GHz.

Now, we apply Algorithm 1 to solve the variational inequality problem **(VIP)** (1) and compare numerical results with other algorithms. In the numerical results reported in the following tables, "Iter." and "Sec." stand for the number of iterations and the cpu time in seconds, respectively.
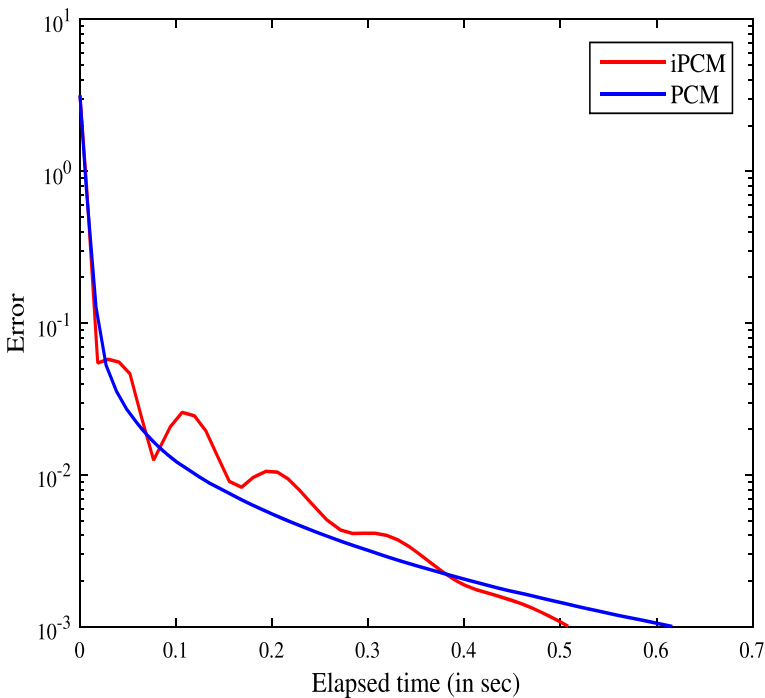

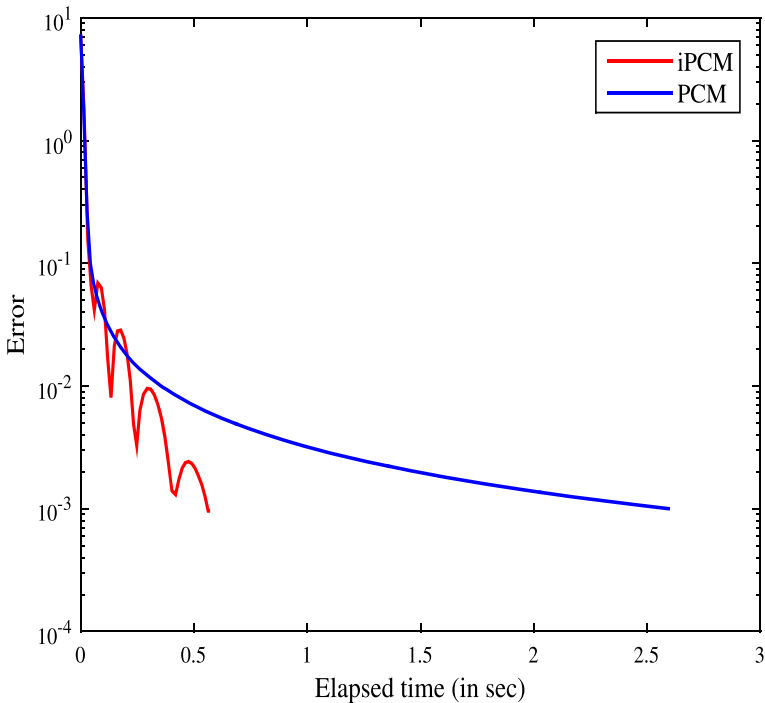**Fig. 4** Comparison of two algorithms in Example 2 with $m = 10$

Fig. 5 Comparison of two algorithms in Example 2 with $m = 50$

Example 1 Suppose that $H = L^2([0, 1])$ with the inner product

$$\langle x, y \rangle := \int_0^1 x(t)y(t)dt, \quad \forall x, y \in H,$$

and the induced norm

$$\|x\| := \left(\int_0^1 |x(t)|^2 dt \right)^{\frac{1}{2}}, \quad \forall x \in H.$$

