




Error bounds for linear complementarity problems of S -QN matrices

Jicheng Li¹ · Ge Li¹ 

Received: 19 June 2018 / Accepted: 17 April 2019 / Published online: 1 May 2019
© Springer Science+Business Media, LLC, part of Springer Nature 2019

Abstract

Linear complementarity problem (LCP) presents many nice properties when the associated matrix belongs to some special matrix classes, especially H -matrices. In this paper, we put forward a new subclass of H -matrices, called S -QN matrices, which is the proper generalization of the QN matrices. We have proved that for a given S -QN matrix A , there exists a diagonal scaling matrix W such that AW is a QN matrix. Then, we present two kinds of error bounds for LCP of S -QN matrices. The Error Bound I generalizes the error bound for LCP of QN matrices. The Error Bound II overcomes the limitation that the Error Bound I cannot be used. Numerical examples illustrate that the Error Bound I is better than other previous bounds for H -matrices in some cases. Moreover, in some special cases, the Error Bound II can improve considerably the Error Bound I.

Keywords Linear complementarity problem · H -matrix · S -SDD matrix · QN matrix · S -QN matrix

1 Introduction

The linear complementarity problem is to find a vector $x \in R^n$ such that

$$x \geq 0, Ax + q \geq 0, x^T(Ax + q) = 0, \quad (1.1)$$

or to show that no such vector x exists, where $A = [a_{ij}] \in R^{n \times n}$ and $q \in R^n$. We abbreviate the problem by $LCP(A, q)$ and denote its solution by x^* . Many problems

✉ Ge Li
lige0312@stu.xjtu.edu.cn

Jicheng Li
jcli@mail.xjtu.edu.cn

¹ School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an, 710049, People's Republic of China

can be described in the form (1.1), such as problems in linear and quadratic programming, the problem of finding a Nash equilibrium point of a bimatrix game, some free boundary problems of fluid mechanics and American options pricing problem (see [1–4]), etc.

It is well-known that the LCP(A, q) has a unique solution for any $q \in R^n$ if and only if A is a P -matrix [2]. When the matrix involved is a P -matrix, Chen and Xiang [5] gave the error bound for the LCP(A, q) as:

$$\|x - x^*\|_\infty \leq \max_{d \in [0,1]^n} \|(I - \Lambda + \Lambda A)^{-1}\|_\infty \|r(x)\|_\infty, \tag{1.2}$$

where x^* is the solution of LCP(A, q), $r(x) = \min\{x, Ax + q\}$, $\Lambda = \text{diag}(d_i)$ with $0 \leq d_i \leq 1$, and the min operator $r(x)$ denotes the componentwise minimum of two vectors. Since real H -matrices with positive diagonal entries form a subclass of P -matrices, the research of special H -matrices has aroused many scholars' interest, such as Nekrasov matrices [10], Ostrowski matrices [13], QN matrices [7], and S -strictly diagonally dominant (S -SDD) matrices [14]. Meanwhile, the error bounds for these subclasses of H -matrices are presented. For example, the corresponding error bounds for LCPs of QN matrices are achieved by Dai et al. in [8] and Gao et al. in [9]. More error bounds for the subclasses of H -matrices can be seen in [12, 16, 17], etc.

The paper uses some basic notations. Denote $N := \{1, \dots, n\}$ by the set of indices, by S a nonempty subset of N , by $\bar{S} := N \setminus S$ the complement of S in N , and by $e := (1, \dots, 1)^T$ the unit column vector of n elements, respectively. We also denote by $r_i(A) := \sum_{k \in N, k \neq i} |a_{ik}|$ i th row sum and by $r_i^S(A) := \sum_{k \in S, k \neq i} |a_{ik}|$ the part of i th row sum, which corresponds to the subset S of the index set N . Obviously, for any subset S of N and any index $i \in N$, we have

$$r_i(A) = r_i^S(A) + r_i^{\bar{S}}(A).$$

A matrix $A = [a_{ij}] \in C^{n \times n}$ is called strictly diagonally dominant (SDD) matrix if for each $i \in N$, it holds that $a_{ii} > r_i(A)$. A matrix $A = [a_{ij}] \in C^{n \times n}$ is called an H -matrix if its comparison matrix $\mathcal{M}(A) = [\tilde{a}_{ij}] \in R^{n \times n}$ defined by the formula

$$a_{ij} = \begin{cases} |a_{ii}|, & i = j, \\ -|a_{ij}|, & i \neq j, \end{cases}$$

is a nonsingular M -matrix. It is well-known that a square matrix A is an H -matrix if and only if there exists a positive diagonal matrix X such that AX is SDD [15].

This paper is organized as follows. In Section 2, we put forward a new subclass of H -matrices, called S -QN matrices. Moreover, by discussing the relationship among the subclasses of H -matrices, we find that the set of QN matrices is the proper subset of S -QN matrices; that is, the conception of S -QN matrix is the proper generalization of the conception of QN matrix. In Section 3, we present two kinds of error bounds for LCP(A, q) when the involved matrix A belongs to the set of S -QN matrices. The Error Bound I generalizes the error bound for LCP of QN matrices in [8] by finding a positive diagonal matrix P , such that matrix AP is an SDD matrix. However, the

Error Bound I cannot be used when its hypotheses are not satisfied. To avoid this case, we propose a new error bound, called the Error Bound II. The main idea of the Error Bound II is the bound of the inverse of the S -QN matrix, which we get by extending the bound of the inverse of the QN matrix to S -QN matrix. Numerical examples show that the Error Bound I is better than the previous error bounds for H -matrices in some cases and the Error Bound II can improve the Error Bound I in some special cases for S -QN matrices. The conclusion is provided in the last section.

2 S-QN matrix

The class of H -matrices plays an important role in linear complementarity problem. Many convergence analyses of the iterative algorithms for LCP have been considered when the coefficient matrices of LCP are H -matrices. Dehghan and Hajarian [18] gave the convergence analysis of SSOR methods for $LCP(A, q)$ when matrix A is an H -matrix. In [19], Li and Dai presented generalized AOR methods and the convergence analysis for LCP of H -matrices. Similarly, Hadjidimos and Tzoumas discussed the convergence of the matrix analogue of the accelerated overrelaxation iterative method for LCP when the coefficient matrices belong to H_+ -matrices [20]. At the same time, the error bounds for LCP of H -matrices or its subclasses also attracted many researchers' interest [6, 8, 11, 13]. Inspired by the S -SDD matrices and QN matrices, we first put forward a new subclass of H -matrices, called S -QN matrices, which is the proper generalization of QN matrices.

Definition 2.1 [10] For any given matrix $A = [a_{ij}] \in C^{n \times n}, n \geq 2$, and for any given nonempty proper subset S of N , then A is called as an S -strictly diagonally dominant (S -SDD) matrix if

- (1) $|a_{ii}| > r_i^s(A)$ for all $i \in S$;
- (2) $(|a_{ii}| - r_i^s(A)) \left(|a_{jj}| - r_j^{\bar{s}}(A) \right) > r_i^{\bar{s}}(A) r_j^s(A)$ for all $i \in S, j \in \bar{S}$.

S -SDD matrices can be characterized in different ways, the following Theorem 2.2 gives out a general characterization by the form of corresponding diagonal scaling matrices, which can be more convenient in some applications. Moreover, these scaling characterizations give us another insight into relation between various subclasses of the class of H -matrices and the class of SDD matrices. The corresponding diagonal scaling matrix W belongs to the set \mathcal{W} , defined as the set of all diagonal matrices whose diagonal entries are either 1 or γ , where γ is an arbitrary positive number, i.e.,

$$\mathcal{W} = \bigcup_{S \subset N} W^S, \\ W^S = \{W = \text{diag}(w_1, \dots, w_n) : w_i = \gamma > 0 \text{ for } i \in S \text{ and } w_i = 1 \text{ for } i \in \bar{S}\}.$$

Theorem 2.2 [10] A matrix A is an S -SDD matrix if and only if there exists a matrix $W \in \mathcal{W}$ such that AW is an SDD matrix.