Let $C := \{x \in H : \|x\| \leq 1\}$ be the unit ball. Define an operator $A : C \to H$ by

$$(Ax)(t) = \max\{0, x(t)\}.$$

It is easy to see that A is 1-Lipschitz continuous and monotone on C. For given C and A, the set of solutions of the variational inequality problem **(VIP)** (1) is given by $\Gamma = \{0\} \neq \emptyset$. It is known that

$$P_C(x) = \begin{cases} \frac{x}{\|x\|_{L^2}}, & \text{if } \|x\|_{L^2} > 1, \\ x, & \text{if } \|x\|_{L^2} \leq 1. \end{cases}$$
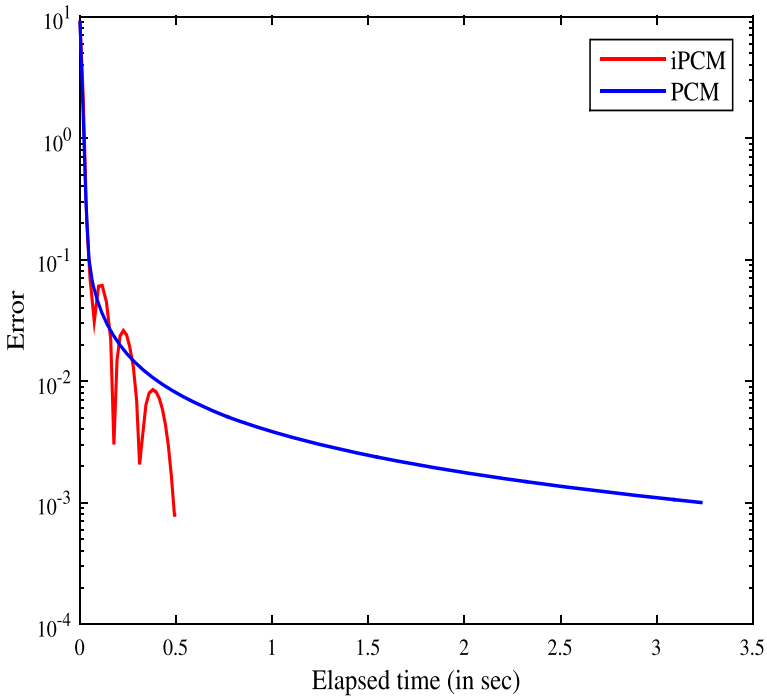
Fig. 6 Comparison of two algorithms in Example 2 with $m = 80$

Now, we apply Algorithm 1 (iPCM), Maingé's algorithm [30] (Algorithm M) and Kraikaew and Saejung's algorithm [26] (Algorithm KR) to solve the variational inequality problem **(VIP)** (1). We use:

(a) The same parameter $\lambda = 0.5$,
(b) The stopping rule $\|x_n - 0\| < 10^{-3}$ for all algorithms,
(c) The same starting point x_0.

Moreover, with respect to Algorithm 1, we take $f(x) = x_1$, $\beta_n = \frac{1}{n}$, and $\alpha = 0.6$. We also choose $\alpha_n = \frac{1}{n}$ for Maingé's algorithm and Kraikaew and Saejung's algorithm. We now make a comparison of three algorithms with different x^0 and report the results in Table 1.

Convergent behavior of Algorithms with different starting point is given in Figs. 1, 2, and 3. In these figures, the value of error $\|x_n - 0\|$ is represented by the y-axis, and number of iterations is represented by the x-axis.

Example 2 Consider the linear operator $A : \mathbb{R}^m \to \mathbb{R}^m$ defined by $A(x) = Mx + q$, which is taken from [24] and has been considered by many authors for numerical experiments (see, for example, [18, 25]), where
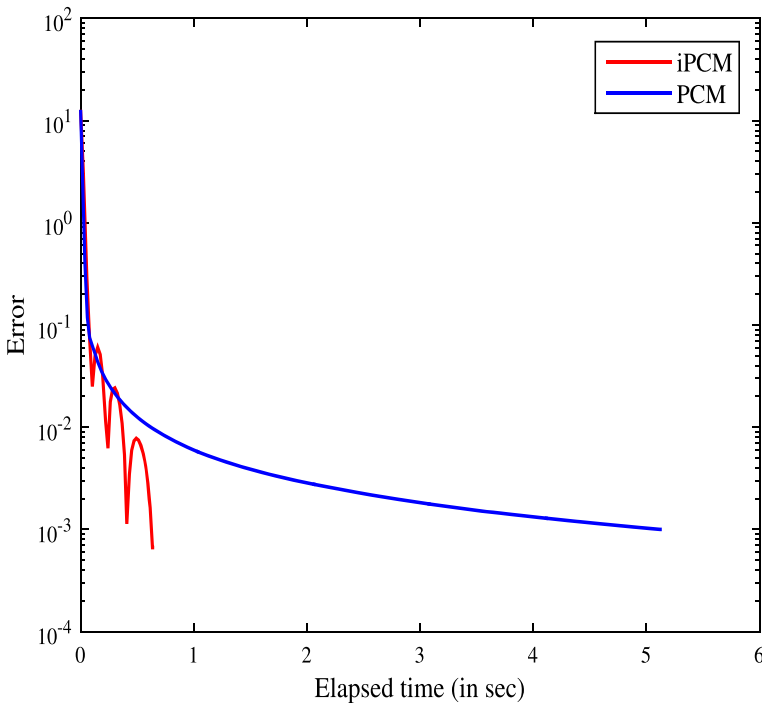
$$M = BB^T + S + D,$$

Fig. 7 Comparison of two algorithms in Example 2 with $m = 150$

B is an $m \times m$ matrix, S is an $m \times m$ skew-symmetric matrix, D is an $m \times m$ diagonal matrix, whose diagonal entries are nonnegative (so M is positive definite), q is a vector in $\mathbb{R}^m$, and

$$C := \left\{ x \in \mathbb{R}^m : -5 \leq x_i \leq 5, \ i = 1, ..., m \right\}.$$

Then, A is monotone and Lipschitz continuous with the Lipschitz constant $L = ||M||$. For $q = 0$, the unique solution of the corresponding variational inequality is $\{0\}$.

Now, we compare our Algorithm 1 (iPCM) with the standard algorithm (Algorithm 1 with $\alpha_k = 0$, shortly, PCM). The starting point is $x_0 = (1, 1, \cdots, 1) \in \mathbb{R}^m$. All entries of the matrices B, S, D are generated randomly (matrices of normally distributed random numbers).

Control parameters and stopping rules are chosen as in Example 1 except $\alpha = 0.9$, $f(x) = 0$, and $\beta_n = \frac{1}{n+2}$. The results are described in Table 2 and Figs. 4, 5, 6, and 7.

In Fig. 8, we illustrate the performances of Algorithm 1 for different choices of the contraction $f(x) = 0.82x, 0.75x, 0.5x, 0.125x$, where $m = 150$ and the stopping criterion is $\|x_n - 0\| < 10^{-4}$.

Computing times for Algorithm 1 are 1.1341, 1.5652, 3.6404, and 7.2754 second for $f(x) = 0.82x, 0.75x, 0.5x, 0.125x$, respectively, and the corresponding number of iterations for Algorithm 1 are 106, 150, 351, and 702.
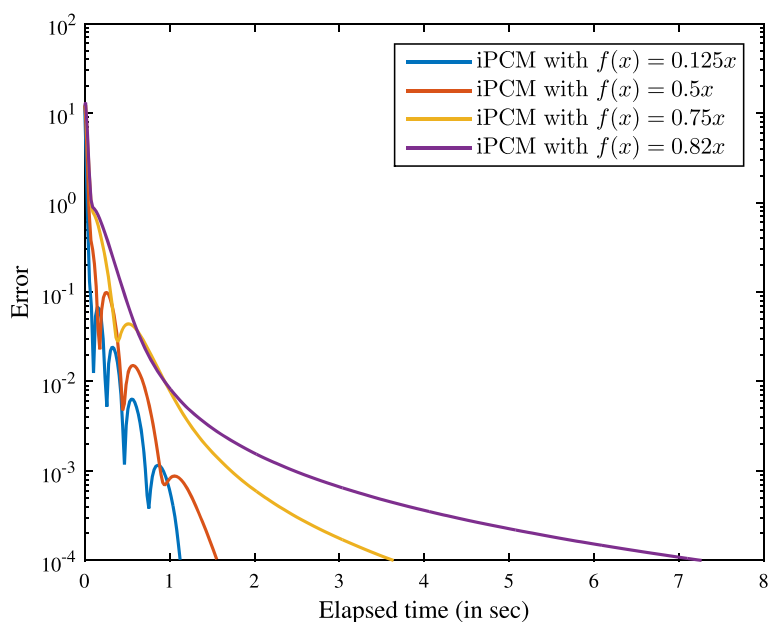
Fig. 8 The performances of Algorithm 1 for different choices of the contraction $f(x) = 0.82x, 0.75x,$ $0.5x, 0.125x$

5 Conclusions

The paper has proposed a new method for solving monotone and Lipschitz VIPs in real Hilbert spaces. Under some suitable conditions imposed on parameters, we have proved the strong convergence of the algorithm. The efficiency of the proposed algorithm has also been illustrated by several numerical experiments.

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