Numerically, if a matrix A is an S -SDD matrix, we can construct the diagonal matrix W by choosing the scaling parameter γ from the interval $(\gamma_1(A), \gamma_2(A))$ with

$$0 \leq \gamma_1(A) := \max_{i \in S} \frac{r_i^{\bar{S}}(A)}{|a_{ii}| - r_i^S(A)}, \gamma_2(A) := \min_{j \in \bar{S}, r_j^S(A)} \frac{|a_{jj}| - r_j^{\bar{S}}(A)}{r_j^S(A)},$$

where, if $r_j^S(A) = 0$ for $j \in \bar{S}$, the final fraction is defined to be $+\infty$.

Assume that $A = D + L + U$ is the standard splitting of a matrix $A \in C^{n \times n}$ into its diagonal (D), strictly lower triangular (L), and strictly upper triangular (U) parts, respectively. Next, we give a characterization of the QN matrices.

Lemma 2.3 [7] *Let $A = [a_{ij}] \in C^{n \times n}$, $n \geq 2$, with $a_{ii} \neq 0, i \in N$. Then A is a QN matrix if and only if*

$$e > M^{-1} |L| |D|^{-1} |U| e. \tag{2.1}$$

where

$$M = (|D| - |L|) |D|^{-1} (|D| - |U|) = \mathcal{M}(A) + |L| |D|^{-1} |U|. \tag{2.2}$$

Obviously, the matrix M is monotone, i.e., M^{-1} is a nonnegative matrix. If we denote

$$G = I - M^{-1} |L| |D|^{-1} |U| = M^{-1} \mathcal{M}(A), \tag{2.3}$$

then, G is a Z -matrix (i.e., its off-diagonal entries are nonpositive) and the condition (2.1) amounts to G is strictly diagonal dominant by rows with positive diagonal entries. Thus, A is a QN matrix if and only if for $i = 1, 2, \dots, n$,

$$[Ge]_i = [\mathcal{M}(G)e]_i > 0. \tag{2.4}$$

Theorem 3.2 in [7] shows that a QN matrix is an H -matrix; however, an H -matrix may be not a QN matrix. In this paper, we introduce a new kind of subclass of H -matrices, called S -QN matrices, which is also the proper generalization of QN matrices. Next, we give out the definition of S -QN matrix.

Definition 2.4 Let a matrix $A = [a_{ij}] \in C^{n \times n}$, $n \geq 2$, with $a_{ii} \neq 0, i \in N$, and any given nonempty proper subset S of N . Then A is called an S -QN matrix if the matrix G defined by (2.3) is an S -SDD matrix with positive diagonal entries.

The following result gives a characterization of S -QN matrices.

Theorem 2.5 *A matrix A is an S -QN matrix if and only if there exists a matrix $W \in \mathcal{W}$ such that AW is a QN matrix.*

Proof At first, we prove the necessity. Let $A = D + L + U$ be an S -QN matrix, by the Definition 2.4, the matrix G defined by (2.3) is an S -SDD matrix and $G = \mathcal{M}(G)$. According to Theorem 2.2, there exists a positive diagonal matrix $W \in \mathcal{W}$ such that

GW is an SDD matrix, i.e., for $i = 1, 2, \dots, n$, we have

$$[(GW)e]_i = [\mathcal{M}(GW)e]_i > 0.$$

The following is to show AW is a QN matrix. According to (2.4), we only need to prove that $\bar{G}e = \mathcal{M}(\bar{G})e > 0$, where $\bar{G} = \bar{M}^{-1}\mathcal{M}(AW) = W^{-1}M^{-1}\mathcal{M}(A)W = W^{-1}GW$ and $\bar{M} = (|DW| - |LW|)|DW|^{-1}(|DW| - |UW|) = MW$. For $i = 1, 2, \dots, n$,

$$[\bar{G}e]_i = [\mathcal{M}(\bar{G})e]_i = [\mathcal{M}(W^{-1}GW)e]_i = [W^{-1}\mathcal{M}(GW)e]_i > 0.$$

Hence, AW is a QN-matrix. The necessity is proved.

Next, we prove the sufficiency. If there exists a positive diagonal matrix $W \in \mathcal{W}$ such that AW is a QN matrix, we need to prove that the matrix A is an S -QN matrix. That is, we need to show the matrix G is an S -SDD matrix with positive diagonal entries.

Since AW is a QN matrix, we have that $\bar{G}e = \mathcal{M}(\bar{G})e = \bar{M}^{-1}\mathcal{M}(AW)e = W^{-1}GWe = (W^{-1}G)We > 0$. By Theorem 2.2, $W^{-1}G$ is an S -SDD matrix. Moreover, $W^{-1}G$ can be expressed as

$$W^{-1}G = \begin{bmatrix} \frac{1}{w_1}g_{11} & \cdots & \frac{1}{w_1}g_{1n} \\ \vdots & \ddots & \vdots \\ \frac{1}{w_n}g_{n1} & \cdots & \frac{1}{w_n}g_{nn} \end{bmatrix}. \tag{2.5}$$

Obviously, the matrix G is also an S -SDD matrix with positive diagonal entries. The sufficiency is proved. □

Remark 2.6 According to the above, the conclusion is that an S -QN matrix is an H -matrix. When the matrix A is an S -QN matrix, then there exists $W \in \mathcal{W}$ such that AW is a QN-matrix. It is well-known that a QN-matrix is also an H -matrix; that is, there exists diagonal matrix X such that $(AW)X$ is an SDD matrix; i.e., there exists a positive diagonal matrix WX , such that $A(WX)$ is an SDD matrix. Hence, an S -QN matrix is also an H -matrix.

Remark 2.7 It is easy to see that the scaling matrix W for an S -QN matrix A is not unique. Since A is an S -QN matrix, then the matrix G is an S -SDD matrix. From the proof of Theorem 2.5, the scaling matrix W for A satisfies that GW is an SDD matrix. Therefore, the parameter γ of matrix W can be taken from the interval $(\gamma_1(G), \gamma_2(G))$. It means that any choice from the interval $(\gamma_1(G), \gamma_2(G))$ can form a scaling matrix W such that AW is a QN matrix.

The following Example 2.1 shows that the class of QN-matrices is the proper subset of the class of S -QN matrices.

Example 2.1 The matrix

$$A = \begin{bmatrix} 1 & -2 & 0 \\ -1 & 3 & - \\ 0 & -1 & 4 \end{bmatrix}$$

is an S -QN matrix with $S = \{2, 3\}$, and it is not a QN matrix.

The following Examples 2.2 and 2.3 illustrate that the set of QN matrices and the set of S -SDD matrices do not contain each other.

Example 2.2 The matrix

$$A = \begin{bmatrix} 1 & 0.5 & 0 & 0.4 \\ 0.5 & 1 & 0.5 & 0 \\ 0 & 0.5 & 1 & 0.5 \\ 0.5 & 0 & 0.5 & 1 \end{bmatrix}$$

is a QN matrix, but not an S -SDD matrix for any subset S .

Example 2.3 The matrix

$$A = \begin{bmatrix} 1 & 0.5 & 0 & 0 \\ 0.5 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0.5 \\ 1 & 2 & 0.3 & 1 \end{bmatrix}$$

is an S -SDD matrix for $S = \{1, 2\}$, but not a QN matrix.

By Theorem 2.2, for an arbitrary S -SDD matrix A , there exists a diagonal matrix $W \in \mathcal{W}$, such that AW is an SDD matrix. It is well-known that an SDD matrix is also a QN matrix, so matrix AW is a QN matrix. Then, according to Theorem 2.5, the matrix A is an S -QN matrix. Therefore, the class of S -QN matrices contains the class of S -SDD matrices.

Figure 1 displays the relationship among these subclasses of H -matrices.

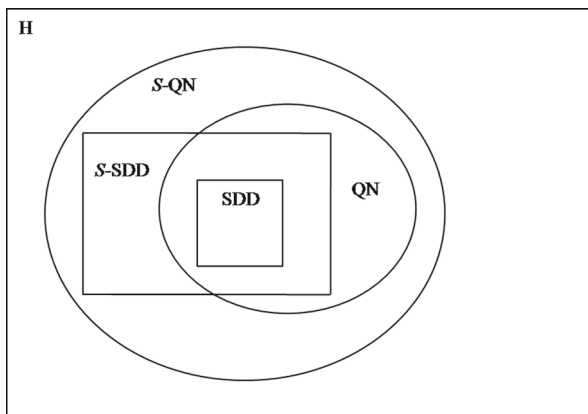


Fig. 1 The relationship about the subclasses of H -matrices

3 Error bounds for LCPs of S-QN matrices

When the matrix involved is a P -matrix, Chen and Xiang proposed the corresponding error bound for $LCP(A, q)$ in (1.2). A big challenge for (1.2) is due to the difficulty for solving $\max_{d \in [0,1]^n} \| (I - \Lambda + \Lambda A)^{-1} \|_\infty$, where $\Lambda = \text{diag}(d_i)$ with $0 \leq d_i \leq 1$. However, if the involved matrix belongs to a subclass of P -matrices, for example, H -matrices with positive diagonals, then many calculable error bounds for the $LCP(A, q)$ can be derived. As a subclass of H -matrices, we present two kinds of error bounds for $LCP(A, q)$ with an S -QN matrix.

3.1 Error bound I for LCPs with an S-QN matrix

García-Esnaola and Peña provided an error bound for $LCP(A, q)$ of H -matrices with positive diagonal entries in [6], it requires to know the diagonal matrix X in advance such that AX is strictly diagonally dominant (SDD). Based on this idea, we first find a diagonal matrix P for an S -QN matrix A such that AP is SDD. Then the error bound of $LCP(A, q)$ when the involved matrix is an S -QN matrix with positive diagonal entries is presented. The following Theorem 3.1 can be found in [6].

Theorem 3.1 [6] *Assume that $A = [a_{ij}] \in R^{n \times n}$ is an H -matrix with positive diagonal entries. Let $\bar{D} = \text{diag}(\bar{d}_i)$ with $\bar{d}_i > 0, i \in N$, be a diagonal matrix such that $A\bar{D}$ is strictly diagonal dominant by rows. For any $i \in N$, let $\bar{\beta}_i = a_{ii}\bar{d}_i - \sum_{j \neq i} |a_{ij}| \bar{d}_j$, then*

$$\max_{d \in [0,1]^n} \left\| (I - \Lambda + \Lambda A)^{-1} \right\|_\infty \leq \max \left\{ \frac{\max_i \{\bar{d}_i\}}{\min_i \{\bar{\beta}_i\}}, \frac{\max_i \{\bar{d}_i\}}{\min_i \{\bar{d}_i\}} \right\}. \tag{3.1}$$

where $\Lambda = \text{diag}(d_i)$ with $0 \leq d_i \leq 1$.

In [8], Dai et al. have found the diagonal matrix K for the QN matrix A such that AK is SDD. Similarly, we can find the corresponding diagonal matrix $P = \text{diag}(p_i)$ for S -QN matrices.

Theorem 3.2 [8] *Let $A = [a_{ij}] \in R^{n \times n}$ be a QN matrix with positive diagonal entries such that, for each $i = 1, \dots, n - 1, a_{ij} \neq 0$ for some $j > i$, and for each $i = 2, \dots, n, a_{ij} \neq 0$ for some $j < i$. Let $\xi := M^{-1} |L| |D|^{-1} |U| e$, where M is defined in (2.2), then the matrix $K = \text{diag}(k_1, \dots, k_n)$ with $k_1 := \xi_1 + \epsilon, \epsilon \in \left(0, \min \left\{ 1 - \xi_1, \min_{2 \leq i \leq n} \frac{[\mathcal{M}(A)\xi]_i}{|a_{i1}|} \right\} \right)$, where $\frac{[\mathcal{M}(A)\xi]_i}{|a_{i1}|} = \infty$ when $a_{i1} = 0$, and $k_i := \xi_i$ for $i = 2, \dots, n$, has positive diagonal entries and it satisfies that AK is SDD.*

Corollary 3.3 *Let $A = [a_{ij}] \in R^{n \times n}$ be an S -QN matrix with positive diagonal entries such that, when $j > i$, there exists $a_{ij} \neq 0$ for each $i = 1, \dots, n -$*

1; and when $j < i$, there exists $a_{ij} \neq 0$ for each $i = 2, \dots, n$. If $W = \text{diag}(w_i) \in \mathcal{W}$ such that AW is a QN matrix, $\bar{\xi} = W^{-1}M^{-1}|L||D|^{-1}|U|We$, where M is defined in (2.2), and $K = \text{diag}(k_1, \dots, k_n)$ with $k_1 := \bar{\xi}_1 + \epsilon$, $\epsilon \in \left(0, \min \left\{1 - \bar{\xi}_1, \min_{2 \leq i \leq n} \frac{[\mathcal{M}(AW)\bar{\xi}]_i}{|a_{i1}|w_1}\right\}\right)$, where $\frac{[\mathcal{M}(AW)\bar{\xi}]_i}{|a_{i1}|w_1} = \infty$ when $a_{i1} = 0$, and $k_i := \bar{\xi}_i$ for $i = 2, \dots, n$. Then the matrix $P = WK = \text{diag}(p_i)$ is the positive diagonal matrix such that AP is SDD.

The following results present an error bound for the linear complementarity problem of S -QN matrices, which generalizes the error bound for QN matrices in [8]. This bound can improve the previous bounds for H -matrices.

Theorem 3.4 *Let $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ be an S -QN matrix with positive diagonal entries such that, when $j > i$, there exists $a_{ij} \neq 0$ for each $i = 1, \dots, n - 1$; and when $j < i$, there exists $a_{ij} \neq 0$ for each $i = 2, \dots, n$. Then*

$$\max_{d \in [0,1]^n} \left\| (I - \Lambda + \Lambda A)^{-1} \right\|_{\infty} \leq \max \left\{ \frac{\max_i \{p_i\}}{\min_i \{t_i\}}, \frac{\max_i \{p_i\}}{\min_i \{p_i\}} \right\}, \tag{3.2}$$

where $t_1 := \epsilon a_{11}w_1$, $t_i := a_{ii}w_i\bar{\xi}_i - \sum_{j \in N \setminus \{i\}} |a_{ij}|w_j\bar{\xi}_j - \epsilon |a_{i1}|w_1$ for each $i = 2, \dots, n$; $\bar{\xi}$, W , P and ϵ are defined in Corollary 3.3.

Proof By Remark 2.6, an S -QN matrix is an H -matrix, we can apply the bound (3.1) for H -matrices to S -QN matrices. Then, we need to prove that $\bar{d}_i = p_i$ and $t_i = \bar{\beta}_i$ for all $i \in N$. Obviously, $\bar{d}_i = p_i$ for all $i \in N$, so we only need to prove that $t_i = \bar{\beta}_i$ for all $i \in N$.

By Theorem 2.5, AW is a QN matrix, and

$$\begin{aligned} \bar{M} &= \mathcal{M}(AW) + |L||D|^{-1}|U|W = MW, \\ \bar{\xi} &= W^{-1}M^{-1}|L||D|^{-1}|U|We, \end{aligned}$$

we can have $\mathcal{M}(AW)\bar{\xi} + |L||D|^{-1}|U|W\bar{\xi} = |L||D|^{-1}|U|We$, in other words, $\mathcal{M}(AW)\bar{\xi} = |L||D|^{-1}|U|W(e - \bar{\xi})$.

Denote $\zeta = (\epsilon, 0, \dots, 0)^T$, then $Ke = (k_1, \dots, k_n)^T = (\bar{\xi}_1 + \epsilon, \bar{\xi}_2, \dots, \bar{\xi}_n)^T = \bar{\xi} + \zeta$,

$$\begin{aligned} \mathcal{M}(AP)e &= \mathcal{M}(AWK)e \\ &= \mathcal{M}(AW)Ke = \mathcal{M}(AW) \cdot (\bar{\xi} + \zeta) \\ &= |L||D|^{-1}|U|W(e - \bar{\xi}) + \mathcal{M}(AW) \cdot \zeta. \end{aligned}$$

It is easy to get that

$$|L| |D|^{-1} |U| = \begin{bmatrix} 0 & 0 & \dots & & 0 \\ 0 & \frac{|a_{21}|}{|a_{11}|} |a_{12}| & \dots & & \frac{|a_{21}|}{|a_{11}|} |a_{1n}| \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & \frac{|\hat{a}_{n1}|}{|\hat{a}_{11}|} |a_{12}| & \dots & \frac{|a_{n1}|}{|a_{11}|} |a_{1n}| + \dots & \frac{|a_{n,n-1}|}{|a_{n-1,n-1}|} |a_{n-1,n}| \end{bmatrix}.$$

For $i = 1$, $\bar{\beta}_1 = [\mathcal{M}(AP) e]_1 = [\mathcal{M}(AWK) e]_1 = [\mathcal{M}(AW) \cdot \zeta]_1 = \epsilon a_{11} w_1 = t_1$.
 For $i = 2, \dots, n$,

$$\begin{aligned} \bar{\beta}_i &= [\mathcal{M}(AP) e]_i = [\mathcal{M}(AWK) e]_i \\ &= \left[\mathcal{M}(AW) (\bar{\xi}_1 + \epsilon, \bar{\xi}_2, \dots, \bar{\xi}_n)^T \right]_i \\ &= a_{ii} w_i \bar{\xi}_i - \sum_{j \in N \setminus \{i\}} |a_{ij}| w_j \bar{\xi}_j - \epsilon |a_{i1}| w_1 \\ &= t_i. \end{aligned}$$

Hence, we can get

$$\max_{d \in [0,1]^n} \left\| (I - \Lambda + \Lambda A)^{-1} \right\|_{\infty} \leq \max \left\{ \frac{\max_i \{p_i\}}{\min_i \{t_i\}}, \frac{\max_i \{p_i\}}{\min_i \{p_i\}} \right\}.$$

□

The error bound given by [5] of $LCP(A, q)$ for an H -matrix $A = [a_{ij}] \in R^{n \times n}$ with positive diagonal entries is the following:

$$\max_{d \in [0,1]^n} \left\| (I - \Lambda + \Lambda A)^{-1} \right\|_{\infty} \leq \left\| \mathcal{M}(A)^{-1} \max(\Gamma, I) \right\|_{\infty}, \tag{3.3}$$

where $\mathcal{M}(A)$ is the comparison matrix of A , $\Gamma = \text{diag}(a_{11}, a_{22}, \dots, a_{nn})$ is the diagonal part of A and $\max(\Gamma, I) = \text{diag}(\max\{a_{11}, 1\}, \dots, \max\{a_{nn}, 1\})$.

The following Example 3.1 illustrates that the bound (3.2) obtained by Theorem 3.4 is better than the bound (3.3) given in [5].

Example 3.1 Consider the following matrix

$$A = \begin{bmatrix} 1 & -2.5 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 4 \end{bmatrix}.$$

It is easy to check that the matrix A is an S -QN matrix for $S = \{2, 3\}$ and $W = \text{diag}(1, 0.4, 0.4)$. By Theorem 3.4, we can get $\xi = (0.9306, 0.9306, 0.2917)^T$, if we choose $\epsilon = 0.0347$, then

$$K = \text{diag}(0.9653, 0.9306, 0.2917),$$

$$P = \text{diag}(0.9653, 0.3722, 0.1167),$$

$$(t_1, t_2, t_3)^T = \mathcal{M}(AP)e = (0.0347, 0.0347, 0.0945)^T.$$

So the bound (3.2) in Theorem 3.4 is

$$\max \left\{ \frac{\max \{p_i\}}{\min \{t_i\}}, \frac{\max \{p_i\}}{\min \{p_i\}} \right\} = 27.8184,$$

which is smaller than the bound (3.3)

$$\left\| \mathcal{M}(A)^{-1} \max(\Gamma, I) \right\|_\infty = 51.$$

3.2 Error bound II for LCPs with an S -QN matrix

In Theorem 3.4, we get an error bound for LCPs of S -QN matrices. Although the bound (3.2) can improve the previous error bound for H -matrices, it is still worth improving. For example, for a given S -QN matrix $A = [a_{ij}] \in R^{n \times n}$ with positive diagonals, if there exists some $i \in N$ such that $a_{ij} = 0$ for any $j > i$ or $j < i$, then the bound (3.2) can not be used to estimate $\max_{d \in [0,1]^n} \|(I - \Lambda + \Lambda A)^{-1}\|_\infty$. Besides, when $\epsilon \rightarrow 0$,

$$t_1 = \epsilon a_{11} w_1 \text{ and } \min_{i \in N} \{t_i\} \rightarrow 0,$$

which implies that

$$\max \left\{ \frac{\max \{p_i\}}{\min \{t_i\}}, \frac{\max \{p_i\}}{\min \{p_i\}} \right\} \rightarrow +\infty.$$

These facts show that when the assumption of Theorem 3.4 is not satisfied or $\epsilon \rightarrow 0$, using the bound (3.2) to estimate $\max_{d \in [0,1]^n} \|(I - \Lambda + \Lambda A)^{-1}\|_\infty$ is inappropriate.

To overcome these drawbacks, we propose a new error bound for S -QN matrices which is better than the bound (3.2) in some cases. Some useful lemmas are listed here.

Lemma 3.5 [11] *Let $\alpha > 0$ and $\eta > 0$. Then for any $x \in [0, 1]$,*

$$\frac{1}{1 - x + \alpha x} \leq \frac{1}{\min \{\alpha, 1\}},$$

and

$$\frac{\eta x}{1 - x + \alpha x} \leq \frac{\eta}{\alpha}.$$

Lemma 3.6 [12] *Let $A = [a_{ij}] \in C^{n \times n}$ be a matrix with $a_{ii} \neq 0$ for $i \in N$ and $\hat{A} = I - \Lambda + \Lambda A = [\hat{a}_{ij}]$, where $\Lambda = \text{diag}(d_i)$ with $0 \leq d_i \leq 1$. Then*

$$z_i(\hat{A}) \leq \eta_i(A)$$

and

$$\frac{z_i(\hat{A})}{\hat{a}_{ii}} \leq \frac{\eta_i(A)}{\min \{a_{ii}, 1\}},$$

where $z_1(\hat{A}) = \eta_1(A) = 1$, $z_i(\hat{A}) = \sum_{j=1}^{i-1} \frac{|\hat{a}_{ij}|}{|\hat{a}_{jj}|} z_j(\hat{A}) + 1$, and

$$\eta_i(A) = \sum_{j=1}^{i-1} \frac{|a_{ij}|}{\min \{|a_{jj}|, 1\}} \eta_j(A) + 1, i = 2, 3, \dots, n.$$

Lemma 3.7 [7] *Let $A = [a_{ij}] \in C^{n \times n}$, $n \geq 2$, be a QN matrix. Then*

$$\|A^{-1}\|_{\infty} \leq \max_{i \in N} \frac{\{M^{-1}e\}_i}{\{M^{-1}\mathcal{M}(A)e\}_i}. \tag{3.4}$$

where M is defined in (2.2).

Next, we will introduce the new error bound for LCPs when the involved matrix is an S -QN matrix. To calculate $\max_{d \in [0,1]^n} \|(I - \Lambda + \Lambda A)^{-1}\|_{\infty}$, we first prove that the matrix $\hat{A} = I - \Lambda + \Lambda A$ is also an S -QN matrix when the matrix A is an S -QN matrix. Then, the error bound is estimated by using the bound of the inverse of the S -QN matrix \hat{A} .

Lemma 3.8 *Let $A = [a_{ij}] \in C^{n \times n}$ be an S -QN matrix with positive diagonal entries, and $\hat{A} = I - \Lambda + \Lambda A = [\hat{a}_{ij}]$, where $\Lambda = \text{diag}(d_i)$ with $0 \leq d_i \leq 1$. Then \hat{A} is an S -QN matrix.*

Proof Since A is an S -QN matrix, there exists a diagonal matrix $W = \text{diag}(w_i) \in \mathcal{W}$ such that $\bar{A} = AW$ is a QN matrix. Let $\bar{\xi} = W^{-1}M^{-1}|L||D|^{-1}|U|We = (\bar{\xi}_1, \bar{\xi}_2, \dots, \bar{\xi}_n)^T$, where M is defined in (2.2), then

$$e > \bar{\xi}. \tag{3.5}$$

Denote

$$\hat{a}_{ij} = \begin{cases} 1 - d_i + d_i a_{ij}, & i = j, \\ d_i a_{ij}, & i \neq j. \end{cases}$$

Since $A = D + L + U$, it follows that \hat{A} and \bar{A} can be split in the form of $\hat{A} = \hat{D} + \hat{L} + \hat{U}$, $\bar{A} = \bar{D} + \bar{L} + \bar{U}$, respectively.

To prove \hat{A} is an S -QN matrix, by Theorem 2.5, we only need to prove that $\hat{A}W$ is a QN matrix. If we denote

$$\hat{\xi} = W^{-1}\hat{M}^{-1}|\hat{L}||\hat{D}^{-1}||\hat{U}|We = (\hat{\xi}_1, \hat{\xi}_2, \dots, \hat{\xi}_n)^T, \tag{3.6}$$

where $\hat{M} = (|\hat{D}| - |\hat{L}|)|\hat{D}^{-1}|(|\hat{D} - \hat{U}|)$, then we only need to prove $e > \hat{\xi}$. Denote

$$\hat{v} = |\hat{L}||\hat{D}^{-1}||\hat{U}|We = (\hat{v}_1, \hat{v}_2, \dots, \hat{v}_n)^T, \tag{3.7}$$

where

$$|\hat{L}||\hat{D}^{-1}||\hat{U}|W = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & \frac{|\hat{a}_{21}|}{|\hat{a}_{11}|} |\hat{a}_{12}| w_2 & \dots & \frac{|\hat{a}_{21}|}{|\hat{a}_{11}|} |\hat{a}_{1n}| w_n \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{|\hat{a}_{n1}|}{|\hat{a}_{11}|} |\hat{a}_{12}| w_2 & \dots & \frac{|\hat{a}_{n1}|}{|\hat{a}_{11}|} |\hat{a}_{1n}| w_n + \dots + \frac{|\hat{a}_{n,n-1}|}{|\hat{a}_{n-1,n-1}|} |\hat{a}_{n-1,n}| w_n \end{bmatrix}.$$

Then we can get

$$\hat{v}_1 = 0, \hat{v}_i = \sum_{j=2}^n \left(\sum_{k=1}^{i-1} \frac{|\hat{a}_{ik}|}{|\hat{a}_{kk}|} |\hat{a}_{kj}| w_j \right) = \sum_{k=1}^{i-1} \left(\frac{|\hat{a}_{ik}|}{|\hat{a}_{kk}|} \sum_{j=k+1}^n |\hat{a}_{kj}| w_j \right), i = 2, \dots, n. \tag{3.8}$$

From (3.6) and (3.7), we have $\hat{\xi} = W^{-1}\hat{M}^{-1}\hat{v} = W^{-1}(|\hat{D}| - |\hat{U}|)^{-1}|\hat{D}|(|\hat{D}| - |\hat{L}|)^{-1}\hat{v}$. If we denote $\hat{\lambda} = (|\hat{D}| - |\hat{L}|)^{-1}\hat{v} = (\hat{\lambda}_1, \dots, \hat{\lambda}_n)^T$, then we get

$$\hat{\xi} = W^{-1}(|\hat{D}| - |\hat{U}|)^{-1}|\hat{D}|\hat{\lambda}, \tag{3.9}$$

and

$$(|\hat{D}| - |\hat{L}|)\hat{\lambda} = \hat{v}. \tag{3.10}$$

By (3.8) and (3.10), we can obtain the entries of $\hat{\lambda}$ as

$$\hat{\lambda}_1 = 0, \hat{\lambda}_i = \frac{\hat{v}_i}{|\hat{a}_{ii}|} + \sum_{j=1}^{i-1} \frac{|\hat{a}_{ij}|}{|\hat{a}_{ii}|} \hat{\lambda}_j, i = 2, \dots, n. \tag{3.11}$$

Moreover, followed by (3.9), we have $(|\hat{D}| - |\hat{U}|)W\hat{\xi} = |\hat{D}|\hat{\lambda}$, this implies the following recursive relations

$$\begin{aligned} \hat{\xi}_n &= \frac{\hat{\lambda}_n}{w_n}, \\ \hat{\xi}_i &= \frac{\hat{\lambda}_i}{w_i} + \sum_{j=i+1}^n \frac{|\hat{a}_{ij}|w_j}{|\hat{a}_{ii}|w_i} \hat{\xi}_j, i = n - 1, \dots, 1. \end{aligned} \tag{3.12}$$

Similarly, if we denote $\bar{v} = |\bar{L}||\bar{D}^{-1}||\bar{U}|We = (\bar{v}_1, \bar{v}_2, \dots, \bar{v}_n)^T$, $\bar{\lambda} = (|\bar{D}| - |\bar{L}|)^{-1}\bar{v} = (\bar{\lambda}_1, \bar{\lambda}_2, \dots, \bar{\lambda}_n)^T$, then

$$\bar{\lambda}_1 = 0, \bar{\lambda}_i = \frac{\bar{v}_i}{|a_{ii}|} + \sum_{j=1}^{i-1} \frac{|a_{ij}|}{|a_{ii}|} \bar{\lambda}_j, i = 2, \dots, n,$$

where

$$\bar{v}_i = \sum_{k=1}^{i-1} \left(\frac{|a_{ik}|}{|a_{kk}|} \sum_{j=k+1}^n |a_{kj}|w_j \right), \tag{3.13}$$

and

$$\begin{aligned} \bar{\xi}_n &= \frac{\bar{\lambda}_n}{w_n}, \\ \bar{\xi}_i &= \frac{\bar{\lambda}_i}{w_i} + \sum_{j=i+1}^n \frac{|a_{ij}|w_j}{|a_{ii}|w_i} \bar{\xi}_j, i = n - 1, \dots, 1. \end{aligned} \tag{3.14}$$

Next, we use mathematical induction to prove $\hat{\lambda}_i \leq \bar{\lambda}_i$ for any $i \in N$.

(1) The conclusion is hold for $i = 1$ and $i = 2$, since for $\hat{\lambda}_1 = \bar{\lambda}_1 = 0$, and

$$\begin{aligned} \hat{\lambda}_2 &= \frac{\hat{v}_2}{|\hat{a}_{22}|} + \frac{|\hat{a}_{21}|}{|\hat{a}_{22}|} \hat{\lambda}_1 = \frac{\hat{v}_2}{|\hat{a}_{22}|} \\ &= \frac{1}{|\hat{a}_{22}|} \cdot \frac{|\hat{a}_{21}|}{|\hat{a}_{11}|} \sum_{j=2}^n |\hat{a}_{1j}|w_j \\ &= \frac{d_2|a_{21}|}{1 - d_2 + d_2|a_{22}|} \cdot \frac{d_2}{1 - d_1 + d_1|a_{11}|} \cdot \sum_{j=2}^n d_1|a_{1j}|w_j \\ &\leq \frac{|a_{21}|}{|a_{22}|} \cdot \frac{1}{|a_{11}|} \cdot \sum_{j=2}^n |a_{1j}|w_j \\ &= \frac{v_2}{|a_{22}|} = \bar{\lambda}_2, \end{aligned}$$

(2) we assume that $\hat{\lambda}_i \leq \bar{\lambda}_i$ for $i = k$, then by (3.8),

$$\begin{aligned} \hat{\lambda}_{k+1} &= \frac{\hat{v}_{k+1}}{|\hat{a}_{k+1,k+1}|} + \sum_{j=1}^k \frac{|\hat{a}_{k+1,j}|}{|\hat{a}_{k+1,k+1}|} \hat{\lambda}_j \\ &= \frac{1}{|\hat{a}_{k+1,k+1}|} \left(\sum_{l=1}^k \left(\frac{|\hat{a}_{k+1,l}|}{|\hat{a}_{ll}|} \sum_{j=l+1}^n |\hat{a}_{lj}| w_j \right) \right) + \sum_{j=1}^k \frac{|\hat{a}_{k+1,j}|}{|\hat{a}_{k+1,k+1}|} \hat{\lambda}_j \\ &= \sum_{l=1}^k \left(\frac{|\hat{a}_{k+1,l}|}{|\hat{a}_{k+1,k+1}|} \sum_{j=l+1}^n \frac{|\hat{a}_{lj}|}{|\hat{a}_{ll}|} w_j \right) + \sum_{j=1}^k \frac{|\hat{a}_{k+1,j}|}{|\hat{a}_{k+1,k+1}|} \hat{\lambda}_j \\ &= \sum_{l=1}^k \left(\frac{d_{k+1} |a_{k+1,l}|}{1 - d_{k+1} + d_{k+1} |a_{k+1,k+1}|} \sum_{j=l+1}^n \frac{d_{lj} |a_{lj}|}{1 - d_l + d_l |a_{ll}|} w_j \right) \\ &\quad + \sum_{j=1}^k \frac{d_{k+1,l} |a_{k+1,j}|}{1 - d_{k+1} + d_{k+1} |a_{k+1,k+1}|} \hat{\lambda}_j \\ &\leq \sum_{l=1}^k \left(\frac{|a_{k+1,l}|}{|a_{k+1,k+1}|} \sum_{j=l+1}^n \frac{|a_{lj}|}{|a_{ll}|} w_j \right) + \sum_{j=1}^k \frac{|a_{k+1,j}|}{|a_{k+1,k+1}|} \bar{\lambda}_j \\ &= \frac{v_{k+1}}{|a_{k+1,k+1}|} + \sum_{j=1}^k \frac{|a_{k+1,j}|}{|a_{k+1,k+1}|} \bar{\lambda}_j \\ &= \bar{\lambda}_{k+1}. \end{aligned}$$

So $\hat{\lambda}_i \leq \bar{\lambda}_i$ is hold for any $i \in N$.

By (3.12), (3.14) and Lemma 3.5, we can get $\hat{\xi} \leq \bar{\xi}$. From (3.5), we can get $e > \hat{\xi}$, i.e., $\hat{A}W$ is a QN matrix. Therefore, $\hat{A} = I - \Lambda + \Lambda A$ is an S-QN matrix. \square

The following lemma extends the bound of the inverse of the QN matrix given by Lemma 3.7 to the S-QN matrix.

Lemma 3.9 *Let $A = [a_{ij}] \in C^{n \times n}$, $n \geq 2$ be an S-QN matrix, if $W = \text{diag}(w_i) \in W$ such that AW is a QN matrix, $\delta = \min\{w_i^{-1}\}$, then*

$$\|A^{-1}\|_{\infty} \leq \frac{1}{\delta} \max_{i \in N} \frac{\{W^{-1}M^{-1}e\}_i}{\{W^{-1}M^{-1}\mathcal{M}(A)We\}_i}, \tag{3.15}$$

where M is defined in (2.2).

Proof Since A is an S-QN matrix, AW is a QN matrix. By Lemma 3.7

$$\|(AW)^{-1}\|_{\infty} \leq \max_{i \in N} \frac{\{W^{-1}M^{-1}e\}_i}{\{W^{-1}M^{-1}\mathcal{M}(A)We\}_i}.$$

Denote $A^{-1} = [a_{ij}^{(-1)}]$ and $R_i(A^{-1}) = \sum_{j=1}^n a_{ij}^{(-1)}$ (the i th row sum), then $\|A^{-1}\|_\infty$ and $\|(AW)^{-1}\|_\infty$ can be expressed as

$$\|A^{-1}\|_\infty = \max_i \{R_i(A^{-1})\}, \quad \|(AW)^{-1}\|_\infty = \max_i \{w_i^{-1} R_i(A^{-1})\},$$

respectively. Let $\delta = \min_i \{w_i^{-1}\}$, then

$$\delta \|A^{-1}\|_\infty = \delta \max_i \{R_i(A^{-1})\} \leq \max_i \{w_i^{-1} R_i(A^{-1})\} = \|(AW)^{-1}\|_\infty.$$

Hence

$$\|A^{-1}\|_\infty \leq \frac{1}{\delta} \max_{i \in N} \frac{\{W^{-1}M^{-1}e\}_i}{\{W^{-1}M^{-1}\mathcal{M}(A)We\}_i}.$$

□

Theorem 3.10 *Let $A = [a_{ij}] \in R^{n \times n}$, and $n \geq 2$ be an S-QN matrix with positive diagonal entries, if $W = \text{diag}(w_i) \in \mathcal{W}$, such that AW is a QN matrix, $\delta = \min\{w_i^{-1}\}$. Then*

$$\max_{d \in [0,1]^n} \|(I - \Lambda + \Lambda A)^{-1}\|_\infty \leq \frac{1}{\delta} \max_{i \in N} \frac{\{W^{-1}\beta\}_i}{\{e - \bar{\xi}\}_i}, \tag{3.16}$$

where $\bar{\xi} = W^{-1}M^{-1}|L||D|^{-1}|U|We$, M is defined in (2.2), $\beta = (\beta_1, \beta_2, \dots, \beta_n)^T$ with $\beta_n = \frac{\eta_n(A)}{\min\{a_{nn}, 1\}}$, $\beta_i = \frac{\eta_i(A)}{\min\{a_{ii}, 1\}} + \sum_{j=i+1}^n \frac{|a_{ij}|}{|a_{ii}|} \beta_j$, $i = n - 1, \dots, 1$, and $\eta_i(A)$ is defined in Lemma 3.6.

Proof By Lemmas 3.8 and 3.9, $\hat{A} = I - \Lambda + \Lambda A$ is an S-QN matrix, and

$$\|(I - \Lambda + \Lambda A)^{-1}\|_\infty \leq \frac{1}{\delta} \max_{i \in N} \frac{\{W^{-1}\hat{M}^{-1}e\}_i}{\{W^{-1}\hat{M}^{-1}\mathcal{M}(\hat{A})We\}_i}. \tag{3.17}$$

By (2.3)

$$\begin{aligned} W^{-1}\hat{M}^{-1}\mathcal{M}(\hat{A})We &= W^{-1}(I - \hat{M}^{-1}|\hat{L}||\hat{D}|^{-1}|\hat{U}|)We \\ &= (I - W^{-1}\hat{M}^{-1}|\hat{L}||\hat{D}|^{-1}|\hat{U}|)We \\ &= e - \hat{\xi}. \end{aligned}$$

Denote $z(\hat{A}) = (z_1(\hat{A}), z_2(\hat{A}), \dots, z_n(\hat{A}))^T$ given in Lemma 3.6. It follows that $|\hat{D}|(|\hat{D}| - |\hat{L}|)^{-1}e = z(\hat{A})$, and $\hat{M}^{-1}e = (|\hat{D}| - |\hat{U}|)^{-1}|\hat{D}|(|\hat{D}| - |\hat{L}|)^{-1}e = (|\hat{D}| - |\hat{U}|)^{-1}z(\hat{A})$, then

$$\|(I - \Lambda + \Lambda A)^{-1}\|_\infty \leq \frac{1}{\delta} \max_{i \in N} \frac{\{W^{-1}(|\hat{D}| - |\hat{U}|)^{-1}z(\hat{A})\}_i}{\{e - \hat{\xi}\}_i}. \tag{3.18}$$

Denote $y = (|\hat{D}| - |\hat{U}|)^{-1}z(\hat{A}) = (y_1, y_2, \dots, y_n)^T$, then $(|\hat{D}| - |\hat{U}|)y = z(\hat{A})$, and we can obtain

$$y_i = \frac{z_i(\hat{A})}{|\hat{a}_{ii}|} + \sum_{j=i+1}^n \frac{|\hat{a}_{ij}|}{|\hat{a}_{jj}|} y_j, j = n - 1, \dots, 1.$$

Next, we prove $y_i \leq \beta_i$ for each $i \in N$. In fact, by Lemma 3.6, for $i = n$,

$$y_n = \frac{z_n(\hat{A})}{\hat{a}_{nn}} \leq \frac{\eta_n(A)}{\min\{a_{nn}, 1\}} = \beta_n.$$

For $i = n - 1$,

$$y_{n-1} = \frac{z_{n-1}(\hat{A})}{\hat{a}_{n-1,n-1}} + \frac{|\hat{a}_{n-1,n}|}{|\hat{a}_{n-1,n-1}|} \cdot y_n \leq \frac{\eta_{n-1}(A)}{\min\{a_{n-1,n-1}, 1\}} + \frac{|a_{n-1,n}|}{|a_{n-1,n-1}|} \cdot \beta_n = \beta_{n-1}.$$

Similarly, for $i = n - 2, \dots, 2, 1$, we also have

$$y_i = \frac{z_i(\hat{A})}{\hat{a}_{ii}} + \sum_{j=i+1}^n \frac{|\hat{a}_{ij}|}{|\hat{a}_{ii}|} y_j \leq \frac{\eta_i(A)}{\min\{a_{ii}, 1\}} + \sum_{j=i+1}^n \frac{|a_{ij}|}{|a_{ii}|} \beta_j = \beta_i.$$

So $y_i \leq \beta_i$ holds for each $i \in N$. If we let $\beta = (\beta_1, \beta_2, \dots, \beta_n)^T$, then

$$W^{-1}\hat{M}^{-1}e = W^{-1}(|\hat{D}| - |\hat{U}|)^{-1}z(\hat{A}) \leq W^{-1}\beta,$$

therefore, by (3.18), we have

$$\begin{aligned} \|(I - \Lambda + \Lambda A)^{-1}\|_\infty &\leq \frac{1}{\delta} \max_{i \in N} \frac{\{W^{-1}\hat{M}^{-1}e\}_i}{\{W^{-1}\hat{M}^{-1}\mathcal{M}(\hat{A})We\}_i} \\ &\leq \frac{1}{\delta} \max_{i \in N} \frac{\{W^{-1}\beta\}_i}{\{e - \hat{\xi}\}_i} \leq \frac{1}{\delta} \max_{i \in N} \frac{\{W^{-1}\beta\}_i}{\{e - \bar{\xi}\}_i}. \end{aligned}$$

□

Remark 3.11 From Theorem 3.4 and Theorem 3.10, the bound (3.2) and the bound (3.16) both require a scaling matrix $W \in \mathcal{W}$, which is closely related with the set of indices S . Hence, for an S -QN matrix, the choice of the set of indices S will leads to different scaling matrices. Moreover, the obtained bounds will be different.

The following example shows that the error bound (3.2) with the parameter ϵ may be smaller than the error bound (3.16) if the proper ϵ is taken. However, it is hard to determine the optimal ϵ .

Example 3.2 Consider the matrix

$$A = \begin{bmatrix} 1 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{5}{7} & 1 & \frac{4}{7} & -\frac{3}{7} \\ 0 & -\frac{1}{2} & \frac{9}{8} & \frac{1}{8} \\ -\frac{1}{3} & 0 & 0 & \frac{4}{3} \end{bmatrix}.$$

It is easy to check that A is an S -QN matrix for $S = \{1, 3, 4\}$ and $W = \text{diag}(0.6, 1, 0.6, 0.6)$. By Theorem 3.4, we can get $\bar{\xi} = (0.5445, 0.7127, 0.7609, 0.2292)^T$, $\epsilon \in (0, 0.2233)$, and

$$P = \text{diag}(0.3267 + 0.6\epsilon, 0.7127, 0.4565, 0.1375).$$

Then, the error bound (3.2) in Theorem 3.4 with respect to ϵ is drawn in Fig. 2. Furthermore, by Theorem 3.10, we can calculate that $\beta = (2.6926, 3.4316, 2.0053, 1.3333)^T$ and $\delta = 1$, so the new error bound (3.16) is

$$\max_{i \in N} \frac{1}{\delta} \cdot \frac{\{W^{-1}\beta\}_i}{\{e - \bar{\xi}\}_i} = 13.9783.$$

The Example 3.3 illustrates that the error bound (3.16) can improve the error bound (3.2) in some special cases.

Example 3.3 Consider the matrix

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 0.5 & 2 & 0.5 \\ 0 & -1 & 1 \end{bmatrix}.$$

Observe that A is an S -QN matrix for $S = \{2, 3\}$ and $W = \text{diag}(1, 0.5, 0.5)$. By Theorem 3.4, we can get $\bar{\xi} = (0.6875, 0.6875, 0.75)^T$, $\epsilon \in (0, 0.3125)$, and

$$P = \text{diag}(0.6875 + \epsilon, 0.3438, 0.3750).$$

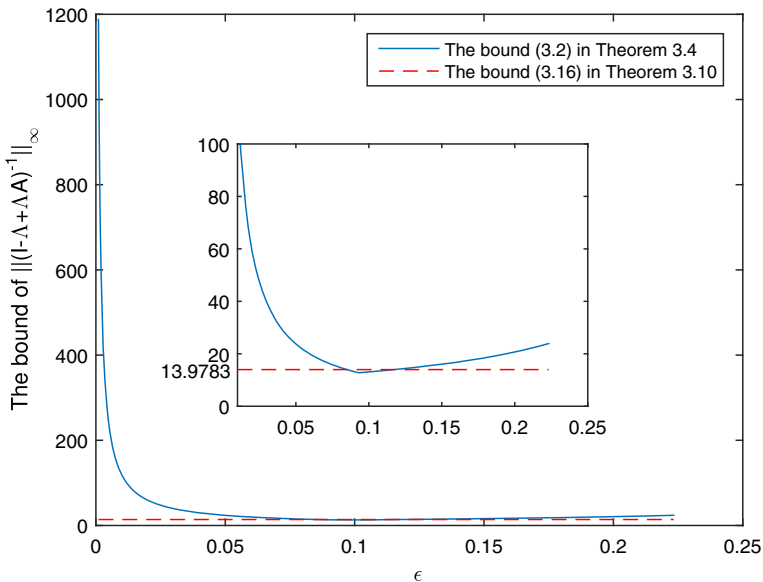


Fig. 2 The bounds of $\max_{d \in [0, 1]^n} \|(I - \Lambda + \Lambda A)^{-1}\|_\infty$

Then the error bound (3.2) in Theorem 3.4 with respect to ϵ is drawn in Fig. 3. By Theorem 3.10, we can calculate that $\beta = (5.2500, 2.1250, 2.500)^T$ and $\delta = 1$, so the new error bound (3.16) is

$$\max_{i \in N} \frac{1}{\delta} \cdot \frac{\{W^{-1}\beta\}_i}{\{e - \bar{\xi}\}_i} = 20,$$

which is smaller than the bound (3.2) for each $\epsilon \in (0, 0.3125)$. So, the bound (3.16) is better than the bound (3.2).

For an S -QN matrix $A = (a_{ij})$, if there exists some $i \in N$ such that $a_{ij} = 0$ for any $j > i$ or $j < i$, then the bound (3.2) can not be used. However, the bound (3.16) may have a nice performance such as the following example.

Example 3.4 Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & -0.5 & 1 \end{bmatrix}.$$

It is easy to verify that A is an S -QN matrix for $S = \{2, 3\}$ and $W = \text{diag}(1, 0.6, 0.6)$, we can calculate that $\bar{\xi} = (0.6000, 0.5000, 0.5000)^T$. Since $a_{21} = 0$, A does not satisfy the hypothesis of Theorem 3.4, and the error bound (3.2) can

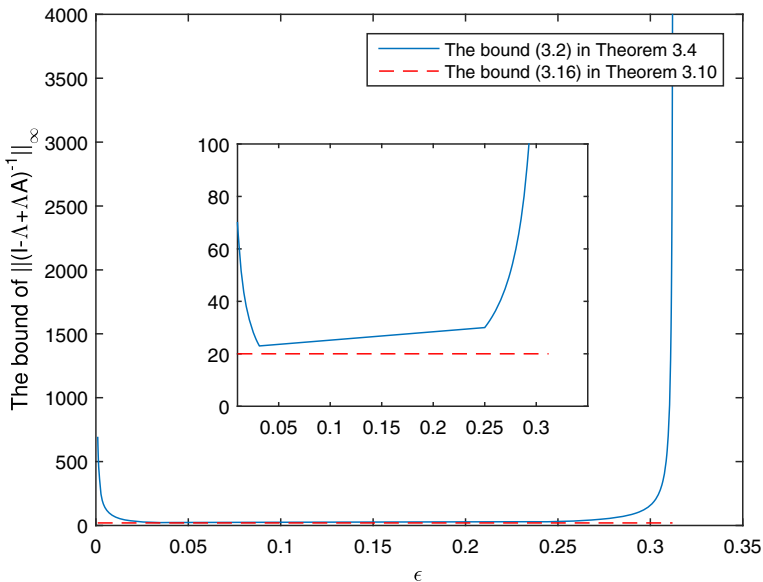


Fig. 3 The bounds of $\max_{\alpha \in [0, 1]^n} \|(I - \Lambda + \Lambda A)^{-1}\|_\infty$

not be used to estimate $\max_{d \in [0,1]^n} \|(I - \Lambda + \Lambda A)^{-1}\|_\infty$. However, by Theorem 3.10, we can get that $\beta = (4, 2.5, 1.5)^T$ and $\delta = 1$, then the error bound (3.16) is

$$\max_{i \in N} \frac{1}{\delta} \cdot \frac{\{W^{-1}\beta\}_i}{\{e - \bar{\xi}\}_i} = 10,$$

which implies that the bound (3.16) in Theorem 3.10 can be more effective and has wider application than the bound (3.2).

It is obvious that the bounds we obtained can be applied to the subclass of S -QN matrices. We give an SDD matrix in the following example and compare the pervious bounds with the bound (3.2) and the bound (3.16). The results illustrate that the bound we obtained have nice performance.

Example 3.5 Consider the matrix

$$A = \begin{bmatrix} 3 & -1.9 & -1 \\ -1 & 3 & -1.9 \\ -2 & 0 & 3 \end{bmatrix}.$$

Observe that A is an SDD matrix, then a Nekrasov matrix, a QN matrix and an S -QN matrix for any set of indices S . Here, we choose $S = \{1, 2\}$ and $W = \text{diag}(1.2, 1.2, 1)$. Since A is an SDD matrix, it is obvious that $I - \Lambda + \Lambda A$ is still an SDD matrix. By Lemma 3.5, we apply the Varah bound [21] to the inverse of $I - \Lambda + \Lambda A$, then the bound is

$$\begin{aligned} \max_{d \in [0,1]^n} \|(I - \Lambda + \Lambda A)^{-1}\|_\infty &\leq \frac{1}{\min_{i \in N} \{1 - d_i + d_i |a_{ii} - d_i r_i(A)\}} \\ &\leq \frac{1}{\min_{i \in N} \{|a_{ii} - r_i(A)\}} = 10. \end{aligned}$$

Since A is a Nekrasov matrix, then $a_{ii} > h_i(A)$ for all $i \in N$, where $h(A) = (2.9, 2.8667, 1.9333)^T$. By Lemma 3.6, we get $\eta(A) = (1, 1.3333, 1.6667)^T$. Then the bound in [12] for Nekrasov matrices is

$$\max_{i \in N} \frac{\eta_i(A)}{\min\{a_{ii} - h_i(A), 1\}} = 10.0023.$$

Moreover, we can calculate that $\xi = (0.6774, 0.7304, 0.6444)^T$ and $\beta = (3.0685, 2.3889, 1.6667)^T$. Then, the bound in [9] for QN matrices is

$$\max_{i \in N} \frac{\beta_i}{\{e - \xi\}_i} = 9.5118.$$

By Theorem 3.10, we get that $\bar{\xi} = (0.6385, 0.6884, 0.7289)^T$ and $\delta = \frac{1}{1.2}$. Then the error bound (3.16) is

$$\max_{i \in N} \frac{1}{\delta} \cdot \frac{\{W^{-1}\beta\}_i}{\{e - \bar{\xi}\}_i} = 8.4882.$$

By Theorem 3.4, if we choose $\epsilon = 0.1808$, then

$$P = \text{diag}(0.9832, 0.8261, 0.7289), (t_1, t_2, t_3)^T = (0.6509, 0.8764, 1.7528)^T.$$

So the bound (3.2) is

$$\max \left\{ \frac{\max_i \{p_i\}}{\min_i \{t_i\}}, \frac{\max_i \{p_i\}}{\min_i \{p_i\}} \right\} = 1.5105.$$

Obviously, the bound (3.2) and the bound (3.16) are smaller than other bounds, which means that the bounds we obtained not only can be applied to a wider class of matrices, but also give a sharper bound for some well-known types of matrices.

4 Conclusions

In this paper, we focus on some classes of special matrices and the error bound for linear complementarity problems. First, we put forward a new subclass of H -matrices, called S -QN matrices, which properly generalizes the class of QN matrices. Then, for an S -QN matrix A , we present two kinds of error bounds for $LCP(A, q)$ to estimate the upper bound of $\max_{d \in [0, 1]^n} \|(I - \Lambda + \Lambda A)^{-1}\|_\infty$, called the Error Bound I and the Error Bound II:

- the Error Bound I: $\max_{d \in [0, 1]^n} \|(I - \Lambda + \Lambda A)^{-1}\|_\infty \leq \max \left\{ \frac{\max_i \{p_i\}}{\min_i \{t_i\}}, \frac{\max_i \{p_i\}}{\min_i \{p_i\}} \right\}$ (in Theorem 3.4);
- the Error Bound II: $\max_{d \in [0, 1]^n} \|(I - \Lambda + \Lambda A)^{-1}\|_\infty \leq \frac{1}{\delta} \max_{i \in N} \frac{\{W^{-1}\beta\}_i}{\{e^{-\frac{\delta}{\xi}}\}_i}$ (in Theorem 3.10).

The Error Bound I is the generalization of the error bound for LCPs of QN matrices but it only applies to some special cases. The Error Bound II can be used for every S -QN matrix. Numerical examples illustrate that the two error bounds have nice performances in some cases.

Acknowledgments The authors are thankful to the anonymous referees for their valuable comments to improve the paper.

Funding information The work was supported by the National Natural Science Foundation of China (11671318).

References

1. Berman, A., Plemmons, R.J.: Nonnegative Matrix in the Mathematical Science. Academic Press, New York (1979)
2. Cottle, R.W., Pang, J.S., Stone, R.E.: The Linear Complementarity Problem. Academic Press, San Diego (1992)
3. Schäfer, U.: A linear complementarity problem with a P -Matrix[J]. Siam Review **46**(2), 189–201 (2004)
4. Feng, L., Linetsky, V., Morales, J.L., et al.: On the solution of complementarity problems arising in American options pricing[J]. Optim. Methods Softw. **26**(4–5), 813–825 (2011)

5. Chen, X., Xiang, S.: Computation of error bounds for p -matrix linear complementarity problems[J]. Math. Program. **106**(3), 513–525 (2006)
6. García-Esnaola, M., Peña, J.M.: A comparison of error bounds for linear complementarity problems of H -matrices. Linear Algebra Appl. **433**, 956–964 (2010)
7. Kolotilina, L.Y.: Bounds for the inverses of generalized Nekrasov matrices[J]. J. Math. Sci. **207**(5), 786–794 (2015)
8. Dai, P.F., Li, C.J., Li, Y.T., Zhang, C.Y.: Error bounds for linear complementarity problems of QN-matrices. Calcolo **53**, 647–657 (2016)
9. Gao, L., Wang, Y., Li, C.: New error bounds for the linear complementarity problem of QN-matrices[J]. Numer. Algorithms, pp. 1–14 (2017)
10. Sulc, T., Cvetković, L., Nedović, M.: Scaling technique for Partition-Nekrasov matrices[J]. Appl. Math. Comput. **271**(C), 201–208 (2015)
11. Li, C.Q., Li, Y.T.: Note on error bounds for linear complementarity problems for B -matrices[J]. Appl. Math. Lett. **57**, 108–113 (2016)
12. Li, C.Q., Dai, P.F., Li, Y.T.: New error bounds for linear complementarity problems of Nekrasov matrices and B -Nekrasov matrices[J]. Numer. Algorithms, pp. 1–13 (2016)
13. Cvetković, L., Kostić, V., Rauski, S.: A new subclass of H -matrices.[J]. Appl. Math. Comput. **208**(1), 206–210 (2009)
14. Cvetković, L., Kostić, V., Varga, R.S.: A new gersgorin-type eigenvalue inclusion set *[J]. Electronic Transactions on Numerical Analysis Etna **302**(5906), 73–80 (2004)
15. Fiedler, M., Ptak, V.: Generalized norms of matrices and the location of the Spectrum[J]. Czechoslov. Math. J. **12**(87), 558–571 (1962)
16. Dai, P., Li, Y.T., Lu, C.J.: Erratum to: error bounds for linear complementarity problems for SB -matrices[J]. Numer. Algorithms **61**(1), 187–187 (2012)
17. García-Esnaola, M., Peña, J.M.: Error bounds for linear complementarity problems of Nekrasov matrices. Numer. Algorithms **67**, 655–667 (2014)
18. Dehghan, M., Hajarian, M.: Convergence of SSOR methods for linear complementarity problems[J]. Oper. Res. Lett. **37**(3), 219–223 (2009)
19. Li, Y., Dai, P.: Generalized AOR methods for linear complementarity problem[J]. Appl. Math. Comput. **188**(1), 7–18 (2007)
20. Hadjidimos, A., Tzoumas, M.: The solution of the linear complementarity problem by the matrix analogue of the accelerated overrelaxation iterative method[J]. Numer. Algorithms, pp. 1–20 (2016)
21. Varah, J.M.: A lower bound for the smallest singular value of a matrix[J]. Linear Algebra Appl. **11**(1), 3–5 (1975)

Publisher's note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